

Super connectivity of direct product of graphs*

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Received 15 July 2012, accepted 14 June 2014, published online 23 January 2015

Abstract

For a graph G , $\kappa(G)$ denotes its connectivity. A graph G is *super connected*, or simply *super- κ* , if every minimum separating set is the neighborhood of a vertex of G , that is, every minimum separating set isolates a vertex. The *direct product* $G_1 \times G_2$ of two graphs G_1 and G_2 is a graph with vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. Let $\Gamma = G \times K_n$, where G is a non-trivial graph and $K_n (n \geq 3)$ is a complete graph on n vertices. In this paper, we show that Γ is not super- κ if and only if either $\kappa(\Gamma) = n\kappa(G)$, or $\Gamma \cong K_{\ell, \ell} \times K_3 (\ell > 0)$.

Keywords: Super connectivity, direct product, vertex-cut.

Math. Subj. Class.: 05C40, 05C76

1 Introduction

Throughout this paper only undirected simple connected graphs without loops and multiple edges are considered. Unless stated otherwise, we follow Bondy and Murty [4] for terminology and definitions.

Let $G = (V(G), E(G))$ be a graph. For two vertices $u, v \in V(G)$, $u \sim v$ means that u is adjacent to v and uv is the edge incident to u and v in G . The set of vertices adjacent to the vertex v is called the *neighborhood* of v and denoted by $N_G(v)$, i.e., $N_G(v) = \{u \mid uv \in E(G)\}$. The *degree* of v is equal to $|N_G(v)|$, denoted by $d_G(v)$. The number $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the *minimum degree* of G . For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. As usual, $K_{m,m}$, (m is a positive integer) denotes the complete bipartite graph; $K_{m,m} - mK_2$ denotes the graph obtained by removing a 1-factor from $K_{m,m}$; K_n denotes the complete graph on n vertices; and \mathbb{Z}_n denotes the ring of integers modulo n .

*This work was supported by the National Natural Science Foundation of China (11271012).

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A *separating set* of a graph G is a set of vertices whose deletion either disconnects G or reduces G to the trivial graph K_1 . The *connectivity* of the graph G is the minimum number of vertices in a separating set of G , and will be denoted by $\kappa(G)$. In particular, $\kappa(K_n) = n - 1$, and $\kappa(G) = 0$ if and only if G is disconnected or a K_1 . Clearly, $\kappa(G) \leq \delta(G)$. A graph G with minimum degree $\delta(G)$ is *maximally connected* if $\delta(G) = \kappa(G)$.

An interconnection network is often modeled as a graph G , where $V(G)$ is the set of processors and $E(G)$ is the set of communication links in the network. The connectivity $\kappa(G)$ of G is an important measurement for fault-tolerance of the network, and the larger $\kappa(G)$ is, the more reliable the network is. As more refined indices of reliability than connectivity, super connectivity was proposed in [2, 3]. A graph G is *super connected*, *super- κ* , for short if every minimum separating set isolates a vertex of G .

The *direct product* $G_1 \times G_2$ of two graphs G_1 and G_2 is defined as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. The direct product is also called the Kronecker product, tensor product, cross product, categorical product, or conjunction. As an operation on binary relations, the direct product was introduced by Whitehead and Russell in their Principia Mathematica [21]. It is also equivalent to the direct product of the adjacency matrices of the graphs (see [20]). As one of the four standard graph products [11], the direct product has been studied from several points of view (see, for example, [1, 6, 8, 12, 13, 15]).

The connectivity of the direct product of graphs has also been investigated in several recent publications. For example, Brešar and Špacapan [7] obtained an upper bound and a lower bound on the edge-connectivity of the direct products with some exceptions, and they also obtained several upper bounds on the vertex-connectivity of the direct products of graphs. Mamut and Vumar [14] proved that $\kappa(K_m \times K_n) = (m - 1)(n - 1)$ where $m \geq n \geq 2$. In [9], it was shown that if $n \geq 3$ and G is a bipartite graph, then $\kappa(G \times K_n) = \min\{n\kappa(G), (n - 1)\delta(G)\}$, and furthermore, the authors also conjectured that this is true for all nontrivial graph G . Later, this conjecture was confirmed independently by Wang and Wu [17] and Wang and Xue [18].

More recently, several papers dealing with the super-connectivity of direct product of graphs were published. Guo et al. [10] showed that for a bipartite graph G with $\kappa(G) = \delta(G)$, $G \times K_n (n \geq 3)$ is super- κ . In [19], the authors generalized this result by showing that for a nonbipartite graph G with $\kappa(G) = \delta(G)$, $G \times K_n (n \geq 3)$ is super- κ . In [19], the authors also pointed out that Guo et al.'s result is not true when $G = K_{\ell, \ell} (\ell \geq 1)$ and $n = 3$, and they also claimed that except for this case, Guo et al.'s statement is true.

