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Assessing What Students Think About Solving Word Problems: The Case of Reversible Reasoning in Students

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Although reversible reasoning is an important strategy in solving mathematical word problems, few studies have examined the link between reversible reasoning and word problem solving. The present study therefore aims to examine students' thought processes in translating statements within word problems by integrating reversible reasoning. A qualitative approach was adopted involving 71 students with diverse backgrounds, genders, educational institutions and achievement levels. A task designed to stimulate reversible reasoning was developed and supporting stimulus questions were formulated. Data were collected through Google Form submissions, think-aloud protocols and interviews. Twelve students who demonstrated indications of reversible reasoning were interviewed to gain in-depth insights. The collected data were analysed using a case study approach comprising three stages: preliminary analysis, open coding and axial coding. Six research themes were identified related to the students' mental activities in reversing temporal sequences, reversing rate relationships, reversing residual perspectives, reversing variable roles, reversing mathematical operations and reversing rate-time concepts. The findings highlight the importance of designing learning experiences that stimulate mental flexibility, conceptual understanding and metacognition as frameworks for fostering reversible reasoning.

Keywords: word problem, problem solving, reversible reasoning

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Ocena mnenja študentov o reševanju besedilnih nalog: primer reverzibilnega razmišljanja pri študentih

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Čeprav je reverzibilno razmišljanje pomembna strategija pri reševanju matematičnih besedilnih nalog, je le malo dozdajšnjih študij preučevalo povezavo med reverzibilnim razmišljanjem in reševanjem besedilnih nalog. Namen te študije je zato preučiti miselne procese študentov pri prevajanju izjav v besedilnih nalogah z vključevanjem reverzibilnega razmišljanja. Uporabljen je bil kvalitativni pristop, v katerega je bilo vključenih 71 študentov, ki prihajajo z različnih ozadij, pripadajo različnim spolom in izobraževalnim ustanovam ter jih odlikujejo različne ravni dosežkov. Zasnovali smo nalogo, namenjeno spodbujanju reverzibilnega razmišljanja, in oblikovali podporna spodbujevalna vprašanja. Podatki so bili zbrani prek obrazcev Google, protokolov razmišljanja na glas in intervjujev. Za pridobitev poglobljenih vpogledov je bilo intervjuvanih dvanajst študentov, ki so pokazali znake reverzibilnega razmišljanja. Zbrani podatki so bili analizirani z uporabo pristopa študije primera, ki je obsegal tri faze: predhodno analizo, odprto kodiranje in aksialno kodiranje. Opredelili smo šest raziskovalnih tem, povezanih z miselnimi dejavnostmi študentov pri obratu časovnega zaporedja, obratu spreminjanja razmerja, obratu vidika ostanka, obratu vloge spremenljivk, obratu matematičnih operacij ter obratu odnosa med deležem in časom. Ugotovitve poudarjajo pomen oblikovanja učnih izkušenj, ki spodbujajo miselno prožnost, konceptualno razumevanje in metakognicijo kot okvire za spodbujanje reverzibilnega razmišljanja.

Ključne besede: besedilna naloga, reševanje problemov, reverzibilno razmišljanje

Introduction

A word problem is a verbal description of a mathematical situation that requires students to interpret, translate and solve the problem using mathematical reasoning and procedures (Csíkos & Szitányi, 2020; Verschaffel et al., 2020)they have been the object of a tremendous amount research over the past 50 years. This opening article gives an overview of the research literature on word problem solving, by pointing to a number of major topics, questions, and debates that have dominated the field. After a short introduction, we begin with research that has conceived word problems primarily as problems of comprehension, and we describe the various ways in which this complex comprehension process has been conceived theoretically as well as the empirical evidence supporting different theoretical models. Next we review research that has focused on strategies for actually solving the word problem. Strengths and weaknesses of informal and formal solution strategies—at various levels of learners' mathematical development (i.e., arithmetic, algebra. Solving word problems engages students in constructing mental and verbal representations and transforming them into mathematical expressions to derive a solution. Word problems are central in mathematics education because of their cognitive complexity and their role in applying mathematics to real-life situations. They are included in mathematics curricula to strengthen conceptual understanding, enhance reasoning skills and develop problem-solving competencies (National Council of Teachers of Mathematics, 2000; Verschaffel et al., 2000).

Assessing students' thinking when solving such problems is essential in order to uncover how they process information, apply strategies and reason through complex tasks. In the present study, "assessment" refers to a qualitative investigation of students' cognitive processes, particularly the ways in which they engage in and demonstrate mathematical reasoning. This type of assessment allows educators and researchers to identify students' strengths, challenges and developmental needs in problem solving. Despite its importance, however, the assessment of students' reasoning, especially reversible reasoning, remains underexplored in current mathematics education research. Reversible reasoning is a form of logical thinking whereby students work backward from known results to infer unknown quantities or previous steps (Ikram et al., 2020; Ramful, 2014). This kind of reasoning plays a crucial role in many mathematical word problems, such as comparing values, compensating for changes and solving inverse problems. The present study focuses specifically on assessing students' reversible reasoning, a higher-order extension of reversible reasoning that involves intentional, logical reconstruction of processes. While reversible reasoning typically refers to a cognitive ability to reverse a sequence of actions (Inhelder & Piaget, 1958; Olive, 2001), reversible reasoning emphasises the reasoning patterns and justifications behind such reversals, especially in symbolic or contextual mathematical tasks.

Despite their significant role in the teaching and learning processes of mathematics, word problems continue to be an area of struggle for students (Fuchs et al., 2021; Kaur, 2019). In the United States, for example, most elementary school students are inadequately prepared to solve word problems, in which they are required to interpret problem situations and integrate words with numbers (Powell et al., 2020). Meanwhile, in Hungary, word problems are typically introduced to elementary school students using simple, non-routine, one-step problems rather than using complex and realistic situations (Csíkos & Szitányi, 2020). The results from the Programme for International Student Assessment (PISA) 2022 show that only 9% of students were able to apply problem-solving strategies to solve complex real-world word problems (Organisation for Economic Co-operation and Development, 2023). Such findings also reflect challenges faced by students in Indonesia, where mathematics instruction heavily emphasises procedural skills and textbook-based exercises, leaving students unprepared to tackle complex word problems that require reasoning, modelling and reflection. National assessments and classroom observations indicate that students often fail to interpret relationships between quantities or apply inverse operations in problem contexts.

Investigating how students approach word problems provides the mathematics education community with valuable insights into students' thinking and ways to support their problem-solving strategies (Goulet-Lyle et al., 2020; Powell et al., 2020). One of the key cognitive abilities relevant to mathematical problem solving is reversible reasoning (Inhelder & Piaget, 1958; Olive & Steffe, 2001). Reversible reasoning refers to working backward in physical or mental actions to move from a known result to an unknown source that produced the result. Working with reversible relationships is challenging for students of all ages (Hackenberg, 2010), so it is essential to examine the strategies used when solving such problems. The present paper focuses on conducting a deep investigation on students' reversible reasoning processes when solving word problems.

