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Classification of minimal Frobenius hypermaps*

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Abstract

In this paper, we give a classification of orientably regular hypermaps with an automorphism group that is a minimal Frobenius group. A Frobenius group G is called minimal if it has no nontrivial normal subgroup N such that G/N is a Frobenius group. An orientably regular hypermap $\mathcal{H}$ is called a Frobenius hypermap if $\mathrm{Aut}(\mathcal{H})$ acting on the hyperfaces is a Frobenius group. A minimal Frobenius hypermap is a Frobenius hypermap whose automorphism group is a minimal Frobenius group with cyclic point stabilizers. Every Frobenius hypermap covers a minimal Frobenius hypermap. The main theorem of this paper generalizes the main result of Breda D'Azevedo and Fernandes in 2011.

Keywords: Frobenius hypermap, Frobenius group.

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1 Introduction

Let $\mathcal{S}$ be a compact and connected orientable surface. A *topological hypermap* $\mathcal{H}$ on $\mathcal{S}$ is a triple $(\mathcal{S}; V; E)$, where V and E denote closed subsets of $\mathcal{S}$ with the following properties:

(1) $B = V \cap E$ is a finite set. Its elements are called the *brins* of $\mathcal{H}$;

(2) $V \cup E$ is connected;

(3) the components of V (called the *hypervertices*) and of E (called the *hyperedges*), are homeomorphic to closed discs;

(4) the components of the complement $\mathcal{S} \backslash (V \cup E)$ are homeomorphic to open discs, and they are called the *hyperfaces* of $\mathcal{H}$.

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The following Figure 1 shows a topological hypermap on torus with 9 brins, 3 hypervertices (black components), 3 hyperedges (grey components) and 3 hyperfaces (white components).

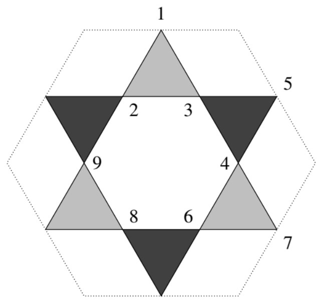

Figure 1: A hypermap on torus.

An important and convenient way to visualize hypermaps was introduced by Walsh in [13]. The *Walsh representation* of a hypermap as a bipartite graph embedding on $\mathcal{S}$ can be described as follows. At the centre of each hypervertex place a white vertex and at the centre of each hyperedge place a black vertex. If a hypervertex intersects a hyperedge then we join the corresponding white vertex and black vertex by an edge. In this way we obtain a bipartite graph. This bipartite graph is said to be the *underlying graph* of $\mathcal{H}$. Figure 2 is the Walsh representation of the hypermap in Figure 1.

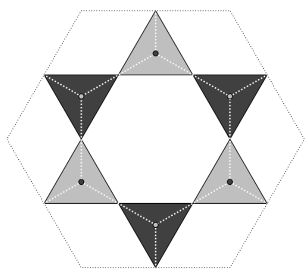

Figure 2: The Walsh representation.

An *algebraic hypermap* is a quadruple $\mathcal{H} = (G, B, \rho_0, \rho_1)$, where G is a finite group which is generated by two elements ρ_0, ρ_1 and acts transitively on a finite set B. By [3], there is a one-to-one correspondence between topological and algebraic hypermaps. The finite group G is the *monodromy group* of $\mathcal{H}$, denoted by $\mathrm{Mon}(\mathcal{H})$. In the Walsh representation, G is a permutation group acting on the set of edges, ρ_0, ρ_1 generate the cyclic permutations of the edges going around the white resp. black vertices in a positive sense, and each cycle of $\rho_0\rho_1$ bounds a hyperface in a negative direction. A permutation α of B is called an *automorphism* of the hypermap $\mathcal{H} = (G, B, \rho_0, \rho_1)$ if it is G-equivariant, i.e. if

$$\alpha(g(b)) = g(\alpha(b))$$

for every $b \in B$ and $g \in G$.