The aim of this article is to determine all graphs G such that $G \times K_n (n \geq 3)$ is not super- κ . The following is the main result.

Theorem 1.1. *Let $\Gamma = G \times K_n$, where $n \geq 3$ and G is a non-trivial graph. Then Γ is not super- κ if and only if one of the following happens.*

- (1) G has a minimum separating set T so that $T \times V(K_n)$ is a minimum separating set of Γ . In particular, $\kappa(\Gamma) = n\kappa(G)$.
- (2) $\Gamma \cong K_{\ell, \ell} \times K_3 (\ell > 0)$.

From Theorem 1.1 we can immediately obtain the following corollaries.

Corollary 1.2. [17, 18] *Let $\Gamma = G \times K_n$, where $n \geq 3$ and G is a non-trivial graph. Then $\kappa(\Gamma) = \min\{n\kappa(G), (n - 1)\delta(G)\}$.*

Corollary 1.3. [10, 19] *For a maximally connected graph G , $G \times K_n (n \geq 3)$ is not super- κ if and only if $n = 3$ and $G \cong K_{\ell, \ell} (\ell > 0)$.*

2 Proof of Theorem 1.1

We start by introducing some notations.

Notations.

- $\Gamma := G \times K_n$, where $n \geq 3$ and G is a non-trivial graph.
- $V(G) := \{u_i \mid i \in \mathbb{Z}_m\}$.
- $V(K_n) := \mathbb{Z}_n$.
- $V_i := \{u_i\} \times V(K_n)$, $i \in \mathbb{Z}_m$.
- S : a minimum separating set of Γ .
- $\Gamma - S := \bigcup_{i=0}^{s-1} \Gamma_i$, where each Γ_i is a connected component of $\Gamma - S$.
- $W_i := V(\Gamma_i)$.

In the following Lemmas 2.1-2.5, we assume that Γ is not super connected, and S is a minimum separating set of Γ with each component Γ_i of $\Gamma - S$ having at least two vertices.

By the definition, we can obtain the following easy facts.

- Lemma 2.1.** (1) $\delta(\Gamma) = (n - 1)\delta(G)$.
- (2) *For any $i \in \mathbb{Z}_m$, V_i is an independent subset of $V(\Gamma)$.*
- (3) *If u_{i_0} is adjacent to v_{i_1} in G , then $(u_{i_0}, j) \sim (u_{i_1}, k)$ (in Γ) if and only if $j \neq k$. In particular, $\Gamma[V_{i_0} \cup V_{i_1}] \cong K_{n,n} - nK_2$.*
- (4) *Let T be a separating set of G . Then $T \times V(K_n)$ is also a separating set of Γ . In particular, $|S| = \kappa(\Gamma) \leq \min\{n\kappa(G), (n - 1)\delta(G)\}$.*
- (5) $s \geq 2$ and $|W_i| \geq 2$ for each $i \in \mathbb{Z}_s$.

Lemma 2.2. *For each $(u_i, j) \in S$, (u_i, j) has at least one neighbor in W_i for each $i \in \mathbb{Z}_s$.*

Proof. Suppose to the contrary that (u_i, j) has no neighbors in W_i for some $i \in \mathbb{Z}_s$. Set $S' = S - \{(u_i, j)\}$. Then W_i must be a component of $\Gamma - S'$. This implies that S' is also a separating set of Γ , contrary to the minimality of S . □

Lemma 2.3. *For two components W_k, W_ℓ , if there exist $(u_i, i') \in W_k$ and $(u_j, j') \in W_\ell$ such that $u_i \sim u_j$ (in G), then $i' = j'$ and $W_k \cap V_i = \{(u_i, i')\}$ and $W_\ell \cap V_j = \{(u_j, j')\}$.*

Proof. Since $u_i \sim u_j$ (in G), it follows from Lemma 2.1 (3) that $\Gamma[V_i \cup V_j] \cong K_{n,n} - nK_2$. As $\Gamma - S$ is disconnected, there are no edges between W_k and W_ℓ . Consequently, $i' = j'$ and $W_k \cap V_i = \{(u_i, i')\}$ and $W_\ell \cap V_j = \{(u_j, j')\}$. □

Lemma 2.4. *Assume that for each V_i there exists at most one W_j such that $V_i \cap W_j \neq \emptyset$. Then $S = T \times V(K_n)$, where T is a minimum separating set of G . In particular, $\kappa(\Gamma) = n\kappa(G)$.*

Proof. We shall first show the following two claims.

Claim 1 If there exists an $i \in \mathbb{Z}_m$ such that $V_i \cap S \neq \emptyset$ and $V_i \not\subseteq S$, then $|V_i \cap S| = n - 1$. Furthermore, for each W_j , there is a V_ℓ such that $|W_j \cap V_\ell| = 1$.

