ARS MATHEMATICA CONTEMPORANEA

Volume 17, Number 1, Fall/Winter 2019, Pages 1–347

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The Society of Mathematicians, Physicists and Astronomers of Slovenia
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The Slovenian Discrete and Applied Mathematics Society

The publication is partially supported by the Slovenian Research Agency from the Call for co-financing of scientific periodical publications.

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In 2018 we launched a purely electronic journal *The Art of Discrete and Applied Mathematics* (ADAM), which we like to see as a sibling of the AMC. Although the journals are similar, in many ways they are complementary. While AMC publishes mostly longer papers, ADAM welcomes shorter papers and notes.

The main reason for introducing the new journal was to relieve pressure of articles submitted to AMC. Currently we publish 80 papers per year in AMC, a great leap from the 20 papers we published in 2008. But even with this increase, the acceptance rate remains quite low: a little less than 28 %.

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When ADAM is covered by the Web of Science, these two journals will indeed become twin journals. This will enable us to transfer papers between the two journals in order to pursue their respective goals and purposes. We hope this will happen in the forseeable future.

Klavdija Kutnar, Dragan Marušič and Tomaž Pisanski
Editors in Chief

Contents

Volume 17, Number 1, Fall/Winter 2019, Pages 1–347

ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 1–35
https://doi.org/10.26493/1855-3974.1800.40c
(Also available at http://amc-journal.eu)

New methods for finding minimum genus embeddings of graphs on orientable and non-orientable surfaces*

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Received 14 September 2018, accepted 4 December 2018, published online 19 June 2019

Abstract

The question of how to find the smallest genus of all embeddings of a given finite connected graph on an orientable (or non-orientable) surface has a long and interesting history. In this paper we introduce four new approaches to help answer this question, in both the orientable and non-orientable cases. One approach involves taking orbits of subgroups of the automorphism group on cycles of particular lengths in the graph as candidates for subsets of the faces of an embedding. Another uses properties of an auxiliary graph defined in terms of compatibility of these cycles. We also present two methods that make use of integer linear programming, to help determine bounds for the minimum genus, and to find minimum genus embeddings. This work was motivated by the problem of finding the minimum genus of the Hoffman-Singleton graph, and succeeded not only in solving that problem but also in answering several other open questions.

Keywords: Graph embedding, genus.

Math. Subj. Class.: 05C10, 05E18, 20B25, 57M15

*The authors are very grateful to Tomaž Pisanski for suggesting that they extend their initial development of the orbit method on the Hoffman-Singleton graph to other graphs, which then led to them to develop the other methods presented here, in order to find the answers to many open questions. The authors are also grateful to the referee for some helpful suggestions about presenting their work. The first author is grateful to the N.Z. Marsden Fund for its support (grant UOA 1626), and acknowledges the use of the MAGMA system [2] for computational experiments and verification of a number of discoveries announced in this paper, as well as Sage [45] in combination with IBM CPLEX for a small number of the ILP computations. The second author acknowledges partial support from the Spanish MEC project ICWT (TIN2016-80250-R) and ARES (CONSOLIDER INGENIO 2010 CSD2007-00004).

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1 Introduction

The question of how to find the smallest genus of those embeddings of a given finite connected graph on an orientable (or non-orientable) surface is a natural extension of determining whether or not a graph is planar, and has a long and interesting history. It is also quite an important question, with applications found in map colouring, topology, finite geometry (configurations and block designs), group theory, number theory and the design of electronic circuits.

Pioneering work was done by Dyck and Heffter in the late 1800s [13, 22], but it was not until the mid-1900s that significant progress was made, leading to the determination by Ringel [38, 39] of the minimum non-orientable genus of the complete graph K_n (for $n > 7$) and the minimum orientable and non-orientable genera of each of the complete bipartite graphs $K_{m,n}$, and then the determination by Ringel and Youngs [40] of the minimum orientable genus of the complete graph K_n (as a key step towards their proof of the Heawood Map Colouring Problem).

Youngs also gave the first proof of the (now) well known fact that every orientable embedding of a connected graph is determined by the rotations of edges at its vertices [52], and this was taken further by Duke [12] to show that the range of genera of embeddings of a given connected finite graph is an unbroken sequence of non-negative integers (from the minimum genus to the maximum genus of the graph). Similar theory was developed by various people for embeddings on non-orientable surfaces; details may be found in [44]. It is worth noting here that a minimum genus non-orientable embedding of a graph is not necessary a 2-cell embedding, but unless the graph is a tree, there is always at least one minimum genus non-orientable embedding which is a 2-cell embedding; see [35].

In the later 1990s, the minimum orientable genus was found for several graphs and families of graphs, some of which are given in [44, Tables I and II]. In many of these families, the graphs have a large degree of symmetry, which can be helpful to a large extent in finding nice embeddings. Various authors developed a range of techniques that can work well for many classes of graphs, involving rotation systems, voltage graphs, edge insertions and deletions, graph contractions, graph amalgamations and graph products. Some of these are described nicely in Gross and Tucker's book on topological graph theory [19].

On the other hand, some other examples proved quite challenging, even when they were vertex-transitive. Notable cases include the Cartesian product $C_3 \square C_3 \square C_3$, a 6-valent graph of order 27 which took some years to deal with (see [32, 4]), the 3-valent Gray graph of order 54 (see [30]), and the associated Doyle-Holt graph, a 4-valent graph of order 27 (considered 13 years ago in [30] and dealt with at last in this paper).

The difficulty is not surprising, even for small graphs, in that a k-valent regular graph of order n has $((k-1)!)^n$ distinct embeddings into an orientable surface. Furthermore, in 1989 it was shown by Thomassen [47] that the problem of finding the minimum orientable genus of a graph is NP-hard, and the problem of determining whether or not the minimum orientable genus of a connected graph is a given non-negative integer g is NP-complete.

Also the problem of deciding whether or not a graph can be embedded in an orientable surface of given genus g has been considered. A polynomial-time algorithm to solve this problem was presented in 1979 by Filotti, Miller and Reif [14], but then shown in 2011 to be flawed, by Myrvold and Kocay [34]. In the meantime, in 1999 Mohar [31] produced an algorithm for this that runs in linear time in the graph order, but doubly-exponential in the genus. In the case where the graph has no such embedding, the latter algorithm returns a minimal subgraph that cannot be embedded in the given surface, and its validity gives a constructive proof of the theorem of Robertson and Seymour [42] for any given closed surface, there are only finitely many minimal forbidden subgraphs.

In contrast, finding the maximum genus of orientable embeddings of graphs is much easier, thanks largely to some work in the 1970s by Xuong, who in [51] gave a formula for this number in terms of the minimum 'deficiency' of spanning trees for the graph. Ten years later Škoviera and Nedela used Xuong's work in [43] to prove that almost every vertex-transitive connected graph is *upper-embeddable* (in the sense of having a maximum genus embedding with just one or two faces), and indeed that this happens whenever the graph has valency or girth greater than 3.

In this paper we make further progress on the problem of finding the minimum genus of graphs (in both the orientable and non-orientable cases). Our work was motivated by a question by the second author about the minimum genus of the Hoffman-Singleton graph, which arose in joint work with Izquierdo on geometries associated with Moore graphs [46].

The Hoffman-Singleton graph is the unique Moore graph of valency 7 and diameter 2 (and indeed the largest known Moore graph of diameter 2), and accordingly, is a 7-valent connected graph of order 50, diameter 2 and girth 5. The properties of this graph, including its order and valency, made it challenging to find the minimum genus using existing methods (as summarised in [50] for example), and so we had to take a new approach. By considering the action of subgroups of the automorphism group of the graph on cycles of small length, we were able to find a minimum genus embedding on a non-orientable surface with pentagonal faces, and then adapt our approach to find a minimum genus orientable embedding as well.

We wrote up an early version of this paper describing our approach and the results, but perplexingly, had difficulty in getting it accepted by a good journal (despite finding a solution to a very challenging problem and developing a significant new approach in order to do that). Then we got some highly astute advice from Tomaž Pisanski, who suggested that we should apply our new approach to more examples, to underline its effectiveness. So we proceeded to do that, and used our new approach to find (for the first time) the minimum orientable or non-orientable genus of several other graphs, and answer a number of open questions about some of these.

The approach we took for the Hoffman-Singleton graph, which we call *the subgroup orbit method*, is useful for finding embeddings of graphs on surfaces with a certain degree of symmetry. The method considers candidates for a subgroup G of suitable order in the automorphism group of the graph such that G induces a group of automorphisms of the embedding, and this helps to reduce the complexity of the search for such an embedding. The automorphism group of a graph embedding is a subgroup of the automorphism group of the underlying graph, and acts semi-regularly on the 'flags' of the embedding (see Subsection 3.1), so $|G|$ must divide the number of flags, which is four times the number of edges of the graph. Orbits of G on closed walks of chosen lengths in the graph are taken as possibilities for the boundaries of faces of the embedding, and then tested for compatibility, completeness and orientability.

The subgroup orbit method works well for finding embeddings with face-transitive automorphism group, but can also work well in other cases where the automorphism group of the embedding has a small number of orbits on faces, and the lengths of those faces are close to the girth of the graph. But of course it is a lot to expect such properties, and indeed for some of the graphs we investigated, there were no such embeddings. For those, we had to develop other methods, which appear to be new as well.

These methods involve a more direct consideration of ways in which cycles in the graph can bound the faces of an embedding. Our second method involves creating an auxiliary graph, with vertices taken as particular cycles in the graph, and adjacency indicating when two such cycles cannot be taken simultaneously as faces of an embedding, and then using the independence number of the auxiliary graph to give an upper bound on the number of faces (and hence a lower bound on the minimum genus). According to Carsten Thomassen (in a private communication), this approach has not been taken before. Our third approach uses (mixed) integer linear programming to achieve the same thing when the auxiliary graph method is not helpful, and our fourth method uses integer linear programming directly for finding the faces of a minimum genus embedding of the graph.

All of these methods are quite general, in the sense that they do not expect the given graph to possess some non-trivial symmetry, even though we developed each of them to deal with graphs that do.

In particular, our new methods enabled us to prove the following:

(a) the minimum non-orientable genus of the Cartesian product graph $C_3 \,\square\, C_3 \,\square\, C_3$ is 13, answering a 1998 question by Brin and Squier [4],

(b) the minimum non-orientable genus of the Gray graph is 13, complementing the determination in 2005 of its minimum orientable genus in [30],

(c) the minimum orientable genus of the Doyle-Holt graph is 5, answering a 2005 question by Marušič, Pisanski and Wilson [30],

(d) the minimum non-orientable genus of the Doyle-Holt graph is 8, complementing (c),

(e) the minimum orientable genus of the dual Menger graph of the Gray (27_3) configuration is 6, answering two more questions from [30], and its minimum non-orientable genus is 11,

(f) the minimum orientable genus of the second smallest semi-symmetric 3-valent graph (which has order 110) is 15, answering the penultimate question in [30], and its minimum non-orientable genus is 28, and

(g) the minimum orientable genus of the Ljubljana graph (which has order 112) is 13, answering the final question in [30], and its minimum non-orientable genus is 27.

We also found the minimum orientable and non-orientable genera for several other interesting graphs, including the Folkman graph and Tutte's 8-cage.

Many of the discoveries mentioned above are described in this paper, in each case to illustrate the particular method(s) we used to make them. Before that, we give some further background in Section 2. Then we describe our 'subgroup orbit' method in Section 3, our 'independence number' approach in Section 4, and our integer linear programming approach in Section 5.

2 Further background

In this section we give further background on graph embeddings, known as maps, and we briefly describe their connection with geometric realisations of certain set systems, and also explain the use of voltage graphs to construct embeddings of particular kinds of graphs.

2.1 Graph embeddings

By an *embedding* of a connected graph X we mean a 2-cell embedding of X on some closed surface S. In particular, such an embedding has the property that when the graph is removed from the surface S, it breaks up S into simply-connected open regions (homeomorphic to open unit disks), called the *faces* of the embedding. (Note here that we do not require the closure of a face to be homeomorphic to a closed unit disk.) Such an embedding of a graph is also called a *map*, and then the graph X is the 1-skeleton of the map M.

Next, if we denote the sets of vertices, edges and faces of the map M by V, E and F respectively, then by the well known Euler-Poincaré formula we have

$$|V| - |E| + |F| = \chi,$$

where χ is the *Euler characteristic* of the surface S. If S is orientable, then $\chi = 2 - 2g$ where g is the genus of S (and of M), and in that case; furthermore, in the special case where $g = 0$ (and $\chi = 2$), the map M is called *planar* or *spherical*, while if $g = 1$ (and $\chi = 0$) then M is *Euclidean* or *toroidal*, and if $g > 1$ (and $\chi < 0$) then M is *hyperbolic*. On the other hand, if S is non-orientable, then $\chi = 2 - p$ where p is the genus of S, with $p = 1$ when S is the projective plane, or $p = 2$ when S is the Klein bottle, and so on.

A given graph X may have several different embeddings, and the Euler characteristic (and hence also the genus) of each one is determined by the number of resulting faces, since the numbers of vertices and edges are exactly the same as for the graph X. In the orientable case, the smallest and largest achievable values of the genus g are called the *minimum orientable genus* and the *maximum orientable genus* of X, respectively. The minimum orientable genus is often called simply the *genus* of X, and denoted by $\gamma(X)$. Similarly, in the non-orientable case, the smallest and largest achievable values of p are the *minimum* and *maximum non-orientable genus* of X, respectively. The former is sometimes also called the *cross-cap number* of X, and is denoted by $\overline{\gamma}(X)$. In both cases, the minimum genus occurs when the number of faces is maximised, or equivalently, when the average face-size is minimised.

As mentioned in the Introduction, every embedding of a connected graph X on an orientable surface is uniquely determined by the cyclic orientation of the edges at each vertex, giving what is known as the 'rotation system' of the embedding. Equivalently, the embedding can be described by giving a set of closed walks (not necessarily simple cycles) bounding the faces, with consistent orientation and folding well around each vertex. For example, if the (anti-clockwise) rotations at the vertices 1 to 4 of K_4 are taken as those which induce the permutations $(2, 3, 4)$, $(1, 4, 3)$, $(1, 2, 4)$ and $(1, 3, 2)$ on their neighbours, respectively, and we trace faces anti-clockwise (by 'turning left' at each successive vertex, then the faces are bounded by the cycles $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 2)$, $(2, 4, 3)$, and this gives an orientable embedding of characteristic $\chi = 4 - 6 + 4 = 2$ and minimum orientable genus 0. If we then replace the rotation at vertex 4 by its inverse, then the faces are bounded by the cycle $(1, 2, 3)$ and the closed walk $(1, 3, 4, 2, 1, 4, 3, 2, 4)$, giving an orientable embedding with $\chi = 4 - 6 + 2 = 0$ and maximum orientable genus 1.

For non-orientable embeddings, the situation is a little more complicated. Any such embedding can also be described by cyclic orientation of the edges at each vertex, or by a set of closed walks bounding the faces, but without consistent orientation. For example, there exists a non-orientable embedding of K_5 with $\chi = 5 - 10 + 6 = 1$ and minimum non-orientable genus 1 with faces bounded by the cycles $(1, 3, 5)$, $(1, 3, 4)$, $(2, 4, 3)$,

$(1,5,2)$, $(2,5,4)$, $(1,4,5,3,2)$, and for this, the local orientations at vertices 1 to 5 are given up to reversal by the cyclic permutations $(2,5,3,4)$, $(1,3,4,5)$, $(1,4,2,5)$, $(1,3,2,5)$ and $(1,2,4,3)$ of their neighbours, but there is no consistent way of orienting these cycles that gives an orientable embedding with the same face-bounding cycles. A connection between the two descriptions above can be made by 'twisting' some edges. Further details are explained in [19, 44] for example.

Finally, before continuing, we make two more points. One is that we may assume that the given connected graph X has no vertices of valency 1 or 2, as their presence does not affect the minimum (or maximum) genus of the graph: in any embedding, a leaf can be added to any vertex, and similarly, a new vertex of valency 2 can be inserted into any edge, without altering the genus. Another is that sometimes for ease of expression we will use F_k to denote the number of faces of size/length k, and F_ℓ to denote the number of faces that are larger than some prescribed integer k.

2.2 Connections with geometric realisations of block designs and configurations

Closely related to the study of embeddings of graphs in surfaces is the study of geometric realisations of set systems, especially block designs and combinatorial configurations.

In 1897, Heffter observed that certain triangular embeddings of graphs in surfaces can be used to construct two-fold triple systems, with the role of the blocks being played by the faces of the map; see [23]. Subsequent work by others took this further, and showed a link between partially balanced incomplete block designs (PBIBDs) and triangular embeddings of strongly regular graphs, for example. Further details can be found in the surveys [16, 17].

A combinatorial configuration is a set system with intersection properties that mimic the properties of geometric configurations of points and lines, or occasionally configurations of other geometric objects such as circles, planes, and so on. Geometric realisations of configurations make up an important and classical area of geometry, described for example in books by Grünbaum [20], Hilbert and Cohn-Vossen [24] and Pisanski and Servatius [37]. Many authors consider embeddings of the Levi graph (incidence graph) of a configuration in a surface to be a geometric embedding of the configuration — see for example the work by Coxeter in [10]. Similarly, geometric realisations of neighbourhood geometries were considered by Van Maldeghem in [48].

On the other hand, any isometric embedding of a graph on a surface gives a geometric realisation of a point-circle configuration, by drawing a circle through the neighbourhood of each vertex of the graph. This was first observed by Gévay and Pisanski for the Euclidean plane [15], and later by Izquierdo and Stokes for other surfaces [46]. Note that this way of realising configurations geometrically is essentially different from the embeddings of block designs described above, because it is not the faces but rather the rotation systems of the embedded graph that constitute the blocks (or circles) of the geometric set system. In particular, isometric embeddings of Moore graphs induce geometric realisations of balanced pentagonal geometries, and this was the motivation for our initial work on embeddings of the Hoffman-Singleton graph, as explained in [46].

2.3 Voltage graphs and covering graphs

Voltage graphs provide a very good way to describe or construct covers of a given smaller graph (or multigraph), and can also be used to construct certain kinds of embeddings of such covering graphs. Here we give a brief summary of some key points about these things, and refer the reader to [18, 19] for further details.

Let X be any finite graph whose automorphism group $A = \mathrm{Aut}(X)$ has a non-trivial subgroup B that acts semi-regularly on $V(X)$ and $E(X)$, meaning that every non-trivial element of B fixes no vertex or edge of X. In this case, all orbits of B on $V(X)$ or $E(X)$ have the same length $n = |B|$. Then we may define a smaller graph Y whose vertices are the orbits of B on $V(X)$, and an edge joins two such vertices if and only if some edge of X joins a pair of vertices in the corresponding orbits. In particular, Y is a quotient of X, and X is a regular cover of Y.

Now choose a set of representatives of the orbits of B on $V(X)$, and let $\overline{v}$ be the representative of the B-orbit containing a vertex v. If $\{v, w\}$ is any edge of X, then so is $\{\overline{v}, \overline{w}^{\beta}\}$ for some $\beta \in B$, and hence so is $\{\overline{v}^{\alpha}, \overline{w}^{\beta\alpha}\}$ for all $\alpha \in B$. Accordingly, there is an arc from $\overline{u}^B$ to $\overline{v}^B$ in the quotient graph Y that we can label with the element β of B. (Also the reverse are could be labelled with β^{-1}, but that is not necessary.) After doing this for an edge from each orbit of B on $E(X)$, we have a directed labelling of the edges of Y that gives enough information to define the covering graph X uniquely, with B considered as a regular permutation group of degree $n = |B|$. When so labelled, the quotient graph Y is called the *voltage graph*, and B is called the *voltage group*, while X is the *derived graph*, constructible from the graph Y and the voltage assignments.

The vertex-set of the derived graph can be regarded as the Cartesian product $V(Y) \times B$, and its edges are of the form $\{(y, \alpha), (z, \beta\alpha)\}$ where $\alpha \in B$, and (y, z) is an arc of Y labelled with $\beta \in B$. To see the connection with constructing Y from the derived graph X, note that y and z may be viewed as $\overline{v}$ and $\overline{w}$, and (y, α) as $v = \overline{v}^{\alpha}$, and $(z, \beta\alpha)$ as $\overline{w}^{\beta\alpha}$.

The voltage graphs described above are also called *regular voltage graphs*, and they correspond to regular coverings of graphs. *Permutation voltage graphs* were introduced by Gross and Tucker in [18], where they proved that it is enough to use permutations from a symmetric group as labels on the (possibly multiple) edges of a voltage graph, to represent an ordinary covering of a given graph. Any regular voltage graph can be expressed as a permutation voltage graph. More generally, a *branched covering of a graph* (which in the literature is also known as a *wrapped quasi-covering of a graph* (see [27, 36])) is a pair of graphs, similar to the pair consisting of a permutation voltage graph and its derived graph, except that branched (or wrapped) vertices are also allowed.

Next, embeddings of the voltage graph Y can also be used to construct embeddings of the derived graph X. To do this, simply assign a cyclic rotation of the edges at each vertex of Y, and then use the voltage assignments to give the analogous rotations at the corresponding vertices of X.

One particularly good feature of this process is that it preserves much of the symmetry of the initial embedding – and indeed there are many cases where a highly symmetric or minimum genus embedding can be described in terms of a voltage graph (see [29]). Not all embeddings of the derived graph X can be obtained in this way, however, as we will see with the Hoffman-Singleton graph. Given a nice embedding of a (branched) cover, it is not certain that the quotient of this embedding is an embedding which is easily recognisable as nice embedding for lifting. In other words, it is not usually clear in advance what kinds of embeddings of the voltage graph (or even what voltage groups and voltage assignments) will result in particularly nice embeddings of the derived graph.

In Section 3.4, we will compare one of our methods for finding graph embeddings with methods that use coverings and voltage graphs.

3 The subgroup orbit method

Here we present the method that we used successfully to find minimum genus embeddings of many of the graphs mentioned in the Introduction. It works well for finding embeddings of a graph with certain degree of non-trivial symmetry. The method uses selected elements of the automorphism group of the graph to construct an embedding which will have an automorphism group featuring at least the selected automorphisms.

3.1 Motivation

This method was inspired by properties of regular maps.

A *flag* of a map M is usually defined as an incident vertex-edge-face triple (v, e, f) in M, but more technically it should be defined as follows, to avoid ambiguity in cases where an edge e lies in just one face. Subdivide each face f of length k in M into $2k$ topological triangles, with the vertices of each triangle being the centre of the face f, a vertex v of M on the boundary of the face f, and the mid-point of an edge e incident with both v and f. We then call each such triangle a flag of M. In this way, every edge of M lies in four flags (with two for each choice of the vertex v).

An automorphism of map M is a bijection from M to itself that preserves its vertex-set, edge-set and face-set, and preserves incidence between these sets. By connectness, every automorphism of M is uniquely determined by its effect on any flag, so the automorphism group of M (denoted by $\mathrm{Aut}(M)$) acts semi-regularly on flags, and it follows that $|\mathrm{Aut}(M)|$ divides the number of flags, namely $4|E(M)|$.

A map M is called *regular* if $\mathrm{Aut}(M)$ is transitive (and hence acts regularly) on the flags of M, or if M is orientable and the group of all orientation-preserving automorphisms of M acts regularly on the arcs of M; see [11] (or [9], for example). These two definitions are not equivalent (indeed the two cases are different, but not mutually exclusive). In both cases the automorphism group of M has a single orbit on faces, and if the face-size is small enough then M can be expected to be a minimum genus embedding of X. (For example, this always happens when all faces of M are triangular.) There are also non-regular maps whose automorphism group has a small number of orbits on faces, and again if the faces are small, then these can give minimum genus embeddings of the underlying graph.

Our method finds minimum genus embeddings for which some non-trivial subgroup of the automorphism group of the graph induces a group of automorphisms of the map, usually with a small number of orbits on faces, when such a subgroup exists.

Before describing it, we repeat the observation that the smallest genus embeddings have the largest possible number of faces (in each of the orientable and non-orientable cases). Also we note the following.

Lemma 3.1. *If X is a connected finite graph of girth g, then in any embedding of X, every face has size at least g, and the number of faces is at most $2|E(X)|/g$.*

Proof. The first conclusion is obvious, and the second follows by counting incident edge-face pairs, which shows that the sum of the sizes of all faces at most $2|E(X)|$. □

The above observations show that it makes sense to consider cycles in the graph of relatively small length (either girth cycles, or 'almost' girth cycles) as possibilities for the closed walks bounding the faces of a small genus embedding. We also use subgroups of the automorphism group of the graph (of order dividing $4|E|$) to reduce the search space.

3.2 Description

Our subgroup orbit method proceeds as follows, for the given connected graph X:

Step 1. Find the set $\mathcal{C}$ of cycles of X of small lengths of interest.

Step 2. Find the automorphism group of X and its conjugacy classes of subgroups.

Step 3. For every representative subgroup G of order dividing $4|E(X)|$ in $\mathrm{Aut}(X)$, taken in decreasing order of G,

- **(a)** find the set $\mathcal{S}$ of orbits of G on the cycles in $\mathcal{C}$,
- **(b)** find subsets of $\mathcal{S}$ whose union forms the set of faces of an embedding of X,
- **(c)** for each such subset, determine the orientability and genus of the resulting map.

Note that Step 3(b) requires checking that the union of the chosen subsets of $\mathcal{S}$ uses every edge exactly twice; in particular, the sum of the lengths of the cycles in the union must be $2|E(X)|$. Also, if some set $\mathcal{S}$ of orbits of G on cycles produces an embedding of X, then G will induce a subgroup of the automorphism group of the resulting map, so its order must divide $4|E(X)|$.

Step 3(b) also requires that the cycles incident with each vertex v fold well around v, providing a cyclic permutation of the edges incident with v. Testing this can be achieved simply by constructing a 'local' graph, representing the vertex-figure on the neighbourhood $X(v)$ of v, with an edge between vertices u and w if and only if the union contains a cycle with edges $\{u, v\}$ and $\{v, w\}$, and then checking that this graph is a k-cycle, where $k = |X(v)|$ is the valency of v. The test for orientability in Step 3(c) then follows on easily from that. Also Step 3(b) can be sped up by use of a backtrack search, adding and removing G-orbits on cycles to and from a union of such orbits, with feasibility tests at each node of the search tree.

In practice, the length of time needed for Steps 1 and 2 is relatively small, while most of the time is required for Step 3. Also the time needed increases as the order of G decreases, because the number of orbits of G on $\mathcal{C}$ increases. But usually we do not conduct Step 3 for every class of subgroups. Indeed we stop the search if it finds an orientable embedding and/or non-orientable embedding of provably minimum genus, since there is then no need to proceed further, and in that case we have found such an embedding (or embeddings) with largest possible automorphism group. Also we can stop the search if it takes too long or requires too much memory, but in principle it can work even when the subgroup G is trivial.

3.3 Application to the Hoffman-Singleton graph

The Hoffman-Singleton graph is the unique Moore graph of valency 7 and diameter 2, and hence has order $1 + 7 + 7 \cdot 6 = 50$ and girth 5.

It has a very nice 'pentagons-and-pentagrams' construction (due to Robertson [41]), which may be described as follows: Take five pentagons P_1, P_2, P_3, P_4, P_5, with each P_i having vertices $u_{i1}, u_{i2}, u_{i3}, u_{i4}$ and u_{i5} and edges $\{u_{i1}, u_{i2}\}$, $\{u_{i2}, u_{i3}\}$, $\{u_{i3}, u_{i4}\}$, $\{u_{i4}, u_{i5}\}$ and $\{u_{i5}, u_{i1}\}$, and five pentagrams (5-pointed stars) Q_1, Q_2, Q_3, Q_4, Q_5, with each Q_i having vertices $v_{i1}, v_{i2}, v_{i3}, v_{i4}$ and v_{i5} and edges $\{v_{i1}, v_{i3}\}$, $\{v_{i3}, v_{i5}\}$, $\{v_{i5}, v_{i2}\}$, $\{v_{i2}, v_{i4}\}$ and $\{v_{i4}, v_{i1}\}$, and then add an edge from vertex u_{ij} to vertex v_{rs} whenever $s \equiv ir + j \pmod 5$.

Equivalently, it may be constructed as the derived graph of a graph T of order 10 whose vertices are $P_1, P_2, P_3, P_4, P_5, Q_1, Q_2, Q_3, Q_4$ and Q_5, with a loop at each vertex and an edge joining each of the 25 pairs of vertices P_i and Q_j, and voltage group $\mathbb{Z}_5$ (under addition). In particular, this makes it a 5-fold cover of T.

For ease of notation, we may re-label the vertices $u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{21}, u_{22}, \ldots, u_{55}$ as 1 to 25, and the vertices $v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{21}, v_{22}, \ldots, v_{55}$ as 26 to 50. Then for example, the neighbours of the vertex 1 are 2, 5, 27, 33, 39, 45 and 46.

The Hoffman-Singleton graph is vertex-transitive. Indeed its automorphism group has order 252 000 and is isomorphic to $\mathrm{P\Sigma U}(3,5)$, which is a semi-direct product of the simple linear group $\mathrm{PSU}(3,5)$ by a cyclic group of order 2 generated by the Frobenius automorphism of $\mathrm{GF}(5^2)$. The stabiliser of a given vertex v is isomorphic to S_7, which acts faithfully on the neighbourhood of v. In particular, the graph is also arc-transitive, or *symmetric*.

An easy computation with the MAGMA system [2] shows that the automorphism group has 148 conjugacy classes of subgroups, of orders 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20, 21, 24, 25, 32, 36, 40, 42, 48, 50, 60, 72, 80, 96, 100, 120, 125, 144, 168, 200, 240, 250, 336, 360, 480, 500, 720, 1000, 1440, 2000, 2520, 5040, 126 000 and 252 000 (with many orders repeated). We can limit our attention to those of order dividing $4|E| = 700$, that is, of order 1, 2, 4, 5, 7, 10, 14, 20, 25, 50 or 100.

It is easy to check that there is no subgroup of order 50 that is complementary to the vertex-stabiliser, and hence the Hoffman-Singleton graph is not a Cayley graph. Moreover, it has no subgroup of order 175, 350 or 700, and hence has no subgroup that acts regularly on the edges or on the arcs of the graph, or on the flags of any embedding. In particular, the Hoffman-Singleton graph is not the underlying graph of a regular map, and this explains why we started thinking about different kinds of embeddings. We collect some of our findings in the following.

Proposition 3.2. *The Hoffman-Singleton is not a Cayley graph, and is not the underlying graph of a regular map.*

Next, by Lemma 3.1, an upper bound on the number of faces of any embedding is $350/5 = 70$, with the bound attained only when all faces are pentagonal.

We implemented our subgroup orbit method in MAGMA, and ran it on an Apple laptop. With $\mathcal{C}$ chosen as the set of all cycles of length 5 (of which there are 1260), it took only minutes to check and eliminate subgroups of order 20 or more, but the computation then slowed down considerably once it reached subgroups of order 10. Because of this, we restricted the search to cyclic subgroups of prime order, and that led us to discover some minimum genus embeddings.

One of the first ones we found (taking only a few minutes in the restricted computation) uses ten orbits on $\mathcal{C}$ of a cyclic subgroup of order 7 in the automorphism group of the graph, generated by the automorphism α that induces the permutation

$$(2,5,27,33,39,45,46)\,(3,26,10,12,37,7,49)\,(4,29,9,36,13,6,47)$$
$$(8,19,48,28,50,30,20)\,(11,40,38,14,35,17,43)\,(15,25,16,41,32,18,23)$$
$$(21,34,44,22,31,24,42)$$

on the re-labelled vertices. Note that this permutation does not act semi-regularly on the vertices, since it fixes the vertex 1. The 70 faces of the embedding are bounded by the ten

5-cycles

$$(1,2,34,32,5),\quad (2,3,29,26,28),\quad (2,3,35,17,47),$$
$$(2,28,23,38,40),\quad (2,34,13,12,47),\quad (2,40,18,44,41),$$
$$(3,35,32,8,48),\quad (3,36,12,44,42),\quad (4,30,16,34,31),$$
$$(4,30,28,23,43),$$

and their images under non-trivial powers of the automorphism α.

This embedding is non-orientable, since the Euler characteristic χ is $50 - 175 + 70 = -55$, which is odd. In particular, it is a non-orientable embedding of minimum genus, and gives the cross-cap number of the graph as $2 - \chi = 57$. The embedding is illustrated in Figure 1, and also in [46].

At this point, we note that the resulting map admits an automorphism of order 7 (acting on the underlying graph in the same way as α above), and also that with the help of MAGMA it is not difficult to show that there are no other map automorphisms apart from powers of α, and so the full automorphism group of this map has order 7.

Another non-orientable embedding we found of the same genus uses 14 orbits of a cyclic subgroup of order 5 generated by the automorphism β that induces the semi-regular permutation

$$(1,6,12,19,22)\,(2,7,13,20,23)\,(3,8,14,16,24)\,(4,9,15,17,25)$$
$$(5,10,11,18,21)\,(26,50,43,40,31)\,(27,46,44,36,32)\,(28,47,45,37,33)$$
$$(29,48,41,38,34)\,(30,49,42,39,35).$$

The 70 faces of this embedding come from the orbits of the following 5-cycles:

$$(1,2,41,20,33),\quad (1,33,23,48,46),\quad (1,46,16,42,45),$$
$$(1,45,13,14,27),\quad (1,27,18,17,39),\quad (1,39,36,38,5),$$
$$(1,5,50,47,2),\quad (2,3,4,37,40),\quad (2,34,31,18,40),$$
$$(3,4,30,8,48),\quad (3,48,18,44,42),\quad (3,42,9,26,29),$$
$$(4,5,32,11,43),\quad (4,31,15,39,37).$$

We later used linear programming (as we will describe in Section 5) to find a large number of non-orientable embeddings of minimum genus with trivial automorphism group, and some further computations using MAGMA showed that 7 is the largest order of the group of automorphisms of any such embedding.

We collect our findings in the following theorem.

Theorem 3.3. *The minimum non-orientable genus of the Hoffman-Singleton graph is* 57*, and occurs for embeddings with* 70 *pentagonal faces. Moreover, the maximum order of a group of automorphisms of such an embedding of this graph is* 7*, and other possibilities for the order are* 1 *and* 5*.*

For orientable embeddings, an upper bound on the number of faces is 69, potentially giving Euler characteristic $\chi = 50 - 175 + 69 = -56$ and genus 29. In theory, this could be achieved in a number of ways: ranging from 68 faces of length 5 and one of length 10, to 64 faces of length 5 and five of length 6.

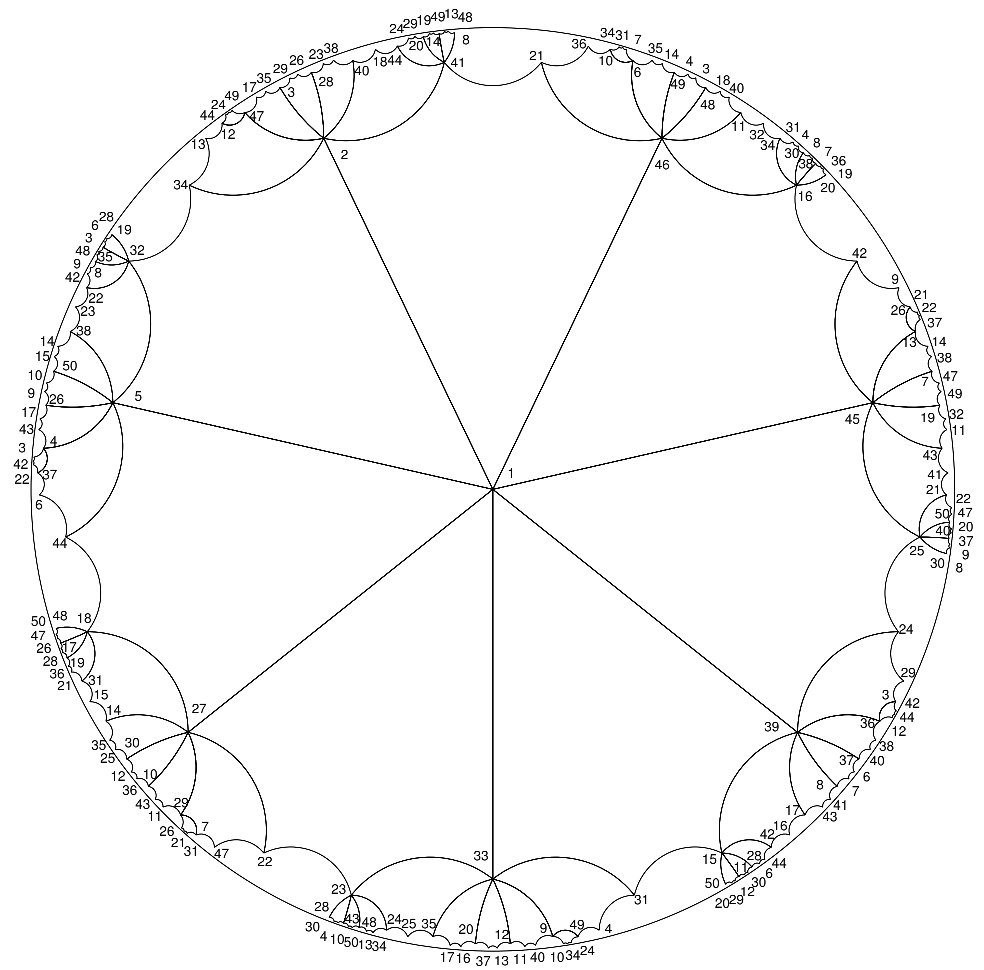

Figure 1: A minimum genus non-orientable embedding of the Hoffman-Singleton graph.

We ran another version of our MAGMA procedure with S chosen as a cyclic subgroup of order 5 in the automorphism group of the graph, and $\mathcal{C}$ chosen as the set of all cycles of length 5 and all the 6-cycles in a single orbit of S, and found an orientable embedding of minimum genus 29, with 64 faces of length 5 and five of length 6. This embedding came from the subgroup of order 5 generated by the obvious automorphism γ of order 5 that induces the semi-regular permutation

$$(1,2,3,4,5)\,(6,7,8,9,10)\,(11,12,13,14,15)\,(16,17,18,19,20)$$
$$(21,22,23,24,25)\,(26,27,28,29,30)\,(31,32,33,34,35)\,(36,37,38,39,40)$$
$$(41,42,43,44,45)\,(46,47,48,49,50).$$

This subgroup is not conjugate to the subgroup generated by the earlier automorphism β mentioned above. The 69 faces of the embedding come from 11 orbits of $\langle\gamma\rangle$ of length 5 on 5-cycles, with representatives

$$\begin{array}{lll} (1,2,34,10,27), & (1,5,38,7,45), & (1,45,13,37,39), \\ (1,46,11,12,33), & (6,35,17,47,7), & (6,37,22,23,28), \\ (6,46,21,41,44), & (11,29,24,34,32), & (11,43,17,26,29), \\ (11,46,48,18,40), & (16,17,43,23,38), & \end{array}$$

plus another nine individual 5-cycles, which are all preserved by γ, namely

$$\begin{array}{lll} (6,7,8,9,10), & (11,15,14,13,12), & (16,20,19,18,17), \\ (21,25,24,23,22), & (26,28,30,27,29), & (31,33,35,32,34), \\ (36,39,37,40,38), & (41,43,45,42,44), & (46,49,47,50,48), \end{array}$$

and a single orbit of $\langle\gamma\rangle$ of length 5 on 6-cycles, with representative

$$(1,27,18,31,21,46).$$

These 69 cycles are consistent, in that they give the rotation system for an orientable embedding. For example, the seven of those 69 cycles that contain the vertex 1 are the six 5-cycles

$$\begin{array}{lll} (1,2,34,10,27), & (5,1,33,9,26), & (1,5,38,7,45), \\ (2,1,39,8,41), & (1,45,13,37,39), & (1,46,11,12,33), \end{array}$$

plus the single 6-cycle

$$(1,27,18,31,21,46),$$

and these are consistent with $\rho = (2,39,45,5,33,46,27)$, which gives a rotation at vertex 1. The rotations at other vertices can be found similarly.

The resulting orientable embedding admits γ as a map automorphism of order 5, and an easy MAGMA computation shows that there are no other automorphisms. In particular, the above embedding is chiral (irreflexible), meaning that it does not admit an orientation-reversing automorphism. Also an extended MAGMA computation showed that 5 is the largest order of any group of automorphisms of an orientable embedding of minimum genus 29.

We collect our findings in the following theorem.

Theorem 3.4. *The minimum orientable genus of the Hoffman-Singleton graph is* 29*, and this is attainable by a chiral embedding with* 69 *faces, of which* 64 *have length* 5 *and five have length* 6*, and with automorphism group of order* 5*. Moreover,* 5 *is the maximum order of a group of automorphisms of any minimum genus orientable embedding of this graph.*

3.4 Comparison with the voltage graph method

Here we make some observations that compare the voltage graph method (as described near the end of Subsection 2.3) with our new subgroup orbit method, in response to a suggestion by Tomaž Pisanski. Each of these two methods involves choice of an eventual group of automorphisms of an embedding (or just a suitable set of permutations), in order to reduce the size of the search space for nice embeddings, and this also makes it easy to describe each embedding found.

But there are many important respects in which they differ.

The voltage graph method involves guessing a way of regarding the given graph as a covering graph of a nice voltage graph (and then choosing suitable voltage assignments, and so on), while the subgroup orbit method does not do this, even though those things can sometimes be the outcome. In this sense, the subgroup orbit method is more systematic than the voltage graph method (even without putting any extra restrictions on the set $\mathcal{C}$ of cycles or the subgroup G, as we did when finding minimum genus orientable embeddings of the Hoffman-Singleton graph). Also the subgroup orbit method can find embeddings that are unlikely to be obtained by the voltage graph method. In particular, this may happen when vertices in some face are identified in the quotient embedding while others are not, but also in other cases where the quotient graph is not obvious or natural.

The minimum genus embeddings we found for the Hoffman-Singleton graph make a good illustration of these arguments. Our orientable embedding with 69 faces can be constructed from an embedding of the quotient graph via the automorphism γ of order 5, and this graph happens to be the very nice voltage graph T from which the Hoffman-Singleton graph is often constructed (as described at the beginning of Section 3.3). On the other hand, the minimum genus non-orientable embeddings that we found have automorphisms that define quite different voltage graphs: the automorphism β of order 5 gives a quotient graph on 10 vertices with multiple edges but no loops, while the automorphism α of order 7 defines a quotient graph on 8 vertices (with one being a branched vertex). Every embedding of one of these quotient graphs will give an embedding of the Hoffman-Singleton graph in some surface. In the third case (using α), however, the quotient graph is not the most obvious one to choose. Also the third case also shows that no particular difficulties need arise from using an automorphism that is not semi-regular.

The next point we make is that it can be difficult to choose the quotient graph and voltage assignments when we want control over the size of the faces and/or the number of faces in the derived embedding, which of course is what we need to do when searching for minimum genus embeddings. Indeed it can be difficult even to guess what lengths the faces should have in the quotient embedding in order to get faces of the desired lengths in the covering graph, without considering also the values of the voltages on the edges, and how they compose. For example, some of the pentagonal faces of the non-orientable embedding obtained from the automorphism β are lifted from closed walks of length 5 in the quotient embedding, consisting of a triangular face together with a closed walk of length 2, but in the voltage graph construction it would not be immediately clear if such a walk would lift to a pentagonal face, or to something larger. In other examples, it may be easy to see how short closed walks will lift in the derived graph, and often they unwind simply as desired (almost by pure luck), but in many cases the situation can be rather complicated, especially when a face is created from a union of smaller closed walks.

Here we feel it is interesting to note that voltage graph methods cannot be used to construct a minimum non-orientable genus embedding of the Hoffman-Singleton graph from the natural 10-vertex voltage graph T (mentioned earlier). Such an embedding must have 70 pentagonal faces, lifted from 14 closed walks of length 5 in T that use each arc exactly once. According to [21], there are four types of cycles of length 5 in the Hoffman-Singleton graph. Cycles of type I are lifted loops, cycles of type II and III are lifts of closed walks of type (v, v, v, u, u), and cycles of type IV are lifts of a cycle of length 4 with an attached loop. Any walk that lifts to a cycle may use each arc no more than once, and it follows that only cycles of types I and IV can be used in a lifted embedding. (Any closed walk of type (v, v, v, u, u) in T could not unwind to a simple cycle of length 5 in the derived graph if the loop at v was taken in both possible directions, and so would have to traverse the loop at v twice in the same direction.) On the other hand, a counting argument shows that we cannot cover each arc in T exactly once using quotient walks of cycles of type I and IV, and so this voltage graph T cannot be used to construct an embedding of the Hoffman-Singleton graph with only pentagonal faces.

The above example shows that the knowledge of a ‘special’ voltage graph does not necessary help when looking for a minimum genus embedding. More generally, if a voltage graph has a large number of vertices or edges, then it can be quite a challenge to find nice embeddings of it, let alone nice embeddings of the derived graph, while the subgroup orbit method is quite capable of easily finding nice embeddings also in those cases. In summary, the subgroup orbit method can produce a greater range of embeddings than the voltage graph method.

On the other hand, the subgroup orbit method works best when the graph has nice embeddings with non-trivial symmetry, while the voltage graph method can be made to work well also in cases where that does not happen (using permutation voltage graphs).

3.5 Some other examples

Example 3.5. The Cartesian product $C_3 \,\square\, C_3 \,\square\, C_3$.

This is an arc-transitive graph of order 27, valency 6, girth 3 and diameter 3 (and is a Cayley graph for the abelian group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$). By Lemma 3.1, any embedding of this graph has at most $162/3 = 54$ faces. In 1985 it was shown to have a genus 7 orientable embedding with 42 faces, by Mohar, Pisanski, Škoviera and White [32], and three years later Brin and Squier proved in [4] that any embedding has as most 43 faces, and thereby showed that the minimum orientable genus of $C_3 \,\square\, C_3 \,\square\, C_3$ is 7, but they left open the question of the minimum non-orientable genus.

With a natural vertex-labelling, our subgroup orbit method implemented in MAGMA takes only a couple of minutes to produce a different and more symmetric orientable embedding of minimum genus than the one found in [32]. This new embedding has automorphism group S of order 36, generated by elements that induce the permutations

$$(2,7)\,(3,4)\,(5,9)\,(10,19)\,(11,25)\,(12,22)\,(13,21)$$
$$(14,27)\,(15,24)\,(16,20)\,(17,26)\,(18,23)$$

and

$$(1, 14, 27)\ (2, 15, 25)\ (3, 13, 26)\ (4, 23, 21, 10, 17, 9)$$
$$(5, 24, 19, 11, 18, 7)\ (6, 22, 20, 12, 16, 8).$$

The resulting map has 18 triangular faces, 18 quadrangular faces and 6 hexagonal faces, coming from the orbits under S of the 3-cycles $(4, 6, 5)$, the 4-cycle $(1, 2, 8, 7)$ and the 6-cycle $(2, 3, 21, 24, 23, 5)$. In particular, the first of the two generators for S given above reverses the 4-cycle $(1, 2, 8, 7)$, and it follows that this embedding is reflexible.

Our subgroup orbit method also quickly finds a non-orientable embedding of minimum genus, with 43 faces, answering the question left open in 1988 by Brin and Squier [4]. This embedding has 24 triangular faces, 12 quadrangular faces, and 7 hexagonal faces, and its automorphism group is a dihedral group of order 12, generated by two elements that induce the permutations

$$(2, 3)\ (4, 7)\ (5, 9)\ (6, 8)\ (10, 19)\ (11, 21)\ (12, 20)\ (13, 25)$$
$$(14, 27)\ (15, 26)\ (16, 22)\ (17, 24)\ (18, 23)$$

and

$$(1, 5, 9)\ (2, 8, 7, 4, 6, 3)\ (10, 14, 18)\ (11, 17, 16, 13, 15, 12)$$
$$(19, 23, 27)\ (20, 26, 25, 22, 24, 21).$$

The 43 faces come from the orbits of the cycles $(1, 2, 3)$, $(2, 11, 20)$, $(10, 11, 12)$, $(1, 2, 11, 10)$, $(10, 12, 15, 24, 22, 19)$ and $(2, 3, 6, 4, 7, 8)$.

Thus we have proved the following improvement of what was achieved in [32] and [4].

Theorem 3.6. *The minimum orientable genus of the Cartesian product $C_3 \,\square\, C_3 \,\square\, C_3$ is 7, and this is attainable by a reflexible embedding with 42 faces, in which there are 18 faces of length 3, plus 18 of length 4, and 6 of length 6, and with automorphism group of order 36. The minimum non-orientable genus of the Cartesian product $C_3 \,\square\, C_3 \,\square\, C_3$ is 13, and this is attainable by an embedding with 43 faces, in which there are 24 faces of length 3, plus 12 of length 4, and 7 of length 6, and with automorphism group of order 12.*

Example 3.7. Tutte's 8-cage.

This is the smallest 5-arc-transitive 3-valent graph. It is bipartite of order 30, with girth 8; indeed it is also the smallest 3-valent graph of girth 8. Its automorphism group is isomorphic to $\mathrm{Aut}(S_6)$, of order 1440.

The number of faces of any embedding is bounded above by $\lfloor 2|E|/8 \rfloor = \lfloor 90/8 \rfloor = 11$. Moreover, if there are exactly 11 faces, and F_8 and F_ℓ are the numbers of faces of length 8 and greater than 8, then $88 + 2F_\ell = 8(F_8 + F_\ell) + 2F_\ell = 8F_8 + 10F_\ell \le 2|E| = 90$ and so $F_\ell \le 1$, which implies that there are ten faces of length 8 and one of length 10.

Our subgroup orbit method quickly gives a minimum genus non-orientable embedding with 11 faces, and cyclic automorphism group of order 10. With a suitable labelling of vertices, the automorphism group is generated by an element inducing the permutation

$$(1, 11, 25, 20, 26, 3, 23, 22, 28, 14)\ (2, 5, 13, 29, 19, 7, 15, 24, 12, 6)$$
$$(4, 27, 17, 8, 18, 9, 16, 10, 21, 30),$$

and the ten faces of length 8 come from the orbit of $(1, 2, 5, 11, 23, 16, 8, 4)$, while the single face of length 10 is bounded by the cycle $(2, 6, 12, 24, 15, 7, 19, 29, 13, 5)$.

Also our subgroup orbit method gives an orientable embedding with 9 faces, and automorphism group of order 3. The automorphism group S is generated by an element inducing the permutation

$$(2, 3, 4)\ (5, 9, 10)\ (6, 7, 8)\ (11, 21, 18)\ (12, 19, 16)$$
$$(13, 17, 22)\ (14, 15, 20)\ (23, 28, 26)\ (24, 29, 30),$$

and the embedding has three faces of length 8, three of length 10 and three of length 12, which come from the orbits under S of the cycles

$(1, 2, 5, 11, 23, 16, 8, 4)$, $(5, 13, 25, 17, 30, 14, 26, 18, 27, 11)$ and
$(2, 6, 14, 30, 16, 23, 15, 7, 19, 29, 13, 5)$.

Our method found no orientable embedding with 11 faces, for a good reason. If there existed one, then there would be ten faces of length 8 and a single face of length 10 (as shown above). By transitivity of the automorphism group of Tutte's 8-cage on 10-cycles, we may choose any 10-cycle C to bound the single face of length 10, and then consider the way the other ten faces wrap around it. By inspection of the edge-set of the graph, it is easy to see that there are exactly four possibilities for a cycle of length 8 containing any edge, and it follows that there are 4^{10} possibilities for how to arrange potential faces of this length around the given 10-cycle C. But then an easy MAGMA computation shows that in all 4^{10} cases, some arc is repeated in two different faces, so this is impossible. (In fact there are only two embeddings that can be found in this way, and both are non-orientable.)