Challenges in Solving Word Problems

Several researchers have investigated the difficulties students face when solving word problems (Vondrová et al., 2019; Xin, 2019). They have found that students struggle to translate verbal representations to mathematical expressions, such as modelling problems through addition, subtraction, multiplication and division operations. It is also challenging for students to comprehend the problem structure and relate different quantities involved in the problem situation to each other (Daroczy et al., 2015; Masriyah, et al., 2024; Mason, 2018). Word problems require students to comprehend the text, utilise their working memory and follow steps to develop solutions (Boonen et al., 2016). In some cases, they also require students to trace a known result back to its unknown source, demonstrating reversible reasoning (Ramful, 2015). Such features of word problems contribute to the challenges students face in solving them.

Students' difficulties in solving word problems also stem from their greater familiarity with numerical problems or computational skills (Fuchs et al., 2020). The limited capacity of students' working memory in allocating attention, sequencing and retrieving information from long-term memory also contributes to their failure in solving these problems (Lee et al., 2018). As a result, they struggle to bridge several gaps when solving word problems, such as: (1) failing to construct a coherent and hierarchical structure to capture key ideas from the problem; (2) being unable to complete problem interpretation to develop a mathematical model of the given situation; and (3) failing to identify appropriate problem models or schemas. These gaps in solving word problems reflect students' understanding and cognitive capacity in applying mathematical concepts.

Traditionally, word problems are often perceived as additional tasks used to practise previously learned concepts and rules (Tirosh et al., 2018; Verschaffel et al. 2020), leading educators to view word problems as mere computational tools (Chapman, 2003). This limited perspective on word problems contributes to students' difficulties in problem solving. Addressing these challenges requires strong reading, reasoning and problem-solving skills (Schoenfeld, 2016). Despite their complexity, word problems have been shown to help students connect mathematics to real-life contexts (Lee et al., 2018). In order to minimise difficulties, students should therefore be familiarised with using multiple representations to model word problems. In addition, they should be given opportunities to propose diverse solutions and guided in using various cognitive skills, such as inductive/deductive reasoning and reversible reasoning.

Supporting Students' Word Problem Solving Skills

Various efforts have been made by mathematics educators to minimise students' difficulties in solving word problems. First, problem-solving instruction through the use of word problems has become a hallmark of instructional practices observed in classrooms (Xin, 2019); students are explicitly taught problem-solving strategies to be used in working with word problems (Csíkos & Szitányi, 2020). However, some teaching strategies should be applied with caution, such as relying on the "keyword strategy" to decide which operation to use. In this approach, students are taught that specific words, such as "add" for addition, "take away" for subtraction and "times" for multiplication, indicate the corresponding mathematical operation. Such an approach does not necessarily lead students to develop conceptual knowledge or mathematical modelling skills.

Second, some curricula emphasise the use of visual representations (such as drawings), whereby students create images based on the problem structure in word problems (Greer, 2012). This approach helps students visualise the problem situation by representing objects or elements described in the word problem. Subsequently, they apply procedural strategies to solve the main problem. The visual representations can be developed by the students themselves or introduced by teachers. In either case, they facilitate the problem-solving process and support students in developing their conceptual understanding of the topics (Corter & Zahner, 2007).

Third, schema-based instruction encourages students to analyse the semantic meaning of word problems and map them into various representations (Hord & Xin, 2013). This approach requires students to first identify the problem situation and then represent it in an appropriate format. Subsequently, they must determine the necessary mathematical operations (addition, subtraction, multiplication or division) required to solve the given problem.

Fourth, students should be able to generate multiple strategies or solutions (e.g., arithmetic and algebraic) when solving word problems (Öçal et al., 2020). In arithmetic-based solutions, answers are derived through a sequence of calculations using numerical values from the problem. In contrast, algebraic solutions require the use of symbols or variables to arrive at the answer. Therefore, it is crucial for schools to adopt a proactive approach in minimising students' difficulties by implementing classroom instruction that effectively integrates word problems. Classroom interventions play a significant role in fostering a deeper understanding of the linguistic meaning embedded in problem situations, ultimately enhancing students' problem-solving abilities.

In order to solve word problems, students must apply their knowledge of mathematical operations so as to understand the mathematical elements embedded in a given problem situation. Additionally, they need to analyse the relationships within the problem context and identify unknown mathematical components. Therefore, a problem-solving model that effectively assesses students' problem-solving processes is essential. The most commonly used model for solving word problems is Polya's model, which consists of four steps: (1)

understand the problem, (2) make a plan, (3) carry out the plan, and (4) look back (Goulet-Lyle et al., 2020; Leong et al., 2012). In the Indonesian curriculum, the importance of Polya's model in problem solving is implicitly recognised and has been introduced at the elementary level as a guideline for helping students to solve problems in a structured and systematic manner.

Reversible Reasoning

Reversibility of thought requires learners to think logically by moving back and forth between input and output (Flanders, 2014). In this kind of thinking, learners can reconstruct the direction of a mental process, allowing them to approach problems flexibly and successfully. For example, in order to answer the question *"The following rectangle is one-third of a shape. Draw the whole shape."*, the student needs to reverse the process of finding one-third of a shape in order to reconstruct the whole. Krutetskii (1976) discussed two processes that are involved in reversible reasoning. The first process relates to the ability to move back and forth between a result and its source, while the second process involves the ability to reverse mental operations in reasoning. Both processes are often used in solving mathematical problems, making reversible reasoning critical for students' success in mathematics (Inhelder & Piaget, 1958; Olive & Steffe, 2001).

Inhelder and Piaget (1958) conceptualised reversibility in two forms: negation/inversion and compensation. Negation/inversion relates to inversing an operation. This form of thinking is similar to the working backward strategy that is widely used in mathematical problem solving (Ramful, 2015). This strategy requires an ability to reason that if quantity A is greater than quantity B, then quantity B is less than quantity A. The common conceptions of "subtraction as the inverse operation of addition" or "division as the inverse operation of multiplication" stem from this type of reversible reasoning.

The compensation form of reversible reasoning relates to the ability to deduce results when changes are made in more than one dimension (Inhelder & Piaget, 1958). For example, when water is poured from a cup into a narrower cup, a child with this skill can notice that the liquid quantity remains the same by compensating for the change in height with the change in width. Ramful (2015) provides an example of a percentage problem to illustrate the concept of compensation. For instance, if an item has a 10% discount and the problem provides the price after the discount, the solver must compensate for the reduction and reason that the final price represents 90% of the original price (100% - 10% = 90%). This form of thinking helps students to use equivalent forms of expressions when solving mathematics problems.