By the assumption, there is a unique $j \in \mathbb{Z}_s$ such that $V_i \cap W_j \neq \emptyset$. By Lemma 2.2, for each vertex, say (u_i, i') , in $V_i \cap S$, there is at least one neighbor, say (u_ℓ, ℓ') , in each W_t with $t \neq j$. By Lemma 2.3, $|V_i \cap W_j| = |V_\ell \cap W_t| = 1$. From our assumption we know that $|V_i \cap S| = |V_\ell \cap S| = n - 1$.

Claim 2 For each $i \in \mathbb{Z}_m$, either $V_i \cap S = \emptyset$ or $V_i \subseteq S$.

Suppose on the contrary that there exists an $i \in \mathbb{Z}_m$ such that $V_i \cap S \neq \emptyset$ and $V_i \not\subseteq S$. For each $j \in \mathbb{Z}_s$, let $\Omega_j = \{\ell \in \mathbb{Z}_m \mid |V_\ell \cap W_j| = 1\}$, and set $n_j = |\Omega_j|$. By Claim 1, $n_j > 0$. Without loss of generality, assume that $n_0 \leq n_1 \leq \dots \leq n_{s-1}$.

Assume that $W_0 \subseteq \bigcup_{\ell \in \Omega_0} V_\ell$. Then for each $(u_i, \ell) \in W_0$, we have $|V_i \cap W_0| = 1$. Combining this with Lemma 2.3, we have for a fixed $(u_i, \ell) \in W_0$, if $u_k \sim u_i$ (in G), then $|V_k \cap S| \geq n - 1$. As $n \geq 3$, one has

$$|S| \geq |V_i \cap S| + \sum_{u_j \in N_G(u_i)} |V_j \cap S| \geq n - 1 + \delta(G)(n - 1) > \kappa(\Gamma).$$

A contradiction occurs.

Now assume that $W_0 \not\subseteq \bigcup_{\ell \in \Omega_0} V_\ell$. Let $U = \bigcup_{V_i \subseteq S} V_i$, $Z_0 = \bigcup_{\ell \in \Omega_0} V_\ell$, and $Z_1 = \bigcup_{\ell \in \Omega_1} V_\ell$. Set $T = U \cup Z_0$. Clearly, $|Z_0 \cap W_0| = n_0$. Since $n_1 \geq n_0$ and $n \geq 3$, one has $|Z_0 \cap W_0| = n_0 < n_1(n - 1) = |Z_1 \cap S|$. Then

$$\begin{aligned} |T| = |U| + |Z_0| &= |U| + |Z_0 \cap S| + |Z_0 \cap W_0| \\ &< |U| + |Z_0 \cap S| + |Z_1 \cap S| \\ &\leq |S|. \end{aligned}$$

Since S is a minimum separating set, $\Gamma - T$ is connected. So there is an edge between $W_0 \setminus T$ and $V(\Gamma) \setminus (T \cup W_0)$. We may assume that $(v_i, j) \in W_0 \setminus T$ is adjacent to $(v_s, k) \in V(\Gamma) \setminus (T \cup W_0)$. Obviously, $(v_s, k) \in S \setminus T$. Since $U = \bigcup_{V_i \subseteq S} V_i$ and $T = U \cup Z_0$, one has $V_s \not\subseteq S$. If $V_s \cap W_0 \neq \emptyset$, then by Claim 1, we must have $|V_s \cap W_0| = 1$ and so $V_s \subseteq Z_0 \subseteq T$. This contradicts the fact that $(v_s, k) \in S \setminus T$. Consequently, $V_s \cap W_0 = \emptyset$. It follows that $V_s \cap W_t \neq \emptyset$ for some $t > 0$. Since $(v_i, j) \sim (v_s, k)$, by Lemma 2.3, $|V_i \cap W_0| = 1$ and so $V_i \subseteq Z_0 \subseteq T$. This contradicts the fact that $(v_i, j) \in W_0 \setminus T$.

Now we are ready to finish the proof. From Claim 2 it follows that $S = T \times V(K_n)$ for some subset T of $V(G)$. Since $n \geq 3$, T is a separating set of G (see [20]). So, $|S| \geq \kappa(G)n$. However, by Lemma 2.1 (4), $|S| \leq n\kappa(G)$. Hence, $|S| = n\kappa(G)$. \square

Lemma 2.5. Assume that there exist a V_i and two different W_{j_0}, W_{j_1} such that $V_i \cap W_k \neq \emptyset$ with $k = j_0, j_1$. Then $n = 3$ and $G \cong K_{\ell, \ell} (\ell > 0)$.