Thus we have the following:

Theorem 3.8. *The minimum orientable genus of Tutte's* 8*-cage is* 4*, attainable by a chiral embedding with* 9 *faces, in which there are three faces of length* 8*, three of length* 10*, and three of length* 12*, and with automorphism group of order* 3*. The minimum non-orientable genus of Tutte's* 8*-cage is* 6*, attainable by an embedding with* 11 *faces, in which there are ten faces of length* 8 *and one of length* 10*, and with cyclic automorphism group of order* 10*.*

Further examples will be met in the next two sections.

4 The independence number approach

4.1 Motivation and description

Lemma 3.1 gives a theoretical upper bound on the number of faces of an embedding, and hence a lower bound on the minimum genus. If an embedding attains that bound, then it will automatically have minimum genus (whether orientable or not). Also if an orientable embedding falls short by just one face, then it will have minimum orientable genus, since in that case the Euler characteristic has to be even.

The two examples considered in Subsection 3.5 (and many other graphs besides those) show that these theoretical upper bounds on the number of faces of an embedding are not always attainable, and in such cases, some other information is required to help decide whether a given embedding has minimum genus. This was already done for $C_3 \,\square\, C_3 \,\square\, C_3$ (in Example 3.5) by Brin and Squier [4], using knowledge of the structure of the graph to reduce the bound from 54 to 43 faces, and similarly, in Example 3.7 we used some particular properties of Tutte's 8-cage to decrease the bound from 11 to 9 in the orientable case. These kinds of approach, however, are not likely to work well in general, so some other approaches are needed.

The main idea of our new approach is that we analyse an appropriate set $\mathcal{C}$ of cycles of the graph that are candidates for the faces, with the aim of finding an upper bound on the number of members of $\mathcal{C}$ that can be combined together to form the faces of an embedding or partial embedding. For example, the set $\mathcal{C}$ could be the set of all girth cycles, or all cycles of length close to the girth.

We then define an *auxiliary graph* $X_{\mathcal{C}}$ with $\mathcal{C}$ as its vertex-set, and with two cycles in $\mathcal{C}$ joined by an edge if and only they cannot occur together in the same embedding.

There are several ways of telling that two cycles cannot occur in the same embedding. Here we use the fact that the local arrangement of neighbours of a vertex requires that any given 2-path lies in at most one face (under the assumption that no vertex of X has valency 2), and accordingly, we define an edge between two members of $\mathcal{C}$ if and only if they have a 2-path in common.

Next, we compute the *independence number* of the auxiliary graph $X_{\mathcal{C}}$. This is the maximum number of pairwise non-adjacent vertices of $X_{\mathcal{C}}$, and can be found (for example) in MAGMA using the `MaximumIndependentSet` command. The resulting number gives an upper bound on the number of cycles from $\mathcal{C}$ that can bound faces of an embedding, and hence can be used to find a lower bound on the average face size, and thereby obtain an improved upper bound on the total number of faces.

The method can be summarised as follows:

Step 1. Choose an appropriate set $\mathcal{C}$ of cycles of interest in the given graph X.

Step 2. Define the *auxiliary graph* $X_{\mathcal{C}}$ on the vertex-set $\mathcal{C}$, with two elements of $\mathcal{C}$ joined by an edge if and only if they cannot occur together in the same embedding.

Step 3. Find the *independence number* of the auxiliary graph $X_{\mathcal{C}}$, which gives an upper bound on the number of the cycles of $\mathcal{C}$ that can occur as faces of any embedding.

This approach works for both orientable and non-orientable embeddings alike, but can be further improved for *orientable* embeddings by taking $\mathcal{C}$ as a suitable set of *oriented* cycles, and by joining two elements of $\mathcal{C}$ by an edge when they have either an *arc* (ordered edge) or an underlying 2-path (or both) in common.

Also at Step 3 in both cases, the `MaximumIndependentSet` command can produce an independent set of maximum size, in case that is helpful.

As the examples below will show, this approach can lead to significant reduction in the upper bound on the number of faces, and then help with determining the minimum genus.

4.2 Some applications

Example 4.1. The Gray graph.

This is the smallest cubic (3-valent) graph that is semi-symmetric, which means regular and edge-transitive but not vertex-transitive; see [8]. It is bipartite with order 54, diameter 6 and girth 8, and has automorphism group of order 1296. An upper bound on the number of faces of any embedding is $\lfloor 162/8 \rfloor = 20$, but this is not sharp. The minimum orientable genus of the Gray graph was found in 2005 by Marušič, Pisanski and Wilson [30] to be 7, via an embedding with only 15 faces, obtained from the embedding of $C_3 \,\square\, C_3 \,\square\, C_3$ on a surface of genus 7 given in [32].

Our new approach also gives both this and the minimum non-orientable genus quite easily. First, the Gray graph has 81 cycles of length 8, but none of length 10. With $\mathcal{C}$ taken as the set of all 8-cycles, our independence number approach gives $F_8 \leq 9$, and then because there are no 10-cycles, all other faces have length at least 12, and so we find that $|F| \leq 9+(2|E|-72)/12 = 9+90/12$, which gives $|F| \leq 16$. Hence every non-orientable embedding of the Gray graph has at most 16 faces, while every orientable embedding has at most 15.

Furthermore, our subgroup orbit method easily finds one of each kind of embedding from a dihedral subgroup S of order 6 in the automorphism group of the graph: an orientable embedding with $F_8 = F_{12} = 6$ and $F_{14} = 3$, and a non-orientable embedding with $F_8 = 9$, $F_{12} = 4$ and $F_{14} = 3$.

With a suitable labelling of the vertices, the dihedral subgroup S can be generated by the two elements of orders 2 and 3 inducing the permutations

$$(1,2)\,(3,15)\,(5,14)\,(6,8)\,(7,9)\,(11,13)\,(12,17)\,(16,18)\,(19,21)\,(20,27)\\ (23,24)\,(25,26)\,(29,31)\,(30,32)\,(34,35)\,(36,37)\,(38,40)\,(39,47)\,(42,43)\\ (44,45)\,(46,48)\,(49,53)\,(50,52)\,(51,54)$$

and

$$(1,2,4)\,(3,9,11)\,(5,6,17)\,(7,15,13)\,(8,14,12)\,(10,27,20)\,(16,26,23)\\ (18,24,25)\,(19,21,22)\,(29,32,34)\,(30,31,35)\,(33,47,39)\,(36,45,42)\\ (37,43,44)\,(38,40,41)\,(46,52,54)\,(48,51,50),$$

and then the faces of the orientable embedding come from the orbits of S containing the 8-cycle

$$(3,29,8,42,23,46,10,33),$$

the 12-cycles

$$(1,29,3,36,14,41,5,37,15,31,2,28),\\ (1,30,5,41,22,51,27,52,21,40,8,29),$$

and the 14-cycle

$$(3,33,15,37,18,53,25,51,22,54,26,49,16,36),$$

while those of the non-orientable embedding come from the orbits of S containing the 8-cycles

$$(1,28,2,31,15,33,3,29), \qquad (3,33,10,46,23,49,16,36),$$

the 12-cycles

$$(5,37,18,50,19,38,12,45,26,54,22,41),\\ (10,48,21,52,27,51,22,54,20,50,19,46),$$

and the 14-cycle

$$(1,29,8,42,11,34,12,38,6,31,15,37,5,30).$$

The above orientable embedding is reflexible, since the 12-cycle

$$(1, 29, 3, 36, 14, 41, 5, 37, 15, 31, 2, 28)$$

is inverted by conjugation by the first generator of S.

Thus we have the following improvement of what was found in [30].

Theorem 4.2. *The minimum orientable genus of the Gray graph is* 7*, attainable by a reflexible embedding with* 15 *faces, of which six have length* 8*, six have length* 12*, and three have length* 14*, and with dihedral automorphism group of order* 6*. The minimum non-orientable genus of the Gray graph is* 13*, attainable by an embedding with* 16 *faces, in which nine have length* 8*, four have length* 12*, and three have length* 14*, and with the same automorphism group of order* 6 *as in the orientable case above.*

Example 4.3. The Ljubljana graph.

This is the third smallest semi-symmetric cubic graph. It is believed to have been first found by R. M. Foster in the 1970s, and first mentioned in [3]. It was later rediscovered in [5], as well as in the computations that produced the list of all small semi-symmetric 3-valent graphs published in [8]. It is bipartite with order 112, diameter 8 and girth 10, and has soluble automorphism group of order 168. Other properties of this graph are described in [7].

The upper bound on the number of faces of any embedding given by Lemma 3.1 is $\lfloor 336/10 \rfloor = 33$, but we can reduce this to 32 using our independence number approach.

If we take $\mathcal{C}$ as the set of all unoriented 10-cycles in the graph (of which there are 168), then the auxiliary graph $X_{\mathcal{C}}$ has independence number 24, and so $F_{10} \leq 24$. Next, since the graph is bipartite, every other face has length 12 or more, and so counting incident edge-face pairs gives $336 = 2|E| \geq 10F_{10} + 12(|F| - F_{10}) = 12|F| - 2F_{10} \geq 12|F| - 48$, and it follows that $|F| \leq (336 + 48)/12 = 384/12 = 32$. Also if there are exactly 32 faces, with 24 of length 10, then the inequality becomes an equality, and then the other eight faces must all have length 12.

Our subgroup orbit method provides an orientable embedding with exactly 32 faces, and automorphism group of order 24, isomorphic to $A_4 \times C_2$. In particular, this embedding has minimum orientable genus.

With a suitable labelling of the vertices, the automorphism group S can be generated by the elements of orders 2 and 3 inducing the permutations

$(1, 39)\ (2, 56)\ (3, 52)\ (4, 45)\ (5, 42)\ (6, 34)\ (7, 15)\ (8, 48)\ (9, 51)\ (10, 30)\ (11, 14)$
$(12, 46)\ (13, 33)\ (16, 43)\ (17, 47)\ (18, 54)\ (19, 31)\ (20, 24)\ (21, 44)\ (22, 41)$
$(23, 29)\ (25, 50)\ (26, 28)\ (27, 40)\ (32, 49)\ (35, 38)\ (36, 55)\ (37, 53)\ (57, 108)$
$(58, 83)\ (59, 89)\ (60, 110)\ (61, 112)\ (62, 104)\ (63, 100)\ (64, 106)\ (65, 75)\ (66, 71)$
$(67, 78)\ (68, 86)\ (69, 94)\ (70, 95)\ (72, 87)\ (73, 93)\ (74, 99)\ (76, 88)\ (77, 111)$
$(79, 82)\ (80, 102)\ (81, 109)\ (84, 98)\ (85, 101)\ (90, 105)\ (91, 107)\ (92, 96)\ (97, 103)$

and

$$(1,50,36)\,(2,26,13)\,(3,18,17)\,(4,38,37)\,(5,21,41)\,(6,45,46)\,(7,19,49)$$
$$(8,10,27)\,(9,35,31)\,(11,51,12)\,(14,32,20)\,(15,53,23)\,(16,39,55)$$
$$(22,30,56)\,(24,34,29)\,(28,54,44)\,(33,40,52)\,(42,47,48)\,(57,103,81)$$
$$(58,85,90)\,(59,93,80)\,(60,75,88)\,(61,74,64)\,(62,91,68)\,(63,82,77)$$
$$(65,107,92)\,(66,96,95)\,(67,108,102)\,(70,100,110)\,(71,86,111)\,(72,78,98)$$
$$(73,97,101)\,(76,79,104)\,(83,109,84)\,(87,89,105)\,(94,99,112),$$

and then the 32 faces of the orientable embedding come from the orbits of S containing the 10-cycle

$$(1,57,2,61,18,91,35,79,11,59)$$

and the 12-cycle

$$(2,57,4,63,10,78,34,95,41,80,12,60).$$

Also the first of the two generators above reverses orientation, so this embedding is reflexible.

Next, by applying a 'twist' to a single edge that is common to the boundary of two distinct faces, we can merge those two faces into one, and thereby obtain a non-orientable embedding with 31 faces (with $F_{10} = 22$, $F_{12} = 8$ and $F_{20} = 1$, or $F_{10} = 23$, $F_{12} = 7$ and $F_{22} = 1$, or $F_{10} = 24$, $F_{12} = 6$ and $F_{24} = 1$). In all cases, the embedding has trivial automorphism group, or in other words, is asymmetric.

It turns out that all of the latter embeddings have minimum non-orientable genus, because there is just one embedding with 32 faces, namely the orientable one described above. To see this, we can use our independence number approach a slightly different way.

First, an easy MAGMA computation shows that the set $\mathcal{C}$ of all 168 cycles of length 10 in the Ljubljana graph forms a single orbit under the action of its automorphism group. Now take any one of these 10-cycles as a representative of $\mathcal{C}$, say C, and let $\mathcal{I}$ be the set of all independent 24-sets in the auxiliary graph $X_{\mathcal{C}}$ that contain C.

Next, partition the remaining 167 cycles from $\mathcal{C}$ into three subsets: *forgettable* 10-cycles, which lie in no 24-set in $\mathcal{I}$, *standard* 10-cycles, which lie in exactly one set in $\mathcal{I}$, and *special* 10-cycles, which lie in more than one set in $\mathcal{I}$. An easy computation shows that there are 82 forgettable 10-cycles, plus 63 standard 10-cycles, and just $167-(82+63) = 22$ special 10-cycles. The forgettable 10-cycles can be ignored, as they cannot bound any face in a 32-face embedding. and so we need only deal with the standard and special 10-cycles.

We do this by considering the ways in which a 2-subset of $\mathcal{C}$ can be extended to an independent 24-set in the auxiliary graph $X_{\mathcal{C}}$. Note that every independent 24-set in $\mathcal{I}$ must contain a standard 10-cycle D, since there are only 22 special 10-cycles. Furthermore, there is just one set 24-set in $\mathcal{I}$ containing a given standard 10-cycle D, and hence just one independent 24-set containing $\{C, D\}$. It follows that we can find all members of $\mathcal{I}$ by letting D run through the set of 63 standard 10-cycles, and for each one, determining the largest independent subset of the induced subgraph of $X_{\mathcal{C}}$ on the set of 10-cycles in $\mathcal{C}$ that are independent of C and D. When this subset has size 22, its union with $\{C, D\}$ is a member of $\mathcal{I}$, and conversely, every member of $\mathcal{I}$ can be found in this way.

In fact, by a MAGMA computation we find that the set $\mathcal{I}$ has only five members, with each standard 10-cycle lying on just one of them, as follows:

- one containing 3 standard 10-cycles and 20 special 10-cycles,

- one containing 11 standard 10-cycles and 12 special 10-cycles,
- one containing 12 standard 10-cycles and 11 special 10-cycles,
- one containing 18 standard 10-cycles and 5 special 10-cycles, and
- one containing 19 standard 10-cycles and 4 special 10-cycles.

A further MAGMA computation shows that the first one gives rise to our known orientable embedding of the Ljubljana graph (with 32 faces), while the other four are mutually equivalent under the action of the full automorphism group of the graph, and give rise to embeddings with fewer than 31 faces. Further details are available if necessary from the first author on request.

In particular, there is just one embedding of the Ljubljana graph with 32 faces, namely the orientable embedding we described earlier, and therefore every non-orientable embedding has at most 31 faces, and $|F| = 31$ gives the minimum non-orientable genus.

Hence we have answered the final question from [30], by proving the following.

Theorem 4.4. *The minimum orientable genus of the Ljubljana graph is* 13*, attainable by a reflexible embedding with* 32 *faces, of which* 24 *have length* 10 *and eight have length* 12*, and with automorphism group of order* 24 *isomorphic to* $A_4 \times C_2$*. The minimum non-orientable genus of the Ljubljana graph is* 27*, attainable by an embedding with* 31 *faces, and trivial automorphism group.*

5 Use of integer linear programming

5.1 Background and description

Our independence number approach can be regarded as a constraint satisfaction problem on a subset of the cycles of the graph, in the sense that it finds the maximum number of cycles from the given set $\mathcal{C}$ that can be considered as bounding cycles for the faces of some embedding. This approach can also be modelled as an integer linear programming (ILP) problem, by using variables x_C for cycles $C \in \mathcal{C}$, with $x_C = 1$ if C is included as a bounding cycle, or 0 if not, and then maximising $\sum_{C \in \mathcal{C}} x_C$ subject to appropriate constraints.

This ILP variant is related to the successful use of the Kramer-Mesner method in the search for block designs, as shown in [28] for example. Incidentally, ILP has been used also to find planar embeddings of graphs [33], and drawings with minimum crossing number in the plane [6]. Also at about the same time as we were using ILP for graph embeddings and beginning to write up this work, another method using ILP was developed by Beyer, Chimani, Hedtke and Kotrbčík [1], but the latter method differs from our one, and we consider our method (and its variants) to be simpler.

In fact we developed four different ILP methods. The first two are particularly easy to describe, and are used to provide lower bounds on the minimum genus of a graph. The other two are modifications of the first two, and are used to construct actual embeddings of a graph in a surface.

In all of them, we will assume that the given connected graph X has no vertices of degree 1 or 2, and that X is is bridgeless (or in other words, 2-edge-connected), so that in any embedding of X, every edge lies in two different faces. Also we use the term 2-*arc* for an ordered triple (u, v, w) of vertices such that u and w are neighbours of v in X, and the term 2-*path* for the same triple when the order of u and w is unimportant.

We now describe our first ILP method, for producing helpful information about the faces of embeddings of a given graph X, whether orientable or not. For this, we let $\mathcal{C}_e$ be the set of all cycles in the set $\mathcal{C}$ containing a given edge e, and $\mathcal{C}_p$ be the set of all cycles in $\mathcal{C}$ containing a given 2-arc $p = (u, v, w)$ or its reverse (w, v, u).

Step 1. Choose a suitable set $\mathcal{C}$ of unoriented cycles of interest in the given graph X, and define a variable x_C for each cycle $C \in \mathcal{C}$, with x_C to take the value 1 if C is included as a bounding cycle of some face of the embedding, or 0 if not.

Step 2. Define the objective function as an linear combination of the variables x_C with appropriate integer coefficients.

Step 3. Set up the constraints as follows:

- $x_C \in \{0, 1\}$ for all $C \in \mathcal{C}$,
- $\sum\limits_{C \in \mathcal{C}_e} x_C \leq 2$ for every edge e of X, and
- $\sum\limits_{C \in \mathcal{C}_p} x_C \leq 1$ for every 2-path p in X.

Step 4. Find the maximum value of the objective function subject to the given constraints.

This gives an upper bound on the number of faces that can be bounded by the cycles in $\mathcal{C}$ in any embedding. For example, if the objective function is the sum $\sum_{C \in \mathcal{C}} x_C$ then the algorithm will search for the maximum number of cycles in the graph satisfying the constraints. Since the faces of any embedding must satisfy these constraints, this gives a simple computational method for bounding the number of faces in an embedding from above. Note that the objective function does not need to be the simple sum $\sum_{C \in \mathcal{C}} x_C$; indeed in the first example below, we take it as a non-trivial weighted combination of the variables x_C for the cycles of interest in $\mathcal{C}$. Also the bound obtained might be less than the number of faces in a minimum genus embedding, for example when we are interested in what is possible for a partial embedding.

Our second ILP method is a modification of the first one, for orientable embeddings only, and the same comments as above apply to it. Steps 1, 2 and 4 are the same, except that $\mathcal{C}$ is taken as a set of *oriented* cycles of interest, and Step 3 is similar to the one above, but we let $\mathcal{C}_a$ be the set of all oriented cycles in the set $\mathcal{C}$ containing a given arc $a = (v, w)$, and then set up the constraints as

- $x_C \in \{0, 1\}$ for all $C \in \mathcal{C}$,
- $\sum\limits_{C \in \mathcal{C}_a} x_C \leq 1$ for every arc a in X, and
- $\sum\limits_{C \in \mathcal{C}_p} x_C \leq 1$ for every 2-path p in X.

Note that these two methods are designed to help provide good upper bounds on the number of faces of an embedding, but are not designed to actually find a minimum genus embedding. Some times they do find one, but as with the independence number approach, it can happen that an optimum solution to the ILP problem does not produce even a partial embedding, because the constraints are necessary but not sufficient. Nevertheless the methods can be very helpful, as the examples in the next subsection will show.

Our other two ILP methods go further, by requiring that cycles are chosen in a way that actually gives an embedding. These methods can be obtained from the first two by modifying the constraints, as we explain below.

First, let $\mathcal{S}_{\{v,k\}}$ be the set of subsets of size k of the neighbourhood $X(v)$ of a vertex v, and for any such subset $S \in \mathcal{S}_{\{v,k\}}$, let $\mathcal{C}_S$ be the set of cycles in $\mathcal{C}$ that contain a 2-arc of the form (a, v, b) such that $a, b \in S$.

We now alter the constraints in our first method (which does not distinguish between non-orientable and orientable embeddings), to the following:

- $x_C \in \{0, 1\}$ for all $C \in \mathcal{C}$,
- $\displaystyle\sum_{C \in \mathcal{C}_e} x_C = 2$ for every edge e of X, and
- $\displaystyle\sum_{C \in \mathcal{C}_S} x_C < k$ for every $S \in \mathcal{S}_{\{v,k\}}$, for every $v \in V(X)$ and $2 \le k \le \left\lfloor \dfrac{\deg(v)}{2} \right\rfloor$.

Similarly, we alter the constraints in our second method (for finding orientable embeddings only), by considering arcs instead of edges in the second constraint. The following lemma explains why these modifications help us find minimum genus embeddings.

Lemma 5.1. *In the third and fourth ILP methods presented above, a feasible region consists of the set of all embeddings and all orientable embeddings of X, respectively, and every feasible solution that maximises the objective function $\sum_{C \in \mathcal{C}} x_C$ gives a minimum genus embedding of X.*

Proof. The constraint $\sum_{C \in \mathcal{C}_e} x_C = 2$ ensures that every edge is used in the embedding exactly twice. In particular, every edge lies in two faces, and so every vertex v occurs $\deg(v)$ times among the set of faces in a feasible solution. Next, the constraints of the form $\sum_{C \in \mathcal{C}_S} x_C < k$ ensure that the faces around each vertex can be arranged into a rotation system. For if that were not possible, then the faces around some vertex v would partition into rotation sub-systems, and at least one of those sub-systems would consist of k faces for some $k \le \lfloor \frac{\deg(v)}{2} \rfloor$, but the relevant constraint makes that impossible.

An arbitrary embedding of X is given by selection of cycles with the property that each edge occurs twice, each vertex occurs $\deg(v)$ times, and there is a rotation system around each vertex. Hence the feasible region consists of all possible embeddings. Moreover, replacing the constraint on edges with the corresponding constraint on arcs is exactly what's needed to reduce the feasible region to orientable embeddings. In particular, in the orientable case only one orientation can be chosen for each cycle. □

Note here that it is also possible to discard the objective function, and instead calculate the feasible region after adding a further constraint of the form $\sum_{C \in \mathcal{C}} x_C = F$, where F is the expected number of faces. Similarly, we may separate this constraint into a number of other constraints specifying the number of cycles that can bound faces of particular lengths.

5.2 Some applications

Example 5.2. The Folkman graph.

This is the smallest semi-symmetric finite graph. It is bipartite of order 20, with valency, diameter and girth 4, and its automorphism group has order 3840. An upper bound on the number of faces of any embedding is $2|E|/4 = 80/4 = 20$. An easy computation shows there are 30 cycles of length 4, and 80 cycles of length 6.

Using our independence number approach with $\mathcal{C}$ taken as the set of all 4-cycles, we find that any embedding (whether orientable or non-orientable) has at most 10 faces of length 4. Then taking $\mathcal{C}$ as the set of all 4-cycles and all 6-cycles, the independence number approach does not help, because the bound it gives is too large. (A reason for this is that it can allow three cycles that are pairwise independent in the auxiliary graph $X_{\mathcal{C}}$ but cannot occur simultaneously as bounding cycles of faces of an embedding.)

On the other hand, the ILP method works very well, and tells us easily that any embedding has at most 15 faces of length up to 6. Together these 15 faces would use up at least $(10 \cdot 4 + 5 \cdot 6)/2 = 35$ of the 40 edges, and it then follows that the number of faces is at most 16. But also our subgroup orbit method finds many embeddings with exactly 16 faces, which are therefore of minimum genus.

The most symmetric non-orientable embeddings of minimum genus have ten faces of length 4, five of length 6 and one of length 10, with a dihedral automorphism group of order 10. With a suitable labelling of the vertices, one such group S is generated by the elements that induce the permutations

$$(2,3)\ (4,5)\ (7,8)\ (9,10)\ (11,19)\ (12,16)\ (14,18)\ (17,20)$$

and

$$(1,4,2,3,5)\ (6,9,7,8,10)\ (11,17,13,20,19)\ (12,14,15,18,16),$$

and then the 16 faces come from the orbits of S containing the 4-cycle

$$(1,11,6,14),$$

the 6-cycle

$$(1,11,9,15,10,19)$$

and the 10-cycle

$$(1,14,3,16,4,15,5,12,2,18).$$

The most symmetric orientable embeddings of minimum genus have ten faces of length 4, four of length 6 and two of length 8, with elementary abelian automorphism group of order 8. With the same vertex-labelling as above, one such group S is generated by the elements inducing the involutions

$$(1,6)\ (2,7)\ (3,8)\ (4,9)\ (5,10),$$

$$(1,2)\ (4,5)\ (6,7)\ (9,10)\ (11,12)\ (13,14)\ (16,20)\ (17,19)$$

and

$$(1,4)\ (2,5)\ (6,9)\ (7,10)\ (13,20)\ (14,16)\ (15,18)\ (17,19),$$

and then the 16 faces come from the orbits of S containing the 4-cycles

$$(1, 11, 6, 14), \qquad (1, 18, 6, 19), \qquad (3, 14, 8, 16),$$

the 6-cycle

$$(1, 14, 3, 13, 2, 18)$$

and the 8-cycle

$$(1, 19, 10, 12, 7, 17, 4, 11).$$

Also the three given generators of S all reverse orientation, so this embedding is reflexible.

Accordingly, we have the following theorem.

Theorem 5.3. *The minimum orientable genus of the Folkman graph is* 3*, attainable by a reflexible embedding with* 16 *faces, of which ten have length* 4*, four have length* 6*, and two have length* 8*, and with elementary abelian automorphism group of order* 8*. The minimum non-orientable genus of the Folkman graph is* 6*, attainable by an embedding with* 16 *faces, in which ten have length* 4*, five have length* 6*, and one has length* 10*, and with dihedral automorphism group of order* 10*.*

Example 5.4. The Doyle-Holt graph.

The Doyle-Holt graph is the smallest finite graph that is *half-arc-transitive*, meaning that it is vertex- and edge-transitive but not arc-transitive. It was discovered independently by Doyle (and mentioned in his Harvard thesis in 1976) and Holt in 1981 (see [25]). This graph has order 27, valency 4, diameter 3 and girth 5, and its automorphism group has order 54. It is also isomorphic to a spanning subgraph of the Menger graph of the dual of the Gray configuration, which we deal with in the next example.

An upper bound on the number of faces of any embedding is $\lfloor 108/5 \rfloor = 21$, and an easy computation shows there are 54 cycles of length 5, and 63 cycles of length 6. Our independence number method gives 27 as an upper bound on the number of faces of length 5, but our ILP method gives an upper bound of 18. Also if an embedding has 21 faces, with F_ℓ of length greater than 5, we have $108 = 2|E| \geq 5F_5 + 6F_\ell = 5(F_5 + F_\ell) + F_\ell = 105 + F_\ell$, and so $F_\ell \leq 3$, and it follows that $(F_5, F_\ell) = (18, 3)$.

Our orbit method gives such a non-orientable embedding with 21 faces, and automorphism group isomorphic to $D_3 \times C_3$, of order 18. With a suitable labelling of the vertices, this group can be generated by the elements inducing the permutations

$$(1, 2, 3)\,(4, 5, 6)\,(7, 8, 9)\,(10, 11, 12)\,(13, 14, 15)\\ (16, 17, 18)\,(19, 20, 21)\,(22, 23, 24)\,(25, 26, 27),$$

$$(2, 3)\,(5, 6)\,(8, 9)\,(10, 21)\,(11, 20)\,(12, 19)\,(13, 24)\\ (14, 23)\,(15, 22)\,(16, 27)\,(17, 26)\,(18, 25)$$

and

$$(1, 4, 7)\,(2, 5, 8)\,(3, 6, 9)\,(10, 13, 16)\,(11, 14, 17)\\ (12, 15, 18)\,(19, 22, 25)\,(20, 23, 26)\,(21, 24, 27),$$

and then the 21 faces of the embedding come from the orbits containing the 5-cycle

$$(1, 13, 8, 19, 18)$$

and the 6-cycle

$$(10, 23, 16, 20, 13, 26).$$

For orientable embeddings on the other hand, the second version of our ILP method (applied to oriented cycles of length 5) shows that the number of faces of length 5 is at most 14. Similarly, the oriented version of our independence number method shows this number is at most 15. In particular, it follows that an orientable embedding cannot have 21 faces (and characteristic -6), so the total number of its faces is no more than 19.

Using our orbit method, we found there exists an orientable embedding with 19 faces, indeed with $F_5 = 14$, $F_6 = 1$ and $F_8 = 4$. With the same vertex-labelling as previously, the automorphism group of this embedding is the group of order 2 generated by the involutory automorphism given for the non-orientable embedding above, and then the faces come from the seven orbits of containing the 5-cycles

$$(1, 13, 20, 5, 25), \quad (1, 18, 19, 8, 13), \quad (2, 14, 21, 7, 22),$$
$$(2, 16, 20, 9, 14), \quad (2, 22, 18, 6, 26), \quad (2, 26, 10, 23, 16),$$
$$(4, 16, 23, 8, 19),$$

the 6-cycle

$$(12, 25, 15, 19, 18, 22),$$

and the orbits of the 8-cycles

$$(4, 12, 22, 7, 10, 5, 20, 16) \quad \text{and} \quad (5, 10, 26, 13, 8, 11, 24, 17).$$

Also the given generator preserves the face bounded by the 6-cycle

$$(12, 25, 15, 19, 18, 22)$$

as well as its orientation, and so this embedding is chiral.

In particular, we have found the minimum orientable genus of the Doyle-Holt graph, thereby answering a question posed in [30] and taking it further, as follows:

Theorem 5.5. *The minimum orientable genus of the Doyle-Holt graph is* 5*, attainable by a chiral embedding with* 19 *faces, of which* 14 *have length* 5*, one has length* 6*, and four have length* 8*, and automorphism group of order* 2*. The minimum non-orientable genus of the Doyle-Holt graph is* 8*, attainable by an embedding with* 21 *faces, of which* 18 *have length* 5 *and three have length* 6*, and automorphism group of order* 18 *isomorphic to* $D_3 \times C_3$.

Example 5.6. The dual Menger graph of the Gray configuration.

The Gray configuration is a configuration of 27 points and 27 lines, which can be realised in 3-dimensional Euclidean space via a $3 \times 3 \times 3$ grid, with the lines as pairwise intersections of 9 planes, partitioned into three triples, each being parallel to one of the three planes with equations $x = 0$, $y = 0$ and $z = 0$.

The Gray graph is the *Levi graph* (or incidence graph) of this configuration, namely the bipartite graph whose vertices are the points and lines and whose edges represent incidence (between points and lines). Also the *Menger graph* of the Gray configuration, which indicates collinearity of points, is isomorphic to the Cartesian product $C_3 \,\square\, C_3 \,\square\, C_3$. On the other hand, the Menger graph of the dual of the Gray configuration, which indicates copunctuality of lines, is another graph of order 27, valency 6, diameter 3 and girth 3, with automorphism group of order 1296. It was studied in some detail in [30].

Let X be this dual Menger graph. As with $C_3 \,\square\, C_3 \,\square\, C_3$, an upper bound on the number of faces of an embedding of X is $162/3 = 54$, but that bound is far from sharp. By inspection (or an easy application of our ILP method) there can be at most 27 faces of length 3, and from this it follows that the number of faces is bounded above by 47. Better still, if we take $\mathcal{C}_j$ as the set of all cycles of length j for $j \in \{3, 4\}$, and then $\mathcal{C} = \mathcal{C}_3 \cup \mathcal{C}_4$, our ILP method gives a maximum value for the objective function $2\sum_{C\,\in\,\mathcal{C}_3} + \sum_{C\,\in\,\mathcal{C}_4}$ as 66, and so $2F_3 + F_4 \le 66$. Hence if F_ℓ denotes the number of faces of length greater than 4, we have

$$162 = 2|E| \ge 3F_3 + 4F_4 + 5F_\ell = 5(F_3 + F_4 + F_\ell) - (2F_3 + F_4) \ge 5|F| - 66,$$

which gives $|F| \le \lfloor (162 + 66)/5 \rfloor = 45$.

Our subgroup orbit method provides a non-orientable embedding with 45 faces, such that $(F_3, F_4) = (18, 27)$. Its automorphism group has order 108, and is isomorphic to a semi-direct product of the non-abelian group of order 27 and exponent 3 by the Klein 4-group V_4. With a suitable labelling of the vertices, this group can be generated by the elements inducing the permutations

$$(1,25)\,(2,11)\,(4,22)\,(5,24)\,(6,9)\,(7,27)\,(8,17)\,(10,26)\,(12,20)\\ (13,21)\,(14,15)\,(16,19)$$

and

$$(1,2,15)\,(3,14,4,5,9,10)\,(6,19,7,11,18,8)\,(12,13,24,26,17,16)\\ (20,23,21,27,25,22),$$

and then the $18 + 27 = 45$ faces of the embedding come from the orbits of the 3-cycle $(1, 4, 8)$ and 4-cycle $(1, 2, 3, 7)$. The characteristic of this embedding is

$$\chi = 27 - 81 + 45 = -9.$$

Next, for orientable embeddings, there can be at most 44 faces (in order to have even characteristic), and our orbit method produces one with $(F_3, F_4, F_6) = (26, 12, 6)$. With the same vertex-labelling as previously, the automorphism group of this one is cyclic of order 6, generated by the second of the two automorphisms given for the non-orientable embedding above. The 44 faces come from the eight orbits containing the five 3-cycles

$$(1,4,8), \qquad (3,8,17), \qquad (3,27,7), \qquad (12,23,20), \qquad (12,24,17),$$

the two 4-cycles

$$(1,8,3,2), \qquad\qquad (3,17,25,27),$$

and the single 6-cycle

$$(4,18,26,16,20,8).$$

Also the given generator reverses orientation, and so this embedding is reflexible.

In particular, we have found the minimum orientable genus of this graph, thereby answering a question posed in [30], and taking it further, as follows:

Theorem 5.7. *The minimum orientable genus of the dual Menger graph of the Gray configuration is* 6*, attainable by a reflexible embedding with* 44 *faces, of which* 26 *have length* 3*, and* 12 *have length* 4*, and* 6 *have length* 6*, with a cyclic automorphism group of order* 6*. The minimum non-orientable genus of the same graph is* 11*, attainable by an embedding with* 45 *faces, of which* 18 *have length* 3*, and* 27 *have length* 4*, and automorphism group of order* 108 *isomorphic to a semi-direct product of the non-abelian group of order* 27 *and exponent* 3 *by the Klein* 4*-group.*

Also in [30] it was noted that if H and D are the Doyle-Holt graph and the Menger graph of the dual Gray configuration, respectively, then $4 \le \gamma(H) \le \gamma(D) \le 7$. We have now shown that $\gamma(H) = 5$ and $\gamma(D) = 6$, so that in fact $4 < \gamma(H) < \gamma(D) < 7$.

Moreover, it was shown in [30, Proposition 1] that if L is the Levi graph and M is the Menger graph of any (v_3) configuration, then $\gamma(M) \le \gamma(L)$, and near the end of [30] the authors asked about finding such a configuration for which that inequality in strict. Our work provides an answer to this question as well, because the Levi graph of the dual of the Gray configuration is the Gray graph (giving $\gamma(L) = 7$), while its Menger graph is the above graph D, with $\gamma(D) = 6 < 7$. Hence we can strengthen Proposition 1 of [30] to the following:

Theorem 5.8. *If L is the Levi graph and M is the Menger graph of any (v_3) configuration, then $\gamma(M) \le \gamma(L)$, and this inequality is strict for the dual of the Gray (27_3) configuration.*

Example 5.9. The second smallest semi-symmetric cubic graph.

This graph is also the smallest 'Iofinova-Ivanov graph', constructed in [26], and so we will call it II_1. It was the most challenging of all the examples we considered in this work. It is a bipartite graph of order 110, diameter 7 and girth 10, with automorphism group of order 1320, isomorphic to PGL(2, 11); see [8] for more details.

The naive upper bound on the number of faces of any embedding of II_1 is $330/10 = 33$, but again this is not sharp. Using our ILP approach on cycles of length 10 and 12, it can be shown that $2F_{10} + F_{12}$ is at most 48, and hence if F_ℓ is the number of faces of length greater than 12, we have

$$\begin{aligned} 330 = 2|E| &\ge 10F_{10} + 12F_{12} + 14F_\ell \\ &= 14(F_{10} + F_{12} + F_\ell) - 2(2F_{10} + F_{12}) \\ &\ge 14|F| - 96, \end{aligned}$$

and therefore $|F| \le \lfloor 426/14 \rfloor = 30$.

Now this improved upper bound is sharp, because our orbit method produces a non-orientable embedding with exactly 30 faces, and $F_{10} = F_{12} = 15$. With a suitable labelling of vertices, the automorphism group S of this embedding is cyclic of order 3, generated by the automorphism inducing the permutation

$$\begin{aligned} &(1,33,46)\,(2,52,18)\,(3,10,24)\,(4,17,13)\,(5,15,41)\,(6,53,32)\,(7,31,27) \\ &\quad (8,26,42)\,(9,47,22)\,(11,38,51)\,(12,44,16)\,(14,34,28)\,(19,36,43)\,(20,40,54) \\ &\quad (21,45,30)\,(23,48,35)\,(25,39,29)\,(49,55,50)\,(56,91,84)\,(57,74,93) \\ &\quad (58,80,103)\,(59,110,65)\,(60,101,94)\,(62,86,77)\,(64,82,92)\,(66,102,104) \\ &\quad (67,95,97)\,(68,72,99)\,(69,73,105)\,(70,79,78)\,(71,90,100)\,(75,85,106) \\ &\quad (76,108,88)\,(83,87,107)\,(89,96,98), \end{aligned}$$

and then the faces of the embedding come from the S-orbits of the five 10-cycles

$$\begin{aligned}&(1, 56, 2, 59, 6, 66, 19, 68, 8, 57),\\&(2, 56, 4, 62, 11, 83, 29, 78, 14, 60),\\&(3, 64, 23, 96, 45, 73, 26, 74, 10, 61),\\&(5, 70, 25, 76, 12, 82, 48, 100, 32, 65),\\&(6, 71, 20, 88, 29, 83, 43, 104, 37, 66),\end{aligned}$$

and the five 12-cycles

$$\begin{aligned}&(1, 56, 4, 63, 13, 77, 16, 88, 20, 67, 7, 58),\\&(1, 57, 3, 64, 16, 77, 51, 106, 34, 70, 5, 58),\\&(2, 59, 15, 80, 31, 99, 42, 105, 47, 81, 9, 60),\\&(7, 67, 30, 98, 49, 109, 55, 106, 51, 107, 36, 72) \quad \text{and}\\&(9, 60, 14, 85, 49, 98, 48, 100, 54, 97, 45, 73).\end{aligned}$$

The characteristic is $\chi = 110 - 165 + 30 = -25$.

Our subgroup orbit method also produces an orientable embedding with 27 faces, indeed with $(F_{10}, F_{12}, F_{14}) = (6, 12, 9)$, from the same cyclic subgroup S of order 3. In particular, this embedding is chiral, because S has odd order. Its faces come from the nine S-orbits containing the two 10-cycles

$$\begin{aligned}&(1, 56, 2, 60, 9, 73, 26, 72, 7, 58),\\&(11, 75, 50, 96, 23, 64, 16, 88, 29, 83),\end{aligned}$$

the four 12-cycles

$$\begin{aligned}&(1, 57, 8, 69, 21, 95, 40, 76, 12, 62, 4, 56),\\&(1, 58, 5, 70, 34, 106, 51, 77, 16, 64, 3, 57),\\&(2, 56, 4, 63, 17, 86, 38, 87, 19, 66, 6, 59),\\&(5, 65, 32, 100, 54, 97, 27, 68, 19, 87, 25, 70),\end{aligned}$$

and the three 14-cycles

$$\begin{aligned}&(2, 59, 15, 80, 31, 95, 21, 89, 55, 109, 49, 85, 14, 60),\\&(3, 64, 23, 90, 53, 102, 37, 104, 43, 99, 42, 93, 24, 61) \quad \text{and}\\&(9, 60, 14, 78, 29, 88, 20, 71, 35, 89, 21, 69, 22, 81).\end{aligned}$$

In turns out that this is an orientable embedding of minimum genus, because there exist none with 29 faces (and characteristic $110 - 165 + 29 = -26$). We were not able to prove that by using the standard forms of our independence number and ILP methods, because the size and girth of the graph create too many cycles for consideration as face boundaries. But we were still able to prove it by a slightly different approach, using restricted forms of those methods, and we now give a brief outline of the proof.

Assume there exists an orientable embedding with 29 faces, and again let F_k denote the number of faces of length k. Then an easy computation of weighted sums shows there are 435 possibilities for the sequence $(F_{10}, F_{12}, F_{14}, F_{16}, F_{18}, \ldots)$, with the integer F_{10} ranging from 9 to 23. These possibilities can be dealt with in groups or individually, with the aim of eliminating all of them.

For example, suppose $F_{12} \geq 1$, and let $\mathcal{C}$ be the set of all directed 10- and 12-cycles in II_1. Now let C and D be any directed 10-cycle and 12-cycle that are independent in the auxiliary graph $X_{\mathcal{C}}$, and let $\mathcal{C}_{(CD)}$ be the subset of $\mathcal{C}$ consisting of all directed 10-cycles that are independent of both C and D in $X_{\mathcal{C}}$. We then compute the independence number of the auxiliary graph $X_{\mathcal{C}_{(CD)}}$ for all such pairs (C, D), up to equivalence in the automorphism group of the graph. This computation shows that the maximum of these independence numbers is 21, from which it follows that $F_{10} \leq 22$ when $F_{12} \geq 1$. This eliminates 70 possibilities for the sequence $(F_{10}, F_{12}, F_{14}, F_{16}, F_{18}, \ldots)$.

Similarly, another 57 possibilities can be eliminated in the case where $F_{12} \geq 2$, for in that case the corresponding independence number computation shows that $F_{10} \leq 21$, and another 71 can be eliminated when $F_{12} \geq 4$, for in that case $F_{10} \leq 19$, and then another 30 when $F_{12} \geq 5$, and another 42 when $F_{14} \geq 1$ (with no assumption on F_{12}). Other cases that help eliminate possibilities include those where both $F_{14} \geq 1$ and $F_{16} \geq 1$, and so on.

This kind of approach reduced the problem to just six possibilities, namely those for which

$$(F_{10}, F_{12}, F_{14}, F_{16}, F_{18}) = (18, 6, 1, 4, 0), (18, 6, 2, 2, 1), (17, 7, 2, 3, 0), \\ (19, 4, 2, 4, 0), (16, 6, 7, 0, 0) \text{ and } (20, 3, 1, 5, 0),$$

all but one of which could be eliminated by similar means. For some of them, we used the ILP method in place of the independence method, when the independence method gave too large a number.

The trickiest case was the last of the above six possibilities, namely the one where $(F_{10}, F_{12}, F_{14}, F_{16}) = (20, 3, 1, 5)$. For this, we took $\mathcal{C}$ be the set of all directed 10-, 12-, 14- and 16-cycles in II_1, and ran through all possibilities for a quintuple Q of independent vertices in the auxiliary graph $X_{\mathcal{C}}$, consisting of three 12-cycles, one 14-cycle and one 16-cycle, and for each one, determined the maximum size of a set T of 10-cycles for which $Q \cup T$ is an independent set in $X_{\mathcal{C}}$. The maximum size found was 18, indicating that if $F_{12} = 3$ and $F_{14} = 1$ and $F_{16} \geq 1$, then $F_{10} \leq 18$. In particular, this shows that $(F_{10}, F_{12}, F_{14}, F_{16})$ cannot be $(20, 3, 1, 5)$.

Hence the number of faces of an orientable embedding of II_1 cannot be 29, and we have an answer to the penultimate open question in [30]. We state this the first part of the following, to complete the paper:

Theorem 5.10. *The minimum orientable genus of the second smallest semi-symmetric cubic graph II_1 is* 15*, attainable by a chiral embedding with* 27 *faces, of which six have length* 10*, twelve have length* 12*, and nine have length* 14*, and with cyclic automorphism group of order* 3*. The minimum non-orientable genus of the same graph is* 27*, attainable by an embedding with* 30 *faces, of which* 15 *have length* 10*, and* 15 *have length* 12*, and with cyclic automorphism group of order* 3 *(acting in the same way on the graph).*

6 Final remarks

In this paper we have presented four new methods that are helpful for determining the minimum genus of embeddings of a graph on a surface. Also we have shown in some detail how counting arguments can be of great use in solving this kind of problem.

Our first one (the *subgroup orbit* method) considers possibilities for a group of automorphisms of the embedding, of suitable order, thereby reducing the search space. A suitable set of candidate faces is formed from closed walks of appropriate lengths in the graph, and then the faces are chosen from orbits of the chosen group on those walks.

The second one (the *independence number* method) uses the independence number of an auxiliary graph to bound the maximum number of faces of given lengths. This method can be very useful when taken in combination with other approaches. In particular, it can be used to determine that no embedding of a given graph can have certain numbers of faces of given lengths, even when counting arguments do not help.

Our third and the fourth methods translate the problem of choosing faces from a set of candidate closed walks into a *linear programming* problem. The third method can help find upper bounds on the number of faces of particular lengths that can be used in an embedding, while the fourth method aims to find an actual embedding of the graph with minimum genus. This is based on an approach that translates the conditions for a set of closed walks of the graph to give an embedding into to a set of linear constraints and an objective function for minimising the genus.

All of these methods use a set of closed walks of the graph (for bounding candidate faces). Since the set of all closed walks of up to given length in the graph can be enormous, it is best to use these methods in combination with counting arguments, to limit (or rule out) the lengths of candidate faces. This is often easily done by hand, but can also be automated. Also our methods can be used more generally to find embeddings of a graph with given face lengths, and are not necessarily restricted to finding minimum genus embeddings.

Our linear programming method for calculating an explicit embedding provides a relatively fast way of finding a minimum genus embedding of a graph, without considering symmetries. It is possible to combine this with prescription of symmetries (indeed, this is a standard trick in linear programming), but the obvious way of doing that involves reducing the problem to finding an embedding of a quotient (voltage) graph. Also it can be difficult to prescribe the lengths of faces of the latter embedding, as seen in Subsection 3.4). The subgroup orbit method is better suited in many cases, because it provides complete control over the lengths of the faces of the cover. Indeed, this has proven very successful in the cases of vertex-, edge- or arc-transitive graphs.

Finally, we have demonstrated how to use these methods with several examples, in which we determined the minimum genus of embeddings of several well-known interesting graphs, either for the first time, or in a different (and sometimes better) way than achieved previously. We have stored details of these minimum genus embeddings at the *AMC* website associated with this article.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 37–49
https://doi.org/10.26493/1855-3974.1496.623
(Also available at http://amc-journal.eu)

Distant sum distinguishing index of graphs with bounded minimum degree

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Received 2 October 2017, accepted 2 January 2019, published online 19 June 2019

Abstract

For any graph $G = (V, E)$ with maximum degree Δ and without isolated edges, and a positive integer r, by $\chi'_{\Sigma,r}(G)$ we denote the r-distant sum distinguishing index of G. This is the least integer k for which a proper edge colouring $c\colon E \to \{1, 2, \dots, k\}$ exists such that $\sum_{e \ni u} c(e) \neq \sum_{e \ni v} c(e)$ for every pair of distinct vertices u, v at distance at most r in G. It was conjectured that $\chi'_{\Sigma,r}(G) \leq (1 + o(1))\Delta^{r-1}$ for every $r \geq 3$. Thus far it has been in particular proved that $\chi'_{\Sigma,r}(G) \leq 6\Delta^{r-1}$ if $r \geq 4$. Combining probabilistic and constructive approach, we show that this can be improved to $\chi'_{\Sigma,r}(G) \leq (4 + o(1))\Delta^{r-1}$ if the minimum degree of G equals at least $\ln^8 \Delta$.

Keywords: Distant sum distinguishing index of a graph, neighbour sum distinguishing index, adjacent strong chromatic index, distant set distinguishing index.

Math. Subj. Class.: 05C15, 05C78

1 Introduction

Integer edge colourings were initiated in the paper of Chartrend et al. [8], where the graph invariant *irregularity strength*, $s(G)$, was introduced as a possible measure of the 'level of irregularity' of a graph G. This referred to the well known phenomenon in graph theory that there are no *irregular graphs*, understood as graphs whose all vertices have pairwise distinct degrees (see also [7] for possible alternative definitions of irregularity in graphs), except the trivial 1-vertex case. For a given graph $G = (V, E)$, $s(G)$ is defined as the least k for which one is able to construct an irregular multigraph (defined analogously as in the

*Financed within the program of the Polish Minister of Science and Higher Education named "Iuventus Plus" in years 2015-2017, project no. IP2014 038873, and partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of Ministry of Science and Higher Education.
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case of graphs above) of G by multiplying some of its edges – each at most k times. In terms of integer colourings, the same value is equivalently defined as the least k so that an edge colouring $c\colon E \to \{1, 2, \ldots, k\}$ exists attributing every vertex $v \in V$ a distinct *weighted degree* defined as:

$$d_c(v) := \sum_{e \ni v} c(e).$$

This we shall also call the *sum at* v, see e.g. [3, 5, 9, 10, 12, 13, 15, 18, 21, 22, 24, 25, 26] for a few out of a vastness of results concerning $s(G)$, which also gave rise to a whole discipline devoted to investigating this and other related problems. One of the most intriguing direct descendants of the irregularity strength is its local correspondent, where we necessarily require an inequality $d_c(u) \neq d_c(v)$ to hold only for adjacent vertices u, v in G. The least k admitting a colouring $c\colon E \to \{1, 2, \ldots, k\}$ with such a feature we shall denote by $s_1(G)$. In the first paper [19] concerning this the authors conjectured that $k = 3$ suffices for every connected graph of order at least 3. This presumption is commonly referred to as the *1–2–3 Conjecture* nowadays. This was investigated e.g. in [1, 2, 35]. The best thus far general result is however the upper bound $s_1(G) \leq 5$ from [17]. A generalization of this concept, forming a link between $s_1(G)$ and $s(G)$, was introduced in [27]. Let $d(u, v)$ denote the distance of vertices u, v in G. We shall call u and v, r-neighbours if $1 \leq d(u, v) \leq r$ in G, where r is a positive integer. For every vertex v in G, the set of its r-neighbours shall be denoted by $N^r(v)$, and we set $d^r(v) = |N^r(v)|$. The least k so that an edge colouring $c\colon E \to \{1, 2, \ldots, k\}$ exists with $d_c(u) \neq d_c(v)$ for every r-neighbours $u, v \in V$ in G is denoted by $s_r(G)$ (note it would be justified to set $s_\infty(G) = s(G)$ in the same spirit), see e.g. [27] and [30] for a few results concerning this concept, which refers to the known distant chromatic numbers (see [20] for a survey of this topic in turn).