Students are likely to use both forms of reversible reasoning when engaging in problem solving that involves reversible situations. Although reversible reasoning is critical for students' mathematical performance, it remains one of the least examined areas in mathematical problem solving. Moreover, the majority of the existing research on reversible reasoning has focused on elementary school students (Pebrianti et al., 2022). There is a need to explore how older students engage in reversible reasoning when solving word problems. The present study aims to contribute to the relevant literature by examining students' reversible reasoning processes in the context of higher education in Indonesia.

The Study Goal and Research Question

International findings have shown that students across countries struggle with mathematical word problems. Similar challenges are evident in Indonesia, where mathematics instruction tends to emphasise procedural routines and textbook-based exercises. Despite the critical role of reversible reasoning in solving word problems, it remains underexplored in the context of higher education. Most existing studies have focused on elementary-level students, leaving a gap in understanding how advanced students engage with problems that demand logical reversibility. The present study aims to address this gap by conducting a fine-grained assessment of students' reversible reasoning strategies in solving mathematical word problems. The research question guiding the study is: *What cognitive strategies related to reversible reasoning do students use when solving word problems?*

Method

Participants

Data were collected from students enrolled in both public and private universities in Indonesia who had completed calculus and algebra courses. These students were still at the foundational level in higher education and were between 19 and 20 years old. The majority of the participants were recruited from the central region of Indonesia, where the authors' colleagues were requested to involve their students in the study. Eligible participants were identified through a screening survey developed by the authors. The inclusion criteria were as follows: (1) they were enrolled in a mathematics education or mathematical sciences programme, (2) they had completed their first year of study (Spring Semester 2020/2021), and (3) they had successfully passed calculus and algebra courses. All of the eligible participants received compensation in the form of study quotas for their participation. The analysis was limited to first-year students because their learning experiences were still recent, allowing them to transfer their prior knowledge from high school. A total of 73 students met the inclusion criteria for the study. A qualitative research approach was employed to address the research question, in which codes were developed representing the students' solutions, their understanding of the problem, and their interpretations in solving word problems.

As shown in Table 1, the study participants came from diverse backgrounds. A high level of mathematical proficiency was demonstrated by 72% of the sample, as indicated by their cumulative grade point average (CGPA), with a standard deviation of 3.73. The remaining 28% were categorised as having moderate proficiency. It should be noted that 90% of the students came from urban schools, most of which were high-achieving institutions. As a result, they were accustomed to solving non-routine mathematical problems.

Table 1
Background of the Participants

Background of the Participants	*N*	Study Samples
Education Programme		
Mathematics	34	46%
Mathematics Science	21	29%
Other	16	25%
Gender		
Female	48	64%
Male	23	36%
University Location		
Urban	64	90%
Rural	7	10%
Cumulative Grade Point Average (GPA)		
Above 3.51	51	72%
Between 3.00 and 3.50	20	28%

Table 1 presents the participating students' backgrounds. Background data such as education programme, gender and university location (urban/rural) were intentionally collected to provide contextual insight into the students' problem-solving experiences. Although these attributes are not treated as independent variables in the analysis, they support the interpretation of cognitive diversity in reversible reasoning. For instance, students from mathematics education programmes may emphasise pedagogical approaches to reasoning, while those from mathematical sciences may be more focused on formal structures. University location (urban vs. rural) may also reflect the students' prior exposure to non-routine problem-solving, which is often associated with opportunities in high-achieving urban universities.

Instrument

The participating students were asked to solve the word problem presented in Table 2. Instructions were provided on how they could submit their solutions via Google Forms. All of the students were obliged to upload their work, even if they felt unable to solve the word problem, i.e., they were still required to submit their responses through the form. This step was necessary in order to identify which parts of the word problem the students found challenging.

The task design also played a crucial role in how the students approached solving the word problem. This aspect was a key focus in a study by Savard & Polotskaia (2017). First, the task should be based on a situation involving an additive/multiplicative relationship between three quantities. Second, it should encourage the students to analyse these relationships. Third, the task should not contain explicit questions that could be directly answered, as this could prevent the students from engaging in deeper problem analysis. Fourth, the students need to learn how to analyse the problem situation from the given task. Fifth, the task should include simple and familiar words and expressions. Finally, the problem situation should integrate representations that align with the students' analysis. These six principles served as a reference in designing the task for the students in the present study.

Table 2

Task Design

Task
Task #1 Three painters, Joni, Deni, and Ari, usually work together. They can paint the exterior of a house in 10 working hours. Deni and Ari have previously painted a similar house together in 15 working hours. One day, the three painters worked together on a similar house for 4 hours. After that, Ari left due to an urgent matter. Joni and Deni needed an additional 8 working hours to complete the painting. Determine the time required for each painter to complete the job individually!

The aim of the designed word problem was to obtain comprehensive information about the students' understanding of operations and the contextual situation presented in the problem. In the present study, Task 1 focused on exploring the students' interpretation of the problem and the representations they produced. The task was refined by modifying and adding specific wording that required critical interpretation. The goal was to examine the extent to which the students interpreted the problem, the representations they generated, and the strategies they employed to produce solutions. Ultimately, the refined word problem also enabled us to assess the extent to which the students correctly applied problem-solving strategies.

The development of the task involved a systematic revision process. The first author initially drafted the task by identifying relevant instruments from previous research. Subsequently, discussions with co-authors and two mathematics education professors were conducted to obtain expert feedback. Based on their suggestions, the task was revised (e.g., by rephrasing ambiguous wording, removing leading prompts and adjusting the question structure) to ensure clarity and to elicit reasoning processes aligned with the study's objectives. To ensure content and construct validity, a formal validation rubric was used and independently evaluated by the two experts. The rubric included criteria such as linguistic clarity, mathematical accuracy, cognitive demand, alignment with reversible-reasoning principles and potential for eliciting varied solution strategies. After incorporating the experts' feedback, the final revised version of Task 1, as presented in Table 2, was administered to 73 students. The student responses were then categorised based on correctness, with 12 students consistently providing accurate answers. Six distinct groups of responses indicated the use of reversible reasoning in solving the problem: (1) reversing the order of time, (2) reversing the rate relationship, (3) reversing the perspective of the remainder, (4) reversing the role of variables, (5) reversing mathematical operations, and (6) reversing the rate-time concept. To further confirm the students' engagement in reversible reasoning, several follow-up questions designed to stimulate such reasoning were developed, as shown in Table 3.

Table 3

Questions Stimulating Students' Reversible Reasoning

Aspect	Questions	Objective
Reversing the Order of Time	What if Joni and Deni worked for 8 hours first, then Ari joined? Would the total time required remain the same?	Encourages the students to consider a different order of time and verify whether the outcome remains consistent.
Reversing the Rate Relationship	If Joni alone takes 30 hours, what is his work rate in houses per hour? How does this help determine Deni's and Ari's rates?	Encourages the students to convert time information into work rate and use it to find other variables.