Proof. Recall that $W_k = V(\Gamma_k)$ with $k = j_0$ or j_1 . We shall finish the proof by the following claims.

Claim 1 $V(\Gamma) = W_{j_0} \cup W_{j_1} \cup S$, $|V_i \cap W_{j_0}| = |V_i \cap W_{j_1}| = 1$ and $|V_i \cap S| = n - 2$.

By Lemma 2.1 (2), V_i is an independent subset, and by Lemma 2.1 (5), $|W_k| \geq 2$ with $k = j_0$ or j_1 . It follows that $V_i \cap W_k \subset W_k$. Since Γ_{j_0} is connected, there exist $(u_i, t_0) \in V_i \cap W_{j_0}$ and $(u_{i_0}, t'_0) \in W_{j_0} \setminus (V_i \cap W_{j_0})$ such that $(u_i, t_0) \sim (u_{i_0}, t'_0)$. Similarly, there exist $(u_i, t_1) \in V_i \cap W_{j_1}$ and $(u_{i_1}, t'_1) \in W_{j_1} \setminus (V_i \cap W_{j_1})$ such that $(u_i, t_1) \sim (u_{i_1}, t'_1)$. From Lemma 2.3 we obtain that $t'_0 = t_1$, $V_i \cap W_{j_0} = \{(u_i, t_0)\}$ and $V_{i_0} \cap W_{j_0} = \{(u_{i_0}, t_1)\}$, and $t'_1 = t_0$, $V_i \cap W_{j_1} = \{(u_i, t_1)\}$ and $V_{i_1} \cap W_{j_1} = \{(u_{i_1}, t_0)\}$ (see Figure 1). In particular, we have $|V_i \cap W_{j_0}| = |V_i \cap W_{j_1}| = 1$.

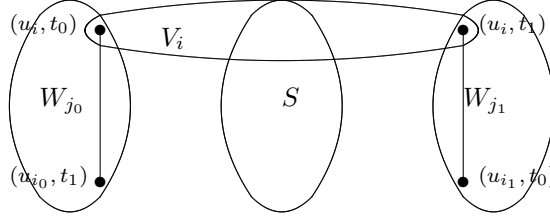


Figure 1: Explanation of the proof of Claim 1

It follows that $|V_i \cap S| \leq n - 2$. If $|V_i \cap S| < n - 2$, then we would have $V_i \cap W_j \neq \emptyset$ for some $j \neq j_0, j_1$. Take $(u_i, t) \in V_i \cap W_j$. Clearly, $t \neq t_0, t_1$. This forces that $(u_i, t) \sim (u_{i_0}, t_1)$, contrary to the fact that Γ_{j_0} and Γ_j are two distinct components. Thus, $|V_i \cap S| = n - 2$.

At last, we shall show that $s = 2$. Suppose to the contrary that $s > 2$. Since $|V_i \cap S| = n - 2 > 0$, we can take $(u_i, j) \in V_i \cap S$. By Lemma 2.2, (u_i, j) has a neighbor, say (u_k, j') in each W_j with $j \neq j_0, j_1$. Since $t_0 \neq t_1$, either $(u_k, j') \sim (u_i, t_0)$ or $(u_k, j') \sim (u_i, t_1)$. This is again contrary to the fact that $\Gamma_{j_0}, \Gamma_{j_1}$ and Γ_j are three distinct components. Thus, $s = 2$ and hence $V(\Gamma) = W_{j_0} \cup W_{j_1} \cup S$.

By Claim 1, we may assume that $V_i \cap W_{j_0} = \{(u_i, t_0)\}$ and $V_i \cap W_{j_1} = \{(u_i, t_1)\}$.

Claim 2 For each $(u_j, \ell) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$, $|V_j \cap S| = n - 1$ or $n - 2$. There is at least one $(u_{i_0}, \ell_0) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$ such that $|V_{i_0} \cap S| = n - 2$.

Take $(u_j, \ell) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$. From Claim 1 we see that $V_i \cap W_{j_1} = \{(u_i, t_1)\}$. By Lemma 2.3, we have $|V_j \cap W_{j_0}| = 1$, implying $|V_j \cap S| \leq n - 1$. If $|V_j \cap S| < n - 1$ then we must have $V_j \cap W_{j_1} \neq \emptyset$. By Claim 1, we have $|V_j \cap S| = n - 2$. Therefore, $|V_j \cap S| = n - 1$ or $n - 2$.

Suppose that for each $(u_j, \ell) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$, we have $|V_j \cap S| = n - 1$. Noting that $n \geq 3$, one has

$$|S| \geq |V_i \cap S| + \sum_{u_j \in N_G(u_i)} |U_j \cap S| \geq n - 2 + \delta(G)(n - 1) > \kappa(\Gamma),$$

a contradiction. Thus, there is at least one $(u_{i_0}, \ell_0) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$ such that $|V_{i_0} \cap S| = n - 2$.

