

# A Comparison of Methods for the Estimation of Weibull Distribution Parameters

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## Abstract

In this paper we study the different methods for estimation of the parameters of the Weibull distribution. These methods are compared in terms of their fits using the mean square error (MSE) and the Kolmogorov-Smirnov (KS) criteria to select the best method. Goodness-of-fit tests show that the Weibull distribution is a good fit to the squared returns series of weekly stock prices of Cornerstone Insurance PLC. Results show that the mean rank (MR) is the best method among the methods in the graphical and analytical procedures. Numerical simulation studies carried out show that the maximum likelihood estimation method (MLE) significantly outperformed other methods.

## 1 Introduction

The Weibull Distribution has been widely studied since its introduction in 1951 by Professor Wallodi Weibull (Weibull, 1951). These studies range from parameter estimation; see for example, Mann et al. (1974), Johnson et al. (1994) and Al-Fawzan (2000) to diverse applications in reliability engineering especially in Tang (2004) and lifetime analysis in Lawless (1982, 2003). The popularity of the distribution is attributable to the fact that it provides a useful description for many different kinds of data, especially in emerging areas such as wind speed and finance (stock prices and actuarial data) in addition to its traditional engineering applications.

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Engineers and statisticians relied mainly on probability plots, referred to as graphical procedure, to analyze life data prior to the advent of desktop computers and reliability analysis software became available. We discuss the three methods; the mean rank (MR), the median rank (MDR) and the symmetric cumulative distribution function (SCDF) in Section 2. Also in Section 2 we review three methods in the objective analytical procedure; the maximum likelihood estimation (MLE), the method of moments (MOM) and the least squares method (LSM). These methods are compared in Section 3, using the mean square error (MSE) and the maximum likelihood (LLH) criteria.

## 2 Methods for parameter estimation

Let  $S_1, S_2, \dots, S_N$  be a random sample of size  $N$  from a population. Define  $r_t = \ln(S_t/S_{t-1})$ ,  $r_t \in (-\infty, \infty)$  as returns of the stock prices (say),  $\{S_t : S_t > 0\}$ . Let  $x_t = r_t^2 \in R^+$  be hereinafter referred to as the squared returns.

### 2.1 The Weibull distribution

The general form of a three-parameter Weibull probability density function (pdf) is given by

$$f(x) = \frac{\beta}{\alpha} \left( \frac{x_t - v}{\alpha} \right)^{\beta-1} \exp \left\{ - \left( \frac{x_t - v}{\alpha} \right)^\beta \right\}, \quad x, v \geq 0; \alpha, \beta > 0 \quad (2.1)$$

where;  $x_t$  is the data vector at time  $t$ ;  $\beta$  is the shape parameter;  $\alpha$  is the scale parameter that indicates the spread of the distribution of sampled data and  $v$  is the location parameter. The Weibull probability density function satisfies the following properties:

- a) If  $0 < \beta < 1$ ,  $f$  is decreasing with  $f(x) \rightarrow \infty$  as  $x \rightarrow 0^+$ .
- b) If  $\beta = 1$ ,  $f$  is decreasing with  $f(x) \rightarrow 1$  as  $x \rightarrow 0^+$ .
- c) If  $\beta > 1$ ,  $f$  at first increases and then decreases, with a maximum value at the mode  $x = \alpha(1 - 1/\beta)^{1/\beta}$ .
- d) For all  $\beta > 0$ ,  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

The cumulative distribution function (cdf) of the Weibull distribution is mathematically given as:

$$F(x_t) = 1 - \exp \left\{ - \left( \frac{x_t - v}{\alpha} \right)^\beta \right\}. \quad (2.2)$$