In this paper we shall investigate a related problem referring to distant chromatic numbers. Given a positive integer r and a graph $G = (V, E)$ without isolated edges, the *r-distant sum distinguishing index* of G, denoted by $\chi'_{\Sigma,r}(G)$, is the least integer k such that there exists a *proper* edge colouring $c\colon E \to \{1, 2, \ldots, k\}$ which *sum-distinguishes* r-neighbours in G, i.e. such that $d_c(u) \neq d_c(v)$ for every $u, v \in V$ with $1 \leq d(u, v) \leq r$. In [31] the following conjecture, approximating the investigated lower bounds discussed e.g. in [27, 31], was posed.

Conjecture 1.1 ([31]). *For every integer $r \geq 3$ and each graph G without isolated edges of maximum degree Δ, $\chi'_{\Sigma,r}(G) \leq (1 + o(1))\Delta^{r-1}$.*

It was also conjectured under the same conditions, that $\chi'_{\Sigma,2}(G) \leq (2 + o(1))\Delta$ [31], and that $\chi'_{\Sigma}(G) = \chi'_{\Sigma,1}(G) \leq \Delta + 2$ for every connected graph G of order at least 3 non-isomorphic to C_5 [14]. Thus far for $r \geq 4$, the following is known.

Theorem 1.2 ([31]). *Let G be a graph without isolated edges and with maximum degree $\Delta \geq 2$, and let $r \geq 4$. Then $\chi'_{\Sigma,r}(G) \leq 6\Delta^{r-1}$.*

Upper bounds of orders conjectured above are also known for $r = 2, 3$, but with slightly worse multiplicative constants than in Theorem 1.2 above, see [31], while the upper bound of the form $\chi'_{\Sigma}(G) \leq (1 + o(1))\Delta(G)$ was proved in [29] and [32], see also [6, 11, 14, 28, 33, 34] for other results concerning the case $r = 1$. In this paper we combine probabilistic approach with a special constructive algorithm in order to provide the following improvements of the best known upper bounds for all $r \geq 4$ from Theorem 1.2, under assumption that the minimum degree of a graph is larger than some poly-logarithmic function of the maximum degree. (The value of this function, which seems unavoidable within our approach, could still be optimized – we did not try to do this for the sake of clarity of the presentation.)

Theorem 1.3. *For every integer $r \geq 4$ there exists a constant Δ_0 such that for each graph G with maximum degree $\Delta \geq \Delta_0$ and minimum degree $\delta \geq \ln^8 \Delta$,*

$$\chi'_{\Sigma,r}(G) < 4\Delta^{r-1}\left(1 + \frac{3}{2\ln \Delta}\right) + 384,$$

hence $\chi'_{\Sigma,r}(G) \leq (4 + o(1))\Delta^{r-1}$ for all graphs with $\delta \geq \ln^8 \Delta$ and without isolated edges.

2 Probabilistic tools and preliminary lemmas

The following standard tools of the probabilistic method shall be applied: the Lovász Local Lemma, see e.g. [4], the Chernoff Bound, see e.g. [16, Theorem 2.1, page 26] and Talagrand's Inequality, see e.g. [23]. Details follow.

Theorem 2.1 (The Local Lemma). *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most D, and that $\mathbf{Pr}(A_i) \leq p$ for all $1 \leq i \leq n$. If*

$$ep(D + 1) \leq 1,$$

then $\mathbf{Pr}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

Theorem 2.2 (Chernoff Bound). *For any $0 \leq t \leq np$,*

$$\mathbf{Pr}(\mathrm{BIN}(n, p) > np + t) < e^{-\frac{t^2}{3np}} \quad \textit{and} \quad \mathbf{Pr}(\mathrm{BIN}(n, p) < np - t) < e^{-\frac{t^2}{2np}} \leq e^{-\frac{t^2}{3np}}$$

where $\mathrm{BIN}(n, p)$ is the sum of n independent Bernoulli variables, each equal to 1 with probability p and 0 otherwise.

Theorem 2.3 (Talagrand's Inequality). *Let X be a non-negative random variable determined by l independent trials $T_1, \ldots, T_l$. Suppose there exist constants $c, k > 0$ such that for every set of possible outcomes of the trials, we have:*

1. *changing the outcome of any one trial can affect X by at most c, and*
2. *for each $s > 0$, if $X \geq s$ then there is a set of at most ks trials whose outcomes certify that $X \geq s$.*

Then for any $t \geq 0$, we have

$$\mathbf{Pr}(|X - \mathbf{E}(X)| > t + 20c\sqrt{k\,\mathbf{E}(X)} + 64c^2k) \leq 4e^{-\frac{t^2}{8c^2k(\mathbf{E}(X)+t)}}. \tag{2.1}$$

We note that knowing that $\mathbf{E}(X) \leq h$ we may also apply Talagrand's Inequality e.g. to the variable $Y = X + h - \mathbf{E}(X)$, with $\mathbf{E}(Y) = h$ to obtain the following counterpart of (2.1) provided that the assumptions of Theorem 2.3 hold for X:

$$\begin{aligned}\mathbf{Pr}(X > h + t + 20c\sqrt{kh} + 64c^2k) &\leq \mathbf{Pr}(Y > h + t + 20c\sqrt{kh} + 64c^2k)\\ &\leq 4e^{-\frac{t^2}{8c^2k(h+t)}}.\end{aligned}$$

Similarly, the Chernoff Bound can be applied e.g. when we know that X is a sum of $n \le k$ random independent Bernoulli variables, each equal to 1 with probability at most q, to prove that

$$\mathbf{Pr}(X > kq + t) < e^{-\frac{t^2}{3kq}}$$

(if $t \le \lfloor k \rfloor q$).

In order to prove our main result we shall need the following observation.

Lemma 2.4. *If Δ is large enough, then for every graph G' of maximum degree $\Delta' \le \Delta$ and with minimum degree $\delta' \ge \frac{1}{2}\ln^5 \Delta$, there exists a spanning subgraph F' of G' with $1 \le d_{F'}(v) \le \frac{d_{G'}(v)}{\ln^3 \Delta}$ for each $v \in V(G')$.*

Proof. We assume that Δ is large enough so that all inequalities within the proof below hold. Independently for every vertex $v \in V(G')$ choose one of its incident edges, each with equal probability, and denote the subgraph induced in G' by the set of all the chosen edges by F'. We shall show that with positive probability such F' complies with our requirements. For every $v \in V(G')$ denote by X_v the random variable representing the number of all edges incident with v and chosen to $E(F')$ by any of the neighbours of v in G', and note that $d_{F'}(v) \le X_v + 1$ (as at most one more edge incident with v in G' might be chosen to $E(F')$ by v itself). Note that for any given vertex $v \in V(G')$ and its neighbour $u \in N_{G'}(v)$, the probability that uv was chosen by u equals $\frac{1}{d_{G'}(u)} \le \frac{2}{\ln^5 \Delta}$, hence

$$\mathbf{E}(X_v) \le \frac{2d_{G'}(v)}{\ln^5 \Delta} \le \frac{d_{G'}(v)}{2\ln^3 \Delta} - \frac{1}{2}.$$

By the Chernoff Bound (with $t = \frac{d_{G'}(v)}{2\ln^3 \Delta} - \frac{1}{2} \ge \frac{\ln^2 \Delta}{5}$) we thus obtain that

$$\mathbf{Pr}(X_v > \frac{d_{G'}(v)}{\ln^3 \Delta} - 1) < e^{-\frac{\ln^2 \Delta}{15}} < \frac{1}{\Delta^3}. \tag{2.2}$$

As any event $X_v > \frac{d_{G'}(v)}{\ln^3 \Delta} - 1$ is mutually independent of all other events $X_{v'} > \frac{d_{G'}(v')}{\ln^3 \Delta} - 1$ with $d(v, v') > 2$, i.e. all except at most $(\Delta')^2 \le \Delta^2$, by (2.2) and the Lovász Local Lemma we may conclude that with positive probability for every $v \in V(G')$, $X_v \le \frac{d_{G'}(v)}{\ln^3 \Delta} - 1$, hence $d_{F'}(v) \le \frac{d_{G'}(v)}{\ln^3 \Delta}$. A desired F' must thus exist. □

We shall also need to guarantee a special ordering of the vertices of a graph $G = (V, E)$. For any linear ordering of V and a vertex $v \in V$, a neighbour or r-neighbour of v which precedes it in the ordering shall be called a *backward neighbour* or *r-neighbour*, resp., of v. The remaining ones in turn shall be referred to as *forward neighbours* or *r-neighbours*, resp., of v, while the edges joining v with its forward or backward neighbours shall be called *forward* or *backward*, resp., as well. For any subset $S \subset V$, let also $N_-(v)$, $N_-^r(v)$, $N_S(v)$, $N_S^r(v)$ denote the sets of all backward neighbours, backward r-neighbours, neighbours in S and r-neighbours in S of v, respectively. Set finally $d_-(v) = |N_-(v)|$, $d_-^r(v) = |N_-^r(v)|$, $d_S(v) = |N_S(v)|$, $d_S^r(v) = |N_S^r(v)|$, and for any subset of edges $E_0 \subseteq E$, $d_{E_0}(v) = |\{u \in N(v) : uv \in E_0\}|$.

The following lemma was proved in [30]. Here we only outline the main ideas behind its proof – the remaining part of the argument can however be reconstructed by an interested reader, as in general it is based on a similar combination of the Chernoff Bound and Local Lemma as the (less complex) proof of Lemma 2.4 above.

Lemma 2.5 ([30])**.** *There exists a constant Δ_0' such that for every graph $G = (V, E)$ with maximum degree $\Delta \geq \Delta_0'$ and minimum degree $\delta \geq \ln^8 \Delta$, there is an assignment attributing every vertex $v \in V$ a distinct real number $X_v \in [0, 1]$ such that if we denote:*

$$A = \left\{ v : X_v < \frac{1}{\ln^2 \Delta} \right\},$$
$$B = \left\{ v : \frac{1}{\ln^2 \Delta} \leq X_v \leq 1 - \frac{1}{\ln^3 \Delta} \right\},$$
$$C = \left\{ v : X_v > 1 - \frac{1}{\ln^3 \Delta} \right\}$$

and order the vertices in V into the sequence $v_1, v_2, \ldots, v_n$ consistently with this assignment, i.e. so that $v_i < v_j$ whenever $X_{v_i} < X_{v_j}$, then for every vertex v in G:

- **(i)** $d_A^r(v) \leq 2\frac{d(v)\Delta^{r-1}}{\ln^2 \Delta}$,
- **(ii)** $d_C^r(v) \leq 2\frac{d(v)\Delta^{r-1}}{\ln^3 \Delta}$,
- **(iii)** $\frac{1}{2}\frac{d(v)}{\ln^2 \Delta} \leq d_A(v) \leq 2\frac{d(v)}{\ln^2 \Delta}$,
- **(iv)** $\frac{1}{2}\frac{d(v)}{\ln^3 \Delta} \leq d_C(v) \leq 2\frac{d(v)}{\ln^3 \Delta}$,
- **(v)** *if $v \in B$, then:* $d_-(v) \geq X_v d(v) - \sqrt{X_v d(v)} \ln \Delta$,
- **(vi)** *if $v \in B$, then:* $d_-^r(v) \leq X_v d(v)\Delta^{r-1} + \sqrt{X_v d(v)\Delta^{r-1}} \ln \Delta$.

Proof. Independently for every $v \in V$ we randomly and uniformly choose a real value $X_v \in [0, 1]$ (i.e., we associate with every v an independent random variable $X_v \sim U[0, 1]$ having the uniform distribution on $[0, 1]$). With probability one, these values are pairwise distinct for all vertices. It is also straightforward to note that for every vertex $v \in V$,

$$\mathbf{E}(d_A^r(v)) \leq \frac{d(v)\Delta^{r-1}}{\ln^2 \Delta},$$
$$\mathbf{E}(d_C^r(v)) \leq \frac{d(v)\Delta^{r-1}}{\ln^3 \Delta},$$
$$\mathbf{E}(d_A(v)) = \frac{d(v)}{\ln^2 \Delta},$$
$$\mathbf{E}(d_C(v)) = \frac{d(v)}{\ln^3 \Delta},$$
$$\mathbf{E}(d_-(v)) = X_v d(v),$$
$$\mathbf{E}(d_-^r(v)) \leq X_v d(v)\Delta^{r-1}.$$

Then one may prove a concentration of all the corresponding random variables using the Chernoff Bound, which implies that the probability of a contradiction of each of the events **(i)** – **(vi)** is bounded from above by Δ^{-3r}. As each of the 6 events associated with v is mutually independent of all other such events associated with vertices at distance exceeding $2r$, analogously as in the previous proof, the thesis is implied by the Lovász Local Lemma, see [30] for details. □

3 Proof of Theorem 1.3

Let $r \geq 4$ be a fixed integer and let $G = (V, E)$ be a graph with maximum degree Δ and with minimum degree $\delta \geq \ln^8 \Delta$. We shall assume that Δ is large enough so that all explicit inequalities below hold and $\Delta \geq \Delta'_0$ (from Lemma 2.5), so we shall not specify its value (but assume in particular that $\delta \geq \ln^8 \Delta \geq 2$, i.e. there are no isolated edges in G).

Let q be the least integer divisible by $3 \cdot 2^5 = 96$ such that

$$\frac{\Delta^{r-1}}{\ln \Delta} \leq q < \frac{\Delta^{r-1}}{\ln \Delta} + 96, \tag{3.1}$$

and let Q be the least integer divisible by q (thus also by 96) such that

$$2\Delta^{r-1} + \frac{\Delta^{r-1}}{\ln \Delta} \leq Q < 2\Delta^{r-1} + 2\frac{\Delta^{r-1}}{\ln \Delta} + 96. \tag{3.2}$$

Fix a vertex ordering $v_1, v_2, \ldots, v_n$ of V consistent with Lemma 2.5 above. Our goal shall be to show that $\chi'_{\Sigma,r}(G) \leq 2Q + 2q$. For every vertex $v \in A \cup B$ we choose one edge joining it with a vertex in C and denote this edge by e_v – it exists by **(iv)** (from Lemma 2.5). A desired colouring shall be constructed via algorithm developed consistently with the fixed vertex ordering, starting from v_1. Prior launching it we first fix an initial proper edge colouring

$$c_0 \colon E \to \{Q + q - \Delta, Q + q - \Delta + 1, \ldots, Q + q\}$$

of G, which exists due to the Vizing's Theorem. Note that this is also a proper edge colouring modulo q (thus also modulo Q), i.e. no two adjacent edges in G have colours congruent modulo q. We shall require this feature within the process of constructing a desired edge colouring from c_0, admitting only temporary deviations from this rule or replacing q with Q in the final part of our argument. While modifying our colouring, by $c(e)$ we shall always mean the contemporary colour of an edge e (hence $d_c(v)$ shall stand for the up-to-date weighted degree of a vertex v), and $d(v)$ shall denote the degree of v in G. In step one of our modifying procedure we shall analyze v_1, in step two we analyze v_2, and so on. In general, in step i we shall be modifying only colours of the edges incident with v_i (via rules specified below). Every vertex v_i, the moment it is analyzed (i.e. in step i), shall be associated with a 2-element set, denoted by S_{v_i}, expressing its two admissible sums, and belonging to the family (of pairwise disjoint sets):

$$\mathcal{S} = \big\{\{l, l+Q\} \mid l \in \mathbb{Z} \wedge \big(l \equiv 0 \pmod{2Q} \vee \\ l \equiv 1 \pmod{2Q} \vee \ldots \vee l \equiv Q-1 \pmod{2Q}\big)\big\}.$$

Starting from the end of step i, we shall require $d_c(v_i) \in S_{v_i}$ till the end of the construction. The key restriction concerning the choice of such set is so that

($*$) S_{v_i} is disjoint with S_{v_j} for every $j < i$ such that $v_j \in N^r(v_i)$.

This shall be strictly required for all $v_i \in A \cup B$.

While modifying colours of the edges, we shall obey the following rules. Suppose a vertex v is being analyzed in a given step. We allow:

(1°) adding Q or subtracting Q (or doing nothing) from the colour of every backward edge of v joining v with a neighbour $u \in A \cup B$ (so that $d_c(u) \in S_u$ afterwards);

(2°) adding 0 or q to the colour of every forward edge of $v \in A \cup B$ except e_v;

(3°) switching the colour of e_v to any integer in $[Q+q, Q+2q]$ for every $v \in A \cup B$, as long as the edge colouring obtained remains proper modulo q.

Note that after introducing such changes we shall always have

$$q - \Delta \leq c(e) \leq 2Q + 2q \tag{3.3}$$

for every $e \in E$ (as desired). Special rules shall be applied to edges e with both ends in C. These however shall be consistent with (3.3), see details below. Let us however note here that by the bounds from (3.1), (3.2) and (3.3) above, since $\frac{2Q+2q}{q-\Delta} < 5 \ln \Delta$, we shall have the following.

Remark 3.1. Any r-neighbours u, v with

$$d(u) \geq d(v) 5 \ln \Delta$$

shall certainly be sum-distinguished in G within our construction.

Suppose now we are about to analyze a consecutive vertex $v \in A$, whose degree we denote by d, and thus far all our rules and requirements have been fulfilled. Note that using admissible modifications (1°), (2°) and (3°) of the colours of the backward and forward edges of v (since less than 2Δ residues modulo q might be blocked for the colour of e_v due to the required properness of edge colouring modulo q), we may obtain more than $d(q - 2\Delta)$ integer sums at v. At least $d(\frac{q}{3} - 2\Delta)$ of these are divisible by 3. The set of these (at least) $d(\frac{q}{3} - 2\Delta)$ integers contains elements (not necessarily both) from no less than $d(\frac{q}{6} - \Delta) > 2\frac{d\Delta^{r-1}}{\ln^2 \Delta}$ pairs from $\mathcal{S}$. On the other hand, by **(i)** (from Lemma 2.5), v has at most $\frac{2d\Delta^{r-1}}{\ln^2 \Delta}$ backward r-neighbours. We may thus perform admissible alterations of the colours of some of the edges incident with v so that afterwards $d_c(v)$ belongs to some pair in $\mathcal{S}$ with elements congruent to 0 modulo 3 which is disjoint with all S_u associated with backward r-neighbours u of v. We set this pair as S_v. We continue in the same manner with all vertices in A.

Suppose now that we have reached a vertex $v \in B$ of degree d, and thus far all our rules and requirements have been fulfilled. Similarly as above, admissible modifications (1°), (2°) and (3°) of colours of the edges incident with v, due to **(iv)** and **(v)**, provide us a list of attainable sums at v of cardinality (where we in particular additionally use the fact that **(iv)** implies that v has at least $\frac{d}{2\ln^3 \Delta} > \frac{Q}{q}$ forward edges):

$$\begin{aligned}
&\left(\frac{Q}{q}\left(X_v d - \sqrt{X_v d} \ln \Delta\right) + \left[d - \left(X_v d - \sqrt{X_v d} \ln \Delta\right)\right]\right)(q - 2\Delta) \\
&\quad \geq \left[2\Delta^{r-1}\left(X_v d - \sqrt{X_v d} \ln \Delta\right) + d\frac{\Delta^{r-1}}{\ln \Delta}\right]\left(1 - \frac{2\Delta}{q}\right) \\
&\quad \geq \left[2\Delta^{r-1}\left(X_v d - \sqrt{X_v d} \ln \Delta\right) + d\frac{\Delta^{r-1}}{\ln \Delta}\right] - \frac{2\Delta}{q} 2\Delta^{r-1} X_v d - \frac{2\Delta}{q} d \frac{\Delta^{r-1}}{\ln \Delta} \\
&\quad \geq \left[2\Delta^{r-1}\left(X_v d - \sqrt{X_v d} \ln \Delta\right) + d\frac{\Delta^{r-1}}{\ln \Delta}\right] - 4\Delta \ln \Delta d - \frac{1}{2} d \frac{\Delta^{r-1}}{\ln \Delta} \\
&\quad \geq 2\Delta^{r-1}\left(X_v d - \sqrt{X_v d} \ln \Delta\right) + \frac{1}{4} d \frac{\Delta^{r-1}}{\ln \Delta}.
\end{aligned}$$

These attainable sums for v contain representatives of at least $\Delta^{r-1}(X_vd - \sqrt{X_vd}\ln\Delta) + d\frac{\Delta^{r-1}}{8\ln\Delta}$ pairs from $\mathcal{S}$. On the other hand, **(vi)** implies that

$$|N_-^r(v)| \leq X_vd\Delta^{r-1} + \sqrt{X_vd\Delta^{r-1}}\ln\Delta,$$

where

$$\Delta^{r-1}(X_vd - \sqrt{X_vd}\ln\Delta) + d\frac{\Delta^{r-1}}{8\ln\Delta} > X_vd\Delta^{r-1} + \sqrt{X_vd\Delta^{r-1}}\ln\Delta.$$

Therefore there is a choice of admissible alterations of the colours of edges incident with v so that afterwards $d_c(v)$ belongs to some set in $\mathcal{S}$ disjoint with S_u for all $u \in N_-^r(v)$. We perform these alterations and set the corresponding set from $\mathcal{S}$ as S_v. We continue in the same manner with all vertices in B.

We are thus left with the analysis of the vertices in C. Let $G' = G[C]$, hence for the maximum degree Δ' of G' we have $\Delta' \leq \Delta$. Note also that by **(iv)**, $\delta' := \delta(G') \geq \frac{1}{2}\frac{\delta}{\ln^3\Delta} \geq \frac{1}{2}\ln^5\Delta$. Therefore, by Lemma 2.4, for Δ sufficiently large, there exists a spanning subgraph F' of G' with $d_{F'}(v) \leq \frac{d_{G'}(v)}{\ln^3\Delta}$ for every $v \in V$. Denote the edges of F' by E' (hence $F' = (C, E')$), and note that for every $v \in C$, $d_C(v) - d_{E'}(v) \geq d_{G'}(v)(1 - \frac{1}{\ln^3\Delta}) \geq 1$ (for Δ sufficiently large), hence the edges in $E'' := E(G') \setminus E' = \{e_1'', e_2'', \ldots, e_m''\}$ also induce a spanning subgraph of G'.

At this point our edge colouring of G is proper modulo q (hence also modulo Q). We shall now admit a temporary deviation from this rule by setting $c(e) = q$ for every $e \in E'$. Next we analyze consecutively all edges $e_1'', \ldots, e_m''$ in E'' (note that their initial colours, defined by c_0, have not been yet altered within our construction, thus all are in the range $[q + Q - \Delta, q + Q]$), and add to a colour of every such subsequent $e_i'' = uv$ an integer in $[0, 6\Delta]$, what is consistent with (3.3), so that the obtained sums at u and v are not congruent to 0 modulo 3 and so that the colour of e_i'' is not congruent to the colours of its adjacent edges in G modulo q. This is always feasible, as the later requirement blocks at most $2(\Delta - 1)$ of at least 2Δ available options in $[0, 6\Delta]$ with an adequate residue modulo 3. After analyzing all edges in E'' (inducing a spanning subgraph of $G[C]$), for every vertex $v \in C$ we have $d_c(v) \equiv 1 \pmod 3$ or $d_c(v) \equiv 2 \pmod 3$ (contrary to the vertices in A). Now we shall randomly adjust the colours of the edges in E' (which are all set to q) to guarantee relatively regular distributions of the sums residues modulo Q in the r-neighbourhoods in C. In particular we shall show the following.

Lemma 3.2. *We may add to the colour of every edge in E' an integer divisible by 3 from the set $\{0, 3, 6, \ldots, Q - 3\}$ so that the obtained edge colouring of G is proper modulo Q, and for each vertex $v \in C$ and every integer $t \in [0, Q - 1]$ which is not congruent to 0 modulo 3, the number of vertices u in $N_C^r(v)$ with $(5\ln\Delta)^{-1}d(v) \leq d(u) \leq d(v)5\ln\Delta$ and with $d_c(u) \equiv t \pmod Q$ is upper-bounded by $6000\frac{d(v)}{\ln^3\Delta}$.*

Proof. We first partition the set $\{0, 3, 6, \ldots, Q - 3\}$ into 32-element sets of consecutive integers congruent to 0 modulo 3: $L_1, L_2, \ldots, L_{Q/96}$ (hence e.g. $L_1 = \{0, 3, 6, \ldots, 93\}$). For every $e \in E'$, as it has less than 2Δ adjacent edges in G (which might block at most 2Δ residues modulo Q for $c(e)$), i.e. less than 2Δ integers in $[0, Q - 1]$ might not be admissible as the additions to the colour of e (equal to q prior to this addition) due to the required properness (modulo Q) of the randomly constructed edge colouring. Thus out of $L_1, L_2, \ldots, L_{Q/96}$, at least $Q/96 - 2\Delta \geq \frac{\Delta^{r-1}}{48}$ lists (sets) are entirely available for e, where a set L_i is called *entirely available* for $e \in E'$ if neither element of $q + L_i$ is congruent modulo Q to the colour of an edge in $E \setminus E'$ adjacent to e in G (we shall distinguish colours of adjacent edges in E' within our construction below). Out of these at least $\frac{\Delta^{r-1}}{48}$ entirely available lists for e we randomly and independently for every edge in E' choose one with uniform probability and denote it by L_e. We also temporarily set $c(e) = \min L_e$.

We claim that at the end of such random procedure, with positive probability, for every $v \in C$ the following event appears:

$\overline{R_v}$: there are at most 31 edges incident with v (and with both ends in C) with a feature that each such edge e is adjacent with an edge e' (with both ends in C) such that $L_e = L_{e'}$.

For this goal we shall estimate the probability of the complement of the above for $v \in C$:

R_v: there exist 32 edges incident with v (and with both ends in C) with a feature that each such edge e is adjacent with an edge e' (with both ends in C) such that $L_e = L_{e'}$.

Fix any $v \in C$ and denote its degree by d. Note first that there are at most $\binom{d}{32} \leq \binom{\Delta}{32}$ ways of choosing 32 distinct edges incident with v. Now for a fixed choice of such 32 edges $B = \{e_1, e_2, \ldots, e_{32}\}$, each of them is supposed to have an adjacent edge coloured the same (with the same list randomly chosen) as itself, so for each edge $e_j \in B$ we choose its adjacent edge e'_j which is supposed to have the same colour as e_j, and estimate the probability of $e_1, \ldots, e_{32}$ being witnesses for R_v to appear, by examining all possible configurations of the choices of their correspondents $e'_1, \ldots, e'_{32}$, which we divide into 33 groups with respect to the number of the edges e'_j belonging to B (note that e'_j does not have to be distinct from e'_l for $j \neq l$). For every $i = 0, \ldots, 32$ (and fixed B), there are at most $\binom{32}{i} 31^i (2\Delta)^{32-i}$ choices of edges $e'_1, e'_2, \ldots, e'_{32}$ so that $|\{j : e'_j \in B\}| = i$. Then for each fixed choice of edges $e'_1, \ldots, e'_{32}$ with this feature, denote $B'' = B \cup \{e'_1, e'_2, \ldots, e'_{32}\}$ (hence $32 \leq |B''| \leq 64 - i$), and let us consider an auxiliary graph H with vertex set B'' and the set of edges: $\{e_l e'_l : l = 1, 2, \ldots, 32\}$. Note that all its components have order at least 2. Fix any subset $B_0 \subset B$ of minimal size such that each component of H has at least one vertex in $(B'' \setminus B) \cup B_0$, and note that $|B_0| \leq \lfloor \frac{i}{2} \rfloor$, as there are $32 - i$ edges $e_l \in B$ (which are vertices of H) adjacent in H with $e'_l \in B'' \setminus B$, while among the remaining at most i edges in B which do not belong to any component including a vertex in $B'' \setminus B$ (which induce the remaining components of H) it is sufficient to choose at most half to form B_0 (one for each of these remaining components of H). Note that edges of G inducing (as vertices of H) any component in this auxiliary graph H must have the same colours (lists) chosen to be witnesses for R_v to take place, hence if we fix colours (lists) for all edges in $(E' \setminus B) \cup B_0$, the probability that independent choices for the remaining at least $32 - \lfloor \frac{i}{2} \rfloor$ edges in E' (from $B \setminus B_0$) shall guarantee R_v is bounded from above by $(\frac{48}{\Delta^{r-1}})^{32 - \lfloor \frac{i}{2} \rfloor}$. By the law of total probability, we thus obtain that:

$$\begin{aligned}\mathbf{Pr}(R_v) &\leq \binom{\Delta}{32} \sum_{i=0}^{32} \binom{32}{i} 31^i (2\Delta)^{32-i} \left(\frac{48}{\Delta^{r-1}} \right)^{32 - \lfloor \frac{i}{2} \rfloor} \\ &\leq 31^{32} \cdot 2^{32} \cdot 48^{32} \Delta^{32} \sum_{i=0}^{32} \Delta^{(32-i) - (32 - \lfloor \frac{i}{2} \rfloor)(r-1)}\end{aligned}$$

$$< 10^{48} \cdot 10^{64}\Delta^{32} \cdot 33\Delta^{-16(r-1)} < 10^{114}\Delta^{-4r-12(r-4)} \le \frac{10^{114}}{\Delta^{4r}} \tag{3.4}$$

(for $r \ge 4$).

Now for each vertex $v \in C$ of degree d in G and every integer $t \in [0, Q-1]$ which is not congruent to 0 modulo 3, let $X_{v,t}$ denote (the random variable expressing) the number of vertices u in $N_C^r(v)$ with $d_c(u) \in [t-31\cdot 93, t+31\cdot 93] \pmod Q$ (where $c(e) = \min L_e$ for every $e \in E'$) and $(5\ln\Delta)^{-1}d \le d(u) \le d\,5\ln\Delta$. In order to prove the thesis we shall also need to guarantee (with non-zero probability) for every $v \in C$ of degree d (in G) and every integer $t \in [0, Q-1]$ which is not congruent to 0 modulo 3 the event:

$\overline{T_{v,t}}$: $X_{v,t} \le 6000\frac{d}{\ln^3\Delta}$.

We thus upper-bound the probability of the complement of this. As to every edge $e \in E'$ we have assigned the colour being the minimal element $\min L_e$ from the randomly chosen list L_e, which may differ by the multiplicity of 96 between distinct lists, there are at most $\lceil(2\cdot 31\cdot 93+1)/96\rceil = 61$ distinct values in the interval $[t-31\cdot 93, t+31\cdot 93]$ the sum at v may possibly attain within our random process. Therefore for every $u \in N_C^r(v)$ with $(5\ln\Delta)^{-1}d \le d(u) \le d\,5\ln\Delta$,

$$\mathbf{Pr}\left(d_c(u) \in [t-31\cdot 93, t+31\cdot 93] \pmod Q\right) \le 61\frac{48}{\Delta^{r-1}}$$

(what can be also easily proved by the law of total probability via analysis of the possible at least $\frac{\Delta^{r-1}}{48}$ choices of lists, hence also additions to the colour, of 'the last edge' in E' incident with u, at most 61 of which might assure that $d_c(u) \in [t-31\cdot 93, t+31\cdot 93] \pmod Q$ regardless of any fixed choices for the remaining edges), by **(ii)** we thus obtain that

$$\mathbf{E}(X_{v,t}) \le \frac{61\cdot 48}{\Delta^{r-1}}\frac{2d\Delta^{r-1}}{\ln^3\Delta} = 5856\frac{d}{\ln^3\Delta}.$$

Note also that a change of choice for any edge in E' may influence $X_{v,t}$ by at most 2. Moreover, for any s, the fact that $X_{v,t} \ge s$ can be certified by the outcomes of at most $s\cdot\frac{10d}{\ln^5\Delta}$ trials, i.e. choices committed on the edges in E' incident with some s r-neighbours u of v in C with $(5\ln\Delta)^{-1}d \le d(u) \le d\,5\ln\Delta$, each of which has at most $\frac{2\frac{d\,5\ln\Delta}{\ln^3\Delta}}{\ln^3\Delta} = \frac{10d}{\ln^5\Delta}$ incident edges in E' by **(iv)** and Lemma 2.4. Thus by Talagrand's Inequality (and comments below it),

$$\begin{aligned}\mathbf{Pr}(T_{v,t}) \le \mathbf{Pr}\Bigg(X_{v,t} > 5856\frac{d}{\ln^3\Delta} + \frac{d}{\ln^3\Delta} + \\ + 20\cdot 2\sqrt{\frac{10d}{\ln^5\Delta}5856\frac{d}{\ln^3\Delta}} + 64\cdot 2^2\frac{10d}{\ln^5\Delta}\Bigg) \\ < 4e^{-\frac{\left(\frac{d}{\ln^3\Delta}\right)^2}{8\cdot 2^2\frac{10d}{\ln^5\Delta}\cdot\left(5856\frac{d}{\ln^3\Delta}+\frac{d}{\ln^3\Delta}\right)}} < \frac{10^{114}}{\Delta^{4r}}.\end{aligned} \tag{3.5}$$

As any event $T_{v,t}$ and R_v is mutually independent of all other events $T_{v',t'}$ and $R_{v'}$ with $d(v,v') > 2r+1$, i.e., all except at most $\Delta^{2r+1}\cdot(\frac{2Q}{3}+1) < \Delta^{3r+1}$ such events, by the Lovász Local Lemma, (3.4) and (3.5) we thus obtain that there is a choice of lists (and additions to the colours) of the edges in E' so that none of the events $T_{v,t}$ and R_v holds for any $v \in C$. This implies among others that each subgraph induced in G' by the edges associated with any fixed list L_i has maximum degree at most 31. Thus by Vizing's Theorem we may arbitrarily recolour *properly* each such subgraph, if necessary, using additions from its corresponding L_i (where $|L_i| = 32$) instead merely the addition $\min L_i$. Note that then the obtained edge colouring of G is proper modulo Q, while colours of some edges could be increased – each by at most 93. As at he same time, every vertex v is by $\overline{R_v}$ incident with at most 31 edges whose colours could be increased, by $\overline{T_{v,t}}$ with $v \in C$ and $t \in \{1, 2, 4, 5, 7, 8, \ldots, Q-2, Q-1\}$ we obtain the thesis. □

We fix any additions to the colours of the edges in E' consistent with the thesis of Lemma 3.2. We shall not alter the colour of any edge with both ends in C anymore, while the remaining ones might be modified by Q. Therefore the edge colouring of G shall remain proper modulo Q, while the sums at vertices in A shall remain distinguished from the sums at vertices in C, as the first ones are congruent to 0 modulo 3, unlike the second ones. As by **(iv)** every vertex in B has a neighbour in C, we may subtract Q if necessary (or do nothing) from the colour of one such edge for every vertex in B so that the weighted degree for every vertex $v \in B$ is set on the smaller element of its associated two-element list S_v. (This is feasible, as prior to these changes, every such edge had its colour between $Q + q - \Delta$ and $Q + 2q$, since it has not been analyzed as a backward edge yet, and therefore (3.3) shall hold for this edge after any of the described changes). The thesis of Lemma 3.2 above obviously still holds afterwards. The sums at vertices in B shall not be altered anymore.

In the final stage of the construction we shall be subsequently analyzing the vertices in C, and modifying colours of the edges joining them with A consistently with (1°) in order to dispose of all the remaining sum-conflicts between vertices in C and their r-neighbours in $B \cup C$. This time however we shall admit placing weighted degrees of two r-neighbours in the same 2-element list from $\mathcal{S}$, but in such a way that these weighted degrees are distinct. Note that for every consecutive $v \in C$ we have available $d_A(v) + 1 \geq \frac{d(v)}{2 \ln^2 \Delta} + 1$ (by **(iii)**) distinct sums, which form an arithmetic progression of difference Q, via admissible changes on the edges joining v with A. These are all congruent to some t modulo Q (not divisible by 3) and include at least $\frac{d(v)}{4 \ln^2 \Delta}$ options which are not fixed as weighted degrees of vertices in B, as these are all set to the smaller elements from their associated lists. So it is sufficient to choose one of such options for v distinct from the contemporary sums at all r-neighbours of v in C with $(5 \ln \Delta)^{-1} d(v) \leq d(u) \leq d(v)\, 5 \ln \Delta$ (cf. Remark 3.1) and with weighted degrees congruent to t modulo Q. This is however feasible, as by Lemma 3.2 above the number of such r-neighbours of v equals at most $6000 \frac{d(v)}{\ln^3 \Delta} < \frac{d(v)}{4 \ln^2 \Delta}$. We choose one of these and perform admissible changes on the edges joining v with A to set it as the sum at v. After analyzing all vertices in C, the construction is completed, while the obtained edge colouring c is proper (even modulo Q), uses colours in $[q - \Delta, 2q + 2Q]$ and guarantees sum-distinction between r-neighbours in G. □

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 51–66
https://doi.org/10.26493/1855-3974.1733.8c6
(Also available at http://amc-journal.eu)

Types of triangle in plane Hamiltonian triangulations and applications to domination and k-walks

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Received 19 June 2018, accepted 29 January 2019, published online 19 June 2019

Abstract

We investigate the minimum number $t_0(G)$ of faces in a Hamiltonian triangulation G so that any Hamiltonian cycle C of G has at least $t_0(G)$ faces that do not contain an edge of C. We prove upper and lower bounds on the maximum of these numbers for all triangulations with a fixed number of facial triangles. Such triangles play an important role when Hamiltonian cycles in triangulations with 3-cuts are constructed from smaller Hamiltonian cycles of 4-connected subgraphs. We also present results linking the number of these triangles to the length of 3-walks in a class of triangulation and to the domination number.

Keywords: Graph, Hamiltonian cycle, domination, 3*-walk.*

Math. Subj. Class.: 05C45, 05C10, 05C38

1 Introduction

In this article all triangulations are simple triangulations of the plane with at least 4 vertices. A triangulation or a graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle. For a triangulation G with a Hamiltonian cycle C of G, a *type-i triangle* with $i \in \{0, 1, 2\}$ is defined as a facial triangle of G which shares exactly i edges with C. We define $t_i(G, C)$

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as the number of type-i triangles. If the triangulation and Hamiltonian cycle are clear from the context, we will also just write t_i.

A triangulation G can be extended by inserting a 4-connected triangulation or polyhedron in a triangle T to obtain a larger graph G'. If there is a Hamiltonian cycle C in G, then we can extend C to a Hamiltonian cycle of G' – unless T is a type-0 triangle. If there is a Hamiltonian cycle C without any type-0 triangles such as in a double wheel or the majority of small 4-connected triangulations (e.g. more than 80% for 4-connected triangulations on 20 vertices), then for the graph G' obtained by inserting a 4-connected triangulation or polyhedron in each triangle in a set of disjoint facial triangles we can extend C to a Hamiltonian cycle of G'. In [3] it is proven that the – still open – question whether all triangulations with at most four 3-cuts are Hamiltonian can be reduced to the question whether for each set of four disjoint triangles in a 4-connected triangulation there is a Hamiltonian cycle so that none of them is a type-0 triangle. More properties of triangulations with a Hamiltonian cycle with few or even without type-0 triangles are described in Section 4. Investigating whether there always exists a Hamiltonian cycle with few type-0 triangles is the main target of this paper.

We denote the number of facial triangles of G by $t(G)$. Euler's formula implies that (with $|G|$ the number of vertices of G), $t(G) = 2|G| - 4$, so it is always an even number. For $i \in \{0, 2\}$ we further define

$$t_i(G) = \min\{t_i(G, C) \mid C \text{ is a Hamiltonian cycle of } G\},$$

and for even $t \geq 4$

$$t_i(t) = \max\{t_i(G) \mid G \text{ is a Hamiltonian triangulation with exactly } t \text{ facial triangles}\}.$$

In some cases we might want to restrict the class to 4- or 5-connected triangulations. Note that there are no 4-connected triangulations G with $t(G) < 8$ and no 5-connected triangulations G with $t(G) < 20$. So for $j = 4$ and even $t \geq 8$, and for $j = 5$ and even $t \geq 20$ we define

$$t_i^j(t) = \max\{t_i(G) \mid G \text{ is a } j\text{-connected triangulation with exactly } t \text{ facial triangles}\}.$$

In this paper, we show the following theorem.

Theorem 1.1. *Let t be an integer. Then the following hold.*

(i) *For $t \geq 8$ we have $t_0(t) \leq \frac{t-8}{3}$, and for $4 \leq t < 8$ we have $t_0(t) = 0$.*
(ii) *For $t \geq 10$ we have $t_0^4(t) \leq \frac{t-10}{3}$, and for $t = 8$ we have $t_0^4(t) = 0$.*
(iii) *For $t \geq 20$ we have $t_0^5(t) \leq \frac{t-12}{3}$.*

In Section 3, we discuss lower bounds on $t_0(t), t_0^4(t)$ and $t_0^5(t)$.

As we will see in Section 4.1, also the number of type-i triangles on one side of a Hamiltonian cycle is relevant, so we also define $\bar{t}_i(G, C)$ as the number of type-i triangles on that side of C with fewer type-i triangles. The numbers $\bar{t}_i(G)$, $\bar{t}_i(t)$, and $\bar{t}_i^j(t)$ are defined correspondingly. By definition

$$\bar{t}_i(G, C) \leq t_i(G, C)/2, \qquad \bar{t}_i(G) \leq t_i(G)/2,$$
$$\bar{t}_i(t) \leq t_i(t)/2 \quad \text{and} \qquad \bar{t}_i^j(t) \leq t_i^j(t)/2$$

for $i \in \{0, 2\}$ and $j \in \{4, 5\}$.

An *outer plane graph* is a plane graph in which all vertices are incident with the outer face. In particular, an outer plane graph with maximal number of edges is called a *maximal outer plane graph*, which is, in other words, an outer plane graph in which all inner faces are triangles. For a triangulation G with a Hamiltonian cycle C, the inside as well as the outside of C together with C form a maximal outer plane graph. For a 2-connected plane graph G, the boundary of the outer face is called the *boundary cycle of* G. In particular, vertices and edges in the boundary cycle of G are *boundary vertices* resp. *boundary edges* in G. A cycle C in a plane graph such that the inside as well as the outside (not including C) contain a vertex is called a *separating cycle*. Note that in a triangulation, any triangle that is not facial is a separating cycle.

Let G be a triangulation with a Hamiltonian cycle C. If we take the dual of the maximal outer plane graph consisting of the inside of C together with C and delete the vertex corresponding to the outer face, then we obtain a subcubic tree in which the vertices of degree $(3 - i)$ correspond to type-i triangles of the triangulation. Using these relations, we get the following proposition.

Proposition 1.2. *Let G be a triangulation with a Hamiltonian cycle C. Then*

$$\bar{t}_2(G, C) = \bar{t}_0(G, C) + 2 \quad \textit{and} \quad t_2(G, C) = t_0(G, C) + 4.$$

Note that the number of facial triangles on the inside is equal to the number of facial triangles on the outside. As $t(G) = t_0(G, C) + t_1(G, C) + t_2(G, C)$, we have

$$t_1(G, C) = t(G) - 2t_0(G, C) - 4.$$

So finding the minimum value for $t_0(G, C)$ is equivalent to finding the minimum value for $t_2(G, C)$, and finding the maximum value for $t_1(G, C)$.

Let G be a triangulation and let C be a Hamiltonian cycle in G. We say that two facial triangles are *adjacent* if they share an edge. An (i, j)-*pair* ($i, j \in \{1, 2\}$) is defined as a pair of adjacent facial triangles consisting of a type-i triangle and a type-j triangle such that the common edge is contained in C. Note that each type-1 triangle is contained in at most one $(1, 2)$-pair.

2 Upper bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$

To prove Theorem 1.1 in this section, we first show some lemmas. A vertex v in a graph G is said to be *dominating* if v is adjacent to all other vertices in G.

If a type-2 triangle T is contained in two $(2, 2)$-pairs, we call the three triangles involved a $(2, 2, 2)$-*triple* and T the *central triangle* of the triple.

Restricted to minimum degree 4 the first part of the following lemma was proven in [13, Lemma 2.1].

Lemma 2.1. *Let G be a triangulation with a Hamiltonian cycle C, but without a dominating vertex. Then there exists a Hamiltonian cycle C' in G such that C' has no $(2, 2, 2)$-triples.*

If G has minimum degree 4, then C' can be chosen in a way that it also has at least as many $(1, 1)$-pairs as C.

Proof. Assume that there is a $(2,2,2)$-triple with central triangle T and let v denote the vertex contained in all three triangles involved. As v is not dominating, there is a first vertex v_0 in counterclockwise orientation from T around v that has a neighbour on C that is not a neighbour of v. Numbering the neighbours of v in clockwise orientation around v as $v_0, v_1, \ldots, v_{\deg(v)-1}$, there is also a first vertex v_k with $k > 0$ and a neighbour on C that is not a neighbour of v. We can reroute the part of C containing $v, v_0, \ldots, v_k$ along the path $v_0, v_1, \ldots, v_{k-1}, v, v_k$. This operation is displayed in Figure 1. Of course the roles of v_0 and v_k are symmetric and we could do the same with their roles interchanged.

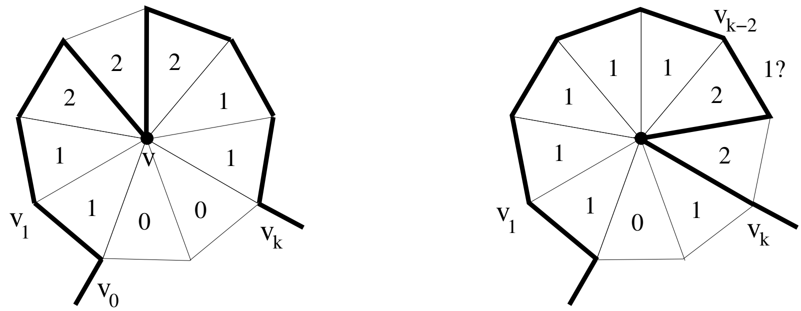

Figure 1: Rerouting a Hamiltonian cycle to remove a $(2,2,2)$-triple.

If all vertices have degree at least 4, any new type-2 triangle contains v and the number of $(2,2,2)$-triples is decreased. Furthermore, no $(1,1)$-pairs without a triangle containing v can be destroyed and after rerouting at least the edges $v_1v_2, v_2v_3, \ldots, v_{k-3}v_{k-2}$ are common edges of a $(1,1)$-pair. These are $k-3$ $(1,1)$-pairs, but note that $k-3$ can be 0. Depending on whether v_0v_1 is the common edge of a $(1,1)$-pair in C, the triangles under discussion can belong to $k-3$ or $k-4$ $(1,1)$-pairs before rerouting – so the number of $(1,1)$-pairs does not decrease.

The vertices v_0 and v_k always have degree at least 4, but if one of $v_1, \ldots, v_{k-2}$ has degree 3, it is contained in a type-2 triangle not containing v. For $v_1, \ldots, v_{k-3}$ (note that this set of vertices can be empty) this type-2 triangle has type-1 triangles on the other side of the edges in the Hamiltonian cycle and is therefore not contained in a $(2,2)$-pair. If v_{k-2} has degree 3 we would produce a $(2,2,2)$-triple. If v_2 has degree larger than 3, we can apply the operation with the role of v_0 and v_k interchanged, so let us assume that v_2 as well as v_{k-2} have degree 3. As no two vertices of degree 3 can be neighbours in a triangulation different from K_4, this implies that $k > 3$.

Let $i > 0$ be minimal so that there is an edge v_iv_{k-1}. Such an i is sure to exist, as $k-3$ is a candidate. We then reroute the cycle along $v_0, v_1, \ldots, v_i, v_{k-1}, v_{k-2}, \ldots, v_{i+1}, v, v_k$ to obtain C'. An example of this rerouting is given in Figure 2.

After rerouting, the only edges that can be the common edge of the two triangles in a new $(2,2)$-pair are $v_{i+1}v_{i+2}$ and v_iv_{k-1}. As v_iv_j is not in C' for any $i < j < k-1$, v_iv_{k-1} can only be in a $(2,2)$-pair if $v_{k-1}v_{k-2}$ is contained in the same triangle, which gives $i = k-3$, so $v_{i+1}v_{i+2} = v_{k-2}v_{k-1}$ is the common edge of a $(2,2)$-pair too and only the case that $v_{i+1}v_{i+2}$ is the common edge of a $(2,2)$-pair remains to be discussed.

Assume that $v_{i+2}v_{i+1}$ is contained in two type-2 triangles — $v_{i+2}v_{i+1}v$ and T'. If the degree of v_{i+2} is 3, then $T' = v_{i+1}v_{i+2}v_{i+3}$ and the second neighbour triangle of T' along C' is a type-1 triangle, so in that case $v_{i+2}v_{i+1}$ is not part of a $(2,2,2)$-triple.

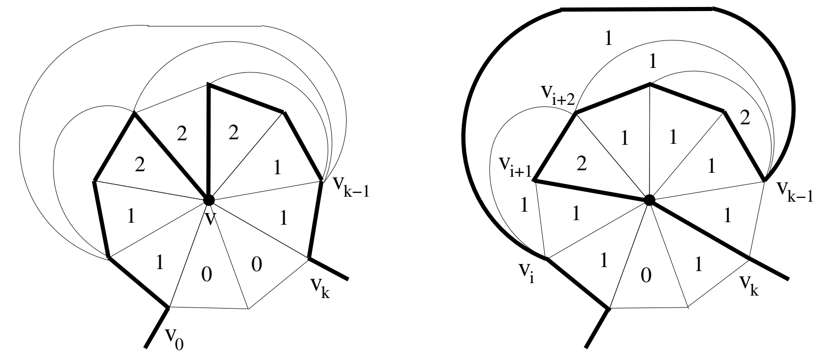

Figure 2: Rerouting a Hamiltonian cycle to remove a $(2,2,2)$-triple if v_{k-2} has degree 3.

If the degree of v_{i+2} is at least 4, the other edge of T' in the Hamiltonian cycle must be $v_{i+2}v_i$, which can only be contained in a type-2 triangle $v_{i+2}v_{i+1}v_i$ if $i+2=k-1$, that is $i=k-3$. In order to be contained in a second type-2 triangle, there must be an edge $v_{k-1}v_{k-4}$. Due to the minimality of i we get $k=4$, so we have the situation depicted in Figure 3 on the left hand side. Rerouting the Hamiltonian cycle along $v_0, v, v_2, v_1, v_3, v_4$ (right hand side of Figure 3) gives a Hamiltonian cycle with one $(2,2,2)$-triple less. □

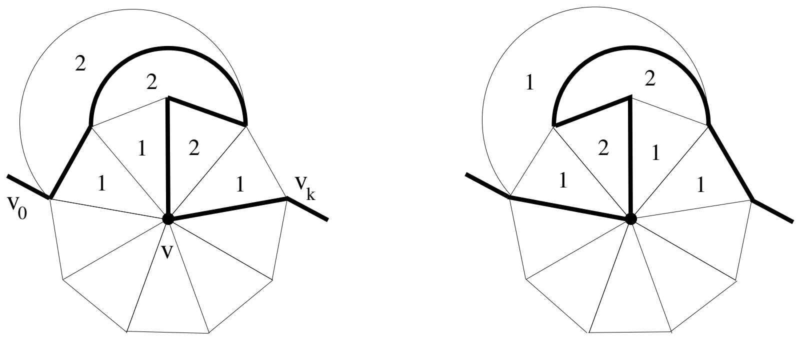

Figure 3: Rerouting a Hamiltonian cycle to remove a $(2,2,2)$-triple if v_{k-2} has degree 3 and the default method produces a $(2,2,2)$-triple.