Since $\alpha\rho_0\alpha^{-1} = \rho_0$ and $\alpha\rho_1\alpha^{-1} = \rho_1$, α induces a permutation on the cycles of ρ_0 and ρ_1. So, in the Walsh representation, $\mathrm{Aut}(\mathcal{H})$ induces a subgroup of the automorphism group of the underlying graph of $\mathcal{H}$, and $\mathrm{Aut}(\mathcal{H})$ preserves the hypervertex set and hyperedge set, respectively. A hypermap is called *regular* if G acts regularly on B. In this case, $\mathrm{Aut}(\mathcal{H})$ is isomorphic to G which acts regularly on B as well.

For a regular hypermap $\mathcal{H} = (G, B, \rho_0, \rho_1)$, the set B can be replaced by G, so that $\mathrm{Mon}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})$ can be viewed as the right and left regular multiplications of G, respectively. So, $\mathcal{H}$ can be denoted by a triple $\mathcal{H} = (G; \rho_0, \rho_1)$, where $G = \langle \rho_0, \rho_1 \rangle$. In this way, the hypervertices (resp. hyperedges and hyperfaces) correspond to right cosets of G relative to $\langle \rho_0 \rangle$, (resp. $\langle \rho_1 \rangle$ and $\langle \rho_0\rho_1 \rangle$). In [4], the hypermap $\mathcal{H} = (G; \rho_0, \rho_1)$ is denoted by $(G; a, b)$ where $a = {\rho_1}^{-1}{\rho_0}^{-1}$ and $b = \rho_0$. From now on, we denote a regular hypermap $\mathcal{H}$ by the triple $\mathcal{H} = (G; a, b)$, and then the hyperfaces (resp. hypervertices and hyperedges) correspond to left cosets of G relative to subgroups $\langle a \rangle$ (resp. $\langle b \rangle$ and $\langle ab \rangle$). Let $\mathcal{H} = (G; a, b)$ and $\mathcal{H}' = (G'; a', b')$ be two orientably regular hypermaps. If there is an epimorphism ρ from G to G' such that $a^\rho = a'$ and $b^\rho = b'$, then $\mathcal{H}$ is called a covering of $\mathcal{H}'$ or $\mathcal{H}$ *covers* $\mathcal{H}'$. Given a group G, $(G; a_1, b_1) \cong (G; a_2, b_2)$ if and only if there exists an automorphism σ of G such that $a_1^\sigma = a_2$ and $b_1^\sigma = b_2$.

A (face-)*primer* hypermap is an orientably regular hypermap whose automorphism group induces faithful actions on its hyperfaces, see [4]. The classification of regular hypermaps with given automorphism groups isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ can be extracted from [12] by Sah. Moreover, Conder, Potočnik and Širáň extended Sah's investigation to reflexible hypermaps, on both orientable and nonorientable surfaces, and provided explicit generating sets for projective linear groups, see [1]. In [2], Conder described all regular hypermaps of genus 2 to 101, and all non-orientable regular hypermaps of genus 3 to 202.

The study of primer hypermaps was initiated by Breda d'Azevedo and Fernandes in 2011. In [4], the authors classified the primer hypermaps with p-hyperfaces for a prime number p, where their automorphism groups are Frobenius groups. Thereafter, they determined all regular hypermaps with p-hyperfaces, see [5]. In [7], Du and Hu classified primer hypermaps with a product of two primes number of hyperfaces. Recently, Du and Yuan characterized primer hypermaps with nilpotent automorphism groups and prime hypervertex valency, see [8].

A Frobenius group is a transitive permutation group G on a set Ω which is not regular on Ω , but has the property that the only element of G which fixes more than one point is the identity element. A Frobenius group G is called *minimal* if it does not have a nontrivial normal subgroup N such that G/N is a Frobenius group. A regular hypermap $\mathcal{H}$ is called a *Frobenius hypermap* if $\mathrm{Aut}(\mathcal{H})$ acting on the hyperfaces is a Frobenius group. Clearly, $\mathcal{H}$ is a primer hypermap. A *minimal Frobenius hypermap* is a Frobenius hypermap whose automorphism group is a minimal Frobenius group with a cyclic point stabilizer. Clearly, every Frobenius hypermap covers a minimal Frobenius hypermap.