In case of  $\nu = 0$ , the pdf in (2.1) reduces to (2.3)

$$f(x_i) = \begin{cases} \left(\frac{\beta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x_i}{\alpha}\right)^\beta\right\}, & x \geq 0; \alpha, \beta > 0 \\ 0, & \text{otherwise} \end{cases} \quad (2.3)$$

with a corresponding cdf as

$$F(x_i) = \begin{cases} 1 - \exp\left\{-\left(\frac{x_i}{\alpha}\right)^\beta\right\}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (2.4)$$

Cheng and Chen (1988) observed that the distribution interpolates between the exponential distribution ( $\beta = 1$ ) and Raleigh distribution ( $\beta = 2$ ). The mean and variance of the Weibull distribution are  $E(X) = \alpha\Gamma(1+1/\beta)$  and  $V(X) = \alpha^2 [\Gamma(1+2/\beta) - \Gamma^2(1+1/\beta)]$  respectively, where  $\Gamma(n)$  is a gamma function evaluated at  $n$ .

## 2.2 Estimation procedures

### 2.2.1 Graphical procedure

If both sides of the cdf in (2.4) are transformed by  $\ln(1/(1-x))$ , we get

$$\ln\left(\frac{1}{1-F(x_i)}\right) = \left(\frac{x_i}{\alpha}\right)^\beta$$

so that

$$\ln\left[\ln\left(\frac{1}{1-F(x_i)}\right)\right] = \beta \ln x_i - \beta \ln \alpha. \quad (2.5)$$

Here,  $x_i$  actually represents the order statistics  $x_{(1)} < x_{(2)} < \dots < x_{(n)}$ .

If we let  $Y = \ln\left[\ln\left(1/(1-F(x_i))\right)\right]$ ,  $X = \ln x_i$  and  $c = -\beta \ln \alpha$ , then (2.5) represents a simple linear regression function corresponding to

$$Y = \beta X + c. \quad (2.6)$$

The unbiased estimate of  $\alpha$ , the scale parameter, is calculated as

$$\hat{\alpha} = \exp\left[-\left(\frac{c}{\beta}\right)\right] \quad (2.7)$$

where  $c$  is the intercept of the linear regression (2.6).

Thus, we perform the estimation of  $\alpha$  and  $\beta$  using the following methods of estimation in Table 1.

**Table 1:** Methods of estimation by graphical procedure

Method	$F(x_i)$
Mean Rank	$i/(n+1)$
Median Rank	$(i-0.3)/(n+0.4)$
Symmetric CDF	$(i-0.5)/n$

We plot  $Y_i$ , which is a function of  $F(x_i)$ , versus  $X_i (= \ln(x_i))$ , using the following procedure:

- Rank the data  $\{x_i\}$  in ascending order of magnitude;
- Estimate  $F(x_i)$  of the  $i$  th rank order; and
- Plot  $Y_i$  versus  $X_i$ .

This plot produces a straight line from which we obtain  $\hat{\beta}$  and  $\hat{\alpha}$  (see (2.6) and (2.7)).

### 2.2.2 Analytical procedure

#### Maximum Likelihood Estimation (MLE)

The method of maximum likelihood estimation is a commonly used procedure for estimating parameters, see, e.g., Cohen (1965) and Harter and Moore (1965). Let  $x_1, x_2, \dots, x_n$  be a random sample of size  $n$  drawn from a population with probability density function  $f(x, \underline{\lambda})$  where  $\underline{\lambda} = (\beta, \alpha)$  is an unknown vector of parameters, so that the likelihood function is defined by

$$L = f(\alpha, \beta) = \prod_{i=1}^n f(x_i, \underline{\lambda}) \quad (2.8)$$

The maximum likelihood of  $\underline{\lambda} = (\beta, \alpha)$ , maximizes  $L$  or equivalently, the logarithm of  $L$  when

$$\frac{\partial \ln L}{\partial \underline{\lambda}} = 0, \quad (2.9)$$

see, for example, Mood et al (1974). Consider the Weibull pdf given in equation (2.3), its likelihood function is given as:

$$\begin{aligned} L(x_1, x_2, \dots, x_n; \beta, \alpha) &= \prod_{i=1}^n \left( \frac{\beta}{\alpha} \right) \left( \frac{x_i}{\alpha} \right)^{\beta-1} \exp \left[ - \left( \frac{x_i}{\alpha} \right)^\beta \right] \\ &= \left( \frac{\beta}{\alpha} \right) \left( \frac{x_i}{\alpha} \right)^{n\beta-n} \sum_{i=1}^n x_i^{(\beta-1)} \exp \left[ - \sum_{i=1}^n \left( \frac{x_i}{\alpha} \right)^\beta \right]. \end{aligned} \quad (2.10)$$

Taking the natural logarithm of both sides yields

$$\ln L = n \ln \left( \frac{\beta}{\alpha} \right) + (\beta - 1) \sum_{i=1}^n \ln x_i - \ln(\alpha^{\beta-1}) - \sum_{i=1}^n \left( \frac{x_i}{\alpha} \right)^\beta \quad (2.11)$$

and differentiating (2.11) partially w.r.t  $\beta$  and  $\alpha$  in turn and equating to zero, we obtain the estimating equations as follows

$$\frac{\partial}{\partial \beta} \ln L = \frac{n}{\beta} + \sum_{i=1}^n \ln x_i - \frac{1}{\alpha} \sum_{i=1}^n x_i^\beta \ln x_i = 0 \quad (2.12)$$

and

$$\frac{\partial}{\partial \alpha} \ln L = -\frac{n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^n x_i^\beta = 0. \quad (2.13)$$

From (2.13) we obtain an estimator of  $\alpha$  as

$$\hat{\alpha}_{mle} = \frac{1}{n} \sum_{i=1}^n x_i^\beta \quad (2.14)$$

and on substitution of (2.14) in (2.12) we obtain

$$\frac{1}{\beta} + \frac{1}{n} \sum_{i=1}^n \ln x_i - \frac{\sum_{i=1}^n x_i^\beta \ln x_i}{\sum_{i=1}^n x_i^\beta} = 0 \quad (2.15)$$

which may be solved to obtain the estimate of  $\beta$  using Newton-Raphson method or any other numerical procedure because (2.15) does not have a closed form solution. When  $\hat{\beta}_{mle}$  is obtained, the value of  $\hat{\alpha}$  follows from (2.14).

### Method of Moments (MOM)

The second procedure we consider here is the MOM which is also commonly used in parameter estimation. Let  $x_1, x_2, \dots, x_n$  represent a set of data for which we seek an unbiased estimator for the  $k^{\text{th}}$  moment. Such an estimator is generally given by

$$\hat{m}_k = \frac{1}{n} \sum_{t=1}^n x_t^k \quad (2.16)$$

where  $\hat{m}_k$  is the estimate of  $k^{\text{th}}$  moment. For the Weibull distribution given in (2.3), the  $k^{\text{th}}$  moment is given by

$$\mu_k = \left( \frac{1}{\alpha^k} \right)^{\frac{k}{\beta}} \Gamma \left( 1 + \frac{k}{\beta} \right) \quad (2.17)$$

where  $\Gamma$  is as defined in subsection 2.1. From (2.17), we can find the 1<sup>st</sup> and 2<sup>nd</sup> moments about zero as follows

$$\hat{m}_1 = \hat{\mu}_1 = \left( \frac{1}{\alpha} \right)^{\frac{1}{\beta}} \Gamma \left( 1 + \frac{1}{\beta} \right) \quad (2.18)$$

and

$$\hat{m}_2 = \hat{\mu}^2 + \hat{\sigma}^2 = \left( \frac{1}{\alpha} \right)^{\frac{2}{\beta}} \left[ \Gamma \left( 1 + \frac{2}{\beta} \right) - \Gamma \left( 1 + \frac{1}{\beta} \right)^2 \right] \quad (2.19)$$