Using a result by Whitney [17], we can prove the existence of a Hamiltonian cycle with at least one $(1,1)$-pair in a 4-connected triangulation. Below we first give the lemma by Whitney, but use a simplified version of the formulation from [7].

Lemma 2.2. *Let G be a 4-connected triangulation. Consider a cycle D in G together with the vertices and edges on one side of D (referred to as the outside of D). Let a and b be two vertices of D dividing D into two paths P_1 and P_2 each of which contains both a and b. If*

- *no two vertices of P_1 are joined by an edge which lies outside of D and*

- *there is a vertex z (distinct from a and b) dividing P_2 into two paths P_3 and P_4 each of which contains z such that no pair of vertices in P_3 and no pair of vertices in P_4 are joined by an edge which lies outside of D,*

then there is a path from a to b using only edges on and outside of D which passes through every vertex on and outside of D.

Using this lemma, we can give the following result. Note that for triangulations being k-connected is equivalent to having no separating cycles of length shorter than k.

Lemma 2.3. *Let G be a 4-connected triangulation which is not isomorphic to the octahedron. There exists a Hamiltonian cycle C in G such that C has at least one $(1, 1)$-pair.*

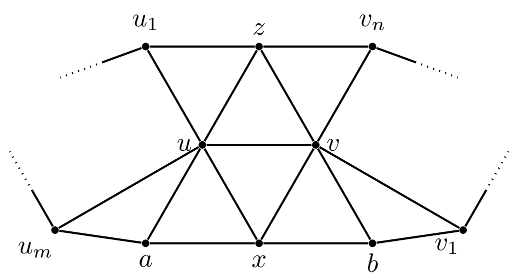

Figure 4: Construction of a Hamiltonian cycle with at least one $(1, 1)$-pair in a 4-connected triangulation.

Proof. As a consequence of the Euler formula and the fact that G is not isomorphic to the octahedron, there exists a vertex x of degree at least 5 in G. Let uvx be an arbitrary triangle containing x. The edge uv is contained in a second triangle, say uvz. Let the vertices adjacent to u (in counterclockwise order) be $v, z, u_1, \ldots, u_m, a, x$ (note that there are no u_i vertices if u has degree 4), and let the vertices adjacent to v be $u, x, b, v_1, \ldots, v_n, z$ (note that there are no v_i vertices if v has degree 4) (see Figure 4).

As G is 4-connected, $D = axbv_1 \cdots v_n z u_1 \cdots u_m a$ is a cycle in G. The vertices a and b partition D into two paths satisfying the conditions of Lemma 2.2 with $P_1 = axb$. Indeed, the path P_2 is divided into P_3 and P_4 by the vertex z. As x has degree at least 5, a and b are not adjacent. All vertices of P_3, resp. P_4, are adjacent to u, resp. v, so any edge which lies outside of D and joins two vertices of P_3 or two vertices of P_4 would be part of a separating triangle.

Let P be the path from a to b described in Lemma 2.2. The Hamiltonian cycle $C = P \cup auvb$ contains the $(1, 1)$-pair (uvx, uvz). □

In the case of 5-connected triangulations, we can prove a slightly stronger result.

Lemma 2.4. *Let G be a 5-connected triangulation. There exists a Hamiltonian cycle C in G such that C has at least two $(1, 1)$-pairs.*

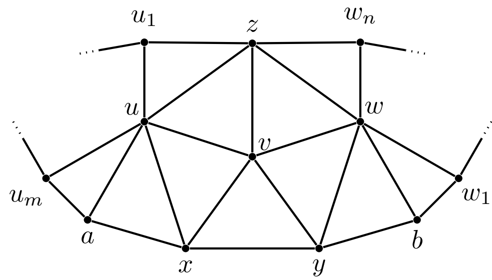

Figure 5: Construction of a Hamiltonian cycle with at least two $(1,1)$-pairs in a 5-connected triangulation.

Proof. Let v be a vertex of G which has degree 5, and let u and w be two neighbouring vertices of v which are not adjacent to each other. Let the vertices adjacent to u be $v, z, u_1, \ldots, u_m, a, x$, and let the vertices adjacent to w be $v, y, b, w_1, \ldots, w_n, z$ (see Figure 5).

As G is 5-connected, $D = axybw_1 \cdots w_n z u_1 \cdots u_m a$ is a cycle in G. The vertices a and b partition D into two paths satisfying the conditions of Lemma 2.2 with $P_1 = axyb$. Indeed, the path P_2 is divided into P_3 and P_4 by the vertex z. As all vertices have degree at least 5, any edge outside of D connecting two vertices of P_1 is contained in a separating triangle or a separating quadrangle. All vertices of P_3, resp. P_4, are adjacent to u, resp. w, so any edge which lies outside of D and joins two vertices of P_3 or P_4 would be part of a separating triangle.

Let P be the path from a to b described in Lemma 2.2. The Hamiltonian cycle $C = P \cup auvwb$ contains the $(1,1)$-pairs (uvx, uvz) and (vwy, vwz). □

Lemma 2.5. *Let G be a triangulation with a dominating vertex v and t triangles. Then $t_0(G) < \frac{t}{4} - 1$ if G is not K_4 and $t_0(K_4) = 0$.*

Proof. We can easily check K_4 by hand, so assume that G is not K_4.

$G - \{v\}$ is an outer plane graph, so it has a vertex w of degree 2. Let w' be a vertex sharing a boundary edge of $G - \{v\}$ with w and let C be the Hamiltonian cycle of G containing $\{v, w\}$, $\{v, w'\}$ and the boundary cycle of $G - \{v\}$ without the edge $\{w, w'\}$. Let $t_{0,\Delta}, t_{1,\Delta}$ and $t_{2,\Delta}$ be the number of facial triangles of type 0, 1 and 2 on the side of C containing the triangle v, w, w'. All triangles on the other side of C contain v and as no type-0 triangle in G contains v, we have $t_0(G) = t_{0,\Delta}$. Since each side of C contains exactly $t(G)/2$ facial triangles, we have $t_{0,\Delta} + t_{1,\Delta} + t_{2,\Delta} = \frac{t(G)}{2}$. Furthermore (as G is not K_4) we have $t_{1,\Delta} \geq 1$ (the unique triangle containing w but not v). So $t_{0,\Delta} + t_{2,\Delta} < t_{0,\Delta} + t_{1,\Delta} + t_{2,\Delta} = \frac{t}{2}$. By Proposition 1.2, we have $t_{2,\Delta} = t_{0,\Delta} + 2$, and hence we get $2t_{0,\Delta} + 2 = 2t_0 + 2 < \frac{t}{2}$ and finally $t_0(G) < \frac{t}{4} - 1$. □

By combining the results above, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For $t < 20$ the theorem was checked by testing all triangulations. The triangulations were generated by the program *plantri* [2] and a straightforward exhaustive search for Hamiltonian cycles with the smallest number of type-0 triangles was performed. Thus, we may assume $t \geq 20$. Let G be a Hamiltonian triangulation with $t \geq 20$ facial triangles.

Suppose that G has a dominating vertex v. Since $G - \{v\}$ has a vertex of degree two, G has a 3-cut, and hence G is not 4-connected. Since $\frac{t}{4} - 1 \leq \frac{t-8}{3}$, Lemma 2.5 implies the result.

Assume now that G has no dominating vertex. Suppose that G has a Hamiltonian cycle with p $(1,1)$-pairs. Lemmas 2.3 and 2.4 imply that $p \geq 1$ if G is 4-connected, and $p \geq 2$ if G is 5-connected. Due to Lemma 2.1, G contains a Hamiltonian cycle C' which has at least p $(1,1)$-pairs and in which each type-2 triangle is contained in at least one $(1,2)$-pair. A type-1 triangle is contained in a $(1,1)$-pair or a $(1,2)$-pair. There are at least $2p$ type-1 triangles in $(1,1)$-pairs of C' and therefore at most $(t_1(G,C') - 2p)$ type-1 triangles in $(1,2)$-pairs. Since each type-2 triangle forms a $(1,2)$-pair with at least one of the type-1 triangles in a $(1,2)$-pair, we get

$$t_2(G,C') \leq t_1(G,C') - 2p.$$

By Proposition 1.2, we have $t_2(G,C') = t_0(G,C') + 4$, and hence

$$t_1(G,C') \geq t_0(G,C') + 4 + 2p.$$

Combining these results with $t(G) = t_0(G,C') + t_1(G,C') + t_2(G,C')$, we get

$$t(G) \geq t_0(G,C') + t_0(G,C') + 4 + 2p + t_0(G,C') + 4.$$

This can be rewritten as

$$t_0(G,C') \leq \frac{t(G) - 8 - 2p}{3},$$

and so we also have

$$t_0(t) \leq \frac{t - 8 - 2p}{3}.$$

Using the values for p from Lemma 2.3 and Lemma 2.4, we get the given bounds. □

3 Lower bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$

In order to prove lower bounds for $t_0(t)$, $t_0^4(t)$ and $t_0^5(t)$, we will construct families of graphs in which each Hamiltonian cycle has at least a certain number of type-0 triangles.

Theorem 3.1.

- *Let $t \geq 16$ be even. Then $t_0(t) \geq \lfloor \frac{t}{3} \rfloor - 5$ and $\bar{t}_0(t) \geq \lfloor \frac{t+2}{6} \rfloor - 3$.*
  We have $t_0(14) = 1$ and $\bar{t}_0(14) = 0$. For $t < 14$ we have $t_0(t) = \bar{t}_0(t) = 0$.
- *Let $t \geq 18$ be even. Then $t_0^4(t) \geq 2(\lfloor \frac{t}{6} \rfloor - 3)$ and $\bar{t}_0^4(t) \geq \lfloor \frac{t}{6} \rfloor - 3$.*
  For $t < 18$ we have $t_0^4(t) = \bar{t}_0^4(t) = 0$.
- *Let $t \geq 20$ be even. Then $t_0^5(t) \geq 2\lfloor \frac{t}{12} \rfloor - 20$.*
  For $t \leq 66$ we have that $\bar{t}_0^5(t) = 0$.

Proof. **$t_0^4(t)$ and $\bar{t}_0^4(t)$:**

The results for $t < 18$ were determined by a computer using the program *plantri* [2] for the generation of all 4-connected triangulations and a straightforward algorithm to compute t_0 and $\bar{t}_0$.

First consider the case where t is a multiple of six, and let $k = \frac{t}{6}$. Consider the fragment B shown in the left part of Figure 6. Take k copies $B_0, \ldots, B_{k-1}$ of B and identify all vertices labelled N and all vertices labelled S, respectively, (we call the resulting vertices the poles) and for $0 \leq i < k$ identify vertex y in B_i with vertex x in $B_{i+1 \pmod k}$. This graph has $6k$ facial triangles, and we denote it by G_k. It is easy to check that G_k is 4-connected.

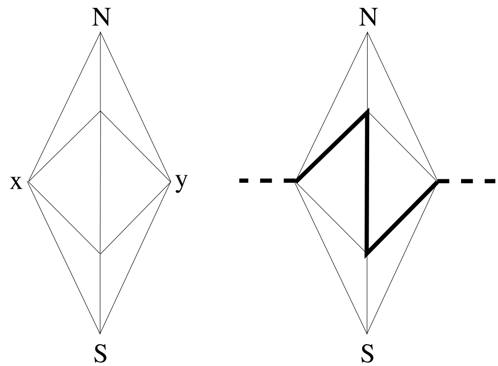

Figure 6: The fragment B used to construct a family of triangulations establishing a lower bound on $t_0^4(t)$ and $\bar{t}_0^4(t)$ and the most common way for a Hamiltonian cycle to pass through this fragment.

We show $t_0(G_k) \geq 2(\frac{t}{6} - 3)$ and $\bar{t}_0(G_k, C) \geq \frac{t}{6} - 3$ by induction on k. Computational results give that for $3 \leq k \leq 8$ we have $t_0(G_k) = 2k - 6$ and $\bar{t}_0(G_k) = k - 3$. Since G_k contains $6k$ triangles, we can also write this as $t_0(G_k) = \frac{t}{3} - 6$ and $\bar{t}_0(G_k) = \frac{t}{6} - 3$, and we are done. So we may assume that $k \geq 9$.

Let C be a Hamiltonian cycle in G_k. An edge of C which is incident to a pole is contained in at most two fragments. Since there are two edges incident to each pole, there are at most 8 fragments that contain an edge of C that is incident to a pole. Since $k \geq 9$, we may assume that C visits the fragment B_{k-1} – up to symmetry – as shown in the right part of Figure 6. This part of the Hamiltonian cycle C produces two type-0 triangles in B_{k-1} – one on each side of C. So, by removing two inner vertices of B_{k-1}, identifying the vertex y in the copy B_{k-2} and the vertex x in the copy B_0, we obtain a Hamiltonian cycle, say C', in G_{k-1}. By the induction hypothesis, $t_0(G_{k-1}, C') \geq 2(\frac{t-6}{6} - 3)$ and $\bar{t}_0(G_{k-1}, C') \geq \frac{t-6}{6} - 3$. Since $t_0(G_k, C) = t_0(G_{k-1}, C') + 2$ and $\bar{t}_0(G_k, C) = \bar{t}_0(G_{k-1}, C') + 1$, we obtain the desired inequality.

For the case where t is not a multiple of six, we let $k = \left\lfloor \frac{t}{6} \right\rfloor$. We apply the same construction, but for a pair of neighbouring fragments we connect the x- and y-vertex by an edge instead of identifying them – see the left part of Figure 7 – or with an extra vertex of degree 4 that is also connected to the poles. This gives 2, resp. 4 extra triangles. Confirming the formulas for these modified triangulations with 3 to 8 fragments with a computer, one can apply the same argumentation as above to prove the equations in the lemma.

$t_0(t)$ and $\bar{t}_0(t)$:

For $t_0(t)$ and $\bar{t}_0(t)$, where 3-cuts are allowed, we use the same fragment and the same constructions as for $t_0^4(t)$ and $\bar{t}_0^4(t)$, but for two fragments we do not identify x and y but instead connect N and S by an edge between these segments – see the right part of Figure 7.

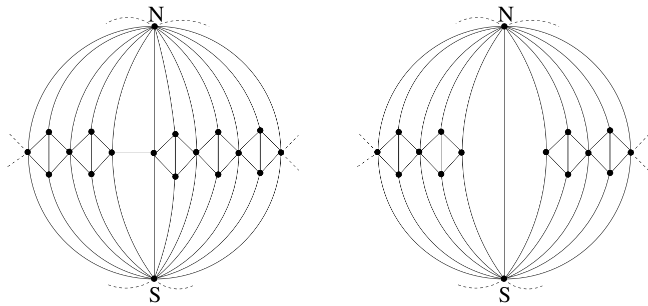

Figure 7: Modifications of the construction for the 4-connected case when t is not a multiple of 6 and for the 3-connected case.

This construction with k fragments gives triangulations with $6k+2$ facial triangles that can be extended to triangulations with $6k+4$ and $6(k+1)$ facial triangles by inserting vertices of degree 3 in one or both triangles containing the edge between the poles.

Computational results for $k \leq 8$ fragments combined with the same reduction argument as before give that $t_0(t) \geq \lfloor \frac{t}{3} \rfloor - 5$ and $\bar{t}_0(t) \geq \lfloor \frac{t+2}{6} \rfloor - 3$.

Remark. For small values of t a double wheel where triangles are subdivided with a vertex of degree 3 alternatingly on both sides of the rim gives a larger result for $t_0(t)$ and $\bar{t}_0(t)$, but the linear factor is only $\frac{1}{4}$, so that the advantage compared to the sequence described is only for small values.

$t_0^5(t)$ and $\bar{t}_0^5(t)$:

For $t \leq 130$ we have that $t_0^5(t) \geq 0 \geq 2\lfloor \frac{t}{12} \rfloor - 20$. So assume that t is even and $t > 130$.

For even $t > 130$ we can construct triangulations in a similar way as for the cases $t_0^4(t)$ and $\bar{t}_0^4(t)$, but use the fragments depicted in Figure 8. We use $r = (t - 12\lfloor \frac{t}{12} \rfloor)/2$ copies $B'_0, \ldots, B'_{r-1}$ of the right fragment with 14 triangles and $l = \lfloor \frac{t}{12} \rfloor - r$ copies $B'_r, \ldots, B'_{r+l-1}$ of the left fragment with 12 triangles.

We identify all vertices labelled N and all vertices labelled S, respectively, and for $0 \leq i < r+l$ identify the vertices y, y' in B'_i with the vertices x, x' in $B'_{i+1 \pmod{(r+l)}}$ respectively. It is easy to check that the resulting graph $G_{r,l}$ is 5-connected.

Checking the different ways how a Hamiltonian cycle can pass the left fragment in Figure 8 without using the poles and saturate the 4 interior vertices (some boundary vertices can also be saturated from outside the segment), gives that each such segment contains at least 2 type-2 triangles. As the fragment on the right hand side of Figure 8 contains the one on the left hand side, the same is true for the fragment on the right hand side too.

So for $t > 130$ and consequently $r+l \geq 11$ any Hamiltonian cycle C in $G_{r,l}$ has at least $r+l-8$ fragments not containing an edge of C incident with a pole and therefore containing at least 2 type-2 triangles. So $t_2(G_{r,l}, C) \geq 2(r+l-8)$ and therefore $t_0(G_{r,l}, C) \geq 2(r+l-8) - 4 = 2(r+l) - 20$. As $r+l = \lfloor \frac{t}{12} \rfloor$ we get $t_0^5(t) \geq 2\lfloor \frac{t}{12} \rfloor - 20$.

The result for $\bar{t}_0^5(t)$ was proven by a computer search testing graphs constructed by the program plantri [2]. All 5-connected triangulations G with up to 66 triangles were found to have $\bar{t}_0(G) = 0$. It should also be noted that the graphs $G_{r,l}$ constructed for the first part all allow a Hamiltonian cycle C with $\bar{t}_0(G, C) = 0$. □

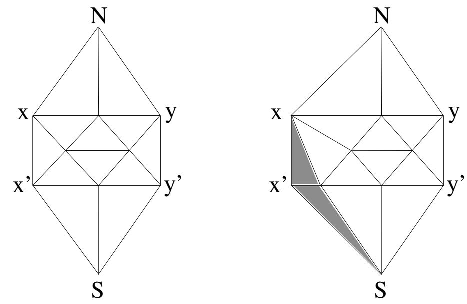

Figure 8: The fragments used for the 5-connected case.

Computational results for $r = 0$ and $l \leq 8$ suggest that $t_0^5(t) \geq 2\lfloor \frac{t}{12} \rfloor - 8$, but a proof similar to the one for $t_0(t)$ and $t_0^4(t)$ is out of reach on the computational side for the basic step in the induction and would be very lengthy on the theoretical side.

For $t_0(t)$, $\bar{t}_0(t)$, $t_0^4(t)$, and $\bar{t}_0^4(t)$ the upper and lower bounds differ only by an additive constant, so there is not much room for improvement. For $t_0^5(t)$, and especially $\bar{t}_0^5(t)$ the upper and lower bounds are far apart and have a different growth rate. In these cases there is not only room, but also need for improvement.

4 Applications different from Hamiltonian cycles

Type-0 triangles are of their own interest in the context of Hamiltonicity of triangulations, as they are the problematic case for the extendability of partial Hamiltonian cycles to the inside of separating triangles (see e.g. [9]), but the number $t_0(G)$ has also an impact on invariants that are not that obviously related to Hamiltonian cycles. In this section, we describe two other topics in graph theory for which the value of $t_0(G)$ is relevant.

4.1 The domination number of a triangulation

A vertex subset S of a graph G is said to be *dominating* if every vertex in $G - S$ has a neighbour in S. The cardinality of a minimum dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. For a triangulation G, Matheson and Tarjan [11] proved that $\gamma(G) \leq \frac{|G|}{3}$ and they conjectured that $\gamma(G) \leq \frac{|G|}{4}$. This conjecture is still open, even when restricted to 4- or 5-connected triangulations.

Plummer, Ye and Zha [13] proved that $\gamma(G) \leq \min\{\lceil \frac{2|G|}{7} \rceil, \lfloor \frac{5|G|}{16} \rfloor\}$ for any 4-connected triangulation G. This is the currently best approach towards the Matheson-Tarjan conjecture. The idea of their inductive proof is to find a Hamiltonian cycle with certain properties of type-2 triangles and to use these for reduction of the graph.

If we can find a Hamiltonian cycle with few type-2 triangles, then (as implicitly used in [13]) we can bound the size of a dominating set as follows: Let C be a Hamiltonian cycle. By symmetry we can assume that the number of type-2 triangles on the inside of C is less than or equal to that on the outside of C. Let G' be the maximal outer plane graph consisting of the inside of C together with C. Note that G' contains $\bar{t}_2(G, C)$ type-2 triangles. It is shown in [5, 16] that any maximal outer plane graph H satisfies $\gamma(H) \leq \frac{|H|+k(H)}{4}$, where $k(H)$ denotes the number of vertices of degree 2 in H. Any vertex of degree two in G' is the common end vertex of two edges of C in a type-2 triangle. Thus, we have

$k(G') = \bar{t}_2(G,C)$. Since $\bar{t}_2(G,C) = \bar{t}_0(G,C) + 2$, we obtain by Proposition 1.2

$$\gamma(G) \le \gamma(G') \le \frac{|G| + k(G')}{4} = \frac{|G| + \bar{t}_2(G,C)}{4} = \frac{|G| + \bar{t}_0(G,C) + 2}{4} \le \frac{2|G| + t_0(G,C) + 4}{8}.$$

So for a given Hamiltonian triangulation, a Hamiltonian cycle C with few type-0 triangles possibly gives a good upper bound on the domination number in that triangulation. In general though, the impact of the values of $t_0(t)$ is a negative one: the lower bounds given in Theorem 3.1 show that at least for 4-connected triangulations a direct application of this method cannot lead to improved bounds for the domination number.

4.2 3-walks with few vertices visited more than once

A *k-tree* of a graph G is a spanning tree of G in which every vertex has degree at most k. A *k-walk* is a spanning closed walk that visits every vertex at most k times. It is well-known that a graph that contains a k-walk also contains a $(k+1)$-tree, see [8] (but the converse does not hold in general). Furthermore, the vertices visited k times in a k-walk correspond to vertices of degree $k+1$ in the $(k+1)$-tree that is constructed.

Every 3-connected planar graph admits a 3-tree [1] and a 2-walk [6]. The result about 3-trees was strengthened in [12] where it is shown that every 3-connected planar graph G admits a 3-tree with at most $\frac{|G|-7}{3}$ vertices of degree 3.

As in the construction of 3-trees from 2-walks in [8], vertices visited twice in a 2-walk correspond to vertices of degree 3 in the 3-tree, it was natural to consider the following problem, which was already mentioned in [12].

Problem 4.1. Is there for every 3-connected planar graph G a 2-walk such that the number of vertices visited twice is at most $\frac{|G|}{3} - c$ for a constant c?

Note that for a 2-walk in a graph G, the number of vertices visited twice is at most t if and only if its length is at most $|G| + t$. With this formulation of the problem in mind, the result that every 3-connected planar graph G contains a spanning closed walk of length at most $\frac{4|G|-4}{3}$ (proven in [10]) can be considered as a first step towards the solution of Problem 4.1. However, a spanning closed walk constructed in [10] may visit a vertex many times, so Problem 4.1 is still open.

In this section we describe a different step towards the solution of Problem 4.1, by limiting the number of times a vertex is visited to 3. The class for which the result is proven is a subclass of all triangulations, but in fact a class containing cases for which Problem 4.1 would hold with equality. Type-0 triangles play an important role in the construction of the walks.

In the language of [9] the triangulations in the class of graphs we will describe now are those triangulations where the so-called decomposition tree is a star. In order not to refer the reader to [9] and to fix notation, we will give an independent description of the class here. To simplify notation, we consider K_4 also as a 4-connected graph in this section. Let $\mathcal{K}$ be the set of all graphs G that can be constructed as follows: Take any 4-connected triangulation H and let F be a subset of facial triangles of H. For each facial triangle $f = xyz \in F$, take a 4-connected plane graph G_f (not necessarily a triangulation) where the outer face is a triangle and let x_f, y_f and z_f be the three boundary vertices of G_f. Then

G is obtained from H by adding G_f inside f for $f \in F$, so that x, y, z are identified with x_f, y_f, z_f, respectively. Except for the case when G is a triangulation with exactly one separating triangle the graph H is uniquely defined for each $G \in \mathcal{K}$ and we write $H(G)$ for it. In the case of one separating triangle there are two possible candidates for H and $H(G)$ denotes an arbitrary one of them.

For example, the face subdivision of a 4-connected triangulation H belongs to $\mathcal{K}$. In the definition above, F is the set of all facial triangles of H and for any face f we have $G_f \simeq K_4$. As in [12, Section 2], the face subdivision of a 4-connected triangulation shows that we cannot decrease the coefficient $\frac{1}{3}$ of $|G|$ in Problem 4.1. So, in this sense, some graphs in $\mathcal{K}$ belong to the most difficult ones for Problem 4.1.

The following result shows that a Hamiltonian cycle C in a 4-connected triangulation T with small $t_0(T, C)$ can be used to construct a 3-walk of short length for the graphs $G \in \mathcal{K}$ with $H(G) = T$. Using Theorem 1.1, in Corollary 4.3 we obtain a general upper bound depending only on the number of vertices in G.

Theorem 4.2. *Let $G \in \mathcal{K}$ be given and C a Hamiltonian cycle in $H = H(G)$. We write $t_0'(H, C)$ (or short t_0') for the number of those type-0 triangles of H that are not faces in G. Then G contains a 3-walk of length at most $\frac{4|G|+t_0'-4}{3}$ which visits each vertex not in H exactly once.*

Proof. Let F, H, and for each facial triangle $f \in F$ also G_f, x_f, y_f, and z_f be as in the definition of $\mathcal{K}$. We denote the length of a walk W by $l(W)$, and let $|R|_- = |R| - 3$ for a plane graph R. With this notation we have $|G| = |H| + \sum_{f \in F} |G_f|_-$.

Claim 4.2.1. *For a 4-connected plane graph R where the outer face is a triangle (including K_4) with vertices x, y, z in the boundary and $a, b \in \{x, y, z\}$ (with possibly $a = b$), there is a (possibly closed) walk $P_{R,a,b}$ of length $|R|_- + 1$ from a to b in R visiting exactly all vertices in R except those in $\{x, y, z\} \setminus \{a, b\}$ and visiting vertices not in the boundary exactly once.*

Proof. The case $G = K_4$ can be easily checked by hand, so assume that G is not K_4.

If $a = b$ (w.l.o.g. $a = b = x$) then according to [15, (3.4)] there exists a Hamiltonian cycle in $G - \{y, z\}$, which is a closed walk with the given properties starting and ending in a.

If $a \neq b$ (w.l.o.g. $a = x$, $b = y$), due to [14, Corollary 2] there is a Hamiltonian cycle C in G through $\{a, z\}$ and $\{b, z\}$. $C - \{\{a, z\}, \{b, z\}\}$ is the walk $P_{G,a,b}$. □

For a given cycle C with a fixed vertex c_1 we define a linear order along one of the directions of C starting from c_1 as $c_1 < c_2 < \cdots < c_n$. For each facial triangle f of H we fix the notation of x_f, y_f, z_f so that $x_f < y_f < z_f$.

With this notation we have:

Claim 4.2.2. *For any two triangles f and f' that belong to the same side of C we have $y_f \neq y_{f'}$.*

Proof. Assume $x_f \leq x_{f'}$. C is divided into three segments by the vertices x_f, y_f and z_f and – as $x_{f'}, y_{f'}$ and $z_{f'}$ are all at least x_f and smaller than c_n, they occur in one of these segments in the order $x_{f'}, y_{f'}, z_{f'}$. This implies that only $x_{f'}$ and $z_{f'}$ can be one of the end vertices of the segment and $y_{f'}$ is in fact different from each of x_f, y_f and z_f. □

We consider the following spanning subgraph H_C^* of the dual of H: The vertex set of H_C^* is the set of triangles of H, and two faces are adjacent in H_C^* if and only if they share an edge in C. Note that for $i \in \{0,1,2\}$, a type-i triangle has degree exactly i in H_C^*. In particular, each component of H_C^* is an isolated vertex, a path or a cycle. We can give an orientation to the edges of such a component P^*, so that each vertex in P^*, except for isolated vertices and one of the end vertices when P^* is a path, has out-degree one. In cases where only one end vertex v of such a path P^* belongs to F, we choose v to have out-degree one.

Recall that F is the set of facial triangles of H into which a graph was inserted. We can partition F into two sets F_0 and F_1. We define for $i \in \{0,1\}$:

$$F_{-} = \{f \in F \mid f \text{ has out-degree exactly } i\}.$$

With $t_1'(H,C)$ (or short t_1') for the number of those type-1 triangles of H that are no faces in G our construction gives $|F_0| \leq t_0' + \frac{t_1'}{2}$.

Now we modify C using Claim 4.2.1 so that for each triangle $f \in F$ it visits each vertex inside G_f exactly once:

- Suppose that $f \in F_0$. Then we add the walk P_{G_f,y_f,y_f} to C. This increases the length of C by $|G_f|_- + 1$.
- Suppose that $f \in F_1$. Let f' be the out-neighbour of f, and let $\{a,b\}$ be the edge in C that is shared by f and f'.

  Then we replace $\{a,b\}$ in C by $P_{G_f,a,b}$. This increases the length of C by only $|G_f|_-$ as one edge in C is also deleted.

The resulting walk C' is a 3-walk because, by Claim 4.2.2, the number of times a vertex is visited is increased by at most 1 for each side of C.

We will first give some equations we will use to compute the length of C'. For the given Hamiltonian cycle C we denote $t_0(H,C)$, $t_1(H,C)$ and $t_2(H,C)$, by t_0, t_1, t_2, respectively.

As $t_0 + t_1 + t_2 = t(H) = 2|H| - 4$ and $t_2 = t_0 + 4$ (by Proposition 1.2), we get $|H| = \frac{2t_0+t_1}{2} + 4 \geq \frac{2t_0'+t_1'}{2} + 4$.

As in each face of F at least one vertex is inserted, we get $|G| \geq |H| + t_0' + t_1'$. So together with the previous equation $|G| \geq \frac{4t_0'+3t_1'}{2} + 4 = \frac{6t_0'+3t_1'}{2} + 4 - t_0'$ which can be rewritten as $t_0' + \frac{t_1'}{2} \leq \frac{|G|+t_0'-4}{3}$ we get

$$\begin{aligned} l(C') &= l(C) + \sum_{f\in F_1} |G_f|_- + \sum_{f \in F_0} (|G_f|_- + 1) = l(C) + \sum_{f\in F} |G_f|_- + |F_0| \\ &= |H| + \sum_{f\in F} |G_f|_- + |F_0| = |G| + |F_0| \leq |G| + t_0' + \frac{t_1'}{2} \\ &\leq |G| + \frac{|G| + t_0' - 4}{3} = \frac{4|G| + t_0' - 4}{3} \end{aligned}$$

This completes the proof of Theorem 4.2. □

Using Theorem 1.1(ii) we obtain the following corollary.

Corollary 4.3. *Except for K_4, any graph $G \in \mathcal{K}$ contains a 3-walk of length at most*

$$\frac{22|G| - 34}{15}.$$

Proof. Applying the construction of a walk from Theorem 4.2, we get that any graph $G \in \mathcal{K}$ with $K_4 = H(G)$ is Hamiltonian, so we just have to check whether $|G| \leq \frac{22|G|-34}{15}$, which is the case for all graphs except K_4 itself (that is if $F = \emptyset$).

Assume now that $t(H(G)) = 8$, so $H(G)$ is the octahedron. If G is Hamiltonian, then there is a walk of length $|G| \leq \frac{22|G|-34}{15}$. Otherwise it follows directly from Theorem 16 in [4] that $|F| \geq 4$. As that result is still unpublished, one can alternatively use our construction of a walk from Theorem 4.2 together with Theorem 4.1 from [9] to obtain that $|F| \geq 4$. Furthermore one can easily find a Hamiltonian cycle with $|F_0| \leq 2$. With $v_i \geq 4$ the number of vertices added inside a triangle of the octahedron, the construction gives a 3-walk with length at most $6 + v_i + 2$ and $6 + v_i + 2 \leq \frac{22(6+v_i)-34}{15}$ for $v_i \geq 4$. From now on assume that $H(G)$ has at least 10 faces.

Let C be a Hamiltonian cycle in $H = H(G)$ with $t_0(H)$ type-0 triangles. Let t_0' denote the number of type-0 triangles of C in H that are not faces in G. As each triangle in F contains at least one vertex, we have that $|G| \geq |H| + t_0'$. By Theorem 1.1(ii), we get $t_0^4(t(H)) \leq \frac{t(H)-10}{3}$ and

$$t_0' \leq t_0^4(t(H)) = t_0^4(2|H| - 4) \leq \frac{2|H| - 14}{3} \leq \frac{2(|G| - t_0') - 14}{3}$$

which implies

$$t_0' \leq \frac{2|G| - 14}{5}.$$

Substituting this into the equation given in Theorem 4.2, we get Corollary 4.3. □

5 Correctness of the computer programs used

The programs constructing Hamiltonian cycles and computing $t_0(\cdot)$ and $\bar{t}_0(\cdot)$ are straightforward branch and bound programs that can be obtained from the authors or be downloaded from `http://caagt.ugent.be/type0/` to check the source code, to check the computational results in this paper, or to be used otherwise. Two independent programs were developed and implemented and the results were compared for each of the around 150 000 000 triangulations with up to 30 triangles generated by *plantri*. There was full agreement. The computation of $\bar{t}_0(\cdot)$ for 5-connected triangulations was done independently up to 60 triangles and for larger values only by the faster of the two programs.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 67–78
https://doi.org/10.26493/1855-3974.1426.212
(Also available at http://amc-journal.eu)

On the generalized Oberwolfach problem

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Received 19 June 2017, accepted 22 April 2019, published online 20 June 2019

Abstract

The generalized Oberwolfach problem $\mathrm{OP}_t(2w+1; N_1, N_2, \ldots, N_t; \alpha_1, \alpha_2, \ldots, \alpha_t)$ asks for a factorization of K_{2w+1} into α_i C_{N_i}-factors (where a C_{N_i}-factor of K_{2w+1} is a spanning subgraph whose components are cycles of length $N_i \geq 3$) for $i = 1, 2, \ldots, t$. Necessarily, $N = \mathrm{lcm}(N_1, N_2, \ldots, N_t)$ is a divisor of $2w+1$ and $w = \sum_{i=1}^{t} \alpha_i$.

For $t = 1$ we have the classic Oberwolfach problem. For $t = 2$ this is the well-studied Hamilton-Waterloo problem, whereas for $t \geq 3$ very little is known.

In this paper, we show, among other things, that the above necessary conditions are sufficient whenever $2w+1 \geq (t+1)N$, $\alpha_i > 1$ for every $i \in \{1, 2, \ldots, t\}$, and $\gcd(N_1, N_2, \ldots, N_t) > 1$. We also provide sufficient conditions for the solvability of the generalized Oberwolfach problem over an arbitrary graph and, in particular, the complete equipartite graph.

Keywords: 2-factorizations, resolvable cycle decompositions, cycle systems, (generalized) Oberwolfach problem, Hamilton-Waterloo problem.

Math. Subj. Class.: 05C51, 05C70

*The author gratefully acknowledges support from NSERC Discovery Grant RGPIN-435898-2013.

†The author gratefully acknowledges support from NSERC Discovery Grant RGPIN-2016-04178.

‡The author gratefully acknowledges the support of a Marie Curie fellowship of the Istituto Nazionale di Alta Matematica, and the support of GNSAGA.

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1 Introduction

We denote by $V(G)$ and $E(G)$ the *vertex set* and the *edge set* of a simple graph G, respectively. Also, we denote by tG the vertex-disjoint union of $t > 0$ copies of G.

A *factor* F of G is a spanning subgraph of G, namely, a subgraph of G such that $V(F) = V(G)$; also, if F is i-regular, we call F an *i-factor*. In particular, a 1-factor of G (also called a *perfect matching*) is the vertex-disjoint union of edges of G whose vertices partition $V(G)$, while a 2-factor of G is the vertex-disjoint union of cycles whose vertices span $V(G)$. A 2-factor of G containing only one cycle is usually called a *Hamiltonian cycle*. We say that a factor is *uniform* when its components are pairwise isomorphic. Hence, a 1-factor is uniform, whereas a 2-factor might not be.

As usual, we denote by K_v the *complete graph* on v vertices; also, we use K_v^* to denote the graph K_v when v is odd and $K_v - I$, where I is a 1-factor of K_v, when v is even. Further, we denote by $K_s[z]$ the *complete equipartite graph* with s parts of size z. Note that, $K_v^* \simeq K_v[1]$ or $K_{v/2}[2]$, according to whether v is odd or even, respectively. Finally, we denote by C_ℓ a cycle of length $\ell \geq 3$ (briefly, an ℓ-cycle), and by $(x_0, x_1 \ldots, x_{\ell-1})$ the ℓ-cycle with edges $x_0x_1, x_1x_2, \ldots, x_{\ell-1}x_0$. A uniform 2-factor whose cycles have all length ℓ is referred to as a *C_ℓ-factor*.

A 2-factorization of a simple graph G is a set $\mathcal{F}$ of 2-factors of G whose edge sets partition $E(G)$. If $\mathcal{F}$ contains only C_ℓ-factors, we speak of a *C_ℓ-factorization* of G. It is well known that a regular graph has a 2-factorization if and only if every vertex has even degree. However, if we specify t 2-factors, say $F_1, F_2, \ldots, F_t$, and ask for the factorization $\mathcal{F}$ to contain α_i factors isomorphic to F_i, then the problem becomes much harder. Much attention has been given to the cases where $t \in \{1, 2\}$ and either $G = K_v^*$ or $G = K_s[z]$.

For $t = 1$, we have the "classic" Oberwolfach problem, which is well known to be hard. A survey of the most relevant results on this problem, updated to 2006, can be found in [15, Section VI.12]. For more recent results we refer the reader to [6, 9, 11, 29].

Although the Oberwolfach problem is still open, it has been completely solved for uniform factors when $G = K_v^*$ [2, 3, 22] or when G is the complete equipartite graph [24]. We recall these results below.

Theorem 1.1 ([2, 3, 22, 24])**.** *Let ℓ, s and z be positive integers with $\ell \geq 3$. There exists a C_ℓ-factorization of $K_s[z]$ if and only if $\ell \mid sz$, $(s-1)z$ is even, further ℓ is even when $s = 2$, and $(\ell, s, z) \notin \{(3,3,2), (3,6,2), (3,3,6), (6,2,6)\}$.*

For $t \geq 1$, we refer to this problem as the *generalized Oberwolfach problem*. More precisely, given a simple graph G, given t 2-factors of G, say $F_1, F_2, \ldots, F_t$, and given t non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_t$, the generalized Oberwolfach problem, denoted by $\mathrm{OP}_t(G; F_1, F_2, \ldots, F_t; \alpha_1, \alpha_2, \ldots, \alpha_t)$, or briefly by $\mathrm{OP}_t(G; (F_i); (\alpha_i))$, asks for a factorization of G into α_i F_i-factors for $i \in \{1, 2, \ldots, t\}$. In the case where each F_i is uniform, namely, F_i is a C_{N_i}-factor, we denote the problem by $\mathrm{OP}_t(G; N_1, N_2, \ldots, N_t; \alpha_1, \alpha_2, \ldots, \alpha_t)$, or briefly by $\mathrm{OP}_t(G; (N_i); (\alpha_i))$. Further, we use v in place of G when $G = K_v^*$. The following necessary conditions are trivial.

Theorem 1.2. *If there exists a solution to $\mathrm{OP}_t(G; (N_i); (\alpha_i))$, then the following conditions hold:*

(1) G is regular of degree $2 \cdot \sum_{i=1}^{t} \alpha_i$,

(2) $\mathrm{lcm}(N_1, N_2, \ldots, N_t)$ is a divisor of the order of G.

The case in which $t = 2$ is known as the Hamilton-Waterloo problem. Although it has received much interest recently, it is still open even in the uniform case. Some of the most important results up to 2006 can be found in [15, Section VI.12]. More recent results can be found in [4, 7, 8, 10, 12, 13, 16, 23, 25]. For more details and some history on the problem, we refer the reader to [12, 13].

Much less is known on $\mathrm{OP}_t(v; (F_i); (\alpha_i))$ when $t > 2$. In [1, 18, 19] the problem is solved for odd orders v up to 17, and even orders v up to 10 (see also [15, Sections VI.12.4 and VII.5.4]). In [6] the problem is settled whenever v is even, each F_i is bipartite (namely, F_i contains only cycles of even length), $\alpha_1 \geq 3$ is odd, and the remaining α_i are even. In [14, 17] the problem is solved whenever $v = p^n$ with p a prime number, $t = n$, and F_i is a C_{p^i}-factor, except possibly when p is odd and the first non-zero integer of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is 1. A partial asymptotic existence result has recently been given in [20], provided that v is sufficiently large and α_1 scales linearly with v. Further results covering specific cases can be found in [5, 26, 28].

In this paper, we focus on the "uniform" generalized Oberwolfach problem $\mathrm{OP}_t(v; (N_i); (\alpha_i))$. In view of Theorem 1.2, for such a problem to be solvable v must be a multiple of each N_i and $\lfloor \frac{v-1}{2} \rfloor = \sum_{i=1}^{t} \alpha_i$; clearly, $1 \leq t \leq \frac{v-1}{2}$. Since $\mathrm{OP}_t(v; (N_i); (\alpha_i))$ has been solved for $t = 1$ (Theorem 1.1), from now on we assume that $t > 1$. Also, we denote by $[a, b]$ the set of integers from a to b inclusive; clearly, $[a, b]$ is empty when $a > b$.

The main result of this paper is the following.

Theorem 1.3. *Let $v \geq 3$ be odd, let $3 \leq N_1 < N_2 < \cdots < N_t$ and set $N = \mathrm{lcm}(N_1, N_2, \ldots, N_t)$ and $g = \gcd(N_1, N_2, \ldots, N_t)$; also, let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be positive integers. Then, $\mathrm{OP}_t(v; (N_i); (\alpha_i))$ has a solution if and only if N is a divisor of v and $\sum_{i=1}^{t} \alpha_i = \frac{v-1}{2}$ except possibly when $t > 1$ and at least one of the following conditions is satisfied:*

(I) $\alpha_i = 1$ *for some* $i \in [1, t]$*;*

(II) $\alpha_i \in [2, \frac{N-3}{2}] \cup \{\frac{N+1}{2}\}$ *for every* $i \in [1, t]$*;*

(III) $g = 1$*;*

(IV) $v = N$.

Given a graph G, $G[n]$ denotes the *lexicographic product* of G with the complement of K_n, namely, $G[n]$ is the graph whose vertex set is $V(G) \times \mathbb{Z}_n$, and two vertices (x, j) and (y, j') are adjacent if and only if x and y are adjacent in G.

The proof of the main theorem relies on the solvability of $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$. More precisely, we prove the following result.

Theorem 1.4. *Let $t \geq 1$ and let $1 \leq n_1 < n_2 < \cdots < n_t \leq n$ be odd integers such that n_i is a divisor of n for each $i \in [1, t]$. Then $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$ has a solution whenever $g \geq 3$, $\sum_{i=1}^{t} \alpha_i = n$, and $\alpha_i \geq 2$ for every $i \in [1, t]$.*

In the next section we introduce some tools and provide some powerful methods which we use in Section 3 where we prove Theorem 1.4. In Section 4 we prove the main results.

2 Preliminary results

We will make use of the notion of a Cayley graph on an additive group Γ, not necessarily abelian. Given $\Omega \subseteq \Gamma \setminus \{0\}$, the *Cayley graph* $\mathrm{Cay}(\Gamma, \Omega)$ is a graph with vertex set

Γ and edge set $\{\gamma(\omega+\gamma) \mid \gamma \in \Gamma, \omega \in \Omega\}$. When $\Gamma = \mathbb{Z}_n$ this graph is known as a *circulant graph*. We note that the edges generated by $\omega \in \Omega$ are the same as those generated by $-\omega \in -\Omega$, so that $\mathrm{Cay}(\Gamma,\Omega) = \mathrm{Cay}(\Gamma, \pm\Omega)$, and that the degree of each point is $|\Omega \cup (-\Omega)|$.

Given a subgraph G of $\mathrm{Cay}(\Gamma,\Omega)$ and an element $\gamma \in \Gamma$, we denote by $G+\gamma$ the *translate of G by γ*, that is, the graph obtained from G by replacing each of its vertices, say x, with $x+\gamma$. It is not difficult to check that $G+\gamma$ is a subgraph of $\mathrm{Cay}(\Gamma,\Omega)$. For a subgroup Σ of Γ, the *orbit of G under Σ* (briefly, the *Σ-orbit of G*) is the set $\mathrm{Orb}_\Sigma(G)$ of all distinct translates of G by an element of Σ, that is, $\mathrm{Orb}_\Sigma(G) = \{G+\sigma \mid \sigma \in \Sigma\}$. The *$\Sigma$-stabilizer of G* is the set $\mathrm{Stab}_\Sigma(G)$ of the elements $\sigma \in \Sigma$ such that $G+\sigma = G$. By the well-known *orbit-stabilizer theorem* (see [27, Theorem 5.7]), $\mathrm{Stab}_\Sigma(G)$ is a subgroup of Σ of index $\mathrm{Orb}_\Sigma(G)$, and therefore $|\mathrm{Orb}_\Sigma(G)| \cdot |\mathrm{Stab}_\Sigma(G)| = |\Sigma|$.

Given a set $\Omega \subseteq \Gamma$, we denote by $C_\ell[\Omega]$ ($\ell \geq 3$) the graph with point set $\mathbb{Z}_\ell \times \Gamma$ and edges $(j,\gamma)(1+j,\omega+\gamma)$, with $j \in \mathbb{Z}_\ell$, $\gamma \in \Gamma$ and $\omega \in \Omega$. In other words, $C_\ell[\Omega] = \mathrm{Cay}(\mathbb{Z}_\ell \times \Gamma, \{1\} \times \Omega)$; hence, it is $2|\Omega|$-regular. It is straightforward to see that if Γ has order n, then $C_\ell[n] \cong C_\ell[\Gamma]$; hence, $C_\ell[\Omega]$ is a subgraph of $C_\ell[n]$. We call the elements of Ω *(mixed) differences*.

Finally, given a set of cycle factors, $\mathcal{C}$, of $C_\ell[n]$, and a set $\Omega \subseteq \Gamma$ we say that $\mathcal{C}$ *exactly covers* Ω, or $C_\ell[\Omega]$, if $\mathcal{C}$ is a factorization of $C_\ell[\Omega]$.

The following result, which generalizes Theorem 2.11 of [13], provides sufficient conditions for the existence of a solution to $\mathrm{OP}_t(C_\ell[\Omega]; (\ell n_i); (\alpha_i))$, where Ω is a subset of an arbitrary group Γ of order n and each n_i is a positive divisor of n.

Theorem 2.1. *Let Γ be an additive group of order n not necessarily abelian, and let $1 \leq n_1 < n_2 < \cdots < n_t \leq n$ be odd integers such that n_i is a divisor of n for each $i \in [1,t]$; also, let Ω be a subset of Γ, and let $\alpha_1, \alpha_2, \ldots \alpha_t$ be non-negative integers such that $\sum_{i=1}^t \alpha_i = |\Omega|$. If there exists an $|\Omega| \times \ell$ matrix A with $\ell \geq 3$ and entries in Ω satisfying the following properties:*

(1) for each $i \in [1,t]$ there are α_i rows of A whose right-to-left sum is an element of order n_i in Γ,

(2) each column of A is a permutation of Ω,

then $\mathrm{OP}_t(C_\ell[\Omega]; (\ell n_i); (\alpha_i))$ has a solution. Moreover, if we also have that

(3) Ω is closed under taking negatives,

then $\mathrm{OP}_t(C_g[\Omega]; (g n_i); (\alpha_i))$ has a solution for any $g \geq \ell$ with $g \equiv \ell \pmod 2$.

Proof. Let $A = [a_{hk}]$ be an $|\Omega| \times \ell$ matrix with entries from $\Omega \subseteq \Gamma$ and satisfying conditions (1) and (2); also, set $\sigma_0 = 0$, $\sigma_i = \sum_{j=1}^i \alpha_j$ and let $\mathcal{R}_i = [\sigma_{i-1}+1, \sigma_i]$ for $i \in [1,t]$. Note that the $\mathcal{R}_i$s partition the interval $[1, |\Omega|]$ since by assumption $\sigma_t = \sum_{j=1}^t \alpha_j = |\Omega|$. By condition (1) and reordering rows if necessary, we can index the rows of A whose right-to-left sum is an element of order n_i by the elements of $\mathcal{R}_i$. Thus, we may assume that the right-to-left sum of the h-th row of A is an element of order n_i if and only if $h \in \mathcal{R}_i$.

For $1 \leq h \leq |\Omega|$ and $1 \leq k \leq \ell$, set $s_{h,0} = 0$ and $s_{h,k} = a_{h,k} + a_{h,k-1} + \cdots + a_{h,1}$. Note that $s_{h,\ell}$ is the right-to-left sum of the h-th row of A and, by the above, $s_{h,\ell}$ has order n_i if and only if $h \in \mathcal{R}_i$; in this case, $n_i s_{h,\ell} = 0$ and $\mu s_{h,\ell} \neq 0$ for any $\mu \in [1, n_i - 1]$. Therefore, for each $i \in [1,t]$ and $h \in \mathcal{R}_i$, the following ℓn_i-cycle is well defined:

$$C^h = (c_0^h, c_1^h, \dots, c_{n_i\ell-1}^h), \text{ where } c_{u+\mu\ell}^h = (u, s_{h,u} + \mu s_{h,\ell}), \text{ for } u \in [0, \ell-1], \ \mu \in [0, n_i - 1].$$

We start by showing that $\mathrm{Orb}_{\overline{\Gamma}}(C^h)$, where $\overline{\Gamma} = \{0\} \times \Gamma$, is a $C_{n_i\ell}$-factor of $C_\ell[n]$. First, note that $C^h + (0, s_{h,\ell}) = C^h$; in fact, $c_w^h + (0, s_{h,\ell}) = c_{w+\ell}^h$, for each $w \in [0, n_i\ell - 1]$, where the subscript $w + \ell$ is taken modulo $n_i\ell$. In other words, addition by $(0, s_{h,\ell})$ is equivalent to a rotation of C^h by ℓ. This means that $(0, s_{h,\ell})$ lies in $\mathrm{Stab}_{\overline{\Gamma}}(C^h)$. Since the order of $(0, s_{h,\ell})$ coincides with the order of $s_{h,\ell}$, which by assumption is n_i, we have that $|\mathrm{Stab}_{\overline{\Gamma}}(C^h)| \geq n_i$. Therefore,

$$|\mathrm{Orb}_{\overline{\Gamma}}(C^h)| = |\overline{\Gamma}|/|\mathrm{Stab}_{\overline{\Gamma}}(C^h)| \leq n/n_i.$$

Hence, $\mathrm{Orb}_{\overline{\Gamma}}(C^h)$ contains at most n/n_i $C_{n_i\ell}$-cycles. To show that $\mathrm{Orb}_{\overline{\Gamma}}(C^h)$ is actually a $C_{n_i\ell}$-factor of $C_\ell[n]$, it is then enough to check that it contains all vertices of $C_\ell[n]$ at least once. Given the point $(u, z) \in \mathbb{Z}_\ell \times \Gamma$, we have that $z = s_{h,u} + x_u$, for a suitable $x_u \in \Gamma$. Therefore, $(u, z) = c_u^h + (0, x_u)$; hence, (u, z) is a vertex of $C^h + (0, x_u) \in \mathrm{Orb}_{\overline{\Gamma}}(C^h)$.