This paper has three sections. In the first section, a quick overview of orientably regular hypermaps is given. In Section 2, we introduce minimal Frobenius groups. In the last section, we give a classification of orientably regular minimal Frobenius hypermaps. Furthermore, the main theorem of this paper generalizes the main result of Breda D'Azevedo and Fernandes, see [4].

2 Minimal Frobenius groups

We refer the readers to [10] for standard notation and results in group theory. Set (r, s) to denote the greatest common divisor of two positive integers r and s. We denote the orders of an element x and of a subgroup H of G as $|x|$ and $|H|$, respectively. A semidirect product of a group N by a group H is denoted by $N : H$. Let $\mathbb{Z}_m = \{0, 1, \cdots, m-1\}$ and $\mathbb{Z}_m^* = \{k \mid k \in \mathbb{Z}_m \text{ and } (k, m) = 1\}$.

Let G be a Frobenius group on Ω. A subgroup K of G is called the *Frobenius kernel* if K acts regularly on Ω. Each point stabilizer is called a *Frobenius complement* of K in G. In the following, we give some interesting results about Frobenius groups and primitive groups.

Proposition 2.1 ([6, P86]). *Let G be a Frobenius group on Ω and $\alpha \in \Omega$, K be the Frobenius kernel, and H be a Frobenius complement. Then:*

(i) *K is a normal and regular subgroup of G.*

(ii) *For each odd prime number p, the Sylow p-subgroups of H are cyclic, and the Sylow 2-subgroups are either cyclic or quaternion groups. If G is not solvable, then it has exactly one nonabelian composition factor, namely A_5.*

(iii) *K is a nilpotent group.*

Proposition 2.2 ([6, Corollary 1.5A.]). *Let G be a group acting transitively on a set Ω with at least two points. Then G is primitive if and only if each point stabilizer G_α is a maximal subgroup of G.*

Lemma 2.3. *Assume $G \leqslant \mathrm{Sym}(\Omega)$ has a regular normal subgroup R, where Ω has at least two points. Then G is primitive if and only if no nontrivial subgroup of R is normalized by G_α, for each α.*

Proof. By Proposition 2.2, G is primitive if and only if G_α is a maximal subgroup of G. Because R is a regular normal subgroup of G, $G = G_\alpha R$ and $G_\alpha \cap R = \{1\}$.

We claim that G_α is maximal if and only if no nontrivial subgroup of R is normalized by G_α. Suppose G_α is not maximal, then there exists a proper subgroup K of G such that $G_\alpha < K$. It follows that $K = K \cap G = K \cap G_\alpha R = G_\alpha(K \cap R)$. In this case, $K \cap R$ is a proper subgroup of R which is normalized by G_α. Conversely, suppose that there exists a proper subgroup H, normalized by G_α, of R. Thus $G_\alpha H$ is a proper subgroup of G and so G_α is not maximal. □

Corollary 2.4 follows directly from Lemma 2.3.

Corollary 2.4. *Assume $G \leqslant \mathrm{Sym}(\Omega)$ has a regular normal subgroup R, where Ω has at least two points. If R is abelian, then G is primitive if and only if no nontrivial normal subgroup of G is contained in R.*

Lemma 2.5. *Let K be the Frobenius kernel of a Frobenius group G which acts on a set Ω. If N is a normal subgroup of G, then either $N \leqslant K$ or $K < N$.*

Proof. Assume that N is not a subgroup of K. Set $\alpha \in \Omega$. Since N is a normal subgroup of G, we have $N = (\bigcup_{g \in K} N_\alpha^g) \cup (N \cap K)$ and so N is a subgroup of $N_\alpha K$. Let $|N_\alpha| =$

m, $|K| = n$ and $|N \cap K| = t$. Then, $|N| = n(m-1)+t$. Since $N \leqslant N_\alpha K$ and $N_\alpha \leqslant N$, we get $N = N \cap N_\alpha K = N_\alpha(N \cap K)$. So, $|N| = mt$ which implies $n(m-1)+t = mt$. Note that $m > 1$, then $n = t$. Therefore, $N \cap K = K$ and K is a proper subgroup of N. □