Aspect	Questions	Objective
Reversing the Perspective of the Remainder	After 4 hours, the completed work is 3/5 of the total. What if 3/5 of the work is considered as the initial work?	Encourages the students to consider the remaining work as the starting point and calculate the required time.
Reversing the Role of Variables	Given the equations $J + D + A = 1/10$ and $D + A = 1/15$, how would you find the value of J? What happens if you solve for D or A first?	Encourages the students to manipulate the equations to find a specific variable and consider the order of solving.
Reversing Mathematical Operations	If you subtract 1/15 and 1/10 to find J, how would you verify that the result is correct?	Encourages the students to verify the mathematical operations by reversing their steps.
Reversing the Rate-Time Concept	If Joni has a work rate 1/30 house per hour, how long would it take to complete 3/5 of the house?	Encourages the students to convert rate into time in order to determine the required duration.

Twelve students with available time were invited to represent the reversible reasoning group. The goal was to have them rework the word problem task while clarifying their responses. They were encouraged to freely reflect on and comment on their answers. Thus, data collection was continued using the think-aloud method and interviews. For the think-aloud process, the students were asked to verbalise their thoughts aloud while solving the task. Their entire mental activity was recorded and later transcribed for analysis. For the interviews, a structured interview protocol was followed by asking the following questions: (1) Can you tell me what you were thinking? (2) What have you done so far? (3) In your own words, what are the key ideas of the problem? (4) Reflect on the concepts related to this task; and (5) How did you decide on this strategy to solve the problem? Additionally, any unclear moments when students remained silent were clarified by asking, "I noticed you paused. What were you thinking at that moment?" The interviews lasted 30–45 minutes. The data from the interviews and the students' written responses were used as references to answer the research question.

Research Design

The study employs a qualitative research approach with a case study design to examine how students understand and solve context-based mathematical word problems. The research specifically focused on identifying patterns in students' reversible reasoning when solving a mathematical task. A case study approach allows for an in-depth exploration of how students engage in problem solving, particularly focusing on their reasoning and strategies. The data were collected from 73 students who completed a context-based mathematical word problem via Google Forms. The students were instructed to submit their solutions regardless of whether they reached a final answer, enabling the researchers to capture a full range of reasoning strategies, including incomplete or incorrect approaches. The word problem used in the study was adapted and refined from prior research, then validated by two experts in mathematics education. The validation process involved assessing content appropriateness, clarity of mathematical structure, relevance to reversible reasoning, and the potential to elicit diverse strategies. Based on their feedback, the wording was revised, possible cues were removed, and the task was modified to be cognitively challenging yet accessible. This expert validation ensured that the instrument was both theoretically grounded and practically effective in eliciting meaningful data.

The transcript data generated from the students' written work and interviews with the first author were analysed using a case study approach (Creswell, 2012; Miles et al., 2014; Yin, 2011). The data analysis procedure involved three stages: preliminary analysis, open coding and axial coding. This procedure was chosen because the primary goal of the study was to develop categorisations, requiring a continuous comparative process for emerging categories. Moreover, since it was not possible to directly access the students' cognitive processes while solving the task, only their interpretations of the word problem could be modelled. Therefore, the analysis represents the best possible attempt to construct a hypothetical model of the students' thinking about the word problem, including key words and critical information in the problem statement. The transcript data was analysed using NVivo, a software tool that facilitates the management of qualitative data. The explanation of each analysis stage is provided below.

Preliminary Analysis

The initial data analysis began immediately after the interviews. At this stage, initial hypotheses were formulated based on the students' verbal expressions while solving the problem. These preliminary assumptions guided the follow-up questions posed by the interviewer. Based on the students' responses to the follow-up questions, the initial hypotheses were refined or adjusted. Additional questions were asked until sufficient data had been collected. After conducting each student interview, the research team convened to discuss the findings. As discussions progressed, the patterns in the students' problem-solving processes became clearer. Specifically, after conducting multiple interviews, the research team identified six distinct tendencies in how the students engaged with the word problem task. This led us to focus more on the situations that triggered the students to interpret problem information and to clarify their verbal expressions from the think-aloud process through subsequent interviews.

Open Coding

The transcripts were analysed by developing codes to describe important and relevant aspects of the students' word problem-solving processes. The initial codes were then refined and expanded to assess the various processes the students used to solve the word problems. This process continued until the responses of the two student groups (those who failed to apply reversible reasoning (n = 59) and those who successfully demonstrated reversible reasoning (n = 12)) were clearly identified. From the latter group, several students were selected for in-depth think-aloud sessions and interviews to gain deeper insights into their reasoning processes. At the end of the open coding process, six prominent themes emerged, broadly characterising the students' understanding while solving the word problem tasks: (1) Reversing the order of time, (2) Reversing the rate relationship, (3) Reversing the perspective of the remainder, (4) Reversing the role of variables, (5) Reversing mathematical operations, and (6) Reversing the rate-time concept. These themes provided a comprehensive representation of how the students conceptualised and approached solving the word problems. The categories were further refined during axial coding by comparing student responses across different triggers and reasoning expressions. NVivo software supported the coding process, enabling systematic categorisation and cross-case comparison. To ensure trustworthiness, the data were triangulated across written work, verbal data and researcher perspectives. Each transcript was independently coded by multiple researchers, followed by consensus meetings. The themes were then validated by external experts in mathematics education, and full verbatim transcripts were maintained to ensure transparency.

Axial Coding

After establishing the six themes, the findings were refined through axial coding. In order to strengthen the definitions of these themes, we compared the situations that triggered the students' responses with their prominent verbal expressions during problem solving. The students' responses were then analysed across the six themes to construct a more comprehensive description. The transcripts were subsequently re-coded using the refined themes. To ensure reliability, each member of the research team independently coded the transcripts, and any discrepancies were resolved through discussion until consensus was reached. Trustworthiness was further enhanced by producing verbatim transcripts of each interview and by validating both the coding process and the identified themes with several experts in mathematics education. The six themes for the word problem task are presented in Table 4.

Table 4
Themes and Codes Related to the Students' Reversible Reasoning

Theme	Code	Example	Description
Reversing the Order of Time	*Ordered_time*	What if Joni & Deni worked for 8 hours first, then Ari joined?	Reversing the order of work execution to explore alternative scenarios.
Reversing the Rate Relationship	*Rate_to_Time*	If Joni alone takes 30 hours, what is his work rate?	Converting time information into rate or vice versa.
Reversing the Perspective of the Remainder	*Quotient_Work*	After 4 hours, the remaining work is 3/5 of the total.	Considering the remaining work as the starting point to calculate the additional time needed.
Reversing the Role of Variables	*Substitution_ Variable*	From $J + D + A = 1/10$ and $D + A = 1/15$, we can find J	Manipulating variables within equations to determine specific values.
Reversing Mathematical Operations	*Operational_ Reverse*	Subtracting 1/15 from 1/10 to find *J*.	Reversing mathematical operations to verify the correctness of the solution.
Reversing the Rate-Time Concept	*Convertion_ Rate_Time*	If Joni's work rate is 1/30, then the total time required is 30 hours.	Transforming the concept of rate into time or vice versa to understand variable relationships.