When we divide the square of  $\hat{m}_1$  by  $\hat{m}_2$ , we get an expression which is a function of only  $\beta$ ,

$$\frac{\hat{\mu}^2}{\hat{\sigma}^2 + \hat{\mu}^2} = \frac{\Gamma \left( 1 + \frac{1}{\beta} \right) \Gamma \left( 1 + \frac{1}{\beta} \right)}{\Gamma \left( 1 + \frac{2}{\beta} \right)} \quad (2.20)$$

where  $\hat{\mu} = E(X_t) = \frac{1}{n} \sum_{t=1}^n x_t$ ,  $\hat{\sigma}^2 = E(X_t^2) - (E(X_t))^2$  and letting  $Z = 1/\beta$  (2.19) is

easily transformed in order to estimate  $\beta$  so that the scale parameter  $\alpha_{mom}$  can be estimated with the following relation

$$\hat{\alpha}_{mom} = \hat{\mu} / \Gamma \left( 1 + \frac{1}{\beta} \right). \quad (2.21)$$

### **The Least Squares Method (LSM)**

The Least Squares method is commonly applied in engineering and mathematics problems that are often not thought of as an estimation problem. We assume that there is a linear relationship between two variables. Assume a dataset that constitute a pair  $(x_t, y_t) = (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  were obtained and plotted. The least squares principle minimizes the vertical distance between the data points

and the straight line fitted to the data, the best fitting line to this data is the straight line:  $y_i = \alpha + \beta x_i$  such that

$$Q(x; \alpha, \beta) = \sum_{i=1}^n (y_i - \alpha - \beta x_i)^2$$

To obtain the estimators of  $\alpha$  and  $\beta$  we differentiate  $Q$  w.r.t  $\alpha$  and  $\beta$ . Equating to zero subsequently yields the following system of equations:

$$\frac{\partial Q}{\partial \alpha} = -2 \sum_{i=1}^n (y_i - \alpha - \beta x_i) \quad (2.22)$$

and

$$\frac{\partial Q}{\partial \beta} = -2 \sum_{i=1}^n (y_i - \alpha - \beta x_i) x_i = 0 \quad (2.23)$$

Expanding and solving equations (2.21) and (2.22) simultaneously, we have

$$\hat{\beta} = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} \quad (2.24)$$

and

$$c = \bar{y} - \hat{\beta} \bar{x}; \quad \hat{\alpha} = \exp\left(-\frac{c}{\hat{\beta}}\right) \quad (2.25)$$

where  $\hat{\alpha}$  and  $\hat{\beta}$  are the unbiased estimators of  $\alpha$  and  $\beta$  respectively.

### 3 Method assessment and selection

#### 3.1 Comparison of estimation methods

The Mean Squared Error (MSE) criterion is given by

$$MSE = \frac{1}{n} \sum_{i=1}^n [\hat{F}(x_i) - F(x_i)]^2 \quad (3.1)$$

where  $\hat{F}(x_i)$  is obtained by substituting the estimates of  $\alpha$  and  $\beta$  (for each method) in (2.4) while  $F(x_i) = i/n$  is the empirical distribution function. The method with the minimum mean squared error ( $MSE_{\min}$ ) becomes the best method for the estimation of Weibull parameters among the candidate methods.

### 3.2 Goodness-of-fit tests

Goodness-of-fit test procedures are intended to detect the existence of a significant difference between the observed (empirical) frequency of occurrence of an item and the theoretical (hypothesized) pattern of occurrence of that item. Here, we assume that the Weibull distribution is a good fit to the given dataset; otherwise, this assumption is nullified if, for this test, the computed statistic is greater or equal to a defined critical value.