Now we know that Claim 2 holds. Since $(u_{i_0}, \ell_0) \in N_\Gamma((u_i, t_0)) \cap W_{j_0}$, it follows from Lemma 2.3 that $V_{i_0} \cap W_{j_0} = \{(u_{i_0}, t_1)\}$ and $V_{i_0} \cap W_{j_1} = \{(u_{i_0}, t_0)\}$ (see Figure 2). By the arbitrariness of V_i , Claims 1,2 also hold if we replace V_i by V_{i_0} .

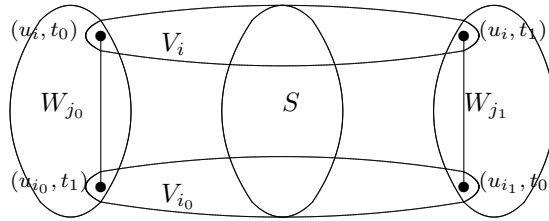


Figure 2: Explanation of Claims 1,2

For the convenience of statement, we shall use the following notations in the remainder of the proof.

Notations

- (1) $N_i = N_\Gamma((u_i, t_0)) \cap W_{j_0}$,
- (2) $N_{i_0} = N_\Gamma((u_{i_0}, t_1)) \cap W_{j_0}$,
- (3) $\Omega_i = \{k \in \mathbb{Z}_m \mid V_k \cap N_i \neq \emptyset\}$,
- (4) $\Omega_{i_0} = \{k \in \mathbb{Z}_m \mid V_k \cap N_{i_0} \neq \emptyset\}$,
- (5) $\Delta_i = \{k \in \mathbb{Z}_m \mid k \notin \Omega_i, V_k \cap N_\Gamma((u_i, t_0)) \neq \emptyset\}$,
- (6) $\Delta_{i_0} = \{k \in \mathbb{Z}_m \mid k \notin \Omega_{i_0}, V_k \cap N_\Gamma((u_{i_0}, t_1)) \neq \emptyset\}$.

It is easy to see that $N_G(u_i) = \{u_k \mid k \in \Omega_i \cup \Delta_i\}$ and $N_G(u_{i_0}) = \{u_k \mid k \in \Omega_{i_0} \cup \Delta_{i_0}\}$. Hence, $|\Omega_i| + |\Delta_i| = d_G(u_i)$ and $|\Omega_{i_0}| + |\Delta_{i_0}| = d_G(u_{i_0})$.

Claim 3 $|N_i| = |\Omega_i|$ and $|N_{i_0}| = |\Omega_{i_0}|$.

By Claim 1, for each $k \in \Omega_i$, we have $|V_k \cap W_{j_0}| = 1$. It follows that $|N_i| = |\Omega_i|$. Similarly, $|N_{i_0}| = |\Omega_{i_0}|$.

Claim 4 Both N_i and N_{i_0} are independent subsets of $V(\Gamma)$.

Take any two vertices, say (u_{i_1}, t) , (u_{i_2}, t') in N_i . Since $V_i \cap W_{j_1} = \{(u_i, t_1)\}$, from Lemma 2.3 it follows that $t = t' = t_1$. So, (u_{i_1}, t) is not adjacent to (u_{i_2}, t') . Therefore, N_i is an independent subset of $V(\Gamma)$. Similarly, N_{i_0} is also an independent subset.

Claim 5 $(\bigcup_{k \in \Omega_i \cup \Delta_i} V_k) \cap (\bigcup_{k \in \Omega_{i_0}} V_k) = \emptyset$ and $(\bigcup_{k \in \Omega_{i_0} \cup \Delta_{i_0}} V_k) \cap (\bigcup_{k \in \Omega_i} V_k) = \emptyset$.

Suppose that $(\bigcup_{k \in \Omega_i \cup \Delta_i} V_k) \cap (\bigcup_{k \in \Omega_{i_0}} V_k) \neq \emptyset$. Take $(u_j, t) \in (\bigcup_{k \in \Omega_i \cup \Delta_i} V_k) \cap (\bigcup_{k \in \Omega_{i_0}} V_k)$. Then $V_j \cap N_{i_0} \neq \emptyset$, implying that $V_j \cap W_{j_0} \neq \emptyset$. Assume $(u_j, t') \in V_j \cap W_{j_0}$. Clearly, u_i and u_{i_0} are neighbors of u_j in G . Since $t_0 \neq t_1$, either $(u_j, t') \sim (u_i, t_1)$ or $(u_j, t') \sim (u_{i_0}, t_0)$. This is contrary to the fact that there are no edges between W_{j_0} and W_{j_1} . Thus, $(\bigcup_{k \in \Omega_i \cup \Delta_i} V_k) \cap (\bigcup_{k \in \Omega_{i_0}} V_k) = \emptyset$. Similarly, we have $(\bigcup_{k \in \Omega_{i_0} \cup \Delta_{i_0}} V_k) \cap (\bigcup_{k \in \Omega_i} V_k) = \emptyset$.