Results

The findings of the study are presented in two sections. The first section focuses on a general overview of how the students solved the word problem. The number of correct and incorrect answers are quantified to provide an initial depiction of the students' problem-solving performance. The second section specifically highlights six student responses that indicate the use of reversible reasoning in solving the problem.

From the students' responses, various approaches to solving the problem were identified. In reversing the rate-time relationship, 13 of the 71 students mistakenly believed that the longer someone works, the less their contribution. This assumption was incorrect, as the longer a person works, the more of the house they complete. Additionally, 8 of the 71 students started with the total time and incorrectly distributed individual contributions; for example, assuming that all of the painters had the same work rate without considering that their work rates differed. The students' inability to correctly reverse the rate-time relationship led to a high rate of incorrect answers. One example of a misconception in reversing rate and time is illustrated in the following excerpt from an interview:

> "If the three painters complete the house in 10 hours, then each painter would also complete the house in 10 hours if working alone."

The next finding concerns the students' errors in reversing the order of information in the problem. Eleven of the 71 students misunderstood the problem situation, assuming that Joni and Deni worked for 4 hours after Ari left. In reality, Ari left after 4 hours, and these students mistakenly reversed the work duration of Joni and Deni. Additionally, 7 of the 71 students misinterpreted the statement that all three painters completed the task in 10 hours, incorrectly assuming that the individual work time could be directly calculated by dividing the total time equally. This assumption indicates that the students treated total work time as individual work time, disregarding differences in work rates. The following is an excerpt from one student's response, illustrating misconceptions in reversing the order of information in the problem:

> "Joni and Deni worked for 4 hours after Ari left, so they completed $4 \times 1/J + D$ of the house."

The next student error occurred when they reversed the calculation steps in solving the problem. Five of the 71 students attempted to reverse the calculation by assuming that each painter contributed equally, using a trial-and-error strategy that lacked systematic reasoning. In this case, the students started with the total work time and tried to divide it proportionally, but without considering work rates. Additionally, 3 students attempted to guess the individual work times for each painter rather than applying a structured approach. The following is an excerpt from one student's response, illustrating misconceptions in reversing the calculation steps in solving the problem:

> "I tried guessing Joni's work time first. If Joni takes 20 hours to complete the house, then I tried assigning 25 hours for Deni and 30 hours for Ari."

The next student error occurred in reversing the roles of individuals within the group. Two key issues related to this mistake were identified. Four of the students assumed that Deni and Ari contributed equally, without considering the given problem data. They misinterpreted the work rate relationship, reasoning that if two people complete the task in 15 hours, then each person alone would take 7.5 hours, failing to account for differing work rates. Two of the students swapped the roles of Joni and Ari in the final stage of the solution. These students incorrectly assumed that Ari continued working, rather than

Joni and Deni. The following is an excerpt from one student's response, illustrating misconceptions in reversing the roles of individuals within the group:

> "Since Deni and Ari complete the house in 15 hours together, that means each of them can complete the house in 7.5 hours."

Finally, the last student error in solving the problem was reversing the verification process of their answers. Six of the 71 students did not check whether their calculated time values correctly produced the expected work rates based on the given information. These students failed to verify their solutions, leading to inconsistent or incorrect results. This aligns with one student's statement in the following excerpt from the interview, demonstrating misconceptions in verifying answers:

> "I found that Joni takes 12 hours, Deni 18 hours, and Ari 24 hours. I am confident that this is correct."

Table 5 summarises common errors made by the students who failed to apply reversible reasoning in solving word problems.

Table 5
Student Errors in Solving Word Problems

No.	Student Error	Reason	Number of Students
1	Reversing the Rate-Time Relationship	Misinterpreting the relationship between work rate and time.	13
		Incorrectly applying the backward approach to determine individual work rates.	8
2	Reversing the Order of Information in the Problem	Assuming Joni and Deni worked for 4 hours *after* Ari left.	11
		Treating total time as individual work time.	7
3	Reversing Calculation Steps	Starting from the total time and attempting to divide it proportionally without considering work rates.	5
		Determining individual work time before establishing the group work rate.	3
4	Reversing the Roles of Individuals in the Group	Assuming that Deni and Ari contributed equally without considering the problem data.	4
		Swapping Joni's and Ari's roles in the final steps of the solution.	2
5	Reversing the Verification Process	Failing to check their answers by comparing the final result with the given problem conditions.	6
	Total		**59**

Table 3 indicates that 59 of the 71 students failed to provide a complete and accurate solution to the word problem. The five types of errors the students made resulted in mistakes that affected their performance, primarily due to a lack of engagement in reversible reasoning. The detailed explanation of these errors is as follows: (1) errors in reversing the rate-time relationship were caused by the students' misunderstanding of the relationship between work rate and total time; (2) errors in reversing the order of information in the problem resulted from the students' inability to logically sequence events and determine each worker's duration of work; (3) errors in reversing calculation steps stemmed from the students starting with the final result or using incorrect strategies to determine work rates; (4) errors in reversing individual roles within the group were due to the students' misinterpretation of each painter's contribution in the collaborative task; (5) errors in reversing the verification process resulted from the students' failure to apply backward strategies to check and correct errors in their answers; and (6) these findings highlight the importance of developing reversible reasoning skills, as errors in logical structuring, sequencing, and verification significantly impact students' problem-solving accuracy and efficiency. Of the 71 students who participated, 59 failed to provide a complete and accurate solution, as they tended to rely on surface-level reversals, such as simply inverting sequences or operations, without considering the conceptual relationships embedded in the problem. For instance, they attempted to reverse time sequences or calculate individual rates from group totals, but failed to maintain consistency across variables or misunderstood the inverse relationship between time and rate. Their strategies lacked coherence and often involved guesswork, incorrect assumptions or fragmented procedures.

Unlike the students who made errors, 12 of the 71 students successfully engaged in reversible reasoning while solving the word problem. In order to gain deeper insights into their mental activities during the problem-solving process, they were prompted with questions designed to stimulate reversible reasoning. These questions helped us to capture and describe in greater detail the thought processes of the students who effectively applied reversible reasoning strategies. This approach allowed us to analyse how these students structured their reasoning, manipulated information and verified their solutions, providing a clearer understanding of their problem-solving strategies. Compared to their peers, they demonstrated a higher level of reasoning maturity, characterised by systematic use of backward logic, strategic manipulation of rate and time relationships, and consistent alignment between operations and the problem context.