#### *Kolmogorov–Smirnov test*

The Kolmogorov-Smirnov test is used to decide if a sample comes from a population with specific distribution. It is based upon a comparison between the empirical distribution function (ECDF) and the theoretical one defined as  $F(x) = \int_{-\infty}^x f(y, \theta) dy$  where  $f(x, \theta)$  is the pdf of the Weibull distribution. Given  $n$  ordered data points  $X_1, X_2, \dots, X_n$ , the ECDF is defined as  $F(X_i) = N(i)/n$  where  $N(i)$  is the number of points less than  $X_i$  ( $X_i$  are ordered from smallest to highest value). The test statistic used is

$$D_n = \sup_{1 \leq i \leq n} \left| \hat{F}(x_i) - F(x_i) \right|. \quad (3.2)$$

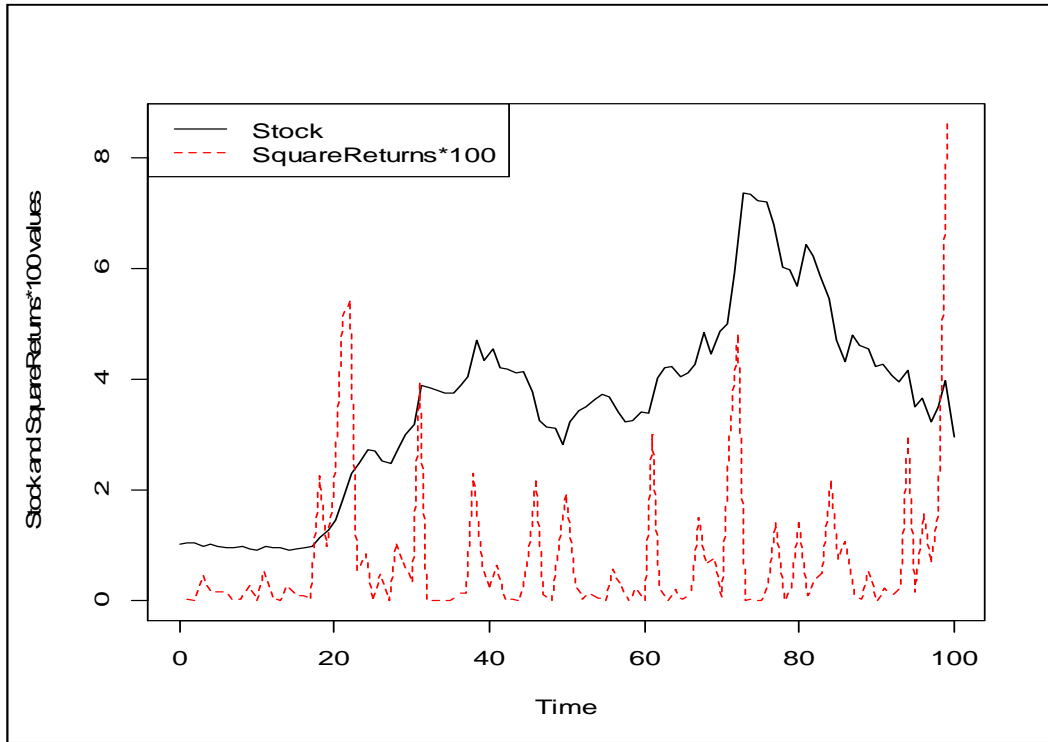
The statistic  $D_n$  converges to zero almost surely as  $n \rightarrow \infty$ .

## 4 Implementation

### 4.1 Data

The data used for this study is the weekly stock prices ( $N = 100$  weeks) collected from Cornerstone Insurance Company PLC, a public liability company listed in the Nigerian Stock Exchange (Appendix I). The squared returns,  $r^2$ , earlier defined in Section 2 are a measure of volatility in the stock prices and are multiplied by 100 without loss of generality. In Figure 1 we present a graphic relationship between the weekly stock prices and its squared returns. We perform the estimation of the parameters using the R software for the graphical and analytical procedures with  $100r^2$  as the dataset and  $r$  is now of length  $n$ . R is a language and environment for statistical computing and graphics (from the R Foundation for Statistical Computing (2013)) ran on the Platform: i386-w64-mingw32/i386 (32-bit).