Claim 6 Let $k \in \Delta_i \cup \Delta_{i_0}$. Then $V_k \cap W_{j_0} = \emptyset$ and $|V_k \cap S| \geq n - 1$.

Assume $k \in \Delta_i$. Then $u_k \in N_G(u_i)$. If $V_k \cap W_{j_0} \neq \emptyset$, then take $(u_k, t) \in V_k \cap W_{j_0}$. Since $k \notin \Omega_i$, (u_k, t) is not adjacent to (u_i, t_0) , and hence $t = t_0$. Consequently, $(u_k, t) \sim (u_i, t_1)$, a contradiction. Thus, $V_k \cap W_{j_0} = \emptyset$. By Lemma 2.3, $|V_k \cap W_{j_1}| \leq 1$, and hence $|V_k \cap S| \geq n - 1$. With a similar argument, we can show that if $k \in \Delta_{i_0}$, then $V_k \cap W_{j_0} = \emptyset$ and $|V_k \cap S| \geq n - 1$.

Claim 7 (1) $n = 3$; (2) $|N_i| = |N_{i_0}| = \delta(G)$; (3) $|\Delta_i| = |\Delta_{i_0}| = 0$; (4) $|S| = 2\delta(G)$; (5) $S = \bigcup_{k \in \Omega_i \cup \Omega_{i_0}} (V_k \cap S)$; (6) for each $k \in \Omega_i \cup \Omega_{i_0}$, $|V_k \cap S| = |V_k \cap W_{j_0}| = |V_k \cap W_{j_1}| = 1$.

By the arbitrariness of V_i and V_{i_0} , we may assume that $|N_i| \leq |N_{i_0}|$. By Claim 5,

$$|S| \geq \left| \bigcup_{k \in \Omega_{i_0}} (V_k \cap S) \right| + \left| \left(\bigcup_{k \in \Omega_i \cup \Delta_i} (V_k \cap S) \right) \right|. \quad (2.1)$$

By Claim 2, if $k \in \Omega_i \cup \Omega_{i_0}$, then $|V_k \cap S| \geq n - 2$, and by Claim 6, if $k \in \Delta_i \cup \Delta_{i_0}$, then $|V_k \cap S| \geq n - 1$. It follows that

$$|S| \geq (n - 2)|\Omega_{i_0}| + (n - 2)|\Omega_i| + (n - 1)|\Delta_i|. \quad (2.2)$$

By Claim 3, we have

$$|S| \geq (n - 2)|N_{i_0}| + (n - 2)|N_i| + (n - 1)|\Delta_i|. \quad (2.3)$$

Since $|N_{i_0}| \geq |N_i|$ and $n \geq 3$, we obtain that

$$|S| \geq 2(n - 2)|N_i| + (n - 1)|\Delta_i| \geq (n - 1)d_G(u_i). \quad (2.4)$$

However, by Lemma 2.1 (4), we have $|S| \leq (n - 1)\delta(G)$. So, in the above four inequalities, “=” must hold. By Eq. (2.4) we obtain that $n = 3$, $|N_i| = |N_{i_0}|$, and $\delta(G) = d_G(u_i)$. Furthermore, for each $k \in \Omega_i \cup \Omega_{i_0}$, $|V_k \cap S| = n - 1$, and for each $k \in \Delta_i$, $|V_k \cap S| = n - 1$. It follows that

$$|S| = 2|N_i| + 2|\Delta_i| = 2\delta(G). \quad (2.5)$$

To show that $|\Delta_i| = |\Delta_{i_0}| = 0$, we shall first show that

$$\left(\bigcup_{k \in \Delta_i} V_k \right) \cap \left(\bigcup_{k \in \Omega_{i_0} \cup \Delta_{i_0}} V_k \right) = \emptyset.$$