First, in reversing the order of work execution, the students imagined an alternative sequence of work different from the problem scenario and calculated its consequences. The students who successfully reversed the sequence of steps (e.g., working backward from the remaining work) demonstrated flexible reasoning skills and systematic problem-solving approaches. They employed heuristic strategies, such as working backward, to arrive at a solution. In this process, the students' mental activities involved mental simulations, whereby they visualised alternative scenarios and reasoned through cause-and-effect relationships to determine how the work sequence influenced the final outcome. This aligns with the following excerpt from one student's interview:

> "The three painters worked for 4 hours. I can calculate how much of the house was painted during that time: 4 x 1/10 = 4/10 of the house. This means the remaining part of the house is 1 – 4/10 = 6/10. I can use this information to determine the combined work rate of Joni and Deni."

From this statement, the reversible reasoning observed involved students moving forward by calculating the painting process, then moving backward by analysing the remaining work to determine the next work rate. This bidirectional reasoning process allowed the students to adjust their approach dynamically, ensuring that their calculations aligned with the given conditions. By integrating both forward and backward reasoning, the students were able to systematically verify and refine their solutions, demonstrating a deep understanding of the problem structure.

Secondly, in reversing the rate-time relationship, the students understood that the relationship between work time and work rate is inverse. If someone requires more time to complete a task, their work rate is lower, and vice versa. This indicates that the students were able to convert time information into work rate (1/*time*) and vice versa. Additionally, they demonstrated an understanding of proportional reasoning and the inverse relationship between rate and time. In this process, the students used mathematical representations to transform time information into rate ($1/T$) and vice versa. They also translated verbal information (time) into mathematical symbols (rate). For example, one student correctly calculated Joni's work rate as 1/30 house per hour from the given time (30 hours). Furthermore, the students engaged in abstraction by recognising the inverse relationship between rate and time. They also applied symbolic manipulation, using mathematical operations to convert units correctly. The following is an excerpt from one student's response, illustrating the mental activity involved in reversing the rate-time relationship:

> "If the three painters can complete the house in 10 hours, their combined work rate is 1/10 house per hour. If Deni and Ari can complete the

> house in 15 hours, their combined work rate is 1/15. I can determine Joni's work rate by reversing this equation: $1/J = 1/10 - 1/15$"

The statement illustrates that the reversible reasoning employed by the students encompasses several key aspects: proportional reasoning, whereby the students recognise and apply scaling relationships between work rate and time; inverse relationships between rate and time, understanding that as one increases, the other decreases; converting time information into rate (1/*time*) and vice versa, demonstrating the ability to shift between verbal descriptions and mathematical representations; and reversing operations to determine individual contributions, thus ensuring that work rates are correctly assigned based on the given conditions. This reasoning process highlights the students' ability to engage in flexible mathematical thinking, systematically applying inverse relationships and symbolic transformations to solve the problem accurately.

Thirdly, in reversing the perspective of the remainder, the students viewed the remaining work as the starting point for solving the problem. In this process, the students engaged in shifting perspectives to better understand the problem and analyse the relationship between the unfinished work and the time required. Additionally, the students modified the scenario (e.g., "What if Deni left instead?"), demonstrating their ability to mentally simulate alternative scenarios and their consequences. This approach allowed them to generate multiple solutions based on different possible conditions. The following is an excerpt from one student's response, illustrating their mental activity in reversing the perspective of the remainder:

> "After 4 hours, the remaining work is 3/5 of the total. If this 3/5 is considered as the initial work, the time required for Joni and Deni is $3/5 \ / \ 1/24 = 14.4$ hours. This shows that different perspectives can be used to solve the problem."

From the statement, the reversible reasoning demonstrated by the students involved the following strategies: shifting perspectives, whereby the students changed their initial viewpoint to see the problem from a different angle (e.g., considering the remaining work as the starting point); recognising the relationship between partial work and time, thus understanding how the unfinished portion of the task affects the overall problem structure; and scenario modification, whereby the students mentally adjusted conditions (e.g., "What if Deni left instead?") to explore alternative solutions. This reasoning process highlights the students' ability to engage in flexible and abstract thinking, allowing them to systematically adjust and restructure the given information to arrive at a solution.

Fourthly, the students reversed the role of variables by changing the sequence of solving for variables in the equation. In this process, the students engaged in equation manipulation to isolate different variables and identify alternative steps to reach a solution. Additionally, the students recognised that individual contributions to the work could be determined by subtracting the contributions of other workers from the total group work. This aligns with the following excerpt from one student's response, illustrating their mental activity in reversing the role of variables:

> "I know that Deni and Ari together can paint the house in 15 hours, meaning they complete 1/15 of the house per hour. Since all three painters work together at a rate of 1/10 house per hour, I can determine Joni's work rate by working backward, subtracting Deni and Ari's contribution from the total work rate."

From the excerpt, the reversible reasoning demonstrated by the students involved the following: rearranging the sequence of solving for variables in the equation, thus showing flexibility in choosing different solution paths; starting with the total contribution of all of the workers, then working backward by subtracting the known contributions to determine the unknown variable; and applying algebraic manipulation to isolate different variables, thus demonstrating an understanding of how individual contributions relate to the total work. This reasoning process highlights the students' ability to engage in systematic and strategic problem solving, using inverse operations to logically deconstruct the given information and derive the correct solution.

Fifthly, the students reversed mathematical operations (e.g., changing subtraction to addition) to verify their solutions. In this process, the students applied a forward approach to obtain an initial result, then worked backward by retracing their calculations to check for consistency and correctness. This aligns with the following excerpt from one student's response, illustrating their mental activity in reversing mathematical operations:

> "I found that Joni needs 20 hours to complete the house alone, Deni needs 30 hours, and Ari needs 40 hours. I will check whether their combined work rates match the problem statement: $1/20 + 1/30 + 1/40 = 1/10$."

In this case, the student performed logical verification to ensure the correctness of their answer and to evaluate the accuracy of the steps taken. Additionally, the students applied a forward-solving approach to obtain a result, then reversed the process to check whether their solution remained consistent with the given initial conditions. This demonstrates reversible reasoning, whereby the students systematically trace their steps backward to verify that their calculations align with the original problem constraints, ensuring solution validity and accuracy.

Sixthly, the students reversed the rate-time concept by converting rate into time or vice versa in order to better understand their relationship. In this process, the students recognised the inverse nature of rate and time and applied this conversion strategically to solve the problem. The following is an excerpt from one student's interview, illustrating their mental activity in reversing the rate-time concept:

> "If Ari has a work rate of 1/40 house per hour, the time needed to complete one house is *40* hours. Conversely, if Ari takes 40 hours, his work rate is 1/40 house per hour."