We claim that $\mathcal{F} = \{\mathrm{Orb}_{\overline{\Gamma}}(C^h) \mid h = 1, 2, \dots, |\Omega|\}$ is a 2-factorization of $C_\ell[\Omega]$. Note that the factors of $\mathcal{F}$ contain between them at most $\ell n|\Omega| = |E(C_\ell[\Omega])|$ edges, counted with their multiplicity. Therefore, it is enough to show that every edge of $C_\ell[\Omega]$ lies in some translate of C^h, for a suitable h. First recall that each edge of $C_\ell[\Omega]$ has the form $(u, x)(1 + u, \omega + x)$ for some $(u, x) \in \mathbb{Z}_\ell \times \Gamma$ and $\omega \in \Omega$. Since, by assumption, any column of $A = [a_{hk}]$ is a permutation of Ω, there is an integer h such that $a_{h,u+1} = \omega$. Note that $(u, s_{h,u})(1 + u, s_{h,u+1}) \in E(C^h)$ and $s_{h,u+1} - s_{h,u} = a_{h,u+1} = \omega$. Therefore, $(u, x)(1 + u, \omega + x)$ is an edge of $C^h + (0, -s_{h,u} + x)$ and the assertion follows.

In order to prove the second part, let $g = \ell + 2q$, $\Omega = \{\omega_1, \omega_2, \dots, \omega_{|\Omega|}\}$, and let A' be the $|\Omega| \times 2q$ matrix defined below:

$$A' = \begin{bmatrix} \omega_1 & -\omega_1 & \dots & \omega_1 & -\omega_1 \\ \omega_2 & -\omega_2 & \dots & \omega_2 & -\omega_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \omega_{|\Omega|} & -\omega_{|\Omega|} & \dots & \omega_{|\Omega|} & -\omega_{|\Omega|} \end{bmatrix}.$$

Since $\Omega = -\Omega$ (condition (3)), it is easy to check that the matrix $\begin{bmatrix} A & A' \end{bmatrix}$ is an $|\Omega| \times g$ matrix satisfying conditions (1) – (2), and this completes the proof. □

We point out that while the above theorem is proved for an arbitrary group Γ, in this paper it is always used when $\Gamma \cong \mathbb{Z}_n$. Also, note that if $t = 1$, then Theorem 2.1 constructs a $C_{\ell n_1}$-factorization of $C_\ell[T]$ or a C_{gn_1}-factorization of $C_g[T]$.

The following corollary is a straightforward consequence of the above theorem by taking $\Omega = \Gamma = \mathbb{Z}_n$.

Corollary 2.2. *Let $t \geq 1$ and let $1 \leq n_1 < n_2 < \dots < n_t \leq n$ be odd integers such that n_i is a divisor of n for any $i \in [1, t]$; also, let $\alpha_1, \alpha_2, \dots, \alpha_t$ be non-negative integers such that $\sum_{i=1}^t \alpha_i = n$. If there exists an $n \times \ell$ matrix A with $\ell \geq 3$ and entries from $\mathbb{Z}_n$ satisfying the following properties:*

(1) for each $i \in [1, t]$, A has α_i rows each of which sums to an element of order n_i in $\mathbb{Z}_n$,

(2) each column of A is a permutation of $\mathbb{Z}_n$,

then $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$ *has a solution for any* $g \geq \ell$ *with* $g \equiv \ell \pmod 2$.

We end this section by recalling the following result proven in [21] which is here stated in a slightly different, but equivalent, form.

Lemma 2.3 ([21]). *Let* $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ *be an additive abelian group of order* n, *and let* $\delta_1, \delta_2, \dots, \delta_n$ *be elements of* Γ, *not necessarily distinct, such that* $\sum_{i=1}^{n} \delta_i = 0$. *Then there exist a permutation* Ψ *of* Γ *and a permutation* π *of the interval* $[1, n]$ *such that* $\Psi(\gamma_i) - \gamma_i = \delta_{\pi(i)}$ *for every* $i \in [1, n]$.

3 Solving $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$

In this section, by exploiting our preliminary results, we provide sufficient conditions for $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$ to be solvable.

Theorem 1.4. *Let* $t \geq 1$ *and let* $1 \leq n_1 < n_2 < \dots < n_t \leq n$ *be odd integers such that* n_i *is a divisor of* n *for each* $i \in [1, t]$. *Then* $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$ *has a solution whenever* $g \geq 3$, $\sum_{i=1}^{t} \alpha_i = n$, *and* $\alpha_i \geq 2$ *for every* $i \in [1, t]$.

Proof. Let $\alpha_i \geq 2$ for $i \in [1, t]$ be integers such that $\sum_{i=1}^{t} \alpha_i = n$. Also, let $\Delta = \{\delta_1, \delta_2, \dots, \delta_n\}$ be the list of elements of $\mathbb{Z}_n$ defined as follows: set $s_0 = 0$, $s_i = \sum_{j=1}^{i} \alpha_j$ for every $i \in [1, t]$, and let

$$(\delta_{s_{i-1}+1}, \delta_{s_{i-1}+2}, \dots, \delta_{s_i}) = \begin{cases} \left(\frac{n}{n_i}, -\frac{n}{n_i}, \dots, \frac{n}{n_i}, -\frac{n}{n_i}\right) & \text{if } \alpha_i \text{ is even,} \\ \left(\underbrace{\frac{n}{n_i}, -\frac{n}{n_i}, \dots, \frac{n}{n_i}, -\frac{n}{n_i}}_{\alpha_i - 3}, \frac{n}{n_i}, \frac{n}{n_i}, -\frac{2n}{n_i}\right) & \text{if } \alpha_i \text{ is odd,} \end{cases}$$

for every $i \in [1, t]$. By recalling that n is odd, we have that $\delta_{s_{i-1}+1}, \delta_{s_{i-1}+2}, \dots, \delta_{s_i}$ are all elements of $\mathbb{Z}_n$ of order n_i, and they sum to 0. It follows that the elements of Δ sum to 0, and Lemma 2.3 guarantees the existence of two permutations Ψ and π of $\mathbb{Z}_n$ such that $\Psi(i) - i = \delta_{\pi(i)}$ for every $i \in \mathbb{Z}_n$.

Now for each $\ell \in \{3, 4\}$, let A_ℓ be the $n \times \ell$ matrix whose i-th row is either

$$\begin{bmatrix} \Psi(i) & -\frac{i}{2} & -\frac{i}{2} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \Psi(i) & i & -i & -i \end{bmatrix}$$

according to whether $\ell = 3$, or 4, respectively. It is not difficult to check that A_3 and A_4 satisfy the following conditions:

(i) for each $i \in [1, t]$, A_3 (resp., A_4) has α_i rows each of which sums to an element of order n_i,

(ii) each column of A_3 (resp., A_4) is a permutation of $\mathbb{Z}_n$.

In other words, A_3 and A_4 satisfy the assumptions of Corollary 2.2 which guarantees the solvability of $\mathrm{OP}_t(C_g[n]; (gn_i); (\alpha_i))$ whenever $g \geq 3$. □

We point out that Theorem 1.4 holds also when $g = 2$. In this case, $C_2[n]$ is taken to be the complete bipartite graph with parts of size n whose edges are taken with multiplicity two. This can be seen by following the proof of Theorem 1.4 but using the matrix $\begin{bmatrix} \Psi(i) & -i \end{bmatrix}$.

4 Solving $\mathrm{OP}_t(v; (N_i); (\alpha_i))$

We say that $\mathrm{OP}_t(G; (N_i); (\alpha_i))$ and $\mathrm{OP}_u(G; (M_j); (\beta_j))$ are equivalent if $\sum_{N_i=x} \alpha_i = \sum_{M_j=x} \beta_j$ for any $x \geq 3$. For example, $\mathrm{OP}_4(G; 4, 4, 5, 7; 4, 6, 8, 2)$ is equivalent to $\mathrm{OP}_5(G; 4, 4, 4, 5, 7; 2, 3, 5, 8, 2)$.

Moreover, for any non-negative integer α we define the integer $f(\alpha)$ as follows:

$$f(\alpha) = \frac{\alpha - \rho}{3}, \quad \text{where } \{0, 2, 4\} \ni \rho \equiv \alpha \pmod 3.$$

Clearly, $\alpha = 3f(\alpha) + \rho$ and $f(\alpha) \equiv \alpha \pmod 2$.

The following result provides sufficient conditions for the existence of a solution to $\mathrm{OP}_t(G; (gn_i); (\alpha_i))$ for an arbitrary graph G.

Theorem 4.1. *Let $t \geq 2$, and let $1 \leq n_1 < n_2 < \cdots < n_t \leq n$ be odd integers such that n_i is a divisor of n for each $i \in [1, t]$. Also, let G be a graph having a factorization into r $C_g[n]$-factors with $g \geq 3$. Then, $\mathrm{OP}_t(G; (gn_i); (\alpha_i))$ has a solution whenever the following conditions simultaneously hold:*

(1) $\sum_{i=1}^{t} \alpha_i = rn$;

(2) $0 \leq \alpha_i \neq 1$ *for every* $i \in [1, t]$;

(3) $\sum_{i=1}^{t} f(\alpha_i) \geq r$;

(4) $|\{i \in [1, t] \mid \alpha_i \text{ is odd}\}| \leq r\left(2\lfloor \frac{n-2}{6} \rfloor + 1\right)$.

Proof. Let $n = 6q + \rho$ where $\rho \in \{3, 5, 7\}$ and let $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ be a factorization of G into r $C_g[n]$-factors. We proceed by induction on r. If $r = 1$, the assertion follows from Theorem 1.4. Now, let $r \geq 2$ and assume that the assertion holds for any graph having a factorization into $r - 1$ $C_g[n]$-factors. It is enough to show that $\mathrm{OP}_t(G; (gn_i); (\alpha_i))$ is equivalent to a problem of the following form:

$$\begin{aligned} \mathrm{OP}_u(G; (N_j); (\beta_j)), \quad & \text{where } \beta_j \in \{2, 3\} \text{ and} \\ & r \leq \delta = |\{j \in [1, u] \mid \beta_j = 3\}| \leq r(2q + 1). \end{aligned} \tag{4.1}$$

In fact, assuming this equivalence, we only need define $\overline{\beta_j}$s so that $\mathrm{OP}_u(F_1; (N_j); (\overline{\beta_j}))$ and $\mathrm{OP}_u(G - F_1; (N_j); (\beta_j - \overline{\beta_j}))$ are solvable; it follows that the problem in (4.1), and hence, the original problem has a solution. We first assume (without loss of generality) that $\beta_j = 3$ if and only if $j \in [1, \delta]$ and consider the following two cases:

1. if $\delta \in [r, r + 2q]$, set

$$\overline{\beta_j} = \begin{cases} \beta_j & \text{if } j \in \{1\} \cup [\delta + 1, \delta + \frac{n-3}{2}]; \\ 0 & \text{otherwise;} \end{cases}$$

2. if $\delta \in [r + 2q + 1, r(2q + 1)]$, we define $\overline{\beta_j}$ as follows,

$$\overline{\beta_j} = \begin{cases} \beta_j & \text{if } j = [1, 2q + 1] \cup [\delta + 1, \delta + \frac{\rho-3}{2}]; \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.4, there exists a solution to $\mathrm{OP}_u(F_1; (N_j); (\overline{\beta_j}))$. It is not difficult to check that $\mathrm{OP}_u(G - F_1; (N_j); (\beta_j - \overline{\beta_j}))$ satisfies all the assumption of this theorem, therefore, by the induction hypothesis, it is solvable.

We now show that $\mathrm{OP}_t(G; (gn_i); (\alpha_i))$ is equivalent to a problem of the form (4.1). We reorder the α_is so that the even α_is appear first. For every $i \in [1, t]$ we define the quadruple of integers $(\gamma_{2i-1}, \gamma_{2i}, N_{2i-1}, N_{2i})$ as follows:

$$(\gamma_{2i-1}, \gamma_{2i}) = \begin{cases} (\alpha_i - 3, 3) & \text{if } \alpha_i \text{ is odd;} \\ (\alpha_i, 0) & \text{if } \alpha_i \text{ is even;} \end{cases} \qquad N_{2i-1} = N_{2i} = gn_i.$$

It follows that $\gamma_1, \gamma_2, \ldots, \gamma_{2t-d}$ are even, whereas $\gamma_i = 3$ for any $i \in [2t - d + 1, 2t]$, where $d = |\{i \in [1, t] \mid \alpha_i \text{ is odd}\}|$ is the number of odd α_is. We point out that $\mathrm{OP}_t(G; (gn_i); (\alpha_i))$ is equivalent to $\mathrm{OP}_{2t}(G; (N_i); (\gamma_i))$; also, since by assumption $\sum_{i=1}^{t} f(\alpha_i) \geq r$, it follows that $\sum_{i=1}^{2t} f(\gamma_i) \geq r$.

We first assume that $d < r$. Now, let $k \in [1, 2t - d]$ be the greatest integer such that $\sum_{i=k}^{2t} f(\gamma_i) \geq r$, and set $r' = \sum_{i=k+1}^{2t} f(\gamma_i)$. Clearly, $r' < r$; also, $r - r'$ is even, since:

$$r \equiv rn = \sum_{i=1}^{k} \gamma_i + \sum_{i=k+1}^{2t} \gamma_i \equiv \sum_{i=k+1}^{2t} \gamma_i \equiv \sum_{i=k+1}^{2t} f(\gamma_i) = r' \pmod 2.$$

We proceed by defining a suitable partition $(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{i,t_i})$ of the integer γ_i such that $\gamma_{ij} \in \{0, 2, 3\}$. First, for each $i \in [k, 2t]$ set $(q_i, \rho_i) = (f(\gamma_i), \gamma_i - 3f(\gamma_i))$ and note that $\rho_i \in \{0, 2, 4\}$. Recall now that $\gamma_k = 3q_k + \rho_k$ is even, hence q_k is even; also, $r - r'$ is even and $q_k \geq r - r'$. Therefore, $\gamma_k = 3(r - r') + 2y$ where $y = \frac{3(q_k - r + r') + \rho_k}{2}$. We now define a partition $(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{i,t_i})$ of γ_i as follows:

- if $i \in [1, k-1]$, set $t_i = \gamma_i/2$ and $\gamma_{ij} = 2$ for any $j \in [1, t_i]$;
- if $i = k$, set $t_i = r - r' + y$ and $\gamma_{ij} = \begin{cases} 3 & \text{if } j \in [1, r - r']; \\ 2 & \text{otherwise.} \end{cases}$
- if $i \in [k + 1, 2t]$, set $t_i = q_i + 2$ and

$$\gamma_{ij} = \begin{cases} 3 & \text{if } j \in [1, q_i]; \\ 0 & \text{if } (j, \rho_i) \in \{(q_i + 1, 0), (q_i + 2, 0), (q_i + 2, 2)\}; \\ 2 & \text{otherwise.} \end{cases}$$

Finally, for any $i \in [1, 2t]$ and $j \in [1, t_i]$ set $N_{ij} = N_i$ and $u = \sum_{i=1}^{2t} t_i$. Clearly, the original problem $\mathrm{OP}_{2t}(G; (N_i); (\gamma_i))$ is equivalent to $\mathrm{OP}_u(G; (N_{ij}); (\gamma_{ij}))$ where $\gamma_{ij} \in \{0, 2, 3\}$ and there are exactly r γ_{ij}s equal to 3. By removing all pairs (N_{ij}, γ_{ij}) with $\gamma_{ij} = 0$, we obtain a problem of the form (4.1).

We finally consider the case where $d \geq r$. As before, we define a partition $(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{i,t_i})$ of the integer γ_i as follows:

$$(t_i, \gamma_{ij}) = \begin{cases} (\frac{\gamma_i}{2}, 2) & \text{if } i \in [1, 2t - d] \text{ and } j \in [1, \frac{\gamma_i}{2}]; \\ (1, 3) & \text{otherwise;} \end{cases}$$

and set $N_{ij} = N_i$ for any $j \in [1, t_i]$, and $u = \sum_{i=1}^{2t} t_i$. Clearly, the original problem $\mathrm{OP}_{2t}(G; (N_i); (\gamma_i))$ is equivalent to $\mathrm{OP}_u(G; (N_{ij}); (\gamma_{ij}))$ where $\gamma_{ij} \in \{2, 3\}$ and there are exactly d γ_{ij}s equal to 3. Since, $d \le r(2q+1)$ by assumption, then $\mathrm{OP}_u(G; (N_{ij}); (\gamma_{ij}))$ is of the form (4.1), and this completes the proof. □

We now provide a result for the complete equipartite graph.

Theorem 4.2. *Let $s, w \ge 3$ be odd integers, let $3 \le N_1 < N_2 < \cdots < N_t$, and let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be positive integers. If $\sum_{i=1}^{t} \alpha_i = \frac{(s-1)w}{2}$ and each N_i is a divisor of w, then $\mathrm{OP}_t(K_s[w]; (N_i); (\alpha_i))$ is solvable, except possibly when $t > 1$ and at least one of the following conditions is satisfied:*

(A) $\alpha_i = 1$ *for some* $i \in [1, t]$*;*

(B) $\gcd(N_1, N_2, \ldots, N_t) = 1$.

Proof. We assume that $t \ge 2$, since the case $t = 1$ is solved in Theorem 1.1.

Now, set $N = \mathrm{lcm}(N_1, N_2, \ldots, N_t)$ and $g = \gcd(N_1, N_2, \ldots, N_t)$; also, let $n_i = N_i/g$, set $n = \mathrm{lcm}(n_1, n_2, \ldots, n_t)$ and note that $N = gn$. By assumption, we have that each N_i is a divisor of w, that is, N is a divisor of w, hence $w = gn\overline{w}$ for some integer $\overline{w} > 0$. By Theorem 1.1, there exists a C_g-factorization of $K_s[g\overline{w}]$ with r C_g-factors, where $r = g\overline{w}(s-1)/2$. By expanding each vertex of this factorization by n, we get a $C_g[n]$-factorization $\mathcal{F}$ of $K_s[g\overline{w}][n] \cong K_s[w]$ with r $C_g[n]$-factors.

We first assume that $n \ge 7$. In this case, to solve $\mathrm{OP}_t(K_s[w]; (N_i); (\alpha_i))$ it is enough to show that conditions (1) – (4) of Theorem 4.1 are satisfied. By assumption $\sum_{i=1}^{t} \alpha_i = \frac{(s-1)w}{2} = rn$, and by exception (A) we have that $\alpha_i \ge 2$ for every $i \in [1, t]$. Further,

$$r\left(2\left\lfloor\frac{n-2}{6}\right\rfloor + 1\right) \ge \frac{r(n-4)}{3} = \frac{g\overline{w}(s-1)}{6}(n-4) \ge n - 4 \ge \frac{n}{3},$$

and since n has at most $\lfloor \frac{n}{3} \rfloor$ distinct divisors, we have that $\frac{n}{3} \ge t$, hence $r\left(2\left\lfloor\frac{n-2}{6}\right\rfloor + 1\right) \ge t$. Finally, we have that

$$rn = \sum_{i=1}^{t} \alpha_i \le \sum_{i=1}^{t} (3f(\alpha_i) + 4) = 4t + 3\sum_{i=1}^{t} f(\alpha_i) < 4r + 3\sum_{i=1}^{t} f(\alpha_i),$$

and since $n \ge 7$, it follows that $\sum_{i=1}^{t} f(\alpha_i) > r(n-4)/3 \ge r$. Therefore, all conditions of Theorem 4.1 are satisfied, hence $\mathrm{OP}_t(K_s[w]; (N_i); (\alpha_i))$ is solvable.

It is left to consider the cases where $n \in \{3, 5\}$. Since N_i is a multiple of g and a divisor of gn, then $N_i \in \{g, gn\}$ for any i. By recalling that $N_1 < N_2 < \cdots < N_t$ and $t \ge 2$, we have that $t = 2$ and $(N_1, N_2) = (g, gn)$. Now, let $\alpha_2 = xn + y$ where $x \ge 0$ and $y \in [0, n-1]$, and since $\alpha_2 \ge 2$ (exception (A)), then $(x, y) \ne (0, 1)$. If $y \ne 1$, we apply Theorem 1.4 to fill x $C_g[n]$-factors of $\mathcal{F}$ with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; 0, n)$, one $C_g[n]$-factor with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; n-y, y)$, and the remaining $r - x - 1$ factors of $\mathcal{F}$ with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; n, 0)$. Similarly, if $y = 1$, since $x > 0$ and $r \ge g \ge 3$ (exception (B)), we again apply Theorem 1.4 and fill $x - 1$ $C_g[n]$-factors of $\mathcal{F}$ with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; 0, n)$, one $C_g[n]$-factor with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; 1, n-1)$, one $C_g[n]$-factor with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; n-2, 2)$, and the remaining $r-x-1$ factors of $\mathcal{F}$ with a solution of $\mathrm{OP}_2(C_g[n]; g, gn; n, 0)$. □

We are now ready to prove the main result of this paper.

Theorem 1.3. *Let $v \geq 3$ be odd, let $3 \leq N_1 < N_2 < \cdots < N_t$ and set $N = \operatorname{lcm}(N_1, N_2, \ldots, N_t)$ and $g = \gcd(N_1, N_2, \ldots, N_t)$; also, let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be positive integers. Then, $\mathrm{OP}_t(v; (N_i); (\alpha_i))$ has a solution if and only if N is a divisor of v and $\sum_{i=1}^{t} \alpha_i = \frac{v-1}{2}$ except possibly when $t > 1$ and at least one of the following conditions is satisfied:*

(I) $\alpha_i = 1$ *for some* $i \in [1, t]$*;*

(II) $\alpha_i \in [2, \frac{N-3}{2}] \cup \{\frac{N+1}{2}\}$ *for every* $i \in [1, t]$*;*

(III) $g = 1$*;*

(IV) $v = N$.

Proof. By Theorem 1.2, if $\mathrm{OP}_t(v; (N_i); (\alpha_i))$ has a solution, then N is a divisor of v and $\sum_{i=1}^{t} \alpha_i = \frac{v-1}{2}$. We now show sufficiency and assume that $t \geq 2$, since the case $t = 1$ is solved in Theorem 1.1. Let $v = Ns$ for a suitable odd integer s. By exception (IV), we have that $s \geq 3$.

We first factorize K_v into $G_0 = sK_N$ and $G_1 = K_s[N]$. By exception (II), there exists $k \in [1, t]$ such that either $\alpha_k = \frac{N-1}{2}$ or $\alpha_i \geq \frac{N+3}{2}$. Then, we apply Theorem 1.1 to fill G_0 with a C_{N_k}-factorization. It remains to solve $\mathrm{OP}_t(G_1; (N_i); (\overline{\alpha_i}))$ where $\overline{\alpha_i} = \alpha_i - \frac{N-1}{2}$ if $i = k$, and $\overline{\alpha_i} = \alpha_i$ otherwise. By taking into account exceptions (I) and (III), we have that:

(a) $\overline{\alpha_i} \neq 1$ for any $i \in [1, t]$, and

(b) $g \geq 3$.

Therefore, Theorem 4.2 guarantees the solvability of $\mathrm{OP}_t(G_1; (N_i); (\overline{\alpha_i}))$ and the assertion is proven. □

Corollary 4.3. *Let $v \geq 3$ be odd, let $3 \leq N_1 < N_2 < \cdots < N_t$, set $N = \operatorname{lcm}(N_1, N_2, \ldots, N_t)$, and let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be positive integers. Then, $\mathrm{OP}_t(v; (N_i); (\alpha_i))$ has a solution whenever N is a divisor of v, $\sum_{i=1}^{t} \alpha_i = \frac{v-1}{2}$, and the following conditions are satisfied:*

(1) $\alpha_i \neq 1$ *for any* $i \in [1, t]$*;*

(2) $\gcd(N_1, N_2, \ldots, N_t) \geq 3$*;*

(3) $v \geq (t+1)N$.

Proof. The case $t = 1$ is solved in Theorem 1.1, therefore, we let $t \geq 2$. By condition (3) and considering that $\sum_{i=1}^{t} \alpha_i = \frac{v-1}{2}$, it follows that there exists $k \in [1, t]$ such that $\alpha_k \geq \frac{N+3}{2}$. If we also take into account conditions (1) and (2), we have that all assumptions of Theorem 1.3 are satisfied, and the assertion follows. □

5 Conclusions

This paper deals with the generalized Oberwolfach problem, denoted by $\mathrm{OP}_t(v; N_1, N_2, \ldots, N_t; \alpha_1, \alpha_2, \ldots, \alpha_t)$, which asks for a 2-factorization of the complete graph K_v into α_i copies of a C_{N_i}-factor, for $i \in \{1, 2, \ldots, t\}$. For a solution of this problem to exist, v must be odd, each N_i must be a divisor of v, and $\sum_i \alpha_i = \frac{v-1}{2}$ (Theorem 1.2).

This problem has been widely studied when $t = 1$ or 2. The case $t = 1$ represents the 'uniform' Oberwolfach problem which has been solved in 1989 [3]. When $t = 2$, this problem is known as the Hamilton-Waterloo problem. Although this version of the problem is still open, by using techniques similar to those adopted in this paper, the current authors were able to make significant progress in the challenging case where the cycle lengths are odd [12, 13].

This paper makes significant progress (Theorem 1.3) on the generalized Oberwolfach problem by showing that the above necessary conditions suffice whenever $v > (t+1)N$, each α_i is greater than 1, and $g \geq 3$, where $g = \gcd(N_1, N_2, \ldots, N_t)$ (Corollary 4.3). This result and its stronger version (Theorem 1.3) rely on Theorem 1.4 which concerns the existence of a factorization of $C_g[n]$ into α_i C_{gn_i}-factors for $i \in \{1, 2, \ldots, t\}$ (that is, the generalized Oberwolfach problem over $C_g[n]$). Theorem 1.4 shows that the trivial necessary conditions suffice whenever $g \geq 3$, and $\alpha_i > 1$ for each i. Clearly, removing this last condition from Theorem 1.4 would automatically yield a similar improvement of our main theorem.

More generally, we provide sufficient conditions (Theorem 4.1) for the solvability of the generalized Oberwolfach problem over an arbitrary graph G. As a consequence, we provide, with Theorem 4.2, a result for the complete equipartite graph, similar to those mentioned above.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 79–88
https://doi.org/10.26493/1855-3974.1508.f8c
(Also available at http://amc-journal.eu)

Block allocation of a sequential resource

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Received 16 October 2017, accepted 27 May 2019, published online 22 June 2019

Abstract

An H-packing of G is a collection of vertex-disjoint subgraphs of G such that each component is isomorphic to H. An H-packing of G is maximal if it cannot be extended to a larger H-packing of G. In this paper we consider problem of random allocation of a sequential resource into blocks of m consecutive units and show how it can be successfully modeled in terms of maximal P_m-packings. We enumerate maximal P_m-packings of P_n of a given cardinality and determine the asymptotic behavior of the enumerating sequences. We also compute the expected size of m-packings and provide a lower bound on the efficiency of block-allocation.

Keywords: Maximal matching, maximal packing.

Math. Subj. Class.: 05C70, 05A15, 05A16

1 Matchings and packings

A *matching* M in a graph G is a collection of edges of G such that no two edges from M have a vertex in common. The number of edges of M is called the *size* of the matching. Small matchings are not interesting – they are easy to find and enumerate. Hence, we are mostly interested in matchings that are as large as possible. There are two ways to quantify the idea of "large" matchings, one of them based on their cardinality, the other based on the set inclusion.

A matching M is *maximum* if there is no matching in G with more edges than M. The cardinality of any maximum matching in G is called the *matching number* of G and denoted by $\nu(G)$. The matching number of a graph on n vertices, obviously, cannot exceed $\lfloor n/2 \rfloor$, since each edge saturates two vertices. A matching that saturates all vertices of G is called a *perfect matching*.

*Partial support of the Croatian Science Foundation (research projects BioAmpMode (Grant no. 8481) and LightMol (Grant no. IP-2016-06-1142)) is gratefully acknowledged. I also thank Damir Vukičević and Kristina Ana Škreb for useful discussions.

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A matching M in G is *maximal* if it cannot be extended to a larger matching in G, i.e., if no other matching in G contains it as a proper subset. Obviously, every maximum matching is also maximal, but the opposite is generally not true. The cardinality of any smallest maximal matching in G, denoted by $s(G)$, is the *saturation number* of G; the largest size of a maximal matching is, of course, $\nu(G)$.

Matchings are natural models for many problems in natural, technical and social sciences. Worth mentioning are applications of perfect matchings in organic chemistry and solid state physics. For a general background on matching theory and terminology we refer the reader to the classical monograph by Lovász and Plummer [14]. For graph theory terms not defined here we also recommend [3, 19].

A closely related concept of packing is a generalization of matching. There are several varieties of packing; we consider here only the simplest case. An *H-packing* of G is a collection of vertex-disjoint subgraphs of G such that each component is isomorphic to H [3]. Hence, a matching of G is a P_2-packing in G, where P_2 denotes a path on 2 vertices. Again, we are interested only in large packings. If a packing is a spanning subgraph, we say that the packing is *perfect*; if no other H-packing has more components, the packing is *maximum*; finally, if an H-packing cannot be extended to a valid H-packing, we say that it is a *maximal H-packing*. The *H-packing number* and *H-saturation number* are defined in the same way as for matchings. When $H = P_m$ we denote these two quantities by $\nu_m(G)$ and $s_m(G)$ and call them the *m-packing number* and *m-saturation number*, respectively. We refer the reader to [12, 13] for some aspects of P_3-packings in claw-free and in subcubic graphs and to [15] for similar problems in directed graphs.

Maximal matchings and packings can serve as models of several physical and technical problems such as the block-allocation of a sequential resource or adsorption of dimers and/or polymers on a structured substrate or a molecule. When that process is random, it is clear that the substrate can become saturated by a number of units much smaller than the theoretical maximum. The respective saturation numbers provide an information on the worst possible case of clogging; they measure how inefficient the adsorption or the allocation process can be. However, in order to assess its efficiency, we also need to know how likely it is that a given number of units will saturate the substrate. Hence, we must study the enumerative aspects of the problem.

For the matching case, the question has been answered in [7]. The main goal of this paper is to contribute to the corpus of knowledge about the enumerative aspects of maximal P_m-packings in paths and cycles. Specifically, we compute the efficiency of block-allocation of length m of a sequential linear or cyclic resource. In some cases we provide explicit formulas for the number of maximal m-packings of a given cardinality, while in other cases we establish the recurrences for the enumerating sequences and then use their uni- and bivariate generating functions to determine their asymptotic behavior.

Finally, in the concluding section we discuss some open problems and indicate some directions of possible future research.

2 Paths and cycles

2.1 Paths

We remind the reader that throughout this paper P_n denotes the path on n vertices, hence of length $n-1$. As a motivation, we consider a parking lot made of n parallel concrete strips such that a car can be parked on any two neighboring strips. In ideal situation, when all drivers take care and park responsibly, the lot can accommodate $\lfloor n/2 \rfloor$ cars. However, if the drivers are careless, the lot can become saturated by a smaller number of cars, as shown in Figure 1. In the worst possible case, it can become saturated by as few as $\lfloor (n+1)/3 \rfloor$

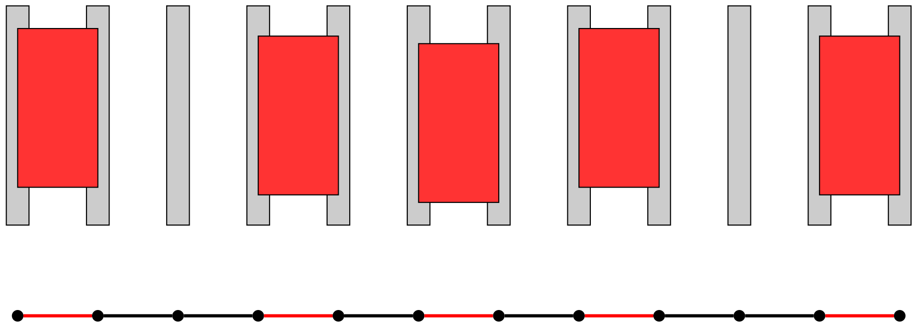

Figure 1: A saturated parking lot and the corresponding maximal matching.

cars. Hence, it is of interest to find out how likely is this to happen, and what is the expected number of cars under the random regime.

In the continuous setting, this problem is known as the random car-parking problem of Rényi [16, 17], while in discrete setting it has a natural representation as a problem of maximal matching in P_n, as shown in Figure 1; it was considered in detail in [7], where its full solution was obtained, including the explicit formulas for the number of different configurations accommodating a given number of cars. Also, the expected number of cars under the random regime was computed, and the asymptotic behavior of the sequence enumerating all possible parking arrangement was determined.

But what happens if we wish to park trucks such that each of them is twice as wide as a car? Each truck will then consume three consecutive strips, as shown in Figure 2, and the corresponding graph-theoretical model will not be a matching, but a packing of copies of

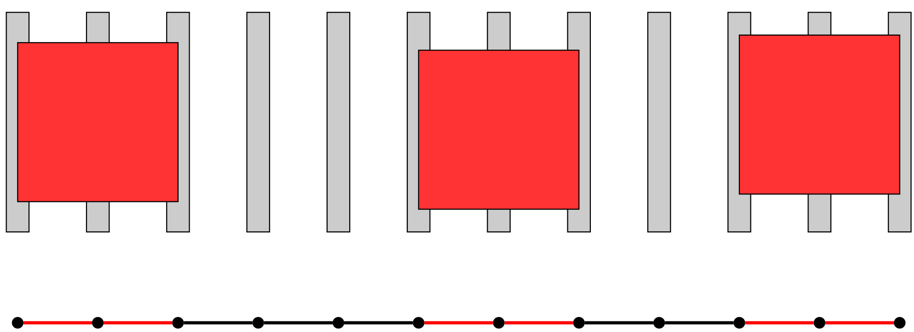

Figure 2: A parking lot saturated with trucks.

P_3 in P_n. Obviously, the structure of the problem remains the same if instead of parking lots and cars and trucks we consider any sequential resource of length n which is allocated in blocks of $m \geq 2$ consecutive units. All such situations can be studied as problems of packing copies of P_m in P_n. We call such a packing an m-packing. In this subsection we consider the enumerative aspects of m-packings in paths. Before counting them, we state (without proof) two results about the smallest and the largest possible size of m-packings in P_n.

Proposition 2.1. *Let P_n be a path on n vertices. Then*

$$s_m(P_n) = \left\lfloor \frac{n+m-1}{2m-1} \right\rfloor \qquad \textit{and} \qquad \nu_m(P_n) = \left\lfloor \frac{n}{m} \right\rfloor.$$

We now start counting all maximal m-packings in P_n. Let $\psi_{n,k}^{(m)}$ denote the total number of maximal m-packings in P_n with exactly k copies of P_m.

Proposition 2.2. *The sequence $\psi_{n,k}^{(m)}$ is given by the recurrence*

$$\psi_{n,k}^{(m)} = \sum_{l=m}^{2m-1} \psi_{n-l,k-1}^{(m)}$$

for $n \geq 2m-1$ and with the initial conditions

$$\psi_{0,0}^{(m)} = \psi_{1,0}^{(m)} = \cdots = \psi_{m-1,0}^{(m)} = 1$$

and $\psi_{l,0}^{(m)} = 0$ for all other values of l.

Proof. Let us label the vertices of P_n by $v_1, \ldots, v_n$. Let v_l be the vertex with the highest label that is covered by a copy of P_m in a maximal m-packing of size k. Clearly, $v_l \in \{v_{n-m+1}, \ldots, v_n\}$ (otherwise there would be enough place to pack one more copy of P_m, contrary to the assumption of maximality), and the remaining $k-1$ copies of P_m must form a valid maximal packing of P_m of size $k-1$ in the remaining portion of P_n, i.e., in P_{l-m+1}. The initial conditions count trivial packings of size zero. □

From the above recurrence one can immediately compute the bivariate generating function for the numbers $\psi_{n,k}^{(m)}$ by multiplying them throughout by $x^n y^k$ and summing over all $n \geq 2m-1$, $k \geq 1$. We state the result omitting the computational details.

Theorem 2.3. *Let $F_m(x,y) = \sum_{n,k\geq 0} \psi_{n,k}^{(m)} x^n y^k$ be the bivariate generating function of $\psi_{n,k}^{(m)}$. Then*

$$F_m(x,y) = \frac{p_m(x)}{1 - y q_m(x)},$$

where $p_m(x) = \frac{1-x^m}{1-x}$ and $q_m(x) = x^m p_m(x)$.

Corollary 2.4. *The bivariate generating function of $\psi_{n,k}^{(m)}$ is given by*

$$F_m(x,y) = \frac{1-x^m}{1-x-x^m(1-x^m)y}.$$

The generating function $F_m(x) = \sum_{n\geq 0} \psi_n^{(m)} x^n$ for the sequence enumerating the total number of m-packings in P_n is now obtained by substituting $y=1$ into the expression for $F_m(x,y)$.

Corollary 2.5. *The generating function of the sequence enumerating the total number of maximal m-packings in P_n is given by*

$$F_m(x) = \frac{1 - x^m}{1 - x - x^m + x^{2m}}.$$

From the above result we can deduce the recurrence satisfied by $\psi_n^{(m)}$.

Corollary 2.6. *The numbers $\psi_n^{(m)}$ satisfy the recurrence*

$$\psi_n^{(m)} = \psi_{n-m}^{(m)} + \cdots + \psi_{n-2m+1}^{(m)}$$

for $n \geq 2m - 1$ with the initial conditions $\psi_0^{(m)} = \cdots = \psi_m^{(m)} = 1$ and $\psi_{m+i}^{(m)} = i + 1$ for $1 \leq i \leq m - 2$.

The numbers $\psi_{n,k}^{(m)}$ form a triangular array with rows indexed by n and columns indexed by k. It can be deduced from the form of the bivariate generating function that the columns are, in fact, shifted rows of the triangle of multinomial (m-nomial) coefficients. Recall that the (p, q)-th m-nomial coefficient

$$t_{p,q}^{(m)} = \sum_{i=0}^{\lfloor q/m \rfloor} (-1)^i \binom{p}{i} \binom{p + q - 1 - im}{p - 1}$$

is the coefficient of x^q in $(1 + x + \cdots + x^{m-1})^p$. (See, for example, sequence A035343 in [18] for $m = 5$.) The observation can be formally stated in the following way.

Corollary 2.7.

$$\psi_{n,k}^{(m)} = t_{k+1,n-mk}^{(m)}.$$

As a consequence, we can obtain formulas for $\psi_{n,k}^{(m)}$ and $\psi_n^{(m)}$. We refer the reader to the *On-Line Encyclopedia of Integer Sequences* for more details on multinomial coefficients [18].

Corollary 2.8.

$$\psi_{n,k}^{(m)} = \sum_{i=0}^{\lfloor \frac{n}{m} - k \rfloor} (-1)^i \binom{k+1}{i} \binom{n + k - m(i + k)}{k};$$

$$\psi_n^{(m)} = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \sum_{i=0}^{\lfloor \frac{n}{m} - k \rfloor} (-1)^i \binom{k+1}{i} \binom{n + k - m(i + k)}{k}.$$

When $m = 2$, the above formulas reduce to known results about the number of maximal matchings [7].

As a further consequence, we note that the number of all maximal m-packings of size k in all paths is given by m^{k+1}.

Our next goal is to determine the asymptotic behavior of the enumerating sequences and then use it to compute the expected size of a maximal m-packing in P_n. We rely on the following version of Darboux's theorem [2].

Theorem A. *If the generating function* $f(x) = \sum_{n\geq 0} a_n x^n$ *of a sequence* (a_n) *can be written in the form* $f(x) = \left(1 - \frac{x}{w}\right)^\alpha h(x)$*, where* w *is the smallest modulus singularity of* f *and* h *is analytic in* w*, then* $a_n \sim \frac{h(w)n^{-\alpha-1}}{\Gamma(-\alpha)w^n}$*, where* Γ *denotes the gamma function.*

As a consequence, the expected size of a maximal m-packing in P_n, $\pi_m(P_n)$, can be computed as

$$\pi_m(P_n) = \frac{[x^n]\frac{\partial F_m(x,y)}{\partial y}\,|_{y=1}}{[x^n]F_m(x,y)\,|_{y=1}},$$

where $[x^n]F(x)$ denotes the coefficient of x^n in the expansion of $F(x)$.

We refer the reader to [2, 20] for more information on obtaining the asymptotics of a sequence from its generating function.

We start by observing that $F_m(x) = F_m(x,y)\,|_{y=1}$ and $\frac{\partial F_m(x,y)}{\partial y}\,|_{y=1}$ can be represented as

$$F_m(x) = \left(1 - \frac{x}{w_m}\right)^{-1} \frac{p_m(x)}{w_m \frac{1-q_m(x)}{w_m - x}} = \left(1 - \frac{x}{w_m}\right)^{-1} g_m(x)$$

and

$$\left.\frac{\partial F_m(x,y)}{\partial y}\right|_{y=1} = \left(1 - \frac{x}{w_m}\right)^{-2} \frac{p_m(x)q_m(x)}{\left[w_m \frac{1-q_m(x)}{w_m - x}\right]^2} = \left(1 - \frac{x}{w_m}\right)^{-2} h_m(x).$$

Here w_m denotes the smallest (and the only) real solution of the equation $q_m(x) = 1$. By plugging this into Theorem A we obtain following results.

Theorem 2.9. *The asymptotics of the number of* m*-packings in* P_n *is given by*

$$\psi_n^{(m)} \sim g_m(w_m) \cdot w_m^{-n}.$$

Theorem 2.10. *The expected size of a maximal* m*-packing in* P_n *is given by*

$$\pi_m(P_n) = \frac{1}{w_m q_m'(w_m)} n,$$

where w_m *is the only real solution of* $q_m(x) = 1$.

Now we can define the *efficiency* of random m-packing in P_n as the quotient of the expected and the optimal size of an m-packing. Since the size of any largest possible m-packing in P_n is $\lfloor n/m \rfloor$, the efficiency is given by

$$\varepsilon(m) = \frac{m}{w_m q_m'(w_m)}.$$

It is, hence, of interest to investigate the behavior of the above quotient for large values of n and m. (We will assume that $n \gg m$, since the opposite case is not very interesting.) Numerical computations indicate that it initially decreases from 0.823 for $m = 2$ and achieves the minimum value of 0.758317 for $m = 9$, and then increases slowly (apparently monotonously) so that for $m = 100$ it has the value of approximately 0.796. In the rest of this subsection we show that $\varepsilon(m)$ remains bounded from below for all values of m.

For the beginning, we transform the expression for $q'_m(x)$ as follows:

$$\begin{aligned} q'_m(x) &= \frac{mx^{m-1}}{1-x}(1-2x^m) + \frac{x^m(1-x^m)}{1-x} \\ &= \frac{x^m(1-x^m)}{1-x}\left[\frac{2m}{x} + \frac{1}{1-x} - \frac{m}{x}\frac{1}{1-x^m}\right] \end{aligned}$$

By plugging in $x = w_m$, the first term on the right-hand side becomes 1, and by multiplying the resulting equation through by w_m, we obtain

$$w_m q_m'(w_m) = \left(2 - \frac{1}{1-w_m^m}\right)m + \frac{w_m}{1-w_m}.$$

We would like to estimate the right-hand side and give some upper bound. The first term never exceeds m; it is enough to note that $w_m > 1/2$ for all $m \geq 2$, and from there it follows $2 - \frac{1}{1-w_m^m} < 2 - \frac{1}{1-2^{-m}} < 1$. In order to bound the second term, we notice that for large enough values of m we must have $w_m < 1 - \frac{3}{m}$. Indeed, this is equivalent to

$$\left(1 - \frac{3}{m}\right)^m - \left(1 - \frac{3}{m}\right)^{2m} > \frac{3}{m},$$

and this is true, since the left-hand side tends to $e^{-3} - e^{-6} \approx 0.047308$, while the right-hand side tends to zero. Numerical computations show that “large enough” here means $m = 68$. By plugging in the upper bound $w_n < 1 - \frac{3}{m}$ into the second term, we obtain $\frac{w_m}{1-w_m} < \frac{m}{3}$. Now the right-hand side can be bounded from above by $\frac{4m}{3}$. This gives us a lower bound on the efficiency.

Proposition 2.11. *The efficiency of m-packings is bounded from below. For all $m \geq 2$,*

$$\varepsilon(m) > \frac{3}{4}.$$

The same argument as above could be used to show that for large enough values of m and for any real $a > 0$, an expression of the type $1 - \frac{a}{m}$ will be an upper bound on w_m. This implies that the right-hand side of the expression for $w_m q'_m(w_m)$ can be bounded from above by $\frac{a+1}{a}m$, and consequently, that $\lim_{m\to\infty} \varepsilon(m) = 1$.

Our results indicate that longer blocks achieve better efficiency of random block allocation of a sequential resource. The dependency is rather mild, and the growth is slow. For example, a hundredfold increase of the block length from $m = 1000$ to $m = 100\,000$ results in the moderate increase of efficiency from $\varepsilon(1000) = 0.844$ to $\varepsilon(100\,000) = 0.903$. Still, the block length of nine seems to be a bad choice.

Before we move to the cycles, we mention that our analysis assumes that all packings are equally probable. It is known for maximal matchings that the efficiency is slightly better if instead one considers dynamics, i.e., the situation where the dimers arrive sequentially and try to bind to the substrate [9]. It would be interesting to see how such approach would affect the efficiency here.

2.2 Cycles

Let us now consider the number of maximal m-packings in a cycle C_n of length $n \geq 3$, $n \geq m$. We denote it by $\varphi_n^{(m)}$, and the number of maximal m-packings in C_n of size k by $\varphi_{n,k}^{(m)}$.

Proposition 2.12. *The numbers* $\varphi_{n,k}^{(m)}$ *are given by*

$$\varphi_{n,k}^{(m)} = m\psi_{n-m,k-1}^{(m)} + \sum_{i=1}^{m-1} i\psi_{n-2m-1,k-2}^{(m)}$$

for $n \geq 3$, $k \geq 2$, *where* $\psi_{n,k}^{(m)}$ *count maximal* m*-packings of size* k *in* P_n*.*

Proof. Let us consider vertex v_n in C_n. If it is not covered by a copy of P_m in an m-packing, then it must be in a "hole" of size i for some $1 \leq i \leq m-1$. At each side of the hole there must be a copy of P_m. Hence the remaining $k-2$ copies of P_m must form a valid m-packing in P_{n-2m-1}, and those are counted by $\psi_{n-2m-1,k-2}^{(m)}$. As there are i holes of size i containing vertex v_n, the second term in the right-hand side of the above expression counts all of them. The first term counts the m-packings in C_n that cover v_n by a copy of P_m. □

Proposition 2.13. *The numbers* $\varphi_n^{(m)}$ *satisfy the same recurrence as the numbers* $\psi_n^{(m)}$*, i.e.,*

$$\varphi_n^{(m)} = \varphi_{n-m}^{(m)} + \cdots + \varphi_{n-2m+1}^{(m)}$$

with the initial conditions

$$\varphi_3^{(m)} = \cdots = \varphi_{m-1}^{(m)} = 1$$

and $\varphi_{m+i}^{(m)} = m + i$ *for* $0 \leq i \leq m-1$*.*

Hence, the asymptotic behavior, the expected size and the efficiency of m-packings in C_n are the same as in P_n.

3 Future developments

This manuscript presents a systematic attempt to address enumerative aspects of maximal P_m-packings in some classes of graphs with simple connectivity patterns. It continues the line of research of a recent paper concerned with maximal matchings [7]. As this is, to the best of my knowledge, the first paper of this type, it leaves unanswered many questions that arise in the course of research. In this last section we outline some of the open problems and suggest some possible directions for future research.

The most natural thing would be to count m-packings in some other families of graphs with repetitive structure that have low connectivity. Examples of such graphs are cactus chains, such as those considered in [5, 6, 7]. Due to their simple structure, it is reasonable to expect that the enumerating sequences will satisfy (rather short) linear recurrences with constant coefficients, yielding thus to the same type of asymptotic analysis as obtained here. Besides finding the asymptotics, an interesting problem would be to find the extremal chains. For maximal matchings ($m = 2$) the problem is solved for hexagonal cacti and it would be interesting to see if the pattern persists for larger values of m.

Another promising class could be the so-called thorny graphs. From a given graph G one obtains the *t-thorny graph* $T_t(G)$ by appending t pendent vertices to every vertex of G. When G has a simple structure, the methods of this paper could be employed to obtain the recurrences for the number of m-packings in $T_t(G)$. As an example, we consider 3-packings in $T_t(P_n)$.

Proposition 3.1. *Let $p_n^{(3)}$ denote the number of 3-packings in $T_t(P_n)$. Then*

$$p_n^{(3)} = \binom{t}{2} p_{n-1}^{(3)} + 2t p_{n-2}^{(3)} + p_{n-3}^{(3)}$$

for $n \geq 3$ with the initial conditions than can be verified by direct computation.

The next step could be to consider linear polymers of connectivity 2. Among them, the most interesting are without doubt the benzenoid chains. Again, there are some results for maximal matchings [6, 7] for benzenoid and polyomino chains, but for other classes of fascia- and rota-graphs [11] not even that case is investigated.

Another direction could be to consider structural and enumerative problems of m-packings in composite graphs, i.e., in graphs that arise from simpler building blocks *via* various binary operations known as graph products. We have considered here one such example of low connectivity (the thorny graph, that could be thought of as the corona product of G and $\overline{K_t}$). However, many interesting operations such as, e.g., the Cartesian product, actually increase the connectivity. It would be too optimistic to expect that complete results of the type presented here could be obtained in general cases, but we believe that the cases when one component is a path or a cycle should be feasible. Another interesting problem would be to determine the m-saturation number of such graphs, in particular for the finite portions of grids and lattices. Also, nanostructures and fullerenes are natural candidates for investigation of structural properties related to m-packings. The results would generalize those for maximal matchings [1, 4].

A graph G is *equimatchable* [10, 14] if every maximal matching in G is also maximum, i.e., if all maximal matchings are of the same size. What can be said about *equipackable* graphs in which every maximal m-packing is also maximum m-packing?

Finally, it would be interesting to see if packing polynomials and maximal packing polynomials, modelled after their matching counterparts [7, 8, 14], would be useful in the study of packing enumeration.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 89–101
https://doi.org/10.26493/1855-3974.1498.77b
(Also available at http://amc-journal.eu)

Direct product of automorphism groups of digraphs

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Received 5 October 2017, accepted 24 January 2019, published online 22 June 2019

Abstract

We study the direct product of automorphism groups of digraphs, where automorphism groups are considered as permutation groups acting on the sets of vertices. By a direct product of permutation groups $(A, V) \times (B, W)$ we mean the group $(A \times B, V \times W)$ acting on the Cartesian product of the respective sets of vertices. We show that, except for the infinite family of permutation groups $S_n \times S_n$, $n \geq 2$, and four other permutation groups, namely $D_4 \times S_2$, $D_4 \times D_4$, $S_4 \times S_2 \times S_2$, and $C_3 \times C_3$, the direct product of automorphism groups of two digraphs is itself the automorphism group of a digraph. In the course of the proof, for each set of conditions on the groups A and B that we consider, we indicate or build a specific digraph product that, when applied to the digraphs representing A and B, yields a digraph whose automorphism group is the direct product of A and B.

Keywords: Digraph, automorphism group, permutation group, direct product.

Math. Subj. Class.: 05E18, 05C20, 20B25

*Supported by the Polish National Science Centre grant No. 2012/07/B/ST1/03318.