**Figure 1:** Plot showing relationship between Weekly Stock Prices and its Squared Returns\*100

## 4.2 Simulation study

We carry out a numerical simulation study in order to investigate the behavior of the shape and scale parameters of the Weibull distribution. In the simulation experiment we set the Weibull distribution on the random variable  $X$  with shape parameter  $\beta = 0.54$  with the aim of mimicking the squared returns ( $100r^2$ ). For the Weibull distribution on  $X$ , generate independently and identically distributed random sample  $(x_1, x_2, \dots, x_n)$  of size  $n$  ( $= 25, 50, 75, 100, 125, 150, 175, 200$ ). Compute the mean of this sample and replicate this process  $N$  times to obtain a series. For each series of size  $n$ , estimate  $\beta$  and  $\alpha$  using the methods described in Section 2, the MSE and the Kolmogorov-Smirnov (KS) statistic. This sequence is of the form  $X_{1, \dots, N}^* = \text{mean}(x_1^*, \dots, x_n^*)_1, \text{mean}(x_1^*, \dots, x_n^*)_2, \dots, \text{mean}(x_1^*, \dots, x_n^*)_N$ ,  $N = 10000$  times; and is accomplished in R for Windows 2013 by the replicate function:  $\text{replicate}(N, \text{mean}(\text{rweibull}(n, \text{shape} = 0.54)))$ .

We remark here that the least squares method (LSM) is related to the graphical procedure in the estimation of Weibull parameters through (2.6), where  $Y = \ln[\ln(1/(1-F(x_i)))]$  is dependent upon the particular graphical method (e.g.,  $F(x_i) = i/(n+1)$  for the mean rank) and  $X = \ln x_i$ ; see also equations (2.7) and (2.25).

### 4.3 Results and Discussion

All computations and simulations in this investigation were done in R version 3.0.0. We relied on the functions `fitdist()` and `fitdistr()` respectively from R packages `fitdistrplus` and `MASS` (see, e.g., Delignette-Muller et al (2013) and Ripley (2013) respectively) for maximum likelihood estimation of the parameters and plots while codes were developed for the other methods. Results for the graphical procedure (MR, MDR and SCDF) were verified using the approach in Dorner (1999) on Microsoft Excel 2013. The R code used for this study is available from the first author on request.

Estimates of the parameters based upon both the graphical and theoretical procedures described in Section 2.2 are presented in Table 2. The shape parameter  $\beta$  lies within the interval (0, 1) which implies, as indicated in Section 2.1, that the function (irrespective of the method) decreases exponentially. We ranked the performance of the methods based on the least MSE criterion. In comparison, the Mean Rank (MR) method has the least MSE ( $3.88 \times 10^{-03}$ ) and at the same time has the least  $D_n$  (0.0563) making it the best among the five methods under study (graphical and analytical procedures) for this particular dataset. The Maximum Likelihood Estimation (MLE) method is, however, superior to Method of Moments in the analytical procedure. From these results the best estimate for the shape and scale parameters are respectively  $(\hat{\beta}, \hat{\alpha}) = (0.5325, 0.4539)$  based on our dataset.