Proposition 2.6 ([11, Lemma 2.3]). *Let K be the Frobenius kernel of a Frobenius group G. If N is a normal subgroup of G and $N < K$, then G/N is a Frobenius group.*

Proposition 2.7 ([11, Corollary 2.6]). *Let $G = KH$ be a Frobenius group, where K is the Frobenius kernel and H is a Frobenius complement. For each $h \in H, h \neq 1$, and for each $k \in K$, the orders of h, kh and hk are equal, that is $|h| = |kh| = |hk|$.*

Based on Lemma 2.5 and Proposition 2.6, we give the following definition of *minimal Frobenius groups*.

Definition 2.8. A Frobenius group G is called *minimal* if it does not have a nontrivial normal subgroup N such that G/N is a Frobenius group.

Lemma 2.9. *If G is a minimal Frobenius group acting on a set Ω with the Frobenius kernel K, then K is an elementary abelian p-group and G is primitive.*

Proof. If G is minimal, then by Proposition 2.6 no nontrivial normal subgroup of G exists in K. Note that K is a nilpotent group. Let P be a Sylow p-group of K, $\Phi(P)$ be the Frattini subgroup of P and L be the p'-Hall group of K. Both $\Phi(P)$ and L are characteristic subgroups of K. So, $L = \Phi(P) = 1$ which implies that K is an elementary abelian p-group.

Because no nontrivial normal subgroup of G is contained in K and K is abelian, it follows that G is primitive by Corollary 2.4. □

Lemma 2.10. *If G is a primitive group acting on a set Ω with non-trivial abelian point stabilizers, then G is a Frobenius group and its Frobenius kernel K is an elementary abelian p-group.*

Proof. It suffices to show that for any two distinct points $\alpha, \beta \in \Omega$, $G_\alpha \cap G_\beta = 1$. Let $J = G_\alpha \cap G_\beta$. Since G is primitive, $G = \langle G_\alpha, G_\beta \rangle$. Note that G_α and G_β are abelian, so J is a normal subgroup of G. Because $\alpha^J = \{\alpha\}$, for any $g \in G$, we have $\alpha^{gJ} = \alpha^{Jg} = \{\alpha^g\}$. That is to say J fixes every point of Ω, so $J = 1$ and G is a Frobenius group. Furthermore, as point stabilizers are maximal, the Frobenius kernel K must be an elementary abelian p-group . □

Corollary 2.11 follows from Lemma 2.9 and 2.10 directly.

Corollary 2.11. *Let G be a permutation group with cyclic point stabilizers. Then, G is a minimal Frobenius group if and only if G is a primitive group.*

For a prime number p and an integer n, an integer m $(m > 1)$ is called a *primitive divisor* of $p^n - 1$ if m divides $p^n - 1$, but it does not divide $p^s - 1$ for any $s < n$.

The following Proposition 2.12 can be obtained from some results in [10, Kapitel II: 3.10, 3.11, 7.3].

Proposition 2.12. *For a prime number p and a positive integer n, set $G = GL(n, p)$.*

(i) *The group G contains a cyclic Singer-Zyklus group $S = \langle x \rangle$ of order $p^n - 1$, and $C_G(S) = S$. Moreover, $N_G(S) = S : \langle y \rangle = \langle x, y \mid x^{p^n-1} = y^n = 1, x^y = x^p \rangle$, and $|N_G(S)| = n(p^n - 1)$. Take an element $g \in S$, if $|g|$ is a primitive divisor of $p^n - 1$, then $N_G(\langle g \rangle) = N_G(S), C_G(\langle g \rangle) = S$ and $\langle g \rangle$ is an irreducible subgroup.*

(ii) *Let L be a cyclic irreducible subgroup of G. Then L is conjugate to a subgroup of S, and $|L|$ is a primitive divisor of $p^n - 1$.*

The following lemma generalizes Lemma 3.3 in [9]. The proof is similar to that of Lemma 3.3, so we omit it.