†Sponsored by the Polish National Science Centre grant No. 2012/05/B/ST1/00626 and by the travel grant PL 08/2016 of the OEAD/DWM.ZWB.183.1.2016.

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The original problem of Kőnig [20], to describe finite abstract groups that are isomorphic to automorphism groups of simple graphs, quickly found an answer due to Frucht [5], namely, each finite group is isomorphic to the automorphism group of some simple graph. A related question, asking which permutation groups on a given set are automorphism groups of graphs on that set of vertices, proved to be much more difficult.

The simplest example of a permutation group that has no graph representation in this sense is the trivial group on two elements. Both simple graphs on two vertices admit the full permutation group S_2 as automorphisms.

In the present paper, we deal with a generalization of the original problem. We study permutation group representability on directed simple graphs (digraphs). Note that the trivial group of the above example, while having no graph representation, obviously does have a digraph representation.

There are, however, groups that have neither graph nor digraph representations. The smallest example is the Klein four group $S_2 \times S_2$ (even symmetries of a square), and that is despite the fact that both factors do have graph representations. This observation led us to study the representability of direct products of representable groups.

Our main result is Theorem 2.1 that says that, given two permutation groups (A, V) and (B, W) that have digraph representations, their direct product $(A \times B, V \times W)$ also has a digraph representation, unless $A \times B$ is one of the four exceptional groups $D_4 \times S_2$, $D_4 \times D_4$, $S_4 \times S_2 \times S_2$, $C_3 \times C_3$, or a member of the infinite family of groups $S_n \times S_n$, $n \geq 2$. It is a digraph counterpart of Theorem 2.10 of [8] by Grech for undirected graphs.

Although it might seem that this generalization should be straightforward, it turns out that we are in need, in addition to the conclusions of the aforementioned paper, of a whole collection of new techniques. The reason is that, as we have already seen in the introduction, there are plenty of permutation groups that are not the automorphism groups of a graph but are the automorphism groups of a digraph with at least one directed edge.

Research on the problem of representability of a permutation group $A = (A, V)$ as the full automorphism group of a digraph (graph) $G = (V, E)$ started with studies of regular permutation groups (see [15, 16, 18, 23, 24, 25, 29, 30], for instance). In particular, it was established that abelian groups and generalized dihedral groups have no simple graph representation. Moreover, 13 other groups with this property were found. The solution of the problem for undirected graphs was completed by Godsil [7] in 1979. He proved that with the exception of the groups mentioned above, all other regular permutation groups are automorphism groups of graphs. For digraphs, L. Babai [1] in 1980 used the result of Godsil, and proved that, except for the groups S_2^2, S_2^3, S_2^4, C_3^2 and the eight element quaternion group Q, each regular permutation group is the automorphism group of a digraph.

The fact that all digraphs and graphs can be interpreted as complete digraphs (graphs) in which the edges and non-edges are distinguished by assigning them one of two colors provides motivation for working with edge-colored digraphs (or graphs) rather than with plain digraphs (graphs). This subject was introduced by H. Wielandt in [32], where permutation groups that are automorphism groups of edge-colored digraphs were called 2-closed, and those that are automorphism groups of edge-colored graphs were referred to as 2^*-closed. In [19] Kisielewicz introduced the notion of graphical complexity of permutation groups and suggested studying products of permutation groups in this context. We

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denote by $\mathrm{DGR}(k)$ ($\mathrm{GR}(k)$) the class of automorphism groups of k-edge-colored digraphs (graphs), and by DGR (GR), the union of all classes $\mathrm{DGR}(k)$ ($\mathrm{GR}(k)$). A k-edge-colored digraph (graph) is a complete digraph (graph) with every edge colored in one of k colors. It is obvious that $\mathrm{GR}(k) \subseteq \mathrm{DGR}(k)$, for every k. Note that the class $\mathrm{DGR}(2)$ ($\mathrm{GR}(2)$) is the class of automorphism groups of digraphs (graphs).

The most general open question in this field is to find all permutation groups that belong to the class DGR. Another problem is to describe all the classes $\mathrm{DGR}(k)$. Several results on $\mathrm{DGR}(k)$ membership for basic classes of permutation groups are known, see for instance [1, 12, 34].

A closely connected topic is research on factorization of digraphs, see [3, 6, 22] and the bibliography given there. The same problem as before is considered, but from a slightly different point of view. Special attention is devoted to homogeneous factorization of complete digraphs [12, 21].

Also, various products of automorphism groups of digraphs were considered, see for instance [10, 11, 14, 28, 31]. In particular, in [10], the direct product of automorphism groups of edge-colored digraphs was studied. One of the results, worked out there, is that, for $k \geq 2$, the direct product $(A \times B, V \times W)$ of two permutation groups (A, V) and (B, W) from the class $\mathrm{DGR}(k)$ belongs to the class $\mathrm{DGR}(k+1)$.

In [9] the study of the direct product was carried on and gave an improvement of the result from [10]. It was shown that for $k \geq 3$, the direct product of two groups from $\mathrm{DGR}(k)$ is either in $\mathrm{DGR}(k)$ or is equal to S_2^3. The same holds for the case of automorphism groups of edge-colored graphs. The result of the present paper can be seen as an extension of the above result for the case $k = 2$.

1 Preliminaries

We assume that the reader has basic knowledge in the areas of graphs and permutation groups, so we omit an introduction to standard terminology. If necessary, additional details can be found in [2, 11, 33].

We recall the most important definitions. A *digraph* G is a pair (V, E), where V is the set of vertices. The set of oriented edges, E, is a subset of $V \times V \setminus \{(v, v) : v \in V\}$ (the set of ordered pairs of different elements of V). By $\overline{G}$ we denote the complement of G. A complete digraph with n vertices is denoted by K_n.

An *undirected edge* is a pair $\{v, w\}$ such that both (v, w) and (w, v) belong to E. By $d_G^1(v)$ we mean the number of undirected edges of the form $\{v, w\}, w \in V$ in a digraph G (the number of 1-neighbors of the vertex v). We define the number of non-neighbors (or 0-neighbors) of a vertex v by $d_G^0(v) = d_{\overline{G}}^1(v)$. If a digraph G is regular, then we denote these numbers $d^1(G)$ and $d^0(G)$, respectively. A *directed edge* is an edge $(v, w) \in E$ such that $(w, v) \notin E$. For every $v \in V$, by $d_G^f(v)$, we denote the number of its forward-neighbors, that is, of directed edges of the form $(v, w), w \in V$ (with $(w, v) \notin E$).

In the case when a digraph G has no directed edges, we say that G is an *undirected graph* (a graph). For a digraph G we let $\mathrm{s}(G)$ denote the undirected graph (shadow graph) that is obtained from G by replacing all directed edges by undirected ones. We will also use the notion of *weak neighbors* of a vertex v in a digraph G, that is, of vertices that are neighbors of v in $\mathrm{s}(G)$. Similarly, a digraph is said to be *weakly connected* if $\mathrm{s}(G)$ is connected.

We define two products of digraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Their

Cartesian product $G_1 \square G_2$ is a digraph $G_1 \square G_2 = (V, E)$, where $V = V_1 \times V_2$, and $((v_1, v_2), (w_1, w_2)) \in E$ if either $(v_1, w_1) \in E_1$ and $v_2 = w_2$, or $v_1 = w_1$ and $(v_2, w_2) \in E_2$. We say that a digraph is *prime* if it is not the Cartesian product of two nontrivial digraphs. It is not hard to show that Cartesian multiplication of graphs is commutative, associative, and that K_1 is a unit.

The second product $G_1 * G_2 = (V, E)$, first studied by Watkins [28], is a digraph where $V = V_1 \times V_2$ and $((v_1, v_2), (w_1, w_2)) \in E$ if and only if either $(v_1, w_1) \in E_1$ and $v_2 = w_2$, or $v_1 \neq w_1$ and $(v_2, w_2) \in E_2$.

For a digraph G with vertex set $V \times W$, the subdigraphs of G induced by sets $V \times \{w\}$ will be called *rows*, and the subdigraphs induced by sets $\{v\} \times W$ will be called *columns*. An edge that belongs neither to a row nor to a column will be called a *slant edge*. When $G = G_1 \square G_2$, for given $v \in V(G)$ and $i \in \{1, 2\}$ we will use the notation *layer* for the row or column (image of G_i) containing v and denote it G_i^v.

A permutation σ of the set V is an automorphism of a digraph $G = (V, E)$ ($\sigma \in \mathrm{Aut}(G)$) if, for $v, w \in V$, a pair $(v, w) \in E$ if and only if $(\sigma(v), \sigma(w)) \in E$. It is obvious that $\mathrm{Aut}(G)$ is a group and that $\mathrm{Aut}(G) = \mathrm{Aut}\left(\overline{G}\right)$.

All groups considered here are groups of permutations. They are considered up to permutation group isomorphism. S_n denotes the full group of permutations of an n-element set. By $C_n, n > 2$, we denote the cyclic group on n elements (i.e. the group generated by the cycle $(1, 2, \ldots, n)$). And finally, by $D_n, n > 2$, we denote the dihedral group acting on an n-element set (i.e. the group generated by $(1, 2, \ldots, n)$ and $(1, n)(2, n-1) \ldots ([n/2], n - [n/2] + 1)$).

We define two kinds of products of permutation groups. Let A and B be permutation groups acting on the sets V and W, respectively. The *direct product* $A \times B$ is the permutation group consisting of the elements $\{(a, b) : a \in A, b \in B\}$ acting on the set $V \times W$ as follows: $(a, b)((v, w)) = (a(v), b(w))$, for $v \in V, w \in W$. The group $A \times A$ is denoted A^2. A *wreath product* $A \,\mathrm{wr}\, B$ acting imprimitively on the set $V \times W$ is the permutation group consisting of the elements $\{(a, b_1, \ldots, b_n) : a \in A, b_i \in B, n = |V|\}$ acting on the set $V \times W$ as follows: $(a, b_1, \ldots, b_n)(i, w) = (a(i), b_i(w))$, where $i \in \{1, \ldots, n\} = V$, $w \in W$. (A acts on the set of columns, B acts on each column independently.)

The class of groups which are the automorphism groups of digraphs with at least one directed edge will be denoted by EDGR.

Lemma 1.1. *Let G be a digraph and $v, w, x, y \in V(G)$, such that the only edges joining any two of them are $(v, w), (y, x) \in E(G)$ and $\{w, y\}, \{v, x\} \in E(G)$. Then, for every cartesian decomposition of the digraph $G = G_1 \square G_2$, there is an $i \in \{1, 2\}$ such that all the arcs between v, w, x, y belong to G_i^v.*

Proof. Without loss of generality assume that the layer G_1^v contains w. Vertex y can now be in the layer $G_1^v = G_1^w$ or in the layer G_2^w. Assume the latter. Then, x has to be at the intersection of G_1^y and G_2^v, as there are no slant arcs in G, but then the orientations of (v, w) and (y, x) are inconsistent with the definition of the cartesian product. Hence, vertex y must be in the layer $G_1^v = G_1^w$. Since the vertex x is a weak neighbor of both y and v which are in a single layer, it also must belong to that layer, because there are no slant arcs. □

In contrast to the undirected case, where Imrich [14] found a short list of exceptional graphs for which both the graph and its complement are connected and not prime, for digraphs with at least one directed edge there are no exceptions, as the following theorem shows:

Theorem 1.2. *For every digraph G with at least one directed edge either G or $\overline{G}$ is weakly connected and prime.*

Proof. Assume the digraph G with at least one directed edge is not prime, that is $G = G_1 \Box G_2$. We have to show that $\overline{G}$ is weakly connected and prime.

Let $(v,w) = ((v_1,v_2),(w_1,w_2)) \in E(G)$ be one of the directed edges of G. Without loss of generality, assume that $(v_1,w_1) \in E(G_1)$ and $v_2 = w_2$. Since the cartesian decomposition is not trivial, there exists a vertex $v_2' \in V(G_2)$, $v_2' \neq v_2$. Then $((v_1,v_2'),(w_1,v_2'))$ is also a directed edge in $E(G)$. If between (v_1,v_2') and (v_1,v_2) there is no edge or there is a directed edge, then it is easy to see that the subdigraph of $\overline{G}$ induced by the vertices (w_1,v_2), (v_1,v_2), (w_1,v_2'), (v_1,v_2') contains edges (directed or undirected) between every pair of vertices, and therefore belongs to a single layer of $\overline{G}$. If there is an undirected edge between (v_1,v_2') and (v_1,v_2) then the same holds by Lemma 1.1. Now, all other vertices of $\overline{G}$ can be split into three categories according to their adjacence in $\overline{G}$ to the vertices (w_1,v_2), (v_1,v_2), (w_1,v_2'), (v_1,v_2'). First, those in G_1^v are neighbors of both (v_1,v_2') and (w_1,v_2'), and those in $G_1^{(v_1,v_2')}$ are neighbors of both (v_1,v_2) and (w_1,v_2). Second, those in G_2^v are neighbors of both w and (w_1,v_2') and those in G_2^w are neighbors of both v and (v_1,v_2'). Third, all other vertices are neighbors of all four vertices (w_1,v_2), (v_1,v_2), (w_1,v_2'), (v_1,v_2').

Because a vertex can be a neighbor of two vertices in one and the same layer only if it also belongs to that layer, we conclude that all vertices in $\overline{G}$ belong to a single layer, so $\overline{G}$ is prime. It is easy to see that it also is weakly connected.

Assume now that G is prime and not weakly connected. Its complement $\overline{G}$ is connected. If $\overline{G}$ were not prime, then, by the previous paragraph, $G = \overline{\overline{G}}$ would have to be weakly connected, contrary to assumption. Thus $\overline{G}$ is weakly connected and prime. □

In what follows we need a result analogous to the Sabidussi-Vizing [26, 27] theorem about the automorphism group of the Cartesian product of connected coprime graphs. To prove it, we use a result on unique prime factorization of digraphs with respect to the Cartesian product. This result can be traced back to Feigenbaum [4], but for an easy proof in a more general setting we refer to the recent paper by Imrich and Peterin [17]:

Theorem 1.3. *Every weakly connected digraph has a unique prime factor decomposition with respect to the Cartesian product.*

We can now state our two simplified versions of the Sabidussi-Vizing theorem for digraphs.

Theorem 1.4. *Let G, H be non-isomorphic weakly connected digraphs, where $|V(G)| \geq |V(H)|$ and G is prime. Then $\mathrm{Aut}(G \Box H) = \mathrm{Aut}(G) \times \mathrm{Aut}(H)$.*

Proof. It is clear that $\mathrm{Aut}(G) \times \mathrm{Aut}(H) \subset \mathrm{Aut}(G \Box H)$. We shall prove the opposite inclusion. To that end, it suffices to show that every $a \in \mathrm{Aut}(G \Box H)$ maps G-layers to G-layers and H-layers to H-layers in $G \Box H$.

We know that $\mathrm{Aut}(G \Box H) \subset \mathrm{Aut}(\mathrm{s}(G \Box H))$ and, in general, the factors of the shadow graph $s(G \Box H) = s(G) \Box s(H)$ need not be prime. Take $a \in \mathrm{Aut}(G \Box H)$. A G-layer in $G \Box H$ has the form $G \Box \{h\}$ for $h \in V(H)$. Consider $\mathrm{s}(G \Box \{h\})$, a cartesian product of subgraphs of $\mathrm{s}(G)$ and $\mathrm{s}(H)$. Using the terms defined in Chapter 6 of [13], it is a convex subgraph of the shadow graph $\mathrm{s}(G \,\square\, H)$, and so, by a corollary that leverages the convexity preserving property of automorphisms, obtained as a step in the proof of Theorem 6.8 therein (first paragraph on page 69), the image of $\mathrm{s}(G \,\square\, \{h\})$ under the automorphism a is again a cartesian product of subgraphs of $\mathrm{s}(G)$ and $\mathrm{s}(H)$, that is, $a(\mathrm{s}(G \,\square\, \{h\})) = \mathrm{s}(G_1) \,\square\, \mathrm{s}(H_1)$, where $G_1 \subset G$ and $H_1 \subset H$. But, since the vertex sets of the shadows are the same as those of the digraphs, we also have that $a(G \,\square\, \{h\}) = G_1 \,\square\, H_1$. Suppose $|V(G_1)| = 1$, that would imply that $H_1 = H$ with $|V(H)| = |V(G)|$ and that G is isomorphic to H, which is contrary to assumption. Now suppose that $1 < |V(G_1)| < |V(G)|$. This would imply that the digraph G has a nontrivial cartesian product decomposition, which is also contrary to assumption. We are, thus, left with the case $|V(G_1)| = |V(G)|$, which proves that a maps G-layers to G-layers.

Because we have no slant arcs and H is weakly conected this means that a maps H-layers into H-layers. □

Theorem 1.5. *Let G be a weakly connected, prime digraph with at least one directed edge. Let H be an undirected and connected graph. Then* $\mathrm{Aut}(G \,\square\, H) = \mathrm{Aut}(G) \times \mathrm{Aut}(H)$.

Proof. Similarly as above, we get that $a(\mathrm{s}(G \,\square\, \{h\})) = \mathrm{s}(G_1) \,\square\, \mathrm{s}(H_1)$. We do not assume that the digraph G has at least as many vertices as H, so we need to exclude the case $|V(G_1)| = 1$ differently. Here this would imply that G is a subgraph of H, but this is not possible as G has a directed edge while H does not. The conclusion follows as above. □

The following proposition is modelled on an observation made in the proof of Theorem 6 of Watkins [28]:

Proposition 1.6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be digraphs where G_2 is weakly connected. Suppose that every automorphism a of the digraph $G = G_1 * G_2$ maps rows onto rows. Then* $\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$.

Proof. Let w_1 and w_2 be weak neighbors in G_2 and let $v \in V_1$ be arbitrarily chosen. Write $a(v, w_i) = (a_1(v, w_i), a_2(v, w_i))$. Since rows are mapped onto rows, a_2 does not depend on v. Hence, $a_2 \in \mathrm{Aut}(G_2)$.

By the definition of the $*$-product, (v, w_2) is the only vertex in $G_1^{(v,w_2)}$ that is not weakly adjacent to (v, w_1). Hence $a(v, w_2) = (a_1(v, w_2), a_2(w_2))$ is the only vertex in $G_1^{a(v,w_2)}$ that is not weakly adjacent to $(a_1(v, w_1), a_2(w_1))$, so $a_1(v, w_1)$ must be equal to $a_1(v, w_2)$. By the weak connectivity of G_2 this means that a_1 only depends on v. It is easily seen that it is an automorphism of G_1. Thus, for any $(v, w) \in V(G)$ we conclude that $a(v, w) = (a_1(v), a_2(w))$, where a_1, a_2 are a automorphisms of G_1, resp. G_2. □

2 Main result

The following theorem settles the problem when the direct product of automorphism groups of digraphs is an automorphism group of a digraph.

Theorem 2.1. *Let $A, B \in \mathrm{DGR}(2)$. Then $A \times B \in \mathrm{DGR}(2)$, unless $A \times B$ is $D_4 \times S_2$, $D_4 \times D_4$, $S_4 \times S_2 \times S_2$, $C_3 \times C_3$, or one of the groups $S_n \times S_n, n \geq 2$.*

The proof is broken up into a series of lemmas. Let us note first that we are given permutation groups $A = (A, V_A)$, $B = (B, V_B)$ and graphs $G_A = (V_A, E_A)$, $G_B = (V_B, E_B)$, where $\mathrm{Aut}\,(G_A) = A$ and $\mathrm{Aut}\,(G_B) = B$. Since $\mathrm{Aut}\,(G) = \mathrm{Aut}\,(\overline{G})$ for any G we may assume without loss of generality that both G_A and G_B are weakly connected. Moreover, by Theorem 1.2 we may also assume that they are prime if they have at least one directed edge.

We begin by extending Theorem 2.10 of [8] by Grech for undirected graphs to directed graphs.

Lemma 2.2. *Let* $A, B \in \mathrm{GR}(2)$*. Then* $A \times B \in \mathrm{DGR}(2)$ *if and only if* $A \times B \in \mathrm{GR}(2)$.

Proof. By Theorem 2.10 of [8], $A \times B \in \mathrm{GR}(2)$, unless $A \times B$ is $D_4 \times S_2$, $D_4 \times D_4$, $S_4 \times S_2 \times S_2$ or $S_n \times S_n$, for $n \geq 2$. In the exceptional cases the pair (v_2, v_1) belongs to the orbit of the pair (v_1, v_2) in the natural action of the group $(A \times B, V)$ on pairs of elements of V. Thus, every digraph G such that $A \times B \subseteq \mathrm{Aut}\,(G)$ has to be an undirected graph. Hence, in all the cases, $A \times B \in \mathrm{DGR}(2)$ would imply $A \times B \in \mathrm{GR}(2)$. Consequently, in the exceptional cases, $A \times B \notin \mathrm{DGR}(2)$. □

Notice that this takes care of all exceptional groups of Theorem 2.1 that are different from $C_3 \times C_3$. The proof also shows that in what follows it suffices to consider only the cases where either A or B admits a digraph representation with at least one directed edge. We can thus assume without loss of generality that $A \in \mathrm{EDGR}$.

Lemma 2.3. *Assume that* A, B *are non-isomorphic groups, where* $A \in \mathrm{EDGR}$ *and* $B \in \mathrm{DGR}(2)$*. Then* $\mathrm{Aut}\,(G_A \,\square\, G_B) = \mathrm{Aut}\,(G_A) \times \mathrm{Aut}\,(G_B)$

Proof. As noted above, G_A and G_B can be chosen to be weakly connected, the complement being taken if necessary, with G_A being prime. Then, if $B \in \mathrm{EDGR}$ so that G_B can also be chosen to be prime, the proof follows from Theorem 1.4, and from Theorem 1.5 otherwise. □

This means that we can assume that $B \cong A$. Moreover, if we are able to find two non-isomorphic weakly connected digraphs, at least one of which is prime, with the same automorphism group A, then Theorem 1.4 also gives us a positive answer.

It therefore remains to consider the case $A \times A$, where A is the automorphism group of a weakly connected prime digraph G_A with at least one directed edge. In other words, we can assume that $A \in \mathrm{EDGR}$ and that G_A is prime.

Lemma 2.4. *Let* $A \in \mathrm{EDGR}$ *with prime* G_A*. If* A *is intransitive, then* $A \times A \in \mathrm{DGR}(2)$.

Proof. We consider two copies $G_r = (V_r, E_r)$ and $G_c = (V_c, E_c)$ of G_A and will define a digraph $G = (V_r \times V_c, E)$ such that $\mathrm{Aut}\,(G) = A \times A$. We call G_r the *row copy* and G_c the *column copy* of G_A.

Since A is intransitive, $G_A \neq K_{|V_A|}$. Let $W \subset V_c$ be one of the orbits of A in its action on G_c. The edge set E of the digraph $G = (V_r \times V_c, E)$ is then defined as the set of all pairs $((v_r, v_c), (w_r, w_c))$ satisfying one of the following conditions:

(a) $(v_c, w_c) \in E_c$ and $v_r = w_r$;

(b) $v_c = w_c$ and

- either $v_c \in W$

- or $v_c \notin W$, and $(v_r, w_r) \in E_r$.

Notice that there are no slant edges and that the subgraphs induced by the columns $\{v_r\} \times V_c$ are isomorphic to G_A, whereas the the subgraphs induced by the rows $V_r \times \{v_c\}$ are isomorphic to $K_{|V_A|}$ if $v_c \in W$, otherwise they are isomorphic to G_A.

In other words, $V_r \times W$ induces the Cartesian product $K_{|V_A|} \,\square\, \langle W \rangle$, where $\langle W \rangle$ denotes the subgraph of G_A induced by W, and $V_r \times \{V_c \setminus W\}$ induces $G_A \,\square\, \langle V_c \setminus W \rangle$.

It is easy to see that $A \times A \subseteq \mathrm{Aut}\,(G)$. We have to prove the converse. To that end it suffices to show that $\mathrm{Aut}\,(G)$ maps rows onto rows and columns onto columns.

Consider a row $V_r \times \{v_c\}$, where $v_c \in W$. The row induces a complete subgraph. Because we have no slant edges, automorphisms can only map it into rows or columns. As all rows and columns have the same number of vertices and since $G_A \neq K_{|V_A|}$, it can only be mapped onto a $V_r \times \{w_c\}$, where $w_c \in W$.

We will now prove that automorphisms of G map columns onto columns. Pick a $v_c \in W$ to single out one of the rows of W, and let (w_r, w_c) be any vertex of G. As there are no slant edges in G, the paths realizing the weak distance of (w_r, w_c) to points (v_r, v_c) in the chosen row will be built of column edges and row edges. By analogy to the reasoning behind the distance formula for the cartesian product, the column edges of any such path projected onto the column graph G_c will form a weak path from w_c to v_c in G_c, just as in a cartesian product, but the row edges can go through regular rows or through $K_{|V_A|}$ rows. When v_r equals w_r, row edges are eliminated. That means that given a vertex (w_r, w_c) there is a unique vertex in the chosen row $V_r \times \{v_c\}$, to which weak distance ρ in G is minimal, this unique vertex (w_r, v_c) is in the same column as (w_r, w_c) and is unique in the above sense for all vertices (w_r, w_c) of that column.

Consider now an automorphism $a \in \mathrm{Aut}\,(G)$. We already know that it will map the row $V_r \times \{v_c\}$ onto some other row $V_r \times \{x_c\}$. If the vertices $(x_r, x_c) = a(w_r, v_c)$ and $(y_r, y_c) = a(w_r, w_c)$ were in different columns, that is if $x_r \neq y_r$, there would be a vertex (y_r, x_c) in row x_c closest to (y_r, y_c) and different than (x_r, x_c):

$$\rho\left((x_r, x_c), (y_r, y_c)\right) > \rho\left((y_r, x_c), (y_r, y_c)\right),$$

while after having applied a^{-1} on both sides we would get

$$\rho\left((w_r, v_c), (w_r, w_c)\right) > \rho\left((w_r', v_c), (w_r, w_c)\right),$$

with $w_r' \neq w_r$ because of $x_r \neq y_r$, but that cannot be true. Hence, any automorphism maps columns onto columns, as vertices of G follow their closest vertices in the chosen row.

Since column edges are mapped by automorphisms onto column edges, row edges are mapped only to row edges, thus, the only way the image of a row can preserve its weak connectedness is for automorphisms to map entire rows onto entire rows. □

Lemma 2.5. *Let $A \in \mathrm{EDGR}$ with prime G_A. If A is transitive and $|V_A| \leq 4$, then $A \times A \in \mathrm{EDGR}$ unless $A = C_3$.*

Proof. The group A is one of C_3 and C_4. By a result of Babai [1], $C_3 \times C_3 \notin \mathrm{EDGR}$. $C_4 \times C_4 \in \mathrm{EDGR}$ by Theorem 1.4 for G_{C_4} and $\overline{G_{C_4}}$. □

Observe that this takes care of the last exceptional case of Theorem 2.1.

Lemma 2.6. *Let $A \in \mathrm{EDGR}$ with prime G_A, where A is transitive and $|V_A| > 4$. If $\overline{G_A}$ is weakly connected, then $A \times A \in \mathrm{EDGR}$.*

Proof. Denote $n = |V_A|$. Since the graph $\overline{G_A}$ is weakly connected, we only need to consider the case $G_A \cong \overline{G_A}$ (otherwise the conclusion follows from Theorem 1.4). This implies that $d^0(G_A) = d^1(G_A)$. Because G_A is not undirected, we infer that $2d^f(G_A) > 1$. Then $d^0(G_A) + d^1(G_A) + 2d^f(G_A) = n - 1$ implies $2d^1(G_A) = 2d^0(G_A) < n - 2$.

We shall now prove that the graph $G = G_r * G_c$, where $G_r = (V_r, E_r)$ and $G_c = (V_c, E_c)$ are copies of G_A, has the property $\mathrm{Aut}\,(G) = A \times A$. To this end, we will show that every undirected edge that is contained in a row is mapped, under the action of $\mathrm{Aut}\,(G)$, onto an undirected edge which is contained in a row, and that the same is true for directed edges.

Let us compare the numbers of the common 1-neighbors of the ends of an undirected edge which is contained in a row, with the same number for the ends of an undirected slant edge. Denote the ends of the edge e by (v_r, v_c) and (w_r, w_c). If e is contained in a row ($v_c = w_c$), then the common 1-neighbors of (v_r, v_c) and (w_r, w_c) are those contained in that row, together with all but two vertices in rows corresponding to 1-neighbors of $v_c = w_c$ in G_c, hence their number is equal to

$$N^1_{G_r}(v_r, w_r) + (n-2)d^1(G_c), \tag{2.1}$$

where $N^1_{G_r}(v_r, w_r)$ is the number of common 1-neighbors of the vertices v_r and w_r (in G_r). If e is a slant edge, the common 1-neighbors of (v_r, v_c) and (w_r, w_c) are the 1-neighbors contained in both rows (excluding the vertex directly in front of the other end if it also is such a 1-neighbor), together with all but two vertices in rows corresponding to common 1-neighbors of both v_c and w_c in G_c. Thus, their number is

$$(n-2)N^1_{G_c}(v_c, w_c) + 2d^1(G_r) - 2\delta, \tag{2.2}$$

where $N^1_{G_c}(v_c, w_c)$ is the number of common 1-neighbors of the vertices v_c and w_c (in G_c), and $\delta \in \{0, 1\}$.

The assumption that the numbers (2.1) and (2.2) are equal, implies

$$(n-2)(d^1(G_c) - N^1_{G_c}(v_c, w_c)) + N^1_{G_r}(v_r, w_r) - 2d^1(G_r) + 2\delta = 0.$$

Since $d^1(G_c) > N^1_{G_c}(v_c, w_c)$ and $2d^1(G_r) < n-2$, it cannot be true. Hence, an undirected edge which is contained in a row cannot be mapped onto a slant undirected edge. Since there are no undirected edges in columns of a $*$-product, the set of the undirected edges that are contained in the rows is preserved by automorphisms.

We continue with a similar calculation for directed edges. Let e be a directed edge with ends as above. If e is contained in a row, then by similar reasoning as in the undirected case, the number of common forward-neighbors of (v_r, v_c) and (w_r, w_c) equals

$$N^f_{G_r}(v_r, w_r) + (n-2)d^f(G_c), \tag{2.3}$$

where $N^f_{G_r}(v_r, w_r)$ is the number of common forward-neighbors of the vertices v_r and w_r (in G_r). If e is a slant edge, then this number is

$$(n-2)N^f_{G_c}(v_c, w_c) + d^f(G_r) - \delta, \tag{2.4}$$

where $N^f_{G_c}(v_c, w_c)$ is the number of common forward-neighbors of the vertices v_c and w_c (in G_c), and $\delta \in \{0, 1\}$.

If it were possible for an automorphism from $\mathrm{Aut}\,(G)$ to map a directed slant edge onto a directed row edge, the numbers (2.3) and (2.4) would need to be the same, which would imply

$$(n-2)(d^f(G_c) - N^f_{G_c}(v_c, w_c)) + N^f_{G_r}(v_r, w_r) - d^f(G_r) + \delta = 0. \tag{2.5}$$

If $e = ((v_r, v_c), (w_r, w_c))$ is a directed slant edge, then (v_c, w_c) is a directed edge of G_c. The equality $d^f(G_c) = N^f_{G_c}(v_c, w_c)$ would then imply that the set of forward-neighbors of each of the vertices v_c and w_c be identical, but this cannot be true, since w_c is a forward-neighbor of v_c but not of itself. Hence, $d^f(G_c) > N^f_{G_c}(v_c, w_c)$. Note that since A is transitive, every vertex has as many backward neighbours as forward neighbours. Therefore since $n > 4$, we infer $d^f(G_r) < n-2$. Thus, equation (2.5) cannot be true and the set of directed edges that are contained in rows is preserved by automorphisms also in this case. Because G_A is weakly connected, it follows by Proposition 1.6 that $\mathrm{Aut}\,(G) = A \times A$. □

Lemma 2.7. *Let* $A \in \mathrm{EDGR}$*, with prime* G_A*, be transitive. If* $\overline{G_A}$ *is disconnected, then* $A \times A \in \mathrm{EDGR}$.

Proof. We first consider the structure of $\overline{G_A}$. Because A is transitive, the subgraphs of $\overline{G_A}$ induced by the vertices belonging to common weakly connected components of $\overline{G_A}$ are isomorphic, so $V_A = W' \times W$, where the weakly connected components of $\overline{G_A}$ are grouped as columns, with column size $s = |W| = n/t$, where $t = |W'| \geq 2$ is the number of weakly connected components of $\overline{G_A}$. Thus, the group A acts on the set of columns as S_t, and on every column independently as some A_1, hence $A = S_t \,\mathrm{wr}\, A_1$. Since there are no edges between columns of $\overline{G_A}$ we infer that $(v, w) \in E(G_A)$ if v and w belong to different columns. Because A is transitive, and G_A is not undirected, we conclude that either $s \geq 4$ or $A_1 = C_3$.

In the latter case, we define $G = G_r * G_c$, where both G_r and G_c are isomorphic to G_A. Then, it is easy to see that the ends of the undirected edges in the rows have common forward-neighbors, and the ends of the undirected slant edges do not. Since the undirected edges in rows form spanning connected subgraphs of the rows, $\mathrm{Aut}\,(G)$ maps rows onto rows. By Proposition 1.6 we conclude that $\mathrm{Aut}\,(G) = A \times B$.

In the case $s \geq 4$, we define a graph $G = (V_r \times V_c, E)$ such that $((v_r, v_c), (w_r, w_c))$ is in E if either $v_c = w_c$ and $(v_r, w_r) \in E(G_r)$ or $(v_c, w_c) \in E(G_c)$, $v_r \neq w_r$, and the vertices v_r and w_r belong to the same weakly connected component in $\overline{G_r}$.

If a connected graph H has a disconnected complement, then the subgraphs of H that are induced by the vertices of the weakly connected components of $\overline{H}$ are sometimes called *Zykov components* of H. Our graph G thus consists of t copies of the $R * G_c$, where R is a Zykov-component of G_r, and the row-edges that are not in a copy of $R * G_c$. We say these row-edges are of type Q.

We wish to show that $\mathrm{Aut}\,(G) = A \times A$. It is easy to check that $A \times A \subseteq \mathrm{Aut}\,(G)$. We have to prove that the converse also holds. To this end, we count the common weak neighbors of the ends of the edges that are contained in a row. These edges have the form $\{(v_r, v_c), (w_r, w_c)\}$, where $v_c = w_c$. If v_r and w_r do not belong to the same Zykov component in G_r, then these edges are of type Q. The number of common weak neighbors of the endpoints of edges of type Q is

$$x = (t-2)s + 2d_W, \tag{2.6}$$

where d_W is the number of those weak neighbors of a vertex in G_r that belong to the same Zykov component of G_r. (The notation d_W is chosen, because all Zykov components are isomorphic to W as defined in the beginning of the proof.) For row-edges that are not of type Q the number of common weak neighbors of their endpoints is

$$y = (t-1)s + N_W(v_r, w_r) + (s-2)\left((t-1)s + d_W\right), \tag{2.7}$$

where $N_W(v_r, w_r)$ is the number of the common weak neighbors of the vertices v_r and w_r in their Zykov component. Since $x < (t-1)s + d_W$, it is obvious that $x < y$.

Moreover, the number of common weak neighbors of the ends of the slant edges of G is

$$2(d_W - \epsilon) + (s-2)N_W(v_c, w_c) + (s-2)(t-1)s$$

for some $\epsilon \in \{0,1\}$ if the endpoints of $\{v_c, w_c\} \in E(G_c)$ belong to the same Zykov component of G_c, and

$$2(d_W - \epsilon) + 2(s-2)d_W + (s-2)(t-2)s$$

for some $\epsilon \in \{0,1\}$ if the endpoints of $\{v_c, w_c\} \in E(G_c)$ belong to different Zykov components of G_c. It is easy to see that under our assumptions both numbers are strictly greater than x. Observe that the graph G has no edges that are contained in its columns.

This calculation implies that $\mathrm{Aut}\,(G)$ preserves the set of edges of type Q. Since these edges form spanning subgraphs for all graphs induced by the rows of G, every $a \in \mathrm{Aut}\,(G)$ maps rows of G onto rows. Moreover, a maps any copy of a Zykov component of G_r that is contained in a row in G onto a copy of a Zykov component of G_r that is contained in the image of that row.

To complete the proof, we have to show that each column of G is mapped onto a column. If we remove edges of type Q we are left with t identical subgraphs $R_i * G_c$ where $i = 1, \ldots, t$. As any automorphim a of G maps rows into rows, it also maps subrows of the form $R_i \times \{v_c\}$ into subrows of the same form $R_j \times \{a(v_c)\}$.

Note that by assumption R has $s \geq 4$ vertices. Thus, every $R_i * G_c$ is weakly connected. From this we infer that automorphisms of G map entire subgraphs $R_i * G_c$ onto entire subgraphs $R_j * G_c$, as in G there are no slant edges between vertices belonging to different Zykov components.

Call subrows $R_i \times \{v_c\}$ and $R_j \times \{w_c\}$ of G adjacent if there is an edge or directed edge between some vertices of them, that is, when there is an edge or a directed edge between v_c and w_c in G_c and $i = j$.

If $R_i \times \{v_c\}$ and $R_i \times \{w_c\}$ are adjacent, so are $R_j \times \{a(v_c)\}$ and $R_j \times \{a(w_c)\}$ and the non-edges between vertices of the subrows are mapped to non-edges between vertices of the images of the subrows. But the non-edges of adjacent subrows span subgraphs that are isomorphic to copies of G_c and whose vertex sets are the columns. Therefore, columns of G are mapped onto columns. □

This also completes the proof of the main theorem.

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ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 103–114
https://doi.org/10.26493/1855-3974.1740.803
(Also available at http://amc-journal.eu)

Integral regular net-balanced signed graphs with vertex degree at most four

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Received 29 June 2018, accepted 14 January 2019, published online 22 August 2019

Abstract

A signed graph is called integral if its spectrum consists entirely of integers, it is r-regular if its underlying graph is regular of degree r, and it is net-balanced if the difference between positive and negative vertex degree is a constant on the vertex set (this constant is called the net-balance and denoted ϱ). We determine all the connected integral 3-regular net-balanced signed graphs. In the next natural step, for $r = 4$, we consider only those whose net-balance is a simple eigenvalue. There, we complete the list of feasible spectra in bipartite case for $\varrho \neq 0$ and prove the non-existence for $\varrho = 0$. Certain existence conditions are established and the existence of some 4-regular (simple) graphs is confirmed. In this study we transferred some results from the theory of graph spectra; in particular, we give a counterpart to the Hoffman polynomial.

Keywords: Signed graph, switching equivalent signed graphs, adjacency matrix, net-balanced signed graph.

Math. Subj. Class.: 05C50, 05C22

1 Introduction

A *signed graph* $\dot{G}$ is obtained from a (simple) graph G by accompanying each edge e by the sign $\sigma(e) \in \{1, -1\}$ (chosen in any way for any edge). The (multiplicative) sign group $\{1, -1\}$ can also be written $\{+, -\}$. We say that G is the underlying graph of $\dot{G}$. The set of vertices of $\dot{G}$ is denoted $V(\dot{G})$. The number of vertices and the number of edges of $\dot{G}$ are denoted n and m, respectively. Clearly, every graph can be interpreted as a signed graph.

*This research is partially supported by the Serbian Ministry of Education, Science and Technological Development, Projects 174033 and 174012. The author is grateful to the anonymous referee for valuable suggestions.
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The $n \times n$ adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0,1)$-adjacency matrix of G by reversing the sign of all 1's which correspond to negative edges. The eigenvalues of $A_{\dot{G}}$ are real and form the spectrum of $\dot{G}$. A detailed introduction to spectra of signed graphs can be found in [7, 10].

A graph is integral if its spectrum consists entirely of integers. The problem of identifying such graphs was posed by Harary and Schwenk [3] in 1974. Since then, a number of results concerning integral graphs have appeared in various references, including research articles, thesis and book chapters (not listed here).

Integral signed graphs are defined in the same way. Transferring the problem to the domain of signed graphs – to the author's knowledge, no one has considered this problem for signed graphs – results in various solutions, some of them having interesting and, at first glance, interesting properties. For example, the signed graph illustrated in Figure 1 is integral and has only two eigenvalues: 2 and -2 (both with multiplicity 3). Moreover, every switching equivalent signed graph is also integral (since it has the same spectrum; see the next section for the details).

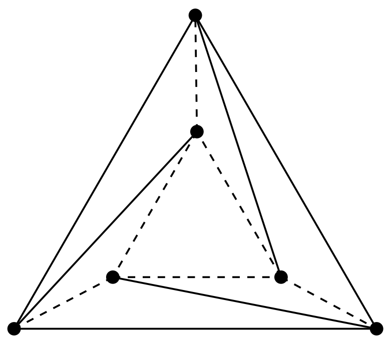

Figure 1: An integral signed graph. (Here and following, negative edges are dashed.)

Our results are announced in the abstract. In Section 2 we introduce the terminology and notation, and give some preliminary results. Sections 3 and 4 are devoted to integral signed graphs which are 3-regular and 4-regular, respectively. An existence condition and certain integral 4-regular (simple) graphs are established in Section 5.

2 Preliminaries

We write d_i^+ and d_i^- for the positive and negative vertex degree (i.e., the number of positive and negative edges incident with i). The existence of a positive (resp. negative) edge between the vertices i and j is designated by $i \overset{+}{\sim} j$ (resp. $i \overset{-}{\sim} j$).

A *walk* in a signed graph is a sequence of alternate vertices and edges such that the consecutive vertices are endpoints of the corresponding edge. A walk is *positive* if the number of its negative edges (with possible repetitions) is not odd. Otherwise, it is negative. Since every cycle in a signed graph is a walk, we may talk about positive or negative cycles, as well.

We say that a signed graph is bipartite or regular (of degree r) if the same holds for its underlying graph. (There is a different approach for regularity in [10].) The spectrum of a bipartite signed graph is symmetric with respect to the origin. The *net-balance* of a vertex

i is defined by $d_i^+ - d_i^-$. We also say that a signed graph is *net-balanced* if the net-balance is a constant on the vertex set; in that case the net-balance is denoted ϱ.

For $U \subset V(\dot{G})$, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of each edge between a vertex in U and a vertex in $V(\dot{G}) \setminus U$. The signed graph $\dot{G}^U$ is said to be *switching equivalent* to $\dot{G}$. Switching equivalent signed graphs share the same spectrum.

The *reverse* $\mathrm{rev}(\dot{G})$ of $\dot{G}$ is obtained by reversing the sign of all edges of $\dot{G}$.

We use the following facts without proofs (for details, see [8, 10]):

- The eigenvalues of $\mathrm{rev}(\dot{G})$ (with repetitions) are obtained by reversing the sign of the eigenvalues of $\dot{G}$.
- Signed graphs $\dot{G}$ and $\mathrm{rev}(\dot{G})$ are switching equivalent if and only if $\dot{G}$ is bipartite.
- A signed graph $\dot{G}$ is switching equivalent to G (the underlying graph) if and only if the vertices of $\dot{G}$ can be divided into two sets (one of them possibly empty) in such a way that an edge is negative if and only if it joins vertices from different sets.
- The spectrum of every net-balanced signed graph contains its net-balance.

It is known that the largest eigenvalue ρ of a signed graph does not exceed the largest eigenvalue of its underlying graph. The proof follows by the next chain of (in)equalities derived on the basis of the Rayleigh principle:

$$\rho(\dot{G}) = 2\left(\sum_{i\overset{+}{\sim}j} x_ix_j - \sum_{i\overset{-}{\sim}j} x_ix_j\right) \leq 2\left(\sum_{i\overset{+}{\sim}j} |x_ix_j| + \sum_{i\overset{-}{\sim}j} |x_ix_j|\right) \leq \rho(G), \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is a unit eigenvector associated with $\rho(\dot{G})$. What we need here is the opposite implication.

Lemma 2.1. *For a connected signed graph $\dot{G}$, if $\rho(\dot{G}) = \rho(G)$ then $\dot{G}$ and G are switching equivalent.*

Proof. Since $\rho(\dot{G}) = \rho(G)$, all the inequalities in (2.1) reduce to equalities. This, in particular, means that $|\mathbf{x}|$ is an eigenvector associated with $\rho(G)$, and so it holds $x_i \neq 0$, for all i. Applying switching with respect to the set of vertices that correspond to negative coordinates of $\mathbf{x}$, we arrive at the signed graph, say $\dot{H}$, such that $|\mathbf{x}|$ is associated with $\rho(\dot{H})$, as well. (The matrix transformation is realized by $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$, D being the diagonal matrix of ± 1 where the sign of a diagonal entry is determined by the sign of the corresponding coordinate of $\mathbf{x}$. Then, $|\mathbf{x}| = D\mathbf{x}$.)

Since $\rho(\dot{H}) = \rho(G)$, it follows (by (2.1), with $\dot{H}$ and $|\mathbf{x}|$ in the roles of $\dot{G}$ and $\mathbf{x}$) that $\dot{H}$ does not contain negative edges, i.e., $\dot{H}$ is a graph isomorphic to G, and we are done. □

Connected integral regular net-balanced signed graphs of vertex degree 0, 1 or 2 are easily determined. In what follows, we move up to $r = 3$ and $r = 4$.

Obviously, if $\dot{G}$ is an integral net-balanced signed graph with $\varrho \geq 0$, then the reverse $\mathrm{rev}(\dot{G})$ is also integral net-balanced with $\varrho \leq 0$, and vice versa. Thus, it is sufficient to consider only those with $\varrho \geq 0$.

3 Case $r = 3$

Recall that connected 3-regular graphs are determined in [2, 4]; there are 13 such graphs. In what follows, we refer to the notation of the latter reference.

By Lemma 2.1, if an r-regular signed graph $\dot{G}$ has r as an eigenvalue, then $\dot{G}$ is switching equivalent to its underlying graph. If r does not belong to the spectrum of $\dot{G}$, but $-r$ does, then $\mathrm{rev}(\dot{G})$ is switching equivalent to its underlying graph. This observation leads to the following result.

Theorem 3.1. *If a connected 3-regular net-balanced signed graph $\dot{G}$ of non-negative net-balance ϱ is integral and at least one of the numbers 3 or -3 is its eigenvalue, then $\dot{G}$ is determined in the following way.*

(a) If 3 is an eigenvalue of $\dot{G}$, then

- *(i) for $\varrho = 3$, $\dot{G}$ is one of the 13 connected 3-regular graphs $G_1, \dots, G_{13}$ obtained in [4];*
- *(ii) for $\varrho = 1$, $\dot{G}$ is switching equivalent to one of the graphs $G_2, G_4, G_5, G_7, G_{10}$ or G_{12} of [4].*

(b) If 3 is not an eigenvalue of $\dot{G}$, then

- *(i) case $\varrho = 3$ cannot occur;*
- *(ii) for $\varrho = 1$, $\dot{G}$ is switching equivalent to G_9 or G_{13} of [4].*

Proof. (a): By Lemma 2.1, $\dot{G}$ is switching equivalent to its underlying graph, and then (a.i) follows directly by the result of the corresponding reference.

(a.ii): Here, $\dot{G}$ is obtained by identifying a perfect matching in one of $G_1, \dots, G_{13}$ and reversing the sign of all edges in the matching. In addition, this reversing must produce a switching equivalent graph. In particular, this means that the vertex set can be partitioned into two sets such that an edge is negative if and only if it joins vertices from different sets. Inspecting all 13 graphs, we conclude that such an action can be performed for the 6 graphs listed. (The procedure is simplified by excluding the signed graphs which do not have the net-balance as an eigenvalue.)

(b): Case (b.i) follows directly.

(b.ii): Here, $\dot{G}$ is non-bipartite and 3 is an eigenvalue of $\mathrm{rev}(\dot{G})$. Considering $G_9, \dots, G_{13}$ as candidates for a graph which is switching equivalent to $\mathrm{rev}(\dot{G})$, we arrive at the 2 solutions: G_9 and G_{13}. □

A transfer of a result from the domain of simple graphs is needed. For signed graphs $\dot{G}_1$ and $\dot{G}_2$, the (tensor) product $\dot{G}_1 \times \dot{G}_2$ is the signed graph with the vertex set $V(\dot{G}_1) \times V(\dot{G}_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if u_i and v_i are adjacent in $\dot{G}_i$, for $1 \leq i \leq 2$. The sign of an edge of $\dot{G}_1 \times \dot{G}_2$ is equal to the product of signs of the corresponding edges of $\dot{G}_1$ and $\dot{G}_2$. The adjacency matrix of $\dot{G}_1 \times \dot{G}_2$ is then identified with the Kronecker product $A_{\dot{G}_1} \otimes A_{\dot{G}_2}$. Accordingly, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $\dot{G}_1$ and $\mu_1, \mu_2, \dots, \mu_m$ are the eigenvalues of $\dot{G}_2$, then the eigenvalues of their product are $\lambda_i \mu_j$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$. In particular, if $\dot{G}$ is a connected integral non-bipartite signed graph, then $\dot{G} \times K_2$ is a connected integral bipartite signed graph, since the eigenvalues of K_2 are 1 and -1. The signed graph $\dot{G} \times K_2$ is called a *bipartite double* (of $\dot{G}$).

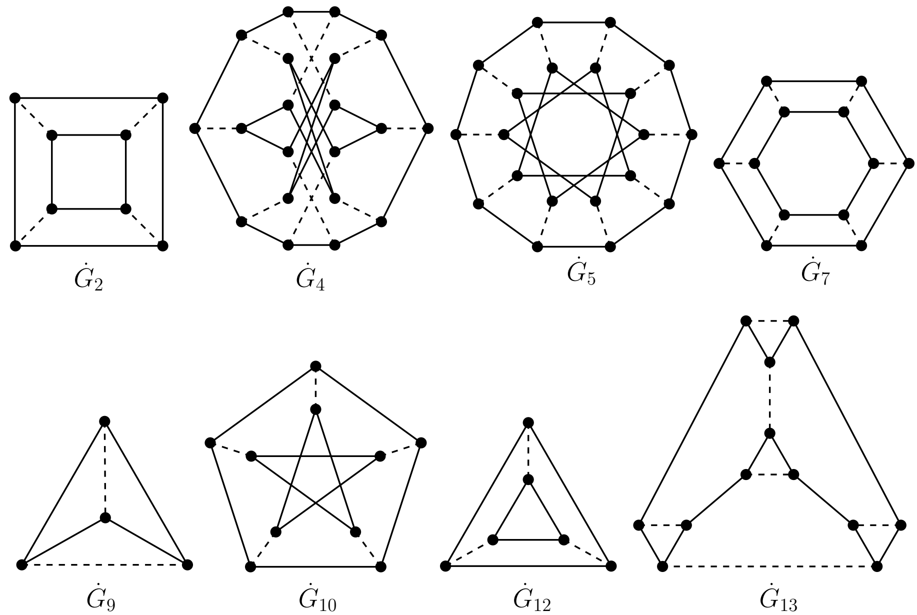

Figure 2: Representatives of signed graphs of Theorem 3.1(a.ii)&(b.ii).

Therefore, any non-bipartite signed graph $\dot{G}$ can be extracted from the decompositions of bipartite ones having the form $\dot{G} \times K_2$, and so the essential part in determining connected integral signed graphs consists of searching for those that are bipartite. If, in addition, a signed graph is regular then it has the same number of vertices in each colour class, and we may assume that the number of its vertices is $2n$.