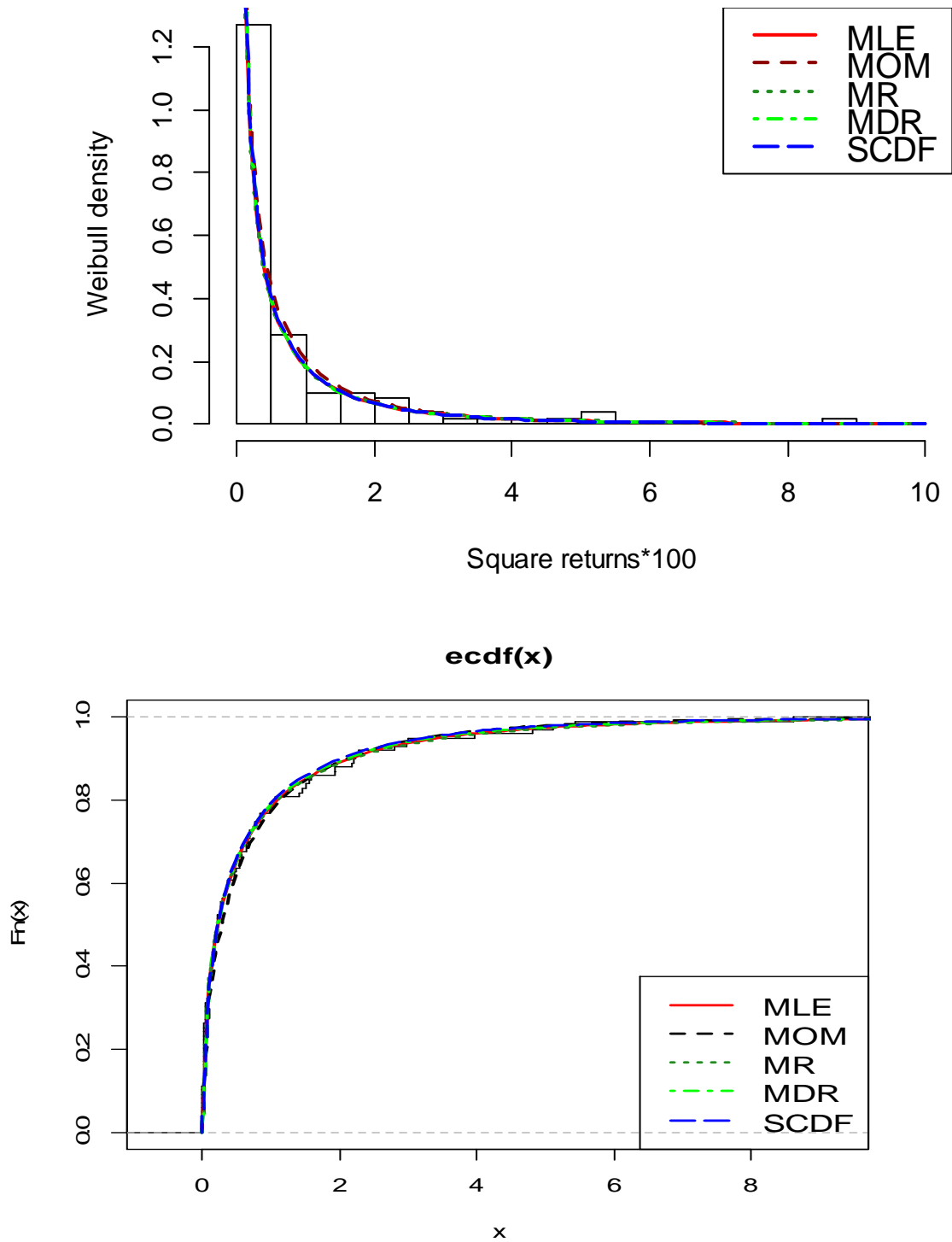
The visual assessments of fit are shown in the histogram (Figure 2(a)) overlaid with the Weibull densities generated from the different methods and in the empirical cumulative distribution function plot of Figure 2(b). The MOM is clearly different from other methods given their MSEs but this difference is not very clear in Figure 2. However, simulation results show (Table 3) that the MLE performed best 86% of the time when the  $n_i$  simulations are run 10,000 times. Similar result was obtained when the KS goodness-of-fit test was conducted to test the adequacy of the Weibull distribution in fitting the simulation data.