From the excerpt, the reversible reasoning demonstrated by the students involved the following: understanding the proportional relationship between rate and time, recognising that they are inversely related; converting rate into time and vice versa, showing flexibility in transitioning between different representations of the problem; and applying inverse operations to verify and refine their solution, ensuring consistency with the given conditions. This reasoning process highlights the students' ability to manipulate mathematical relationships dynamically, reinforcing their conceptual understanding of how rate and time interact in problem solving.

A summary of these six findings is presented in Table 6.

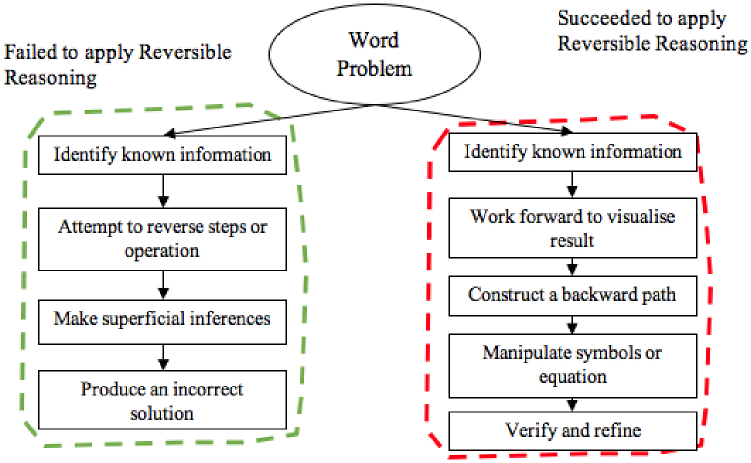
Table 6
Summary of Research Findings

Theme	Reversible Reasoning and Cognitive Activity	Student Response
Reversing the Order of Time	Modifying the sequence of events to observe their effects, involving mental simulation and cause-effect reasoning.	"The total time will be different because the sequence affects the work."
Reversing the Rate-Time Relationship	Converting time to rate or vice versa, involving abstraction and symbolic manipulation.	"Joni's rate = 1/30 house/hour, solo time = 30 hours."
Reversing the Perspective of the Remainder	Viewing remaining work as the starting point, engaging in cognitive flexibility and proportional reasoning.	"If 3/5 is considered the initial work, additional time = 14.4 hours."
Reversing the Role of Variables	Changing the order of solving variables, involving variable substitution.	If A = 1/40, then D = 1/24
Reversing Mathematical Operations	Reversing operations for verification, engaging in logical verification and metacognition.	"1/10 – 1/15 = 1/30 then 1/30 + 1/15 = 1/10"
Reversing the Rate-Time Concept	Converting rate to time or vice versa, involving proportional relationships.	"Ari's rate = 1/40 house/hour, so time = 40 hours."

Figure 1 provides a side-by-side schematic of the cognitive paths used by the students in the two groups. The path on the left in the diagram shows common reasoning detours, errors and fragmented attempts among the students who failed to solve the problem, such as incorrect reversals, superficial inferences and lack of verification. In contrast, the path on the right demonstrates the strategic and coherent reasoning sequence followed by the students who succeeded, starting with identifying known elements, followed by applying forward and backward reasoning iteratively, engaging in symbolic manipulation and performing logical verification. This figure visually synthesises the differences in reversible reasoning maturity and provides a concise framework for understanding the students' diverse problem-solving behaviours.

Figure 1

Comparison of thinking processes: unsuccessful versus successful application of reversible reasoning in solving word problems.



Discussion

The present study investigates how students think when solving word problems involving reversible reasoning. This research serves as a reinforcement of previous studies that have contributed to the literature on strategies that students develop when solving problems. For the activity of reversing the order of time, a study by Verschaffel et al. (2020) found that the ability of students to manipulate the sequence of time in mathematical word problems is closely related to their understanding of temporal reasoning. In the present study, we found that students who could imagine alternative scenarios (e.g., changing the order of tasks) tended to have a stronger grasp of causal relationships within problems. However, many students struggled with this due to a tendency towards linear thinking (Van Dooren et al., 2018). With regard to the house painting problem, a study by Star and Rittle-Johnson (2009) indicated that practising problems that require students to "reverse" scenarios enhances cognitive flexibility and problem-solving abilities.

The concept of rate as the inverse of time has been extensively studied in the context of proportional reasoning. Lamon (2007) highlighted the fact that students often struggle to convert between rate and time due to a lack of conceptual understanding of their inverse relationship. However, context-based interventions (e.g., work-sharing problems) have been shown to enhance this understanding (Cramer & Post, 1993). Next, the mental activity of reversing the perspective of the remainder aligns with research on perspective-taking in mathematics. Goldin (2000) found that the ability of students to view problems from different perspectives (e.g., focusing on the remaining work) is closely related to metacognitive skills. Students trained to "think backward" demonstrated significant improvements in solving complex problems (Selden & Selden, 2005). In the current context, students who were able to use the working backward strategy were more successful in solving the research task than those who relied solely on procedural approaches.

Reversing the role of variables requires an ability to manipulate variables within equations, which is a key aspect of algebraic flexibility. Blöte et al. (2001) found that students who could isolate different variables in an equation demonstrated a deeper understanding of algebra. However, many students tend to become trapped in procedural routines without truly understanding the underlying structure of equations (Kieran, 1992).

In reversing mathematical operations, the habit of verifying solutions by applying inverse operations serves as an indicator of metacognitive skills. Lucangeli and Cornoldi (1997) found that students who routinely check their answers using inverse operations demonstrated higher accuracy levels. However, many students tend to skip this step due to haste and time constraints (Santagata, 2005). In reversing the rate-time concept, a conceptual understanding of the rate-time relationship serves as a fundamental skill for topics in physics and economics. Thompson (1994) emphasised that students need to develop an intuitive understanding of rate as "amount per unit time", rather than merely applying formulas. Multimodal representations (such as graphs, tables and equations) have been shown to be effective in fostering this understanding (Ainsworth, 2006).

Compared to the majority of the students in the present study, who relied on procedural routines, the 12 students who successfully applied reversible reasoning demonstrated a more coherent and integrated problem-solving strategy. They were able to navigate between forward and backward reasoning paths, linking symbolic expressions with visual models and verifying their solutions through inverse operations. In contrast, the 59 students who failed to provide accurate solutions tended to employ surface-level reversals or isolated steps without connecting the underlying concepts, often resulting in guesswork, fragmented logic and inconsistent answers. This contrast highlights differing levels of reversible reasoning maturity and conceptual depth.

The six identified themes are not discrete, but rather interconnected elements of students' reasoning patterns. For instance, difficulties in reversing the order of time often co-occurred with errors in interpreting the rate-time relationship, suggesting a cascading effect between temporal sequencing and proportional reasoning. Similarly, students who reversed operations successfully were often those who had previously engaged in reversing roles or perspectives. These observations indicate that a weakness or strength in one type of reversible activity can influence success in others, reinforcing the idea that reversible reasoning is a coordinated network of cognitive actions rather than isolated skills (Ikram et al., 2020).