Suppose on the contrary that for some $k \in \Delta_i$, $V_k \cap \left(\bigcup_{k \in \Omega_{i_0} \cup \Delta_{i_0}} V_k \right) \neq \emptyset$. Since $|V_k \cap S| = n - 1$, from Claim 6 it follows that $|V_k \cap W_{j_1}| = 1$. Take $(u_k, t) \in V_k \cap W_{j_1}$. Then u_i, u_{i_0} are neighbors of u_k in G . Since $t_0 \neq t_1$, either $(u_k, t) \sim (u_i, t_1)$ or $(u_k, t) \sim (u_{i_0}, t_0)$. This is contrary to the fact that there are no edges between W_0 and W_1 . Thus, $\left(\bigcup_{k \in \Delta_i} V_k \right) \cap \left(\bigcup_{k \in \Omega_{i_0} \cup \Delta_{i_0}} V_k \right) = \emptyset$. It follows that

$$\begin{aligned} |S| &\geq \left| \bigcup_{k \in \Omega_{i_0} \cup \Delta_{i_0}} (V_k \cap S) \right| + \left| \bigcup_{k \in \Omega_i \cup \Delta_i} (V_k \cap S) \right| \\ &= \left| \bigcup_{k \in \Omega_{i_0}} (V_k \cap S) \right| + \left| \bigcup_{k \in \Omega_i \cup \Delta_i} (V_k \cap S) \right| + \left| \bigcup_{k \in \Delta_{i_0}} (V_k \cap S) \right| \\ &\geq 2|N_i| + 2|\Delta_i| + 2|\Delta_{i_0}| \\ &= 2\delta(G) + 2|\Delta_{i_0}|. \end{aligned}$$

Combining this with Eq. (2.5) we obtain that $|\Delta_{i_0}| = 0$. Since $d_G(u_i) = \delta(G)$, we have $d_G(u_{i_0}) \geq d_G(u_i)$. Recall that $|N_i| + |\Delta_i| = d_G(u_i)$ and $|N_{i_0}| + |\Delta_{i_0}| = d_G(u_{i_0})$. Since $|N_i| = |N_{i_0}|$, one has $|\Delta_{i_0}| \geq |\Delta_i|$, implying $|\Delta_i| = 0$.

At last, from Eq. (2.1) it can be deduced that

$$S = \left(\bigcup_{k \in \Omega_{i_0}} (V_k \cap S) \right) \cup \left(\bigcup_{k \in \Omega_i} (V_k \cap S) \right). \quad (2.6)$$

Claim 8 $W_{j_0} = N_i \cup N_{i_0}$.

Suppose that $W_{j_0} \neq N_i \cup N_{i_0}$. Since Γ_0 is a component of $\Gamma - S$, we can take a vertex, say (v_{k_1}, t) , in $W_{j_0} - (N_i \cup N_{i_0})$ such that (v_{k_1}, t) is adjacent to some vertex, say (v_{k_2}, t') , in $N_i \cup N_{i_0}$. Since $(v_{k_2}, t') \in N_i \cup N_{i_0}$, by Claim 7 $|V_{k_2} \cap W_{j_1}| = 1$. By Lemma 2.3, we have $|V_{k_1} \cap W_{j_0}| = 1$. By Claim 1, we have $V_{k_1} \cap S \neq \emptyset$. From Eq. (2.6) we see that $k_1 \in \Omega_{i_0} \cup \Omega_i$, and hence $(v_{k_1}, t) \in N_i \cup N_{i_0}$, a contradiction. Thus, $W_{j_0} = N_i \cup N_{i_0}$.

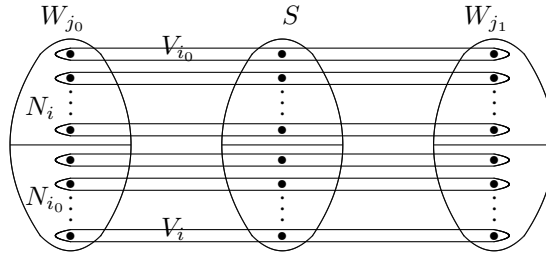


Figure 3: Explanation of Claims 8,9

Claim 9 $m = |G| = 2\delta(G)$.

By Claim 8, $|W_{j_0}| = 2\delta(G)$. So, $m \geq |W_{j_0}| = 2\delta(G)$. Suppose that $m > 2\delta(G)$. By Claim 7, for each $k \in \Omega_i \cup \Omega_{i_0}$, $|V_k \cap S| = |V_k \cap W_{j_0}| = |V_k \cap W_{j_1}| = 1$, and $S = \bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap S)$. This implies that $\bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$ is a proper subset of W_{j_1} . By the connectedness of Γ_1 , take an edge e in Γ_1 such that one end, say (u_{k_1}, t) , of e is in $\bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$ and the other end, say (u_{k_2}, t') , is in $W_{j_1} \setminus \bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$. By Claim 7, $|V_{k_1} \cap W_{j_0}| = 1$, and by Lemma 2.3, we have $|V_{k_2} \cap W_{j_1}| = 1$. By Claim 1, $|V_{k_2} \cap S| \geq 1$. It follows from Eq. (2.6) that $k_2 \in \Omega_{i_0} \cup \Omega_i$. This forces that $(u_{k_2}, t') \in \bigcup_{k \in \Omega_{i_0} \cup \Omega_i} (V_k \cap W_{j_1})$, a contradiction.