Returning to connected integral 3-regular net-balanced signed graphs, it remains to determine those that avoid ± 3 in the spectrum. According to the previous observation, we may consider the bipartite ones, so those with $2n$ vertices and the spectrum

$$\left[2^{m_2}, 1^{m1}, 0^{2m_0}, (-1)^{m_1}, (-2)^{m_2}\right],$$

where the exponents stand for the multiplicities.

Counting the difference between the numbers of positive and negative closed walks and considering the spectral moments, we get $m_2 + m_1 + m_0 = n, 4m_2 + m_1 = 3n$ and $16m_2 + m_1 = 15n + 4q$, where q is the difference between the numbers of positive and negative quadrangles contained.

At this point, one could continue by the spectral moments of higher order to obtain the feasible spectra, but this situation can easily be resolved in a different way. Solving the previous system, we get $m_1 = -(n + \frac{4}{3}q)$ (and certain parametrizations of other multiplicities which are not important). The last equality implies that q must be negative, that is our signed graph, say $\dot{G}$, must contain a negative quadrangle. If every vertex is at distance at most 1 from such a quadrangle, then $\dot{G}$ has at most 8 vertices. Otherwise, if there exists a vertex at distance 2 from a fixed negative quadrangle, then the vertex between them is adjacent to only one vertex of the quadrangle (otherwise, the largest eigenvalue of that signed subgraph would be greater than 2, and then the same would hold for $\dot{G}$ as follows by the interlacing argument). So, $\dot{G}$ contains a negative quadrangle with a pendant path of length 2. Considering the possible neighbourhoods of the vertices of such a subgraph and bearing in mind the other conditions (3-regularity and net-balancedness), we conclude by hand that the largest eigenvalue of $\dot{G}$ must be greater than 2.

Therefore, it remains to consider connected signed graphs with at most 8 vertices. This is easily performed by reversing the signs of edges of connected bipartite 3-regular graphs with 4, 6 and 8 vertices (in each case, there is only one such graph). Obviously, the net-balance cannot be equal to 3, and since $\dot{G}$ is bipartite, it cannot be equal to -3, either. The result is summarized in the next theorem.

Theorem 3.2. *If $\dot{G}$ is a connected integral bipartite 3-regular net-balanced signed graph avoiding ± 3 in the spectrum, then $\dot{G}$ is the signed graph illustrated in Figure 3 or its reverse.*

Clearly, signed graphs obtained in the previous theorem are switching equivalent and none of them is a bipartite double.

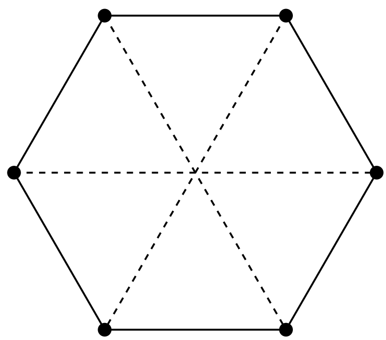

Figure 3: Integral signed graph from Theorem 3.2.

4 Case $r = 4$

The next natural case is made up of connected integral 4-regular net-balanced signed graphs. We first recall that connected integral 4-regular graphs (so, signed graphs with net-balance equal to 4) are not fully determined. What we do know are the feasible spectra of such bipartite graphs, and [9] lists 828 such spectra (in the future this number could be decreased). The existence of the corresponding graphs is confirmed for a small number of those spectra – by the same reference, in 19 cases; in the next section we confirm 2 more.

Accordingly, it would be illusory to expect all the integral bipartite 4-regular signed graphs to be identified, even if we impose that they are net-balanced. Considering the feasible spectra, if ± 4 is an eigenvalue, then they are the same as those listed in [9]. For an integral 4-regular graph, the corresponding signed graphs can be obtained as in Theorem 3.1. This will not be performed here; instead, we determine the feasible spectra of our signed graphs that avoid ± 4 in the spectrum, but with the additional condition that they contain the net-balance as a simple eigenvalue. It occurs that there is a comparatively small number of such spectra. Before that, we prove a 'signed' variant of the result concerning the Hoffman polynomial of a graph [6, Theorem 2.1.6].

Theorem 4.1. *For a signed graph $\dot{G}$, if there exists a polynomial P such that $P(A_{\dot{G}}) = J$ (J being an all-1 matrix), then $\dot{G}$ is a connected net-balanced signed graph.*

Conversely, if in addition the net-balance ϱ is a simple eigenvalue of $\dot{G}$, then the polynomial exists and has the form

$$P(x) = n \prod \frac{x - \lambda}{\varrho - \lambda}, \tag{4.1}$$

where n is the number of vertices and the product goes over all distinct eigenvalues, excluding ϱ.

Proof. The proof is similar to that of the original result, along with certain adaptations tailored for signed graphs. For the sake of completeness, we give all the details. Denote $A = A_{\dot{G}}$. The existence of P implies the identity $AJ = JA$. Since the (i, j)-entry of AJ is $d_i^+ - d_i^-$ and the same entry of JA is $d_j^+ - d_j^-$, we have that $d_i^+ - d_i^-$ is a constant on the vertex set.

Clearly, $\dot{G}$ cannot be disconnected, since for i and j belonging to different components the (i, j)-entry of any power of A would be zero, giving $P(A) \neq J$.

Denote $W(x) = \frac{\mu(x)}{x-\varrho}$, where μ stands for the minimal polynomial of A. As $\mu(A) = O$, it follows $(A - \varrho I)W(A) = O$, giving $AW(A) = \varrho W(A)$. Now, the unit vector $\mathbf{j}$ is an eigenvector associated with ϱ, but since this is the simple eigenvalue, the dimension of its eigenspace is 1, and by the last identity, every column of $W(A)$ is a multiple of $\mathbf{j}$. Moreover, the symmetry of $W(A)$ implies

$$W(A) = cJ, \tag{4.2}$$

for some $c \neq 0$. Thus, the polynomial exists and, so far, has the form $P(x) = \frac{1}{c}W(x)$.

Further, the identity $a_k A^k \mathbf{j} = a_k \varrho^k \mathbf{j}$ (for a real a_k) yields $W(A)\mathbf{j} = W(\varrho)\mathbf{j}$, which together with (4.2) gives $cJ\mathbf{j} = W(\varrho)\mathbf{j}$, i.e., $nc = W(\varrho)$, which finally gives (4.1). □

This result covers net-balanced signed graphs which need not be regular. The converse statement requires the multiplicity of ϱ to be 1, which will be an essential condition in our further considerations. Another consequence of (4.1), when $\dot{G}$ is additionally integral, is that n divides the product $\prod(\varrho - \lambda)$; this condition was exploited in many searches for integral regular graphs.

Let

$$\left[3^{m_3}, 2^{m_2}, 1^{m_1}, 0^{2m_0}, (-1)^{m_1}, (-2)^{m_2}, (-3)^{m_3}\right]$$

denote the spectrum of our connected bipartite signed graph that avoids the eigenvalues 4 and -4.

Theorem 4.2. *If $\dot{G}$ is a connected integral bipartite 4-regular signed graph with $2n$ vertices, then the multiplicities of its eigenvalues are parametrized by*

$$m_3 = \frac{1}{180}(54n + 26q + 3h), \qquad m_2 = \frac{1}{30}(6n - 16q - 3h),$$
$$m_1 = \frac{1}{4}(2n + h) + \frac{5}{6}q, \qquad m_0 = -\frac{1}{18}(8q + 3h),$$

where q and h respectively denote the difference between the numbers of positive and negative quadrangles and hexagons contained.

Proof. Considering the spectral moments up to the 6th order and counting the differences between the numbers of positive and negative closed walks, we arrive at the following Diophantine system:

$$\begin{aligned}\sum_{i=-3}^{3} m_{|i|} i^0 &= 2n,\\ \sum_{i=-3}^{3} m_{|i|} i^2 &= 8n,\\ \sum_{i=-3}^{3} m_{|i|} i^4 &= 56n + 8q,\\ \sum_{i=-3}^{3} m_{|i|} i^6 &= 464n + 144q + 12h.\end{aligned} \tag{4.3}$$

where, say for the 3rd equation, the first term on the right-hand side counts closed walks of length 4 traversing along at most 2 distinct edges. Observe that such walks are positive independently of the edge signature. Similarly, the second term counts the same walks traversing along quadrangles.

Solving this system for $m_3, \ldots, m_0$, we get the result. □

Table 1: Feasible parameters of signed graphs described in Theorem 4.3.

n	m	m_3	m_2	m_1	m_0	q	h
6	24	2	1	2	1	3	−14
10	40	4	1	0	5	15	−70
15	60	6	1	2	6	21	−92
20	80	8	1	4	7	27	−114
30	120	11	1	17	1	21	−62
30	120	12	1	8	9	39	−158
60	240	23	1	29	7	57	−194
60	240	24	1	20	15	75	−290
60	240	25	1	11	23	93	−386
60	240	26	1	2	31	111	−482

Include now the announced condition.

Theorem 4.3. *A connected integral bipartite* 4*-regular net-balanced signed graph whose net-balance is* 2 *and appears as a simple eigenvalue has one of the spectra shown in Table* 1*. Each row contains one half of the number of vertices, the number of edges, the multiplicities of positive eigenvalues, one half of the multiplicity of* 0 *and previously defined parameters* q *and* h*.*

Proof. Solving the system (4.3) for $m_2 = 1$, we get

$$m_3 = \frac{1}{18}(6n + q - 3), \qquad m_1 = \frac{1}{2}(2n - q - 5),$$
$$m_0 = \frac{1}{9}(-3n + 4q + 15) \quad \text{and} \qquad h = 2n - \frac{16}{3}q - 10.$$

Now, since the net-balance is a simple eigenvalue, it follows that $2n$ divides

$$\prod_{i \neq 2,\ i=-3}^{3} (2-i) = 120,$$

giving a finite number of possibilities for n. Observing the expressions of m_1 and m_0, we conclude that q satisfies

$$\frac{3}{4}(n-5) \leq q \leq 2n-5,$$

giving a finite range for this parameter. Concerning the possible values for n and using the previous inequalities, we arrive at solutions listed in the table. □

The existence of the signed graph with data as in the first row is confirmed by hand. Namely, we have considered connected bipartite 4-regular graphs with 12 vertices (there are 4) and arguing as in the proof of Theorem 3.1(a.ii). The resulting solution is illustrated in Figure 4.

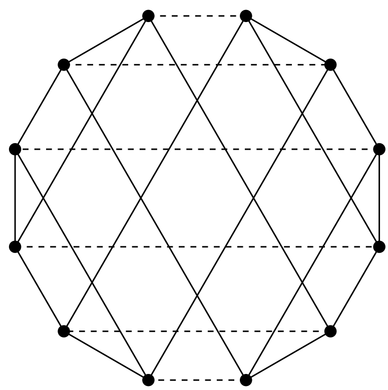

Figure 4: Integral signed graph with spectrum $[3^2, 2, 1^2, 0^2, (-1)^2, -2, (-3)^2]$.

Remark 4.4. What about non-bipartite signed graphs? Let $\dot{G}$ be such a signed graph (with all the remaining conditions as in Theorem 4.3). Although the multiplicity of 2 in the spectrum of $\dot{G}$ is 1, the same eigenvalue does not need to be simple in $\dot{G} \times K_2$. Therefore, the feasible parameters of the bipartite double are given by Theorem 4.2. The number of vertices (of $\dot{G} \times K_2$) now divides 240, and the numbers of quadrangles and hexagons are bounded in terms of n (see [1]) giving the magnitudes for our parameters q and h. Therefore, the sets of feasible spectra can be obtained, but there are many, and we skip their presentation. Alternatively, one may consider the spectra of $\dot{G}$ directly, and include the odd spectral moments, but again, there are many.

Set now $\varrho = 0$.

Theorem 4.5. *There is no connected integral* 4*-regular net-balanced signed graph avoiding* ± 4 *in the spectrum whose net-balance is* 0 *and appears as a simple eigenvalue.*

Proof. First, such a signed graph cannot be bipartite (see the multiplicity of 0). Let $\dot{G}$ be a non-bipartite signed graph described in this theorem (if any). Then the number of its vertices (denoted n) divides

$$\prod_{i\neq 0,\ i=-3}^{3}(0-i)=36,$$

i.e., the number of vertices of its bipartite double divides 72. In addition, 0 is an eigenvalue of $\dot{G}\times K_2$ of multiplicity 2. Solving the system (4.3) for $m_0 = 2$ and every possible n, we arrive at exactly 9 solutions: one for $n = 6$, one for $n = 9$, one for $n = 12$, 2 for $n = 18$ and 4 for $n = 36$.

There is another condition to be included: the parameter q of $\dot{G}\times K_2$ cannot be odd. Indeed, by definition of the product, every quadrangle of $\dot{G}$ produces its two copies in $\dot{G}\times K_2$, and vice versa, every quadrangle of $\dot{G}\times K_2$ has a copy and both of them arise from the isomorphic quadrangle of $\dot{G}$. Therefore, the parameter q of $\dot{G}\times K_2$ is twice the same parameter of $\dot{G}$, i.e., it cannot be odd.

Now, only the solution for $n = 9$ passes this test for q (with $q = -6$), which means that $\dot{G}$ could only have 9 vertices. Since for $\dot{G}$, $q = -3$, the number of quadrangles in the underlying graph G must be odd, and this is not satisfied in the case of 6 (out of 16) connected 4-regular graphs with 9 vertices. For the remaining 10, one may choose between a computation by hand (which reduces to searching for cyclic decompositions and checking the spectra) or brute force performed by computer. In any case, there are no solutions (note that some of those underlying graphs produce signed graphs satisfying all but the last condition of the theorem – regarding the simplicity of 0). □

Remark 4.6. In this section we restricted ourselves to net-balanced signed graphs whose net-balance appears as a simple eigenvalue. Of course, there are examples of those that are connected integral 4-regular and net-balanced, yet the net-balance is not a simple eigenvalue; at the end of the proof of Theorem 4.5, we mentioned that we met some of them. Another example is a net-balanced 4-dimensional cube with negative quadrangles. Indeed, its adjacency matrix can be written as

$$\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix},$$

where A is the adjacency matrix of the cycle C_8 and A^* is obtained from A by reversing the sign of all the entries corresponding to 4 non-adjacent edges. We have $\varrho = 2$, and the spectrum is given by $[2^8, (-2)^8]$.

5 An existence condition for bipartite regular graphs

Following our idea of [5], we establish an existence condition for bipartite r-regular graphs with $q = 0$. Namely, the adjacency matrix of such a graph can be written in the form

$$A_G = \begin{pmatrix} O & B^T \\ B & O \end{pmatrix},$$

and then the top-left block $B^TB - rI_n$ of $A_G^2 - rI_{2n}$ represents the adjacency matrix of an $r(r-1)$-regular graph, say H. Moreover, if $r, \lambda_2, \dots, \lambda_k$ are (distinct) non-negative eigenvalues of G, then the eigenvalues of H belong to $\{r^2 - r, \lambda_2^2 - r, \dots, \lambda_k^2 - r\}$.

Therefore, in the search for G, we may check whether H exists or not, and if it does not then G does not exist, either. Conversely, if H exists then G can be reconstructed from it.

Accordingly, we confirm the existence of bipartite 4-regular graphs with 2 spectra listed in [9].

Theorem 5.1. *There exists a connected bipartite* 4*-regular graph with data*

$$(15, 0, 10, 4, 0, 0, 210) \quad \textit{and} \quad (16, 0, 12, 0, 3, 0, 192)$$

(both given in the form $(n, m_3, m_2, m_1,\ m_0, q, h)$*, where the parameters are defined in Theorem* 4.3*).*

Proof. Considering the first data for our graph, say G, we arrive at a putative 12-regular graph H with 15 vertices and whose eigenvalues belong to $\{12, 0, -3\}$. Thus, H is strongly regular (see [6, Theorem 3.4.7]). Moreover, since 0 is one of the eigenvalues, it must be complete multipartite [6, Theorem 3.4.9]. This graph is unique, and we can use it to construct G by obeying the following rules:

- the vertices of H correspond to the vertices of one colour class of G;
- the vertices from the same colour class of H do not have common neighbours in G;
- every two vertices from different colour classes of H have exactly one common neighbour in G.

Finally, G is the incidence graph of a block design with points $1, 2, \dots, 15$ arranged into the following 15 blocks:

4 7 10 13	1 8 12 13	1 5 10 15	1 6 7 14	1 4 9 11
5 8 11 14	2 9 10 14	2 6 11 13	2 4 8 15	2 5 7 12
6 9 12 15	3 7 11 15	3 4 12 14	3 5 9 13	3 6 8 10

The second data is considered in the same way. Now, H has 16 vertices, its eigenvalues belong to $\{12, 0, -4\}$, and thus H is again a unique complete multipartite graph. Using the same method, we arrive at G determined by the following blocking:

1 5 9 13	1 2 7 8	1 3 10 12	1 4 14 15
2 6 10 14	3 4 5 6	2 4 9 11	2 3 13 16
3 7 11 15	9 10 15 16	5 7 14 16	5 8 10 11
4 8 12 16	11 12 13 14	6 8 13 15	6 7 9 12

The proof is complete. □

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 115–124
https://doi.org/10.26493/1855-3974.1528.d41
(Also available at http://amc-journal.eu)

A family of multigraphs with large palette index

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Received 13 November 2017, accepted 1 April 2019, published online 25 August 2019

Abstract

Given a proper edge-coloring of a loopless multigraph, the palette of a vertex is defined as the set of colors of the edges which are incident with it. The palette index of a multigraph is defined as the minimum number of distinct palettes occurring among the vertices, taken over all proper edge-colorings of the multigraph itself. In this framework, the palette pseudograph of an edge-colored multigraph is defined in this paper and some of its properties are investigated. We show that these properties can be applied in a natural way in order to produce the first known family of multigraphs whose palette index is expressed in terms of the maximum degree by a quadratic polynomial. We also attempt an analysis of our result in connection with some related questions.

Keywords: Palette index, edge-coloring, interval edge-coloring.

Math. Subj. Class.: 05C15

1 Introduction

Generally speaking, as soon as a chromatic parameter for graphs is introduced, the first piece of information that is retrieved is whether some universal meaningful upper or lower bound holds for it. This circumstance is probably best exemplified by mentioning, say, Brooks' theorem for the chromatic number and Vizing's theorem for the chromatic index. In either instance the maximum degree Δ is involved and that probably explains the trend

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to consider Δ as a somewhat natural parameter, in terms of which bounds for other chromatic parameters should be expressed. In the current paper we make no exception to this trend and use the maximum degree Δ as a reference value for the recently introduced chromatic parameter known as the *palette index*. To this purpose we introduce an additional tool, that we call the *palette pseudograph*, which can be defined from a given multigraph with a proper edge-coloring. Some properties of the palette pseudograph are investigated in Section 2 and we feel they might be of interest in their own right. In the current context, we use these properties in connection with an attempt of finding a polynomial upper bound in terms of Δ for the palette index of a multigraph with maximum degree Δ. As a consequence of our main construction in Section 3, we can assert that if such a polynomial bound exists at all then it must be at least quadratic.

Throughout the paper, following a standard terminology (see for instance [7]), we use the term *multigraph* to denote an undirected graph with multiple edges but no loops, while we use the term *pseudograph* for a graph admitting both multiple edges and loops. For any given multigraph G, we always denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. We further denote by G_s the underlying graph of G, that is the simple graph obtained from G by shrinking to a single edge any set of multiple edges joining two given vertices.

By a *coloring* of a multigraph G we always mean a proper edge-coloring of G. A coloring of G is thus a mapping $c\colon E(G) \to C$, where C is a finite set whose elements are designated as colors, with the property that adjacent edges always receive distinct colors. We shall often say that (G, c) is a colored multigraph, meaning that c is a coloring of the multigraph G.

Given a colored multigraph (G, c), the *palette* $P_c(x)$ of a vertex x of G is the set of colors that c assigns to the edges which are incident with x.

The palette index $\check{s}(G)$ of a simple graph G is defined in [9] as the minimum number of distinct palettes occurring among the vertices, taken over all proper edge-colorings of the graph G. The definition can be extended verbatim to multigraphs. The exact value of the palette index is known for some classes of simple graphs.

- A graph has palette index 1 if and only if it is a class 1 regular graph [9, Proposition 1].
- A connected class 2 cubic graph has palette index 3 or 4 according as it does or it does not possess a perfect matching, respectively [9, Theorem 9].
- If n is odd, $n \geq 3$ then $\check{s}(K_n)$ is 3 or 4 depending on $n \equiv 3$ or $1 \pmod 4$, respectively [9, Theorem 4].
- The palette index of complete bipartite graphs was determined in [8] in many instances.

The quoted result for complete graphs shows that it is possible to find a family of graphs, for which the maximum degree can become arbitrarily large, and yet the palette index admits a constant upper bound, namely 4 in this case.

As it was remarked in [4], the fact that a class 2 regular graph of degree Δ always admits a $(\Delta+1)$-coloring forces $\Delta+1$ to be an upper bound for the palette index of such a graph (namely, $\Delta+1$ is the number of Δ-subsets of a $(\Delta+1)$-set of colors).

That is definitely not the case for non-regular graphs: it was shown in [3] that for each positive integer Δ there exists a tree with maximum degree Δ whose palette index grows asymptotically as $\Delta \ln(\Delta)$.

Consequently, one cannot expect for the palette index any analogue of, say, Vizing's theorem for the chromatic index: the palette index of graphs of maximum degree Δ cannot admit a linear polynomial in Δ as a universal upper bound.

It is the main purpose of the present paper to produce an infinite family of multigraphs, whose palette index grows asymptotically as Δ^2, see Section 3. Our method relies essentially on a tool that we define in Section 2, namely the palette pseudograph of a colored multigraph. This concept is strictly related to the notion of palette index and it appears to yield a somewhat natural approach to the study of this chromatic parameter.

2 The palette pseudograph of a colored multigraph

For any given finite set X and positive integer t we denote by $t \cdot X$ the multiset in which each element of X is repeated t times.

The next definition will play a crucial role for our construction in Section 3. Given a colored multigraph (G, c), we define its *palette pseudograph* $\Gamma_c(G)$ as follows.

The vertex-set of $\Gamma_c(G)$ is $V(\Gamma_c(G)) = \{P_c(v) : v \in V(G)\}$. In other words the vertices of $\Gamma_c(G)$ are all pairwise distinct palettes of (G, c).

For any given pair of adjacent vertices x and y of G, we declare the (not necessarily distinct) palettes $P_c(x)$ and $P_c(y)$ to be adjacent and define the corresponding edge in the palette pseudograph $\Gamma_c(G)$.

More precisely, if x and y are adjacent vertices in G such that their palettes $P_c(x)$ and $P_c(y)$ are distinct, then $P_c(x)$ and $P_c(y)$ yield two distinct vertices connected by an ordinary edge in the palette pseudograph $\Gamma_c(G)$, see vertices x_1 and x_2 in Figure 1. If, instead, x and y are adjacent vertices in G with equal palettes $P_c(x)$ and $P_c(y)$, these form a single vertex with a loop in the palette pseudograph $\Gamma_c(G)$, see vertices x_2 and x_3 in Figure 1.

If two (equal or unequal) palettes appear on several pairs of adjacent vertices of G, then each such pair yields one edge in $\Gamma_c(G)$ (either a loop or an ordinary edge). It is thus quite possible that the palette pseudograph $\Gamma_c(G)$ presents multiple (ordinary) edges between two given distinct vertices as well as multiple loops at a given vertex.

An example of a pair (G, c) and the corresponding palette pseudograph $\Gamma_c(G)$ is presented in Figure 1.

The number of vertices of the palette pseudograph $\Gamma_c(G)$ is thus equal to the number of distinct palettes in the colored multigraph (G, c), while the number of edges (loops and ordinary edges) in $\Gamma_c(G)$ is equal to the number of edges in the underlying simple graph G_s.

The following proposition is also an easy consequence of the definition of the palette pseudograph: note that each loop in $\Gamma_c(G)$ contributes 2 to the degree of its vertex.

Proposition 2.1. *For any given colored multigraph (G, c), the degree of a vertex $P_c(x)$ in the palette pseudograph $\Gamma_c(G)$ is equal to the sum of the degrees in the underlying simple graph G_s of all vertices whose palettes in (G, c) are equal to $P_c(x)$.*

3 The main construction

The main purpose of this Section is the construction of a multigraph G^{Δ} with maximum degree Δ, whose palette index is expressed by a quadratic polynomial in Δ.

For the sake of brevity we shall assume Δ even, $\Delta \geq 2$: a slight modification of our

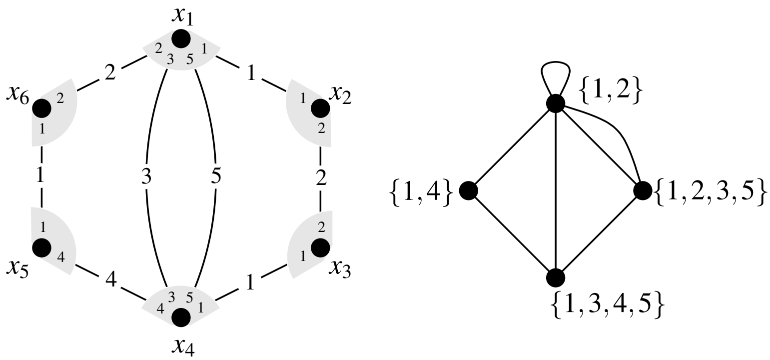

Figure 1: A multigraph G with a proper edge-colouring and the associated palette pseudograph.

construction yields the same result for odd values of Δ. Even though our graph G^Δ is not connected, a connected example can easily be obtained from G^Δ as follows. Introduce a new vertex ∞ which is declared to be adjacent to each vertex of degree Δ in G^Δ. The resulting multigraph $\widetilde{G}^\Delta$ is connected with maximum degree $\Delta + 1$. The palette index of the multigraphs $\widetilde{G}^\Delta$ is again bounded from below by a quadratic polynomial in Δ. We feel appropriate at this stage to stress a peculiar property of the palette index, in comparison with other chromatic parameters: it is not true in general that the palette index of a multigraph is equal to the maximum of the palette indices of its connected components (see Proposition 3 in [4]). This says that there is no particular reason to prefer connected examples to disconnected ones in this context.

The multigraph G^Δ is obtained as the disjoint union of multigraphs H_t^Δ, for $t = 1, 2, \dots, \Delta - 2$, which are defined as follows.

Let H^Δ be the simple graph with vertices $u, v^0, v^1, \dots, v^{\Delta-1}$ and edges $uv^0, uv^1, \dots, uv^{\Delta-1}, v^0v^1, v^2v^3, \dots, v^{\Delta-2}v^{\Delta-1}$. The graph H^Δ is sometimes called a windmill graph [6] and can also be described as being obtained from the wheel W_Δ (see [2]) by alternately deleting edges on the outer cycle.

The multigraph H_t^Δ is obtained by replacing each edge v^jv^{j+1} which is not incident with the central vertex u with t parallel edges between the same vertices v^j and v^{j+1}.

In detail, define for $t = 1, 2, \dots, \Delta - 2$

$$\begin{aligned} V(H_t^\Delta) &= \{u_t, v_t^0, v_t^1, \dots, v_t^{\Delta-1}\} \\ E(H_t^\Delta) &= t \cdot \{v_t^j v_t^{j+1} : j \in \{0, 2, 4, \dots, \Delta - 2\}\} \cup \{u_t v_t^j : j \in \{0, 1, 2, \dots, \Delta - 1\}\} \\ H_t^\Delta &= \big(V(H_t^\Delta), E(H_t^\Delta)\big) \end{aligned}$$

For $j = 0, 1, \dots, \Delta-1$ we denote the edge $u_t v_t^j$ by e_t^j or simply by e^j once t is understood. Furthermore, for any index $j \in \{0, 1, \dots, \Delta - 1\}$ there is a uniquely determined index $j' \in \{0, 1, \dots, \Delta - 1\}$, $j \neq j'$ such that $v_t^{j'}$ is the unique vertex, other than u_t, which is adjacent to v_t^j in H_t^Δ.

The submultigraph of H_t^Δ which is induced by the vertices $u_t, v_t^j, v_t^{j'}$ will be denoted by L^j. The edges of L^j are e^j, $e^{j'}$ and the t repeated edges having v_t^j and $v_t^{j'}$ as endvertices. By definition, we have $L^j = L^{j'}$.

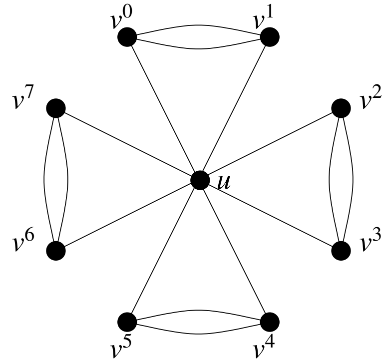

Figure 2: The graph H_2^8.

We now assume that a k-edge-coloring $c\colon E(H_t^\Delta) \to C = \{c_0, c_1, \ldots, c_{k-1}\}$ is given and study some properties of the palette pseudograph $\Gamma_c(H_t^\Delta)$. Since the central vertex u_t has degree Δ in H_t^Δ we have $\Delta \leq k$ and may assume, with no loss of generality, that $c(e^j) = c_j$ holds for $j = 0, 1, \ldots, \Delta - 1$. The inequality $t \leq \Delta - 2$ yields in turn $t + 1 < \Delta$. Consequently, since each non central vertex v_t^j has degree $t + 1$, we see that the palette $P_c(u_t) = \{0, 1, \ldots, \Delta - 1\}$ is distinct from every other palette $P_c(v_t^j)$. For that reason, rather than looking at the palette pseudograph $\Gamma_c(H_t^\Delta)$ we consider the subpseudograph $\Gamma_c^-(H_t^\Delta) = \Gamma_c(H_t^\Delta) \setminus P_c(u_t)$ obtained by removing the palette $P_c(u_t)$ (as a vertex of the palette pseudograph).

Lemma 3.1. *The pseudograph $\Gamma_c^-(H_t^\Delta)$ is a simple graph and is a forest.*

Proof. We prove first of all that $\Gamma_c^-(H_t^\Delta)$ has no loop, that is $P_c(v_t^j) \neq P_c(v_t^{j'})$ for all j. Consider the two adjacent vertices v_t^j and $v_t^{j'}$. The corresponding edges e^j and $e^{j'}$ have distinct colors c_j and $c_{j'}$ in $\{0, \ldots, \Delta - 1\}$, respectively. The color c_j cannot appear on one of the edges between v_t^j and $v_t^{j'}$, since c is a proper coloring. Hence, c_j belongs to $P_c(v_t^j)$ and does not belong to $P_c(v_t^{j'})$, and the two palettes are distinct, as claimed.

Next, we prove that $\Gamma_c^-(H_t^\Delta)$ has no multiple edges, by showing that if $P_c(v_t^j) = P_c(v_t^h)$ for $h \neq j, j'$, then $P_c(v_t^{j'}) \neq P_c(v_t^{h'})$. Suppose the vertices v_t^j and v_t^h share the same palette. The edges e^j and e^h are colored with colors c_j and c_h, respectively. Hence $\{c_j, c_h\} \subset P_c(v_t^j)$ $(= P_c(v_t^h))$. In particular, one of the edges between v_t^h and $v_t^{h'}$ has color c_j and so we have $c_j \in P_c(v_t^{h'})$. On the other hand, c_j does not belong to $P_c(v_t^{j'})$ because c is a proper coloring, and the claim follows.

In order to complete our proof, we need to prove that $\Gamma_c^-(H_t^\Delta)$ has no cycle and is thus a forest.

Assume, by contradiction, that $\Gamma_c^-(H_t^\Delta)$ has a cycle Γ. Without loss of generality, we may assume that Γ contains the vertices $P_c(v_t^0)$ and $P_c(v_t^1)$ of $\Gamma_c^-(H_t^\Delta)$. Since $P_c(v_t^0)$ has degree at least two in $\Gamma_c^-(H_t^\Delta)$, there exists $h \neq 0$ such that $P_c(v_t^0) = P_c(v_t^h)$ and $P_c(v_t^{h'})$ belongs to Γ. Recall that e_0 has colour c_0 in c. Therefore, the colour c_0 belongs to both palettes $P_c(v_t^0)$ and $P_c(v_t^h)$, since they are the same palette. Furthermore, the edge e_h has colour c_h, different from c_0. Then c_0 is the colour of one of the edges between v_t^h and $v_t^{h'}$. Hence, the colour c_0 also belongs to the palette $P_c(v_t^{h'})$. Repeating the same argument, we obtain that c_0 belongs to each palette of the cycle Γ. That is a contradiction, since c_0 does not belong to the palette $P_c(v_t^1)$. □

Lemma 3.2. *The degree of a vertex $P_c(v_t^j)$ in $\Gamma_c^-(H_t^\Delta)$ is exactly equal to the number of vertices of H_t^Δ having the same palette $P_c(v_t^j)$ in the colouring c.*

Proof. The underlying simple graph of $H_t^\Delta \setminus \{u_t\}$ is the disjoint union of isolated edges, that is every vertex has degree exactly 1 in the underlying simple graph. It follows from Proposition 2.1 that when a given palette P is viewed as a vertex in $\Gamma_c^-(H_t^\Delta)$, then its degree is equal to the number of vertices in H_t^Δ sharing the palette P. □

The next Proposition states a well-known property of forests.

Proposition 3.3. *The average degree of a forest is strictly less than* 2.

Proof. Suppose that the forest F has n vertices. Then F has at most $n-1$ edges and

$$\sum_{v \in V(F)} d(v) = 2|E(F)| \leq 2(n-1)$$

so that the average degree is

$$\frac{1}{n} \sum_{v \in V(F)} d(v) \leq \frac{2(n-1)}{n} < 2.$$ □

By the previous proposition and Lemma 3.2, the average number of vertices in H_t^Δ sharing the same palette is less than 2 and that implies the following lower bound for the palette index of H_t^Δ:

$$\check{s}(H_t^\Delta) > \frac{\Delta}{2} + 1.$$

Theorem 3.4.

$$\frac{\Delta}{2}(\Delta - 2) < \check{s}(G^\Delta) < (\Delta + 1)(\Delta - 2) \tag{3.1}$$

Proof. The second inequality is an immediate consequence of the fact that the number of vertices in G^Δ is $(\Delta + 1)(\Delta - 2)$.

For the first inequality, it is sufficient to observe that all vertices of degree $t+1$ in G^Δ belong to the subgraph H_t^Δ, so they cannot share the same palette with a vertex in another subgraph $H_{t'}^\Delta$, with $t' \neq t$. On the other hand, the vertex u_t of degree Δ in H_t^Δ could share the same palette with every other vertex of degree Δ, one in each subgraph H_t^Δ. We obtain

$$\check{s}(G^\Delta) \geq \sum_t (\check{s}(H_t^\Delta) - 1) > (\Delta - 2)\frac{\Delta}{2}.$$ □

4 Some considerations on a related parameter

We introduce a new natural parameter related to the palette index of a multigraph. Consider an edge-coloring c of G which minimizes the number of palettes, that is the number of palettes is exactly $\check{s}(G)$: how many colors does c require? More precisely, we consider the minimum k such that there exists a k-edge-coloring of G with $\check{s}(G)$ palettes. We will denote such a minimum by $\chi'_{\check{s}}(G)$. Obviously, $\chi'_{\check{s}}(G) \geq \chi'(G)$ because we need at least the number of colors in a proper edge-coloring. In [9], the authors remark that in some cases this number is strictly larger than the chromatic index of the graph. How much larger could it be?

An upper bound for the value of $\chi'_{\check{s}}(G)$ for some classes of graphs can be deduced from an analysis of the proofs of the corresponding results for the palette index.

- [9] Let K_n be a complete graph with $n > 1$ vertices. Then,

$$\chi'_{\check{s}}(K_n) = \Delta \quad \text{if} \quad n \equiv 0 \pmod 2$$
$$\chi'_{\check{s}}(K_n) \leq \frac{3\Delta}{2} \quad \text{if} \quad n \equiv 1 \pmod 2$$

  In particular, if $n = 4k+3$ then it is proved that the palette index is equal to 3 and the proof is obtained by using three sets of colors of cardinality $2k+1$. If $n = 4k+5$, the proof works by using three sets of colors of cardinality $2k+1$ and three additional colors, that is $6(k+1)$ colors. The number of colors is exactly $\frac{3\Delta}{2}$ in both cases.

- [9] Let G be a cubic graph. Then,

$$\chi'_{\check{s}}(G) \leq 5.$$

  In particular, five colors are necessary if G is not 3-edge-colorable and has no perfect matching.

- [4] Let G be a 4-regular graph. Then,

$$\chi'_{\check{s}}(G) \leq 6.$$

  In particular, six colors are used in some examples with palette index 3 (see the proof of Proposition 11 in [4]).

- [3] Let G be a forest. Then,

$$\chi'_{\check{s}}(G) = \Delta.$$

- [8] Let $K_{m,n}$ be a complete bipartite graph with $1 \leq m \leq n$. This situation is a little more involved, in the sense that we cannot always obtain a good upper bound for $\chi'_{\check{s}}(K_{m,n})$ using the proofs of the results in [8]. In some cases, see for instance Proposition 11 in [8], the number of colors is twice the maximum degree Δ (recall that minimizing the number of colors was not important in that context). Nevertheless, we analyze some small cases and obtain the same number of palettes (the minimum) by using a smaller number of colors.

  One such example is obtained by considering the graph $K_{5,6}$ (i.e. case $k = 3$ in Proposition 11 of [8]). Denote by $\{u_1, \dots, u_5\}$ and $\{v_1, \dots, v_6\}$ the bipartition of the vertex-set of $K_{5,6}$. The proof of Proposition 11 in [8] furnishes an edge-coloring with 12 colors and 6 palettes. Following the notation used in [8] we represent the coloring with a matrix, where the element in position (i, j) is the color of the edge $u_i v_j$.

$$M_{5,6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 7 & 12 & 11 & 10 & 9 \end{pmatrix}$$

The following coloring has only 8 colors and again 6 palettes.

$$M'_{5,6} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 4 & 5 & 7 & 8 & 1 & 2 \\ 5 & 4 & 8 & 7 & 2 & 1 \end{pmatrix}$$

We would like to stress that, even if we can obtain similar colorings for some other sporadic cases, we are not able to generalize our results to all infinite families considered in [8].

All previous results and the study of some sporadic cases suggest that $\chi'_{\check{s}}(G)$ cannot be too large with respect to Δ. In particular, we believe there exists a linear upper bound for $\chi'_{\check{s}}(G)$ in terms of Δ. The following is thus an even stronger conjecture.

Conjecture 4.1. *Let G be a (simple) graph. Then,*

$$\chi'_{\check{s}}(G) \leq \lceil \frac{3}{2}\Delta \rceil.$$

As far as we know, this conjecture is new and completely open. We believe any progress in that direction could be useful for a deeper understanding of the behavior of the palette index of general graphs.

5 Concluding remarks and open problems

In this final Section we propose some further open questions and indicate a few connections with other known problems.

In Section 3, we have presented a family of multigraphs whose palette index is expressed by a quadratic polynomial in Δ. We were not able to find a family of simple graphs with such a property and so we leave the existence of such a family as an open problem.

Problem 5.1. For $\Delta = 3, 4, \ldots$, does there exist a simple graph with maximum degree Δ whose palette index is quadratic in Δ?

As far as we know, the best general upper bound in terms of Δ for the palette index of a simple graph G is the trivial one, which is obtained from a $(\Delta + 1)$-edge-colouring c of G: in principle, each non-empty proper subset of the set of colours could occur as a palette of (G, c), whence $\check{s}(G) \leq 2^{\Delta+1} - 2$. On the other hand, all known examples suggest that this upper bound is far from being tight. In particular, we raise the question whether a polynomial upper bound holding for general multigraphs may exist at all.

Problem 5.2. Prove the existence of a polynomial $p(\Delta)$ such that $\check{s}(G) \leq p(\Delta)$ for every multigraph G with maximum degree Δ.

We slightly suspect that if a polynomial p solving Problem 5.2 can be found at all, then some quadratic polynomial will do as well.

Finally, we would like to stress how this kind of problems on the palette index is somehow related to another well-known type of edge-colorings, namely interval edge-colorings, introduced by Asratian and Kamalian in [1].

Definition 5.3. A proper edge-coloring c of a graph with colors $\{c_1, c_2, \ldots, c_t\}$ is called an *interval edge-colouring* if all colours are actually used, and the palette of each vertex is an interval of consecutive colors.

The following relaxed version of the previous concept was first studied in [10] and then explicitly introduced in [5].

Definition 5.4. A proper edge-colouring c of a graph with colors $\{c_1, c_2, \ldots, c_t\}$ is called an *interval cyclic edge-colouring* if all colours are used and the palette of each vertex is either an interval of consecutive colors or its complement.

Both interval and interval cyclic edge-colorings are thus proper edge-colourings with severe restrictions on the set of admissible palettes.

There are many more results on interval edge-colourings (see among others [12]). In particular, it is known that not all graphs admit an interval edge-colouring. Furthermore, it is proved in [11] that if a multigraph of maximum degree Δ admits an interval edge-colouring then it also admits an interval cyclic Δ-edge-colouring.

The following holds:

Proposition 5.5. *Let G be a multigraph of maximum degree Δ admitting an interval edge-colouring. Then, $\check{s}(G) \leq \Delta^2 - \Delta + 1$.*

Proof. Since G admits an interval edge-colouring, then it also admits an interval cyclic Δ-edge-colouring c (see [11]). Each palette of (G, c) is thus an interval of colors in the set $\{c_1, c_2, \ldots, c_\Delta\}$ or its complement is one such interval. For $t = 1, \ldots, \Delta - 1$, there are exactly Δ such subsets of cardinality t, and a unique one for $t = \Delta$. We have thus at most $\Delta(\Delta - 1) + 1$ distinct palettes in (G, c), that is $\check{s}(G) \leq \Delta^2 - \Delta + 1$. □

In other words, the previous Proposition assures that a putative example of a family of multigraphs whose palette index grows more than quadratically in Δ should be searched for within the class of multigraphs without an interval edge-colouring.

In this paper, we also introduce the palette pseudograph of a colored multigraph (G, c). A precise characterization of the palette pseudograph of the family introduced in Section 3 is the key point of our main proof. It suggests that a study of palette pseudographs in a general setting could increase our knowledge of the palette index. Possibly, it could also help in the search for an answer to some of the previous problems. Hence, we conclude our paper with the following:

Problem 5.6. Let H be a pseudograph. Determine whether a colored multigraph (G, c) exists, such that H is the palette pseudograph of (G, c).

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ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 125–139
https://doi.org/10.26493/1855-3974.1526.b8d
(Also available at http://amc-journal.eu)

Total positivity of Toeplitz matrices of recursive hypersequences*

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Received 9 November 2017, accepted 12 March 2019, published online 6 September 2019

Abstract

We present a new class of totally positive Toeplitz matrices composed of recently introduced hyperfibonacci numbers of the r-th generation. As a consequence, we obtain that all sequences $F_n^{(r)}$ of hyperfibonacci numbers of r-th generation are log-concave for $r \geq 1$ and large enough n.

Keywords: Total positivity, totally positive matrix, Toeplitz matrix, Hankel matrix, hyperfibonacci sequence, log-concavity.

Math. Subj. Class.: 15B36, 15A45

1 Introduction and preliminary results

A matrix M is *totally positive* if all its minors are positive real numbers. When it is allowed that minors are zero, then M is said to be *totally non-negative*. Such matrices appears in many areas having numerous applications including, among other topics, graph theory,

*Partial support of the Croatian Science Foundation through project BioAmpMode (Grant no. 8481) is gratefully acknowledged by the first author. All authors gratefully acknowledge partial support of Croatian-Slovenian bilateral project “Modeling adsorption on nanostructures: A graph-theoretical approach” BI-HR/16-17-046. The third author is partially supported by Slovenian research agency ARRS, program no. P1-0383.

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Pólya frequency sequences, oscillatory motion, symmetric functions and quantum groups among these areas [1, 2, 12, 13, 18]. The notion of total positivity is closely related with log-concavity and more on this one can find in a paper by Stanley [21]. A classical result by Whitney, Loewer and Cryer [8] says that any totally non-negative matrix M can be factored as a product of totally non-negative matrices $M = L_1 \cdots L_m D U_1 \cdots U_m$, where D is a diagonal matrix with non-negative elements, L_i is a matrix of the form $I + cE_{j+1,j}$, U_i is a matrix of the form $I + cE_{j,j+1}$ and $E_{k,l}$ is the matrix which has a 1 on the k, l position and zeros elsewhere. There is also a connection between totally non-negative matrices and planar networks proved by Karlin and McGregor [15], and Lindström [16]. The famous Lindström lemma gives combinatorial interpretation of a minor through the weights of collections of vertex-disjoint paths in a planar network.

An important notion when testing a matrix on total positivity is *initial minor*. We let I, J denote column set and row set, respectively. A minor $\Delta_{I,J}$ where both I and J consist of several consecutive indices and where $I \cup J$ contain 1, is called *initial*. Thus, each matrix entry is the lower-right corner of exactly one initial minor. In this work we use Theorem 1.1, which is proved by Gasca and Peña [14].

Theorem 1.1. *A square matrix is totally positive if and only if all its initial minors are positive.*

The notion of total positivity can be refined as follows. A matrix M is said to be *totally positive of order* p (or TP$_p$, in short) if all its minors of all orders $\leq p$ are positive.

The concept of total positivity extends in a straightforward manner also to (semi)infinite matrices. It turns out that many such triangular matrices appearing in combinatorics are indeed TP [3]. Recently, Wang and Wang proved total positivity of Catalan triangle via Aissen-Schonberg-Whitney theorem [22]. Further general results on triangular matrices and Riordan array have been obtained by Chen, Liang and Wang [5, 6] as well as Zhao and Yan [23], while Pan and Zeng give combinatorial interpretation of results on total positivity of Catalan-Stieltjes matrices [20].

A *Toeplitz* matrix $T = [t_{i,j}]$ is a (finite or infinite) matrix whose entries satisfy $t_{i,j} = t_{i+1,j+1}$. In finite case,

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & t_0 & \cdots & t_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{pmatrix}.$$

In words, elements of a Toeplitz matrix are constant along diagonals descending from left to right. If the elements of a matrix are constant along diagonals ascending from left to right, the matrix is called a *Hankel matrix*. An example is given here,

$$H = \begin{pmatrix} t_0 & t_1 & \cdots & t_{n-1} \\ t_1 & t_2 & \cdots & t_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_{2n-2} \end{pmatrix}.$$

Obviously, each Toeplitz (or Hankel) matrix of order n gives rise to a unique sequence (of length $2n - 1$ in the finite case) of its elements. The connection also works the other way:

Given an (infinite) sequence (a_n) and given integers n_0 and m, we can construct a Toeplitz (or a Hankel) matrix of order m having a_{n_0} in the upper left corner. In what follows we present a class of totally positive Toeplitz matrices whose entries are hyperfibonacci numbers [4, 17, 24]. These sequences of numbers were recently introduced by Dil and Mező in a study of a symmetric algorithm for hyperharmonic and some other integer sequences [9].

Definition 1.2. The hyperfibonacci sequence of the r-th generation $(F_n^{(r)})_{n\geq 0}$ is a sequence arising from the recurrence relation

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \tag{1.1}$$

where $r \in \mathbb{N}$ and F_n is the n-th term of the Fibonacci sequence, $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0, F_1 = 1$.

Proposition 1.3 gives some basic identities for hyperfibonacci sequences [7].

Proposition 1.3. *For hyperfibonacci sequence $(F_n^{(r)})_{n\geq 0}$ we have*

(i)

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)} \tag{1.2}$$

(ii)

$$F_n^{(1)^2} - F_{n-1}^{(1)} F_{n+1}^{(1)} = F_{n-3}^{(1)} + 1 + (-1)^{n+1}$$

(iii)

$$F_n^{(1)} F_{n+1}^{(1)} - F_{n-1}^{(1)} F_{n+2}^{(1)} = F_{n-2}^{(1)} + 1 - (-1)^{n+1}$$

(iv)

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}. \tag{1.3}$$

Explicit formula for determinant of the Hankel matrix of hyperfibonacci sequence of r-th generation

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

has been obtained in [19] and here we state it in Theorem 1.4. We will find it useful in establishing our main result, the total positivity of the Toeplitz matrix of the same sequence with odd-indexed hyperfibonacci number in the upper left corner.

Theorem 1.4. *For the sequence $(F_k^{(r)})_{k\geq 0}$, $r \in \mathbb{N}$ and $n \in \mathbb{N}$ a determinant of a matrix $A_{r,n}$ takes values ± 1,*

$$\det(A_{r,n}) = (-1)^{n+\left\lfloor \frac{r+3}{2} \right\rfloor}.$$

The TP_2 property of Toeplitz and Hankel matrices is closely related to log-concavity and log-convexity, respectively, of the associated sequences. Recall that a sequence (a_n) of positive numbers is *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ holds for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If the inequality is reversed, the sequence is *log-convex*. The literature on log-concavity and log-convexity is vast. Besides already mentioned classical papers by Stanley [21] and Brenti [3], we refer the reader also to [10, 11, 20, 22] for some recently developed techniques. In particular, the log-concavity of hyperfibonacci numbers of all generations $r \geq 1$ has been established in [24] by using recurrence relations. Here we proceed to prove more general claims that will imply the log-concavity results of reference [24].

2 Positivity of hyperfibonacci determinant

We let $B_{m,n}^{(r)} = [b_{i,j}]$ denote the matrix of order m consisting of hyperfibonacci numbers of the r-th generation,

$$B_{m,n}^{(r)} := \begin{pmatrix} F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_{n-m+1}^{(r)} \\ F_{n+1}^{(r)} & F_n^{(r)} & \cdots & F_{n-m+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+m-1}^{(r)} & F_{n+m-2}^{(r)} & \cdots & F_n^{(r)} \end{pmatrix}$$

with the constraint $r \geq m-1$. In what follows we will show that there exist $q(r) \in \mathbb{N}$ such that $\det(B_{m,n}^{(r)})$ is positive for $n \geq q(r)$.

From the elementary properties of the Fibonacci sequence known as Cassini identity we immediately have that the matrix

$$M = \begin{pmatrix} F_{2n+1} & F_{2n+2} \\ F_{2n+2} & F_{2n+3} \end{pmatrix}$$

is positive for $n \in \mathbb{N}_0$ and the matrix

$$M' = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n+2} & F_{2n+1} \end{pmatrix}$$

is positive for $n \in \mathbb{N}$. In Proposition 2.1 we extend the property of positivity to matrices of order 2 consisting from first generation of hyperfibonacci numbers while a general result, involving r-th generation of hyperfibonacci numbers is given in Theorem 3.5.