In order to clarify these differences and highlight the dynamic interplay between reasoning types, Figure 1 presents a comparative schematic of students' thought processes. It visually distinguishes between the linear and fragmented approach of unsuccessful students – marked by procedural shortcuts, incorrect reversals and lack of verification – and the flexible, bidirectional approach of successful students, who employed mental simulations, symbolic manipulations and logical verifications (Hackenberg, 2010; Ikram et al., 2020; Ramful, 2015). This visualisation emphasises the cognitive progression and reasoning quality that differentiates high-performing students.

The present study offers a novel contribution to the existing literature by synthesising the verbal, symbolic, visual and gestural dimensions of reversible reasoning within a single framework. While previous research has explored individual aspects, such as working backward or rate-time relationships, this study reveals how multiple modalities interact dynamically during complex word problem solving. The inclusion of gesture analysis and representational transitions (e.g., from verbal to geometric forms) provides deeper insight into students' internal cognitive restructuring processes, thus expanding the understanding of how reversible reasoning emerges and operates in real classroom contexts.

In this study, 12 of the 71 participating students demonstrated reversible processes while solving the word problem. During problem-solving, the students' minds were filled with self-generated questions, which triggered them to restructure the problem situation. It was observed that the students perceived the problem based on the initial situation, leading to a reconstruction of the structural relationship between the initial conditions and the final result. In other words, a transformation occurred from the result back to the source, following the same path from the source to the result (Hackenberg, 2010; Ikram et al., 2020; Ramful, 2015). This finding also suggests that the students constructed both forward and backward processes in order to establish analogous conditions related to the given problem (Hackenberg & Lee, 2015). Additionally, the students' reasoning was influenced by their perspective on the problem situation, as interconnected elements shaped their understanding (Paoletti, 2020).

Particular attention was devoted to the students who engaged in reversible processes. During problem-solving, the students activated reversible reasoning in the initial identification stage. We suspect that the students exhibited a familiarity with representing problems in algebraic form, but felt challenged to express the problem in alternative ways. In other words, from the outset, the students perceived the problem visually without considering other aspects (Haciomeroglu, 2015). The students' visual perspective stemmed from their awareness of the relationship between symbolic representation and visual-spatial representation in the problem they encountered (Natsheh & Karsenty, 2014). Visual thinking also emerged, as the students recognised the connection between the symbol-sense and visual-spatial aspects of the problem (Hong & Thomas, 2015). Moreover, the individual's mental model is inherently analytical, enabling them to execute visual thinking in problem representation (Hoffkamp, 2011); therefore, this visual awareness encourages students to express the problem in a visual format.

The students' awareness of the core problem, which they represented in a reversed manner, was closely linked to their identification of the problem situation. In other words, they already understood the objects involved in the problem and anticipated the expected outcomes, often representing them as concrete objects (e.g., a rectangular diagram) (Seah & Horne, 2019). The connection between the representations used to express the problem visually and the individual's ability to visualise it also served as a trigger for this type of thinking (Duval, 2006). This thinking process was further influenced by the internal representations individuals constructed to solve the problem (Fujita et al., 2017). Moreover, visual awareness played a crucial role in shaping the individual's thinking and prompted them to express problems geometrically (Mamolo et al., 2015). Geometric awareness, in turn, was influenced by the individual's ability to visualise the problem situation (Yao, 2020). Therefore, visual thinking serves as a foundation that stimulates individuals to engage in geometric reasoning.

The findings of this study indicate that the meaning of verbal information in the problem situation leads students to engage in in-depth analysis, ultimately enabling them to draw conclusions to express the problem. In other words, the verbal information contained in the problem influences the individual's perspective in interpreting the situation (Kar, 2015). From a semantic perspective, verbal information is processed mentally, generating ideas that are later used to solve the problem (Swidan & Fried, 2021). Additionally, when individuals interpret the verbal information of a problem, they assimilate this information into words, symbols, expressions or statements, forming a mental schema (Hong & Thomas, 2015). Therefore, the meaning of verbal information serves as an initial trigger, prompting students to immediately engage in visual thinking when solving the problem.

Conclusion

The results of this study reveal six types of students' mental activities when solving word problems involving reversible reasoning. These findings imply that students who can reverse time sequences, rate relationships, perspectives, variables, operations and rate-time concepts exhibit cognitive flexibility by viewing problems from multiple perspectives. Additionally, their reasoning involves conceptual understanding of the relationship between rate and time, as well as its inverse form. Moreover, these students demonstrate a habit of verifying solutions and reflecting on their thought processes.

The study highlights the need for instructional approaches that emphasise reversible reasoning. Notably, 59 of the 71 participating students failed to engage in reversible reasoning, particularly in working backward, variable substitution and using diverse representations to enhance learning outcomes. However, the study faced limitations in examining the extent of the instructional interventions experienced by the students who successfully engaged in reversible reasoning. Therefore, we recommend further investigation of the role of learning activities that contribute to students' development of reversible reasoning, thus providing deeper insights into its growth and progression. Furthermore, we recommend a long-term study to explore the development of students' reversible reasoning, particularly during critical periods of mathematical concept acquisition.

The findings of the present study suggest several practical applications in mathematics education. First, educators can design instructional activities that explicitly incorporate reversible reasoning to enhance students' problem-solving skills. For example, teachers can introduce structured problem-solving tasks that require students to work backward or explore inverse relationships, thus reinforcing their ability to analyse problems flexibly. Additionally, incorporating visual representations, such as diagrams, graphs and interactive models, can help students better understand the structural relationships within mathematical problems. Moreover, curriculum developers can integrate reversible reasoning strategies into learning modules, particularly in algebra, proportional reasoning and problem-solving exercises. Providing students with metacognitive training, whereby they actively reflect on their reasoning processes and solution strategies, can further strengthen their ability to engage in reversible reasoning. Finally, professional development programmes for teachers should emphasise instructional methods that promote cognitive flexibility, thus equipping educators with effective strategies to foster reversible reasoning skills in students. By implementing these applications, mathematics instruction can better support students in developing higher-order thinking skills, ultimately improving their ability to approach complex mathematical problems with greater adaptability and conceptual understanding.

Ethical statement

This research was conducted in accordance with ethical standards for pedagogical research. The study was reviewed and approved by the Universitas Cokroaminoto Palopo, Indonesia, and Universitas Negeri Makassar, Indonesia Ethical Research Committee, which confirmed the ethical suitability of the research. All of the procedures followed ethical guidelines to ensure the protection of participants' rights and data privacy.

Disclosure statement

None of the authors have any conflict of interest to declare. No financial or non-financial interests influenced the design, implementation or reporting of this research.

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