Claim 10 $G \cong K_{\ell, \ell}$, where $\ell = \delta(G)$.

Clearly, $\{u_k \mid k \in \Omega_i \cup \Omega_{i_0}\} \subseteq V(G)$. By Claim 9, $m = |G| = 2\delta(G)$. It follows that $V(G) = \{u_k \mid k \in \Omega_i \cup \Omega_{i_0}\}$. Set $B_0 = \{u_k \mid k \in \Omega_i\}$ and $B_1 = \{u_k \mid k \in \Omega_{i_0}\}$. Take any two vertices, say u_{k_1} and u_{k_2} , in B_0 . Suppose $u_{k_1} \sim u_{k_2}$. By Claim 7, we may assume that $V_{k_i} \cap W_{j_0} = \{(u_{k_i}, d_i)\}$ with $i = 1$ or 2 . From Claim 4 we obtain that (u_{k_1}, d_1) is not adjacent to (u_{k_2}, d_2) , and hence $d_1 = d_2$. Since $u_{k_1} \sim u_{k_2}$, (u_{k_1}, d_1) is adjacent to all the remaining vertices in V_{k_2} . Again, by Claim 7, we get that $|V_{k_2} \cap W_{j_1}| = 1$. This implies that there is an edge between W_{j_0} and W_{j_1} , a contradiction. Therefore, u_{k_1} and u_{k_2} are nonadjacent. By the arbitrariness of u_{k_1} and u_{k_2} , we get that B_0 is an independent subset of $V(G)$. Similarly, B_1 is also an independent subset of $V(G)$. It follows that G must be a bipartite graph with two partition sets B_0 and B_1 . By Claims 3,7, we know that $|B_0| = |B_1| = \delta(G)$. This means that $G \cong K_{\ell, \ell}$, where $\ell = \delta(G)$. \square

Lemma 2.6. Let ℓ be a positive integer. Then $K_{\ell, \ell} \times K_3$ is not super- κ .

Proof. Let $B_0 = \{v_i \mid i \in \mathbb{Z}_\ell\}$ and $B_1 = \{u_i \mid i \in \mathbb{Z}_\ell\}$ be the two partition sets of $K_{\ell, \ell}$. Set $V(K_3) = \mathbb{Z}_3$. Let $S = V(K_{\ell, \ell}) \times \{1\}$. Clearly, $|S| = 2\ell$. By [9], $\kappa(K_{\ell, \ell} \times K_3) = 2\ell$.

Set $W_0 = (B_0 \times \{0\}) \cup (B_1 \times \{2\})$ and $W_1 = (B_0 \times \{2\}) \cup (B_1 \times \{0\})$. Clearly, $V(K_{\ell, \ell} \times K_3) = S \cup W_0 \cup W_1$. It is also easy to see that $\Gamma[W_i] \cong K_{\ell, \ell}$ for $i = 0, 1$.

Furthermore, in $K_{\ell,\ell} \otimes K_3$ there are no edges between W_0 and W_1 . It follows that $K_{\ell,\ell} \times K_3 - S$ is disconnected with no isolated vertices. Therefore, $K_{\ell,\ell} \times K_3$ is not super- κ . \square

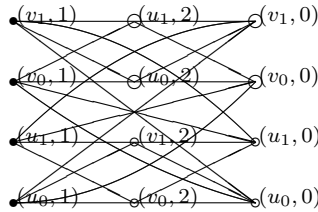


Figure 4: $K_{2,2} \times K_3$

Proof of Theorem 1.1. By Lemmas 2.4 and 2.5, we can get the necessity. For the sufficiency, by Lemma 2.6, $K_{\ell,\ell} \times K_3$ is not super- κ .

Now assume that $\kappa(\Gamma) = n\kappa(G)$. Suppose to the contrary that Γ is super- κ . Then $\kappa(\Gamma) = \delta(\Gamma) = (n - 1)\delta(G)$, and hence $(n - 1)\delta(G) = n\kappa(G)$. So, $\kappa(G) < \delta(G)$. Let T be a minimum separating set of G . Then $G - T$ has no isolated vertices. By Lemma 2.1 (4), $T \times V(K_n)$ is a separating set of Γ . Clearly, $|T \times V(K_n)| = n\kappa(G)$. So, $T \times V(K_n)$ is also a minimum separating set of G . Since Γ is super- κ , $T \times V(K_n)$ must be the neighborhood of some vertex, say (u_i, j) . Let $u_k \in T$. Then $(u_k, j) \in T \times V(K_n)$, and hence $(u_i, j) \sim (u_k, j)$. This is clearly impossible by the definition of the direct product of graphs. Thus, Γ is not super- κ . \square

Acknowledgements: The author is indebted to the anonymous referee for many valuable comments and constructive suggestions.

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