**Table 2:** Summary of results and comparison of methods for Weibull parameter estimation

Procedure	Method	$\hat{\alpha}$	$\hat{\beta}$	MSE	KS
Graphical	MR	0.4539	0.5325	$3.88 \times 10^{-03}$	0.0563
	MDR	0.4494	0.5452	$4.21 \times 10^{-03}$	0.0615
	SCDF	0.4461	0.5553	$4.49 \times 10^{-03}$	0.0656
Analytical	MLE	0.4563	0.5421	$6.59 \times 10^{-03}$	0.0617
	MOM	0.5244	0.6026	$1.18 \times 10^{-01}$	0.1055

**Table 3** Simulation results (based on 10,000 iterations)

<i>n</i>	Measure	Method				
		MR	MDR	SCDF	MLE	MOM
25	MSE	3.5726	3.5815	3.5837	<b>1.2557</b>	1.6770
	KS	0.0600	0.0600	0.0601	<b>0.0501</b>	0.9821
50	MSE	4.6281	4.6323	4.6282	<b>1.4930</b>	3.5122
	KS	0.0681	0.0682	0.0683	<b>0.0540</b>	0.9596
75	MSE	4.9234	4.9502	4.9407	<b>1.5438</b>	4.2108
	KS	0.0683	0.0684	0.0684	<b>0.0563</b>	0.9741
100	MSE	4.8839	4.9119	4.8985	<b>1.3216</b>	4.4869
	KS	0.0653	0.0654	0.0654	<b>0.0587</b>	0.0964
125	MSE	5.2496	5.2389	5.2598	<b>1.4261</b>	4.9398
	KS	0.0750	0.0750	0.0751	<b>0.0590</b>	0.9600
150	MSE	5.4266	5.4118	5.4341	<b>1.4671</b>	5.2043
	KS	0.0672	0.0671	0.0673	<b>0.0604</b>	0.9665
175	MSE	6.4067	6.3872	6.4096	<b>1.7235</b>	6.0586
	KS	0.0726	0.0726	0.0726	<b>0.0657</b>	0.9720
200	MSE	5.1548	5.1831	<b>1.3525</b>	1.4170	5.0833
	KS	0.0674	0.0675	0.0818	<b>0.0614</b>	0.9816



**Figure 2:** Fit of different methods (a) Density and Histogram (b) ECDF

## 5 Conclusion

The performances of five methods in the estimation of the parameters of the Weibull distribution were compared in this study. The MR was selected as the best method that gives the best estimates of the two-parameter model for square returns dataset, while the MLE is preferred over the MOM for the analytical procedure. These decisions were based on the minimum MSE criterion. When these methods were compared based upon simulation results, the maximum likelihood estimate method showed superiority over other methods. The least squares method (LSM), we remark, is also known as the rank regression method (RRM) because the estimation of the parameters of the Weibull distribution is dependent upon regressing some form of log and rank transformations of a given dataset according to the rank plotting position.

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## Appendix

**Table A1:** Weekly stock prices (read row-wise)

1.03	1.06	0.99	1.03	0.99	0.95	0.96	0.98	0.93	1.05
0.92	0.99	0.97	0.96	0.91	0.94	0.97	0.99	1.15	1.27
1.46	1.83	2.31	2.49	2.73	2.70	2.52	2.49	2.76	3.00
3.18	3.88	3.84	3.79	3.76	3.75	3.89	4.04	4.70	4.34
4.55	4.20	4.19	4.12	4.13	3.77	3.25	3.14	3.12	2.82
3.24	3.44	3.50	3.64	3.72	3.68	3.41	3.24	3.26	3.42
3.38	4.02	4.21	4.23	4.04	4.11	4.28	4.84	4.46	4.87
5.00	5.91	7.36	7.34	7.23	7.19	6.79	6.03	5.97	5.69
6.42	6.23	5.86	5.46	4.71	4.32	4.79	4.62	4.54	4.22
4.28	4.08	3.95	4.16	3.50	3.65	3.22	3.50	3.97	2.96