Lemma 2.13. *Let $X = T : \langle x \rangle$ and $Y = T : \langle y \rangle$ be two subgroups of $A = AGL(n, p) = T : G$, where $G = GL(n, p)$, T is the translation subgroup, and x, y are nontrivial elements in G. If σ is an isomorphism from X to Y mapping $\langle x \rangle$ to $\langle y \rangle$, then, there exists an element $u \in G$ such that $\sigma = I(u)|_X$, where $I(u)$ is the inner automorphism of A induced by u. In particular, $u \in N_G(\langle x \rangle)$ if $\langle x \rangle = \langle y \rangle$.*

3 Classification of minimal Frobenius hypermaps

For a prime number p, an integer $n \geqslant 1$ ($n \geqslant 2$ if $p = 2$) and a primitive divisor m of $p^n - 1$, let S be the cyclic Singer-Zyklus group of $GL(n, p)$, $\langle a \rangle$ be a subgroup of S with order m and T be the translation subgroup of $AGL(n, p)$. Define a group M of order mp^n as

$$M = T : \langle a \rangle \leqslant T : S \leqslant AGL(n, p) = T : GL(n, p).$$

By Proposition 2.12, $\langle a \rangle$ is an irreducible subgroup. Hence M is a primitive group, and consequently M is a Frobenius group by Lemma 2.10.

Let F be a minimal Frobenius group acting on a set Ω ($|\Omega| > 2$) with cyclic point stabilizers, and K be its Frobenius kernel. By Lemma 2.9, K is an elementary abelian p-group and F is a primitive group. Set $|K| = p^n$, and then $|\Omega| = p^n$. Take an element $\alpha \in \Omega$ and assume $|F_\alpha| = k$. By Proposition 2.12, k is a primitive divisor of $p^n - 1$, and $GL(n, p)$ has only one conjugacy class of irreducible cyclic subgroups of order k. Hence $AGL(n, p)$ has only one conjugacy class of subgroups isomorphic to F which implies $F \cong M = T : \langle a \rangle$ when $k = m$. These discussions give the following Theorem 3.1.

Theorem 3.1. *Let F be a minimal Frobenius group with cyclic point stabilizers of order m. Then, $F \cong T : \langle a \rangle$, where T is elementary abelian of order p^n for some prime number p and an integer $n \geqslant 1$, m is a primitive divisor of $p^n - 1$ and $|\langle a \rangle| = m$. Clearly, $|F| = mp^n$.*

Lemma 3.2. *Let $M = T : \langle a \rangle$ be the group defined as in the first paragraph of this section. If $\mathcal{H} = (M; R, L)$ is a Frobenius hypermap, then $\mathcal{H}$ is isomorphic to*

$$\mathcal{H}(p, n, m, i, j) = (M; a^i, a^j b),$$

where $1 \neq b \in T$, m is a primitive divisor of $p^n - 1$, $j \in \mathbb{Z}_m$, $i \in \mathbb{Z}_m^*$ and $(i, p) = 1$. Moreover, different parameter pairs (i, j) give non-isomorphic hypermaps with p^n hyperfaces, each of valency m. Furthermore, there are $\frac{m\phi(m)}{n}$ non-isomorphic hypermaps, where ϕ is the Euler's totient function.

Proof. Let $G = GL(n,p)$ and then $M \leqslant AGL(n,p) = T : G$. Since M is a Frobenius group, M has only one conjugacy class of subgroups of order m. So we can assume $R = a^i$ for some $i \in \mathbb{Z}_m^*$. Remember that S is the cyclic Singer-Zyklus group of $GL(n,p)$ and $\langle a \rangle$ is a subgroup of S. So, M is a normal subgroup of $T : S$. Since S fixes a and acts transitively on $T \backslash \{1\}$ by conjugation, we may fix $L = a^j b$, where j is calculated modular m.

If there exists an automorphism σ of M such that $(a^i)^\sigma = a^{i'}$ and $(a^j b)^\sigma = a^{j'} b$, then $b^\sigma = a^\epsilon b$ for some $\epsilon \in \mathbb{Z}_m$. Clearly, the orders of b and $a^\epsilon b$ are equal. While according to Proposition 2.7, the two elements $a^\epsilon b$ and a^ϵ have the same order which is coprime with that of b if $\epsilon \neq 0$ modulo m. So, $b^\sigma = b$. By Lemma 2.13, there exists an element $u \in G$ such that $\sigma = I(u)|_F$, where $u \in N_G(\langle a \rangle)$. According to Proposition 2.12,

$$N_G(\langle a \rangle) = S : \langle y \rangle = \langle x, y \mid x^{p^n - 1} = y^n = 1, x^y = x^p \rangle,$$

where $S = \langle x \rangle$. Because $b^\sigma = b$, it follows that $u = y^t$, where t is calculated modular n. So, $a^\sigma = a^{y^t} = a^{p^t}$. As a result, we may assume $(i,p) = 1$ in $R = a^i$. As a result, we get $\frac{m\phi(m)}{n}$ non-isomorphic hypermaps $(M; a^i, a^j b)$, where ϕ is the Euler's totient function. Clearly, $(M; a^i, a^j b)$ has p^n hyperfaces, each of valency m. □

By Theorem 3.1, the automorphism group of a minimal Frobenius hypermap is isomorphic to $M = T : \langle a \rangle$, where $|T| = p^n$ and $|\langle a \rangle| = m$. Consequently, we give the following classification theorem of minimal Frobenius hypermaps.

Theorem 3.3. *$\mathcal{H}$ is a minimal Frobenius hypermap if and only if $\mathcal{H}$ is isomorphic to*

$$\mathcal{H}(p,n,m,i,j) = (M; a^i, a^j b),$$

where M is a group defined as in the first paragraph of this section, m is a primitive divisor of $p^n - 1$, $j \in \mathbb{Z}_m$, $i \in \mathbb{Z}_m^*$ and $(i,p) = 1$. Moreover, different parameter pairs (i,j) give non-isomorphic hypermaps with p^n hyperfaces, each of valency m. And, there are $\frac{m\phi(m)}{n}$ non-isomorphic minimal Frobenius hypermaps, where ϕ is the Euler's totient function.

According to Corollary 2.11, we have the following Proposition 3.4.

Proposition 3.4. *If $\mathcal{H}$ is a regular hypermap, then $\mathcal{H}$ is a minimal Frobenius hypermap if and only if* $\mathrm{Aut}(\mathcal{H})$ *acts primitively on the hyperfaces.*

The next Proposition 3.5 follows from Lemma 2.5.

Proposition 3.5. *Every Frobenius hypermap covers a minimal Frobenius hypermap.*

The H-sequence of a hypermap $\mathcal{H}$ is a sequence $[|v|, |e|, |f|; V, E, F; |\mathrm{Aut}(\mathcal{H})|]$, where $|v|, |e|, |f|, V, E$ and F stand for the hypervertex valency, hyperedge valency, hyperface valency, number of hypervertices, number of hyperedges and number of hyperfaces of $\mathcal{H}$, respectively.

Corollary 3.6. *The H-sequence of the minimal Frobenius hypermap $\mathcal{H}(p,n,m,i,j) = (M; a^i, a^j b)$ is*

(i) $[p, m, m; mp^{n-1}, p^n, p^n; mp^n]$ *for $j = 0$;*

(ii) $[m, p, m; p^n, mp^{n-1}, p^n; mp^n]$ *for $j = m - i$;*

(iii) $[\frac{m}{(m,j)}, \frac{m}{(m,i+j)}, m; (m,j)p^n, (m,i+j)p^n, p^n; mp^n]$ *for* $j \neq 0$ *and* $j \neq m - i$.

Proof. The sequence is determined by the first three entries, namely $|a^j b|$, $|a^{i+j} b|$ and $|a^i|$. These entries can be easily calculated according to Proposition 2.7. □

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