Proposition 2.1. *For $n, r \in \mathbb{N}$ determinant of the matrix $B_{2,n}^{(1)}$ is positive,*

$$\det(B_{2,n}^{(1)}) = \det \begin{pmatrix} F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} \end{pmatrix} > 0.$$

Proof. We apply relations presented in Proposition 1.3 to get $F_n^{(1)} - F_{n-1}^{(1)} = F_n$. Now, by the properties of determinant (column subtraction and then row subtraction) we obtain

$$\det \begin{pmatrix} F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} \end{pmatrix} = \det \begin{pmatrix} F_n & F_{n-1}^{(1)} \\ F_{n+1} & F_n^{(1)} \end{pmatrix} = \det \begin{pmatrix} F_n & F_{n-1}^{(1)} \\ F_{n-1} & F_n \end{pmatrix}$$
$$= \det \begin{pmatrix} F_n & F_{n+1} - 1 \\ F_{n-1} & F_n \end{pmatrix} > 0. \qquad \square$$

Theorem 2.2. *Let $m \in \mathbb{N}$. Then there is $n_m \in \mathbb{N}$ such that $\det\left(B_{m,n}^{(m-1)}\right) > 0$ for all $n \geq n_m$.*

Proof. Employing elementary transformation on matrices and using relation (1.2) we get

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} F_n & F_{n-1}^{(1)} & F_{n-2}^{(2)} & \cdots & F_{n-m+1}^{(m-1)} \\ F_{n+1} & F_n^{(1)} & F_{n-1}^{(2)} & \cdots & F_{n-m+2}^{(m-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n+m-1} & F_{n+m-2}^{(1)} & F_{n+m-3}^{(2)} & \cdots & F_n^{(m-1)} \end{pmatrix}$$

$$= \det\begin{pmatrix} F_n & F_{n-1}^{(1)} & F_{n-2}^{(2)} & \cdots & F_{n-m+1}^{(m-1)} \\ F_{n-1} & F_n & F_{n-1}^{(1)} & \cdots & F_{n-m+2}^{(m-2)} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n-m+2} & F_{n-m+3} & F_{n-m+4} & \cdots & F_{n-1}^{(1)} \\ F_{n-m+1} & F_{n-m+2} & F_{n-m+3} & \cdots & F_n \end{pmatrix}. \tag{2.1}$$

Having in mind relation (1.3) we immediately obtain

$$F_{n-r}^{(r)} = F_{n+r} - \sum_{k=0}^{r-1} \binom{n+k}{r-1-k}$$

and furthermore

$$F_{n-r}^{(r)} = F_{n+r} - S_r, \tag{2.2}$$

where

$$S_r := \sum_{k=0}^{r-1} \binom{n+k}{r-1-k}.$$

Thus, $S_1 = 1$, $S_2 = n+1$, $S_3 = \frac{n(n-1)}{2} + n + 2$, $S_4 = \frac{n^3+5n}{6} + n + 3$, etc. Now, we substitute entries in (2.1) according to (2.2) to get

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} F_n & F_{n+1} - S_1 & F_{n+2} - S_2 & \cdots & F_{n+m-1} - S_{m-1} \\ F_{n-1} & F_n & F_{n+1} - S_1 & \cdots & F_{n+m-2} - S_{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n-m+1} & F_{n-m+2} & F_{n-m+3} & \cdots & F_n \end{pmatrix}. \tag{2.3}$$

In the following steps of this proof we let Δ_1, Δ_2, Δ_3 denote matrices we deal with. We will show that determinants of these matrices are equal to each other. In order to make the proof more readable, the elements of the last two columns of Δ_1, Δ_2, Δ_3 are denoted by $c_{i,j}$, $c'_{i,j}$, $c''_{i,j}$, respectively. On the other hand, the elements of the first $m-2$ columns of these matrices are denoted by $b_{i,j}$ and they do not change their values under performed transformation.

When performing elementary transformations on matrix columns of (2.3) we obtain

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} S_2-S_1 & S_3-S_2-S_1 & \cdots & F_{n+m-2}-S_{m-2} & F_{n+m-1}-S_{m-1} \\ S_1 & S_2-S_1 & \cdots & F_{n+m-3}-S_{m-3} & F_{n+m-2}-S_{m-2} \\ 0 & S_1 & \cdots & F_{n+m-4}-S_{m-4} & F_{n+m-3}-S_{m-3} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & F_n & F_{n+1}-S_1 \\ 0 & 0 & \cdots & F_{n-1} & F_n \end{pmatrix}$$
$$= \det(\Delta_1)$$

where we get $\Delta_1 = [b_{i,j}]$ by similar transformation on rows,

$$\Delta_1 = \begin{pmatrix} S_2-2S_1 & S_3-2S_2-S_1 & \cdots & -S_{m-1}+S_{m-2}+S_{m-3} \\ S_1 & S_2-2S_1 & \cdots & -S_{m-2}+S_{m-3}+S_{m-4} \\ 0 & S_1 & \cdots & -S_{m-3}+S_{m-4}+S_{m-5} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & S_2-S_1 \\ 0 & 0 & \cdots & F_{n+1}-S_1 \\ 0 & 0 & \cdots & F_n \end{pmatrix},$$

$$\begin{aligned} b_{i,j} &= b_{i+1,j+1}, \;\; i=1,\dots,m-1, \;\; j=1,\dots,m-3, \\ b_{i,j} &= c_{i,j}, \;\; i=1,\dots,m, \;\; j=m-1,m, \\ c_{i,m-1} &= c_{i+1,m}, \;\; i=1,\dots,m-3 \end{aligned}$$

and where entries $b_{i,j}$ get values

$$\begin{aligned} b_{1,1} &= S_2-2S_1 \\ b_{1,2} &= S_3-2S_2-S_1 \\ b_{1,3} &= S_4-2S_3-S_2+2S_1 \\ b_{1,4} &= S_5-2S_4-S_3+2S_2+S_1 \\ b_{1,5} &= S_6-2S_5-S_4+2S_3+S_2 \\ &\vdots \\ b_{1,m-2} &= S_{m-1}-2S_{m-2}-S_{m-3}+2S_{m-4}+S_{m-5}, \end{aligned}$$

while for entries $c_{i,j}$ we have

$$\begin{aligned} c_{1,m-1} &= -S_{m-2}+S_{m-3}+S_{m-4} \\ c_{2,m-1} &= -S_{m-3}+S_{m-4}+S_{m-5} \\ &\vdots \\ c_{m-3,m-1} &= -S_2+S_1 \\ c_{m-2,m-1} &= -S_1 \\ c_{m-1,m-1} &= F_n \\ c_{m,m-1} &= F_{n+1}, \end{aligned}$$

and

$$\begin{aligned} c_{m-1,m} &= F_{n+1} - S_1 \\ c_{m,m} &= F_n. \end{aligned}$$

Furthermore, we form matrix $\Delta_2 = [b_{i,j}]$ with $b_{i,j} = c'_{i,j},\ \ i = 1, \ldots, m,\ \ j = m-1, m,$ by performing row transformations

$$\begin{aligned} c'_{i,m-1} &= c_{i,m-1} + \sum_{j=1}^{m-3} b_{i,j},\ \ i = 1, \ldots, m \\ c'_{i,m} &= c_{i,m} + \sum_{j=1}^{m-2} b_{i,j},\ \ i = 1, \ldots, m. \end{aligned}$$

As a consequence of these two operations for the last two columns of Δ_2 we obtain

$$\begin{pmatrix} -S_{m-4} + S_{m-6} + S_{m-7} + \cdots + S_2 & -S_{m-3} + S_{m-5} + S_{m-6} + \cdots + S_2 \\ \vdots & \vdots \\ -S_4 + S_2 & -S_5 + S_3 + S_2 \\ -S_3 & -S_4 + S_2 \\ -S_2 & -S_3 \\ -S_1 & -S_2 \\ 0 & -S_1 \\ 0 & 0 \\ F_n & F_{n+1} \\ F_{n-1} & F_n \end{pmatrix}.$$

(while the other entries of Δ_2 are equal to those of Δ_1). Clearly, $\det(\Delta_1) = \det(\Delta_2)$. Furthermore, we perform row transformations

$$\begin{aligned} c''_{i,m-1} &= c'_{i,m-1} + b_{i,m-5} + 2b_{i,m-6} + 4b_{i,m-7} + \cdots + (F_{m-3} - 1)b_{i,1} \\ c''_{i,m} &= c'_{i,m-1} + b_{i,m-4} + 2b_{i,m-5} + 4b_{i,m-6} + \cdots + (F_{m-2} - 1)b_{i,1} \end{aligned}$$

to get matrix $\Delta_3 = [b_{i,j}]$ where $b_{i,j} = c''_{i,j},\ \ i = 1, \ldots, m,\ \ j = m-1, m$. Then, the last two columns of Δ_3 are

$$\begin{pmatrix} -F_{m-2} & -F_{m-1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ F_n & F_{n+1} \\ F_{n-1} & F_n \end{pmatrix}.$$

Namely, a straighforward but tedious algebraic manipulation give us a nice value for $c''_{1,m-1}$,

$$\begin{aligned} c''_{1,m-1} &= (F_{m-6} - 1)S_1 + (F_{m-5} - 1)2S_1 - (F_{m-4} - 1)S_1 - (F_{m-3} - 1)2S_1 \\ &= -F_{m-2}. \end{aligned}$$

In the same fashion one can prove that $c''_{1,m} = -F_{m-1}$ and $c''_{i,j} = 0$, $i = 2, \dots, m-2$, $j = m-1, m$. Again, determinant is not affected under these transformations, $\det(\Delta_3) = \det(\Delta_2)$.

We shall now separately treat the matrix Δ_3, for even and odd n. Using the Fibonacci recurrence relation, for even n we immediately obtain

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m-2} & -F_{m-2} & -F_{m-1} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & b_{3,2} & \cdots & b_{3,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & F_{n-1} & 0 & 1 \\ 0 & 0 & \cdots & -F_{n-2} & 1 & 1 \end{pmatrix}$$

$$= -\det\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b'_{1,m-2} & -F_{m-3} & -F_{m-2} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & b_{3,2} & \cdots & b_{3,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

where $b'_{1,n-2} = b_{1,m-2} + F_{m-3}F_{n-1} - F_{m-2}F_{n-2}$. This determinant can be represented as the sum of the upper triangular determinants. Now we use the fact that there is $q \in \mathbb{N}$ such that the Fibonacci number F_q is bigger that the value $P(q)$, $F_q > P(q)$, where $P(n)$ is a polynomial of any degree. The only element in the matrix above containing Fibonacci numbers is $b'_{1,m-2}$. The fact that the term $F_{n-1}F_{m-3}$ has a positive contribution in the determinant completes the proof for case when n is even.

When n is odd we have

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m-2} & -F_{m-2} & -F_{m-1} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & b_{3,2} & \cdots & b_{3,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & F_{n-2} & 1 & 1 \\ 0 & 0 & \cdots & -F_{n-1} & 0 & 1 \end{pmatrix}.$$

Now, analogue arguments as when n is even completes the proof. □

In particular, when $m = 4$ we have

$$\det\left(B_{4,n}^{(3)}\right) = \det\begin{pmatrix} S_2 - 2 & S_3 - 2S_2 - 1 & -1 & -2 \\ 1 & S_2 - 2 & 0 & 0 \\ 0 & 1 & F_n & F_{n+1} \\ 0 & 0 & F_{n-1} & F_n \end{pmatrix}.$$

When n is even then

$$\det\left(B_{4,n}^{(3)}\right) = \det\begin{pmatrix} S_2-2 & S_3-2S_2-1 & -1 & -2 \\ 1 & S_2-2 & 0 & 0 \\ 0 & F_{n-1} & 0 & 1 \\ 0 & -F_{n-2} & 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} S_2-2 & S_3-2S_2-1-F_{n-2}+F_{n-1} & -1 & -1 \\ 1 & S_2-2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= -(S_2-2)\begin{pmatrix} S_2-2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} S_3-2S_2-1-F_{n-2}+F_{n-1} & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= -(n-1)^2 + \frac{n(n-1)}{2} - n - 1 + F_{n-3}.$$

The inequality

$$F_{n-3} > (n-1)^2 - \frac{n(n-1)}{2} + n + 1$$

holds true for $n \geq 15$ and consequently $\det\left(B_{4,n}^{(3)}\right) > 0$ for $n \geq 15$ when n is even. Similarly, when n is odd

$$\det\left(B_{4,n}^{(3)}\right) = (S_2-2)\begin{pmatrix} S_2-2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} S_3-2S_2-1-F_{n-3} & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (n-1)^2 - \frac{n(n-1)}{2} + n + 1 + F_{n-3}.$$

Thus, it follows from these two cases that $\det\left(B_{4,n}^{(3)}\right) > 0$ for $n \geq 15$.

Note that the proof of Theorem 2.2 can be used to efficient calculation of determinants of matrices $B_{m,n}^{(m-1)}$. We will illustrate this on the example for $m = 4$ and $n = 5$. In that case, when applying the proof of Theorem 2.2 we have

$$\det\left(B_{4,5}^{(3)}\right) = \det\begin{pmatrix} 51 & 25 & 11 & 4 \\ 97 & 51 & 25 & 11 \\ 176 & 97 & 51 & 25 \\ 309 & 176 & 97 & 51 \end{pmatrix} = \det\begin{pmatrix} 6 & 11 & -1 & -1 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 24 - 11 = 13.$$

Corollary 2.3. *Let $m, n, r \in \mathbb{N}$ and $r \geq m-1$. Then there is $q \in \mathbb{N}$ such that determinant of the matrix $B_{m,n}^{(r)}$ is positive for all $n \geq q$,*

$$\det\left(B_{m,n}^{(r)}\right) > 0.$$

Proof. We proceed by induction on r. The base case, $r = m - 1$, is provided by Theorem 2.2. Let us now assume that the claim is true for $m - 1 \leq p \leq r - 1$. Our task is to show that the determinant

$$\det(B^{(r)}_{m,n}) = \det \begin{pmatrix} F^{(r)}_{n} & F^{(r)}_{n-1} & \cdots & F^{(r)}_{n-m+1} \\ F^{(r)}_{n+1} & F^{(r)}_{n} & \cdots & F^{(r)}_{n-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r)}_{n+m-1} & F^{(r)}_{n+m-2} & \cdots & F^{(r)}_{n} \end{pmatrix}$$

is also positive. We first recall (1.2) and then start subtracting rows of $B^{(r)}_{m,n}$. We subtract $(m-1)$-st row from m-th, then $(m-2)$-nd from $(m-1)$-st, and continue all the way down till we subtract the first row from the second. Since the determinant remains unchanged, we obtain

$$\det(B^{(r)}_{m,n}) = \det \begin{pmatrix} F^{(r)}_{n} & F^{(r)}_{n-1} & \cdots & F^{(r)}_{n-m+1} \\ F^{(r-1)}_{n+1} & F^{(r-1)}_{n} & \cdots & F^{(r-1)}_{n-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r-1)}_{n+m-1} & F^{(r-1)}_{n+m-2} & \cdots & F^{(r-1)}_{n} \end{pmatrix}.$$

We expand the determinant on the right hand side over the elements of the first row.

$$\begin{aligned} \det(B^{(r)}_{m,n}) &= F^{(r)}_{n}\Delta_1 + \cdots + F^{(r)}_{n-m+1}\Delta_m \\ &= \frac{F^{(r)}_{n}}{F^{(r-1)}_{n}} F^{(r-1)}_{n}\Delta_1 + \cdots + \frac{F^{(r)}_{n-m+1}}{F^{(r-1)}_{n-m+1}} F^{(r-1)}_{n-m+1}\Delta_m, \end{aligned}$$

where Δ_i denotes the determinant obtained from $\det(B^{(r)}_{m,n})$ by omitting the first row and i-th column for $1 \leq i \leq m$. Let us denote $x_i = \frac{F^{(r)}_{n-i+1}}{F^{(r-1)}_{n-i+1}}$ and define a function $f\colon \mathbb{R}^m \to \mathbb{R}$ by

$$f(x_1, \ldots, x_m) = \sum_{i=0}^{m} x_{i+1} F^{(r-1)}_{n-i} \Delta_i.$$

Obviously, $f(1, \ldots, 1) = \det(B^{(r-1)}_{m,n}) > 0$, and hence $f(c, \ldots, c) = c \cdot \det(B^{(r-1)}_{m,n}) > 0$, for any positive constant c. In particular, $f(\phi^2, \ldots, \phi^2) > 0$, where $\phi^2 = \frac{3+\sqrt{5}}{2}$.

Since f is continuous, there must exist a neighborhood

$$W = (\phi^2 - \delta_1, \phi^2 + \delta_1) \times \cdots \times (\phi^2 - \delta_m, \phi^2 + \delta_m)$$

such that f is positive on W. Now we use the explicit expression

$$F^{(r)}_{n} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}$$

from Proposition 1.3. By dividing it through by analogous expression for $F^{(r-1)}_{n}$ and passing to limit when $n \to \infty$, one readily obtains

$$\lim_{n\to\infty} \frac{F^{(r)}_{n}}{F^{(r-1)}_{n}} = \phi^2.$$

That further implies that, for large enough n, the coefficient $x_i = \frac{F^{(r)}_{n-i+1}}{F^{(r-1)}_{n-i+1}}$ falls into $(\phi^2 - \delta_i, \phi^2 + \delta_i)$ for all i, and hence

$$f\left(\frac{F^{(r)}_{n}}{F^{(r-1)}_{n}}, \dots, \frac{F^{(r)}_{n-m+1}}{F^{(r-1)}_{n-m+1}}\right) = \det(B^{(r)}_{m,n}) > 0.$$

That completes the proof. □

3 Main results

We let $T_{r,n}$ denote the matrix of order $r+2$ consisting of hyperfibonacci numbers of the r-th generation,

$$T_{r,n} := \begin{pmatrix} F^{(r)}_{2n+1} & F^{(r)}_{2n} & \cdots & F^{(r)}_{2n-r} \\ F^{(r)}_{2n+2} & F^{(r)}_{2n+1} & \cdots & F^{(r)}_{2n-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r)}_{2n+r+2} & F^{(r)}_{2n+r+1} & \cdots & F^{(r)}_{2n+1} \end{pmatrix}.$$

Lemma 3.1. *For $n \in \mathbb{N}$ and the hyperfibonacci sequence $\left(F^{(1)}_n\right)_{n\geq 0}$ the matrix*

$$T_{1,n} = \begin{pmatrix} F^{(1)}_{2n+1} & F^{(1)}_{2n} & F^{(1)}_{2n-1} \\ F^{(1)}_{2n+2} & F^{(1)}_{2n+1} & F^{(1)}_{2n} \\ F^{(1)}_{2n+3} & F^{(1)}_{2n+2} & F^{(1)}_{2n+1} \end{pmatrix}$$

is totally positive.

Proof. According to Proposition 2.1 the three initial minors of order 2 of $T_{1,n}$ are positive. It is immediately seen from Theorem 1.4 that determinant $\det(T_{1,n})$ is positive. These facts complete the proof. □

Note that the matrix $T_{1,n} = [t_{i,j}]$ is a Toeplitz matrix, with the element $t_{1,1}$ being hyperfibonacci number of the first generation having odd index. If we allow both even and odd indices for $t_{1,1}$ then the property of total positivity is lost. Such determinant of order 3 in not positive for even indices (by Theorem 1.4), while it keeps the positivity of minors of order 2. We express this fact, that follows from the proof of Lemma 3.1, in Corollary 3.2.

Corollary 3.2. *For $n \in \mathbb{N}$ and the hyperfibonacci sequence $\left(F^{(1)}_n\right)_{n\geq 0}$ the matrix*

$$T'_{1,n} = \begin{pmatrix} F^{(1)}_{n} & F^{(1)}_{n-1} & F^{(1)}_{n-2} \\ F^{(1)}_{n+1} & F^{(1)}_{n} & F^{(1)}_{n-1} \\ F^{(1)}_{n+2} & F^{(1)}_{n+1} & F^{(1)}_{n} \end{pmatrix}$$

is TP_2.

Lemma 3.3. *For $n \geq 4$ and the hyperfibonacci sequence $\left(F^{(2)}_n\right)_{n\geq 0}$ the matrix*

$$T_{2,n} = \begin{pmatrix} F^{(2)}_{2n+1} & F^{(2)}_{2n} & F^{(2)}_{2n-1} & F^{(2)}_{2n-2} \\ F^{(2)}_{2n+2} & F^{(2)}_{2n+1} & F^{(2)}_{2n} & F^{(2)}_{2n-1} \\ F^{(2)}_{2n+3} & F^{(2)}_{2n+2} & F^{(2)}_{2n+1} & F^{(2)}_{2n} \\ F^{(2)}_{2n+4} & F^{(2)}_{2n+3} & F^{(2)}_{2n+2} & F^{(2)}_{2n+1} \end{pmatrix}$$

is totally positive.

Proof. According to Proposition 2.1 the five initial minors of order 2 of $T_{2,n}$ are positive. Furthermore, the three initial minors of order 3 are positive when $n \geq 3$ by Corollary 2.3. However, when $n = 3$ determinant $\det(T_{2,n})$ is negative (by Theorem 1.4) so the matrix $T_{2,n}$ is totally positive for $n \geq 4$. □

Having in mind Proposition 2.1 and the fact that the matrix $B^{(2)}_{3,n}$ has positive determinant for $n \geq 7$ we immediately derive Corollary 3.4.

Corollary 3.4. *For $n \geq 8$ and the hyperfibonacci sequence $\left(F^{(2)}_n\right)_{n\geq 0}$ the matrix*

$$T'_{2,n} = \begin{pmatrix} F^{(2)}_n & F^{(2)}_{n-1} & F^{(2)}_{n-2} & F^{(2)}_{n-3} \\ F^{(2)}_{n+1} & F^{(2)}_n & F^{(2)}_{n-1} & F^{(2)}_{n-2} \\ F^{(2)}_{n+2} & F^{(2)}_{2n+1} & F^{(2)}_n & F^{(2)}_{n-1} \\ F^{(2)}_{n+3} & F^{(2)}_{2n+2} & F^{(2)}_{2n+1} & F^{(2)}_n \end{pmatrix}$$

is TP_3.

Furthermore, it holds true that

$$\det(B^{(3)}_{4,n}) > 0, \quad n \geq 15$$
$$\det(B^{(4)}_{4,n}) > 0, \quad n \geq 5.$$

When $r \geq 5$ there is no constraint on the value of n when asking for positivity of $\det(B^{(r)}_{4,n})$.

Theorem 3.5. *For the hyperfibonacci sequence $\left(F^{(r)}_n\right)_{n\geq 0}$ there is $q \in \mathbb{N}$ such that the matrix $T_{r,n}$ of order $r+2$*

$$T_{r,n} = \begin{pmatrix} F^{(r)}_{2n+1} & F^{(r)}_{2n} & \cdots & F^{(r)}_{2n-r} \\ F^{(r)}_{2n+2} & F^{(r)}_{2n+1} & \cdots & F^{(r)}_{n-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r)}_{n+r+2} & F^{(r)}_{2n+r+1} & \cdots & F^{(r)}_{2n+1} \end{pmatrix}$$

is totally positive for $n \geq q$.

Proof. First we prove that $2n+1$ initial minors of order 2 are positive. These submatrices are of the form $B^{(r)}_{2,m_2}$ where $m_2 > 2n - r$, so there they have positive determinant for $r \geq 1$ and $n \geq 1$, according to Corollary 2.3. Obviously, another initial minors are of the form

$$B^{(r)}_{3,m_3}, B^{(r)}_{4,m_4}, \ldots, B^{(r)}_{r+1,m_{r+1}}.$$

According to Corollary 2.3 there exist numbers $q_3, q_4, \ldots, q_{r+1} \in \mathbb{N}$ such that

$$\det(B^{(r)}_{3,m_3}) > 0, \quad m_3 \geq q_3$$
$$\det(B^{(r)}_{4,m_4}) > 0, \quad m_4 \geq q_4$$
$$\vdots$$
$$\det(B^{(r)}_{r+1,m_{r+1}}) > 0, \quad m_{r+1} \geq q_{r+1}.$$

It remains to show that $\det(T_{r,n})$ is itself positive. We start by noticing that $T_{r,n}$ can be obtained from $A_{r,2n-r}$ by reversing the order of columns. That corresponds to right multiplication of $A_{r,2n-r}$ by U_{r+2}, where U_{r+2} is a square matrix of order $r+2$ whose elements are $(U_{r+2})_{i,j} = 1$ if $i+j = r+3$ and zero otherwise. It is immediately seen that $\det(U_{r+2}) = (-1)^{\lfloor (r+2)/2 \rfloor}$. Now we have $\det(T_{r,n}) = \det(A_{r,2n-r})\det(U_{r+2})$, and Theorem 1.4 implies

$$\det(T_{r,n}) = (-1)^{2n-r+\lfloor (r+3)/2 \rfloor + \lfloor (r+2)/2 \rfloor} = (-1)^2 = 1,$$

for all r. That completes the proof. □

We conclude the section with another result that follows directly from Corollary 3.4.

Corollary 3.6. *For the hyperfibonacci sequence* $\left(F_n^{(r)}\right)_{n\geq 0}$ *there is* $q \in \mathbb{N}$ *such that the matrix* $T'_{r,n}$ *of order* $r+2$

$$T'_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_{n-r-1}^{(r)} \\ F_{n+1}^{(r)} & F_n^{(r)} & \cdots & F_{n-r}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r}^{(r)} & \cdots & F_n^{(r)} \end{pmatrix}$$

is TP$_{r+1}$ *for* $n \geq q$.

4 Concluding remarks

In this paper we have considered several classes of Toeplitz matrices associated to sequences of hyperfibonacci numbers of given generation. We have established various positivity results for such matrices. In particular, we showed that such matrices with odd-indexed hyperfibonacci numbers on the main diagonal are totally positive for large enough values of index n. When the restriction to odd-valued indices is omitted, the total positivity is not preserved, but we established that those matrices are TP$_{r+1}$ for a given generation r and large enough n. That implies (at least asymptotical) log-concavity of hyperfibonacci numbers of all generations $r \geq 1$. Our results thus extend and strengthen results of reference [24] established by a different approach. It would be interesting to have combinatorial proofs of log-concavity of $F_n^{(r)}$ for $r \geq 1$; at the moment, we are not aware of any.

We have also tried to explore the form of dependence of q_r on r. The numerical evidence, collected in Table 1, suggests that $2q_r+1$, the index in the upper left corner, behaves as $7r-5$ for even r and $7r-4$ for r odd. It would be interesting to examine whether the

Table 1: Some values of parameter q_r in Theorem 3.5.

r	1	2	3	4	5	6	7	8	9	10	11
$2q_r+1$	5	9	17	23	31	37	45	51	59	65	73

pattern (or at least a linear dependence) persists for larger r, and if it does, to find some explanation.

We are fairly confident that the methods and results presented here could be extended so as to encompass also other sequences defined by two-term recurrences and their iterated partial sums. It would be worthwhile to explore whether the same approach could be applicable to the sequences defined by longer linear recurrences with constant coefficients, such as the sequence of tribonacci numbers.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 141–151
https://doi.org/10.26493/1855-3974.1517.e42
(Also available at http://amc-journal.eu)

Graph characterization of fully indecomposable nonconvertible $(0, 1)$-matrices with minimal number of ones*

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Received 27 October 2017, accepted 15 July 2019, published online 10 September 2019

Abstract

Let A be a $(0, 1)$-matrix such that PA is indecomposable for every permutation matrix P and there are $2n + 3$ positive entries in A. Assume that A is also nonconvertible in a sense that no change of signs of matrix entries, satisfies the condition that the permanent of A equals to the determinant of the changed matrix.

We characterized all matrices with the above properties in terms of bipartite graphs. Here $2n + 3$ is known to be the smallest integer for which nonconvertible fully indecomposable matrices do exist. So, our result provides the complete characterization of extremal matrices in this class.

*The work of the second and the fourth authors was partially supported by Slovenian Research Agency (research core fundings No. P1-0288, No. P1-0222, and by grant BI-RU/16-18-033). The work of the first and the third authors is supported by Russian Scientific Foundation grant 17-11-01124.

The authors are especially thankful to the referee for communicated to them the gap which existed in Remark 3.15 of the original draft.

Keywords: Permanent, indecomposable matrices, graphs.

Math. Subj. Class.: 05C40, 15A27, 15A04, 05C50

1 Introduction

Let $M_{m,n}(\Sigma)$ denote the set of matrices of size $m \times n$ with entries from a certain algebraic set Σ. Unless explicitly stated otherwise, $\Sigma \subseteq \mathbb{Z}$ is a subset of integers. Typically $\Sigma = \{0,1\}$ or $\Sigma = \{-1,1\}$ and in these two cases we will write $M_{m,n}(0,1)$ or $M_{m,n}(\pm 1)$, and if $m = n$, then we write shortly $M_{n,n}(\Sigma) = M_n(\Sigma)$. We consider two well known functions of matrices, permanent and determinant, which are defined by formulas:

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \qquad \det A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \operatorname{sgn}(\sigma) a_{i\sigma(i)},$$

where S_n is the group of permutations of order n and $\operatorname{sgn}(\sigma)$ is a sign of permutation σ.

Permanent is a good counting function in combinatorics and applications, but there is no fast algorithms known for computing the permanent function itself on arbitrary matrices. Ryser formula which requires $O(n2^{n-1})$ multiplication operations is still one of the best known algorithms, for details see [1] or [9]. Moreover, Valiant proved that computing even a permanent of $(0,1)$-matrix is #P-complete problem ([12]). Recent investigations of permanents of $(0,1)$ and $(-1,1)$ matrices can be found in [6] and [3], correspondingly, and references therein. In comparison, the determinant which is very similar to permanent can be easily computed by Gauss elimination algorithm. One of the possible approaches to compute permanent is to convert it by a certain transformation to the determinant. The sign-conversion is one of the classical possibilities to construct such a transformation.

We say that matrix $A \in M_n(0,1)$ is sign convertible or just convertible if there is matrix $X \in M_n(\pm 1)$ such that $\operatorname{per} A = \det(A \circ X)$, where operation $\circ$ is the Hadamard, i.e., entrywise product. The notion of convertibility was presented by Pólya in [10] and studied by different mathematicians (for details see [4, 5, 9]). Convertibility of $(0,1)$-matrices is equivalent to many problems in graph theory (for details see [7, 8, 11, 13]). Thus the class of $(0,1)$-matrices is particularly important.

In [4] different notions of bounds of convertibility were presented. We say that integer Ω_n is an upper bound for convertibility if for any $A \in M_n(0,1)$ with $\operatorname{per} A > 0$ and with more than Ω_n nonzero entries it follows that A is not convertible. We say that ω_n is a lower bound for convertibility if any matrix $A \in M_n(0,1)$ with less than ω_n positive entries is convertible. It is known that $\Omega_n = \frac{n^2+3n-2}{2}$ (see [5]) and $\omega_n = n + 6$ (see [4]).

In [2] lower bounds for convertibility were found under additional assumption that matrices are indecomposable or fully indecomposable. Note that instead of indecomposable some authors use other terminology like irreducible, see a book by Brualdi and Ryser [1]. Since the present paper is a continuation of our previous work [2] we use the same terminology as in [2]. Notice that the term ”fully indecomposable” is also used in the same monograph (see [1, page 112]). Let us state the corresponding definitions below.

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Definition 1.1. A matrix $A \in M_n(0,1)$ is called *decomposable* if there exists permutation matrix $P \in M_n(0,1)$ such that

$$A = P \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} P^t,$$

where B, D are square matrices and C is possibly a rectangular matrix. If A is not decomposable, it is called *indecomposable*.

Definition 1.2. A matrix $A \in M_n(0,1)$ is called *partially decomposable* if there exist permutation matrices $P, Q \in M_n(0,1)$ such that

$$A = P \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} Q,$$

where B, D are square matrices and C is possibly a rectangular matrix. If A is not partially decomposable, it is called *fully indecomposable*.

Remark 1.3. One observes easily that $A \in M_n(0,1)$ is not fully indecomposable if and only if for some integer $p \in \{1, \ldots, n-1\}$ there exists a zero block of size $p \times (n-p)$ in A.

Remark 1.4. We note that a fully indecomposable matrix is always indecomposable, but the converse may not be true. Observe that in each row and in each column of a fully indecomposable matrix there are at least 2 positive entries.

In [2, Example 4.3] we showed that lower bound for indecomposable matrices equals $n+6$ and can not be improved. For fully indecomposable matrices better lower bound was found in the same paper.

Theorem 1.5 ([2])**.** *Let $A \in M_n(0,1)$ be a fully indecomposable matrix with less than $2n+3$ positive entries. Then matrix A is convertible.*

Our aim is to describe extremal case of Theorem 1.5. Namely, we classify all fully indecomposable matrices with $2n+3$ positive entries which are nonconvertible. Our paper is organized as follows. In Section 2 we reformulate the notion of convolution (introduced in [2]) in terms of bipartite graphs and describe the properties of this operation. In Section 3 we prove our main result Theorem 3.13 on the characterization of the extremal case using the language of the graph theory.

2 Convolution via bipartite graphs

The following notion of convolution was presented in [2].

Definition 2.1. Let $A \in M_n(0,1)$ and let the first row of A has exactly two non-zero entries a_{11}, a_{12}. Then the *convolution* of A by the first row is the following matrix $S_1(A) \in M_{n-1}(0,1)$,

$$S_1(A) = \begin{pmatrix} \max(a_{21}, a_{22}) & a_{23} & \cdots & a_{2n} \\ \max(a_{31}, a_{32}) & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, a_{n2}) & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Here we delete the first row and take the maximum between the corresponding elements in the first and second columns.

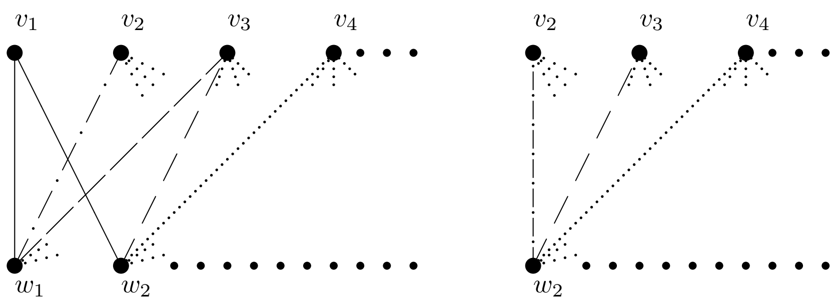
Similarly, if the i-th row of A has exactly two nonzero entries a_{ij}, a_{ik}, $j < k$, the *convolution* $S_i(A) \in M_{n-1}(0,1)$ of A by the i-th row is defined as the matrix obtained from A by deleting the i-th row and k-th column and exchanging the j-th column by the maximum of j-th and k-th columns.

Notation 2.2. Let $A \in M_{m,n}(\Sigma)$, $\alpha \subseteq \{1,\dots,m\}$ and $\beta \subseteq \{1,\dots,n\}$. By $A(\alpha|\beta)$ we denote the matrix obtained from A by removing rows with indexes from α and columns with indexes from β. By $A[\alpha|\beta]$ we denote the submatrix of A located on intersection of rows with indexes from α and columns with indexes from β. We will write shortly $A(|1,2)$ instead of $A(\{\}|\{1,2\})$ etc.

Our main goal in this section is to present the notion of convolution with the help of graphs. Let $\Gamma = \Gamma(V,W,E)$ be a simple bipartite graph with $V \cup W$ as the set of vertices and E as the set of edges. Write $V = \{v_1,\dots,v_m\}$ and $W = \{w_1,\dots,w_n\}$. We say that matrix $A \in M_{m,n}(0,1)$ is biadjacency matrix of Γ if the following holds: $a_{ij} = 1$ if and only if $\{v_i, w_j\} \in E$. Thus $|V|$ is equal to the number of rows in A and $|W|$ is equal to the number of columns in A. The number of edges of a vertex v is a valency of this vertex. Since we study square $(0,1)$-matrices we will consider only bipartite graphs with $|V| = |W|$.

Remark 2.3. Let $\Gamma = \Gamma(V,W,E)$ be a simple bipartite graph and $A \in M_n(0,1)$ its biadjacency matrix. Then permutation of rows of A corresponds to renumbering of vertices in V, permutation of columns of A corresponds to renumbering of vertices in W and transposition of A corresponds to exchange of sets V and W. Thus these transformations do not change the structure of the graph.

Suppose that convolution can be applied to a matrix $A \in M_n(0,1)$, i.e., suppose A has a row with exactly two nonzero entries. By Remark 2.3 we can assume that A has two positive elements a_{11} and a_{12} in the first row and $S_1(A)$ is a convolution of A by the first row. Let A be biadjacency matrix of $\Gamma = \Gamma(V,W,E)$, see Figure 1(a), and $S_1(A)$ be biadjacency matrix of Γ_1, see Figure 1(b).



(a) the original graph (b) the result of convolution

Figure 1: Convolution.

Lemma 2.4. *Let $A \in M_n(0,1)$. Let the first row of A has exactly two non-zero entries a_{11}, a_{12}, and let $S_1(A)$ be the convolution of A. Then bipartite graph Γ_1 with biadjacency matrix $S_1(A)$ is constructed from bipartite graph Γ with biadjacency matrix A by the following steps:*

(1) Vertices v_1 and w_1 are removed.

(2) Every edge in Γ of the form $\{x, w_1\}$ for $x \in \{v_2, \ldots, v_n\}$ is replaced by an edge in Γ_1 of the form $\{x, w_2\}$.

Proof. To obtain $S_1(A)$ from A the following transformations are done.

1. The first row and the first column of A are removed. Thus vertices $v_1 \in V$ and $w_1 \in W$ are removed from Γ.

2. Since $A(1|1,2) = S_1(A)(|1)$ the corresponding subgraphs in Γ and Γ_1 coincide.

3. In $S_1(A)$ elements of the first column are represented by $\max(a_{i1}, a_{i2})$, where $i = 2, \ldots, n$. Since we consider $(0,1)$-matrices there are four possible options.

3.1. Suppose $a_{i1} = a_{i2} = 0$. Then $\max(a_{i1}, a_{i2}) = 0$ and no edges in Γ and Γ_1 correspond to these entries of A and $S_1(A)$.

3.2. Suppose $a_{i1} = 1$ and $a_{i2} = 0$. Then there is an edge $\{v_i, w_1\}$ in Γ. Since $\max(a_{i1}, a_{i2}) = 1$ this edge in Γ_1 is replaced by $\{v_i, w_2\}$. For $i = 2$ this case is represented in Figure 1(a) for Γ and in Figure 1(b) for Γ_1 by dash-dotted edges.

3.3. Suppose $a_{i1} = 0$ and $a_{i2} = 1$. Then there is an edge $\{v_i, w_2\}$ in Γ. Since $\max(a_{i1}, a_{i2}) = 1$ this edge remains also in Γ_1. For $i = 4$ this case is represented in Figure 1(a) for Γ and in Figure 1(b) for Γ_1 by dotted edges.

3.4. Suppose $a_{i1} = a_{i2} = 1$. Then there are edges $\{v_i, w_1\}$ and $\{v_i, w_2\}$ in Γ. Since $\max(a_{i1}, a_{i2}) = 1$ these edges are replaced by the edge $\{v_i, w_2\}$ in Γ_1. For $i = 3$ this case is represented in Figure 1(a) for Γ and in Figure 1(b) for Γ_1 by dashed edges. In this case we will say that edges are *merged*. □

3 Main result

We will use the following results obtained in [2].

Theorem 3.1 ([2, Theorem 3.6]). *Let $A \in M_n(0,1)$. Let the first row of A have exactly two nonzero entries a_{11} and a_{12}, and let $S_1(A)$ be the convolution of A. Then A is convertible if and only if $S_1(A)$ is convertible.*

Theorem 3.2 ([2, Theorem 3.8]). *Let $A \in M_n(0,1)$ be a fully indecomposable matrix with at most $2n+2$ positive entries. Then A is convertible.*

Now we prove that the convolution of a fully indecomposable matrix is fully indecomposable.

Lemma 3.3. *Let $A \in M_n(0,1)$. Let the first row of A have exactly two nonzero entries a_{11} and a_{12}, and let $S_1(A)$ be the convolution of A. Let A be fully indecomposable. Then $S_1(A)$ is fully indecomposable.*

Proof. Assume on the contrary that $S_1(A)$ is partially decomposable. Then there exists a $k \times (n-k-1)$ zero submatrix $B = S_1(A)[i_1, \ldots, i_k | j_1, \ldots, j_{n-k-1}]$ for some $1 \leq k \leq n-2$ and some $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k-1}$. We consider two cases depending on whether B includes the first column of $S_1(A)$ or not.

1. Suppose $j_1 > 1$. Since $A(1|1,2) = S_1(A)(|1)$ then B is a submatrix of A as well, i.e., $B = A[i_1+1, \ldots, i_k+1 | j_1+1, \ldots, j_{n-k-1}+1]$. Since $a_{1,l} = 0$ for $l > 2$ and since $j_1 + 1 > 2$ it follows that $A[1, i_1+1, \ldots, i_k+1 | j_1+1, \ldots, j_{n-k-1}+1]$ is a $(k+1)\times(n-k-1)$ zero submatrix. So A is partially decomposable, a contradiction.
2. Suppose $j_1 = 1$. Let $S_1(A) = (s_{ij})$. Since $0 = s_{i_l,1} = \max(a_{i_l+1,1}, a_{i_l+1,2})$ for any $l = 1, \ldots, k$ it follows that $A[i_1+1, \ldots, i_k+1 | 1, j_1+1, \ldots, j_{n-k-1}+1]$ is a $k \times (n-k)$ zero submatrix. So A is partially decomposable, a contradiction. □

The following example shows that the converse does not hold, i.e., if $S_1(A)$ is fully indecomposable, then A is not necessarily a fully indecomposable.

Example 3.4. The matrix A, defined below, is partially decomposable while $S_1(A)$ is fully indecomposable.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Notation 3.5. Let $A \in M_n(0,1)$. By $\nu(A)$ we denote the number of positive entries of A. By $J_k \in M_k(0,1)$ we denote the k-by-k matrix with all entries equal to 1.

Lemma 3.6. *Let $A \in M_n(0,1)$, $n > 3$, be a fully indecomposable nonconvertible matrix with $\nu(A) = 2n+3$. Then the convolution can be applied recursively to obtain J_3. On step k of the process we obtain fully indecomposable, nonconvertible matrix of order $(n-k)$ with $2(n-k)+3$ positive entries.*

Proof. By Remark 1.4 in each row of A there are at least two positive elements. Since $\nu(A) = 2n+3$ by Pigeonhole principle there is a row in A with exactly 2 positive entries. With no loss of generality these entries are a_{11} and a_{12}. Since the convolution S_1 removes the first row of A it follows that $\nu(S_1(A)) \leq 2(n-1)+3$. By Theorem 3.1, $S_1(A)$ is nonconvertible and by Lemma 3.3, $S_1(A)$ is fully indecomposable. Thus by Theorem 3.2, $\nu(S_1(A)) \geq 2(n-1)+3$.

Combining both inequalities we obtain $\nu(S_1(A)) = 2(n-1)+3$ and matrix $S_1(A)$ meets all the conditions of this lemma. Repeating the arguments $n-3$ times we obtain J_3. □

Lemma 3.7. *Let $A \in M_n(0,1)$, $n > 3$, be a fully indecomposable nonconvertible matrix with $\nu(A) = 2n+3$ and with exactly two positive entries $a_{11} = a_{12} = 1$ in the first row. Let A and $S_1(A)$ be the biadjacency matrices of bipartite graphs Γ and Γ_1, respectively. Then Γ_1 is constructed from Γ without merging edges.*

Proof. Suppose the edges $\{x, w_1\}$ and $\{x, w_2\}$ of Γ are merged by convolution. It means that there is $i > 1$ such that $a_{i1} = a_{i2} = 1$. These two positive entries are replaced by one in matrix $S_1(A)$. Thus $\nu(S_1(A)) \leq 2n+3-3 = 2(n-1)+2$, which contradicts Lemma 3.6. □

Lemma 3.8. *Let $A \in M_n(0,1)$, $n \geq 3$, be a fully indecomposable nonconvertible matrix with $\nu(A) = 2n+3$. Then in A there are $n-3$ columns (rows) with exactly two positive entries and 3 columns (rows) with exactly three positive entries.*

Proof. By Remark 1.4 in each row of A there are at least two positive entries. By Lemma 3.6 we can construct sequence of $n-3$ convolutions to obtain matrix J_3. By Lemma 3.7 there are no merges of edges, hence after applying a convolution the number of positive entries in non-deleted rows does not change.

To prove the statement for columns we transpose the matrix and repeat our arguments. □

A chain of three edges is any sequence of edges of the form $\{a, v_1\}, \{v_1, v_2\}, \{v_2, b\}$ which constitute a path of length 3 for some vertices a, v_1, v_2, b.

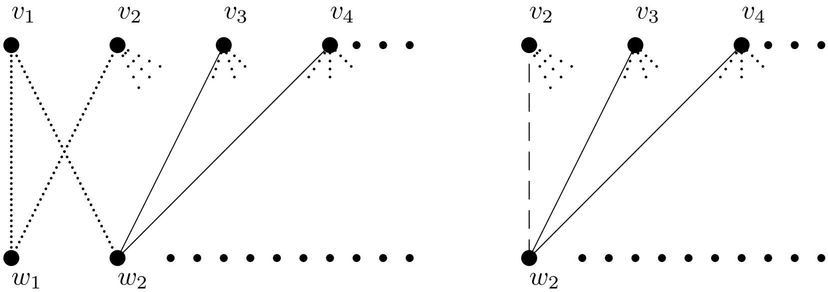
Lemma 3.9. *Let $A \in M_n(0,1)$, $n > 3$, be a fully indecomposable nonconvertible matrix with $\nu(A) = 2n+3$ and with exactly two positive entries $a_{11} = a_{12} = 1$ in the first row. Then the first or the second column (or both) contains exactly two nonzero entries. Moreover, suppose the first column of A contains exactly two nonzero entries and let A and $S_1(A)$ be the biadjacency matrices of bipartite graphs Γ and Γ_1, respectively. Then Γ_1 is obtained from Γ by replacing a chain of three edges by a single edge and deleting the two intermediate vertices of this chain.*

Remark 3.10. No generality is lost in assuming that first column contains exactly two nonzero entries — we can always swap the first two columns to achieve this.

Remark 3.11. Conversely, under the assumptions and notations of Lemma 3.9, Γ is obtained from Γ_1 by subdividing an edge with two additional vertices. Note that this procedure preserves bipartiteness of graphs.

Proof of Lemma 3.9. By Lemma 3.8 in each column of A there are either 2 or 3 positive entries. Since permutation of columns does not change the structure of the graph we consider three cases.

1. Suppose that in the first and in the second columns of A there are three positive entries. By Lemma 3.7 no edges were merged in $S_1(A)$. Thus there are four positive entries in the first column of $S_1(A)$. Note that by Lemma 3.6, $S_1(A)$ is fully indecomposable nonconvertible matrix of order $n-1$ and $\nu(S_1(A)) = 2(n-1)+3$, so by Lemma 3.8 in each column of $S_1(A)$ there are at most three positive entries, a contradiction.
2. Suppose there are two and three positive entries in the first and in the second column of the matrix. With no loss of generality we can permute columns of the matrix to obtain two positive entries in the first column and three positive entries in the second column. By Lemma 3.7 no edges are merged thus $a_{i1}a_{i2} = 0$ for any $i \geq 2$. We may assume that $a_{11} = a_{21} = 1$ in the first column and $a_{12} = a_{32} = a_{42} = 1$ in the second column. The structure of the graph is represented in Figure 2(a). By Lemma 2.4 convolution $S_1(A)$ remove vertices v_1 and w_1 and edges $\{v_1, w_1\}$ and $\{v_1, w_2\}$ and the edge $\{v_2, w_1\}$ is replaced by the edge $\{v_2, w_2\}$. The resulted graph is represented in Figure 2(b). The removed elements of Γ are represented by dotted edges (Figure 2(a)) the added element of Γ_1 are represented by dashed edge (Figure 2(b)). Thus the chain $\{w_2, v_1\}, \{v_1, w_1\}, \{w_1, v_2\}$ is replaced by the edge $\{w_2, v_2\}$ to obtain graph Γ_1. The lemma is proved in this case.

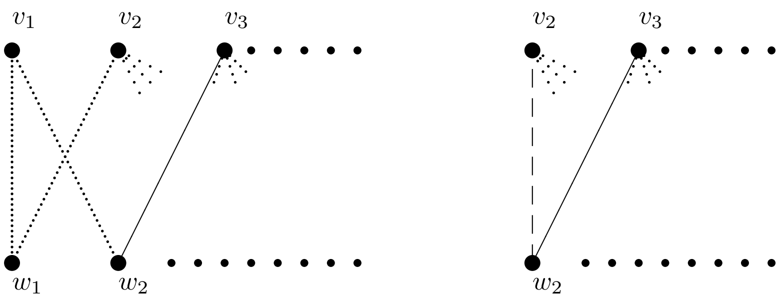


(a) the chain before convolution is marked by the dotted edges

(b) the same chain after convolution is the dashed edge

Figure 2: Convolution of matrix with 3 positive entries in 1st column and 2 positive entries in 2nd column.

3. Suppose there are two positive entries in the first column and two positive entries in the second column. By Lemma 3.7 no edges are merged thus $a_{i1}a_{i2} = 0$ for any $i \geq 2$. We may assume that $a_{11} = a_{21} = 1$ in the first column and $a_{12} = a_{32} = 1$ in the second column. The structure of the graph is represented in Figure 3(a). By Lemma 2.4 convolution $S_1(A)$ remove vertices v_1 and w_1 and edges $\{v_1, w_1\}$ and $\{v_1, w_2\}$ and the edge $\{v_2, w_1\}$ is replaced by the edge $\{v_2, w_2\}$. The resulted graph is represented in Figure 3(b). Thus the chain $\{w_2, v_1\}, \{v_1, w_1\}, \{w_1, v_2\}$ (dotted edges, Figure 3(a)) is replaced by the edge $\{w_2, v_2\}$ (dashed edge, Figure 3(b)) to obtain graph Γ_1. The lemma is proved. □



(a) the chain before convolution is marked by the dotted edges

(b) the same chain after convolution is the dashed edge

Figure 3: Convolution of matrix with 2 positive entries in 1st column and 2 positive entries in 2nd column.

Lemma 3.12. *Let Γ be a graph obtained from the bipartite graph Γ_1 by subdividing one or more its edges with even number of points. Let $A(\Gamma_1)$ and $A(\Gamma)$ be the corresponding biadjancency matrices. If $A(\Gamma_1)$ is fully indecomposable then same holds for $A(\Gamma)$.*

Proof. We use the notation from the proof of Remark 3.11. It suffices, by induction, to consider the case when Γ is obtained from Γ_1 by subdividing only one of its edges with two vertices. Without loss of generality we may assume that the subdivided edge is $\{v_1, w_1\}$ and that we are adding vertices v_0, w_0. Then, the matrix corresponding to Γ has the form

$$A(\Gamma) = \begin{array}{c} \\ w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{cccccc} v_0 & v_1 & v_2 & \cdots & v_n \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & \bigstar & \dots & \bigstar \\ 0 & \bigstar & \bigstar & \dots & \bigstar \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \bigstar & \bigstar & \dots & \bigstar \end{bmatrix} \end{array}$$

where $\bigstar$ denote the entries of the biadjacency matrix $A(\Gamma_1)$. It follows from Remark 1.3 that $A(\Gamma)$ is fully indecomposable if and only if it does not contain a zero block of size $p \times (n + 1 - p)$ for some $p = 1, \dots, n$ where $n + 1$ is the size of $A(\Gamma)$. Now, by the induction, the $n \times n$ matrix $A(\Gamma_1)$ is fully indecomposable so it does not contain a zero block of size $1 \times (n - 1)$. It follows that the $(n + 1) \times (n + 1)$ matrix $A(\Gamma)$ has at least two ones in each row, i.e. has no zero block of size $1 \times n$. The first row of $A(\Gamma)$ contains $n-1$ zeros. However, at the corresponding columns (2)–($n+1$) (the starting column being indexed by 0), the other rows of $A(\Gamma)$ consists of elements of $A(\Gamma_1)$ so cannot have $n - 1$ zero entries. That is, $A(\Gamma)$ does not contain a zero block of size $2 \times (n - 1)$. Likewise we see that inside columns (3)–(n) the matrix $A(\Gamma_1)$ does not contain a zero $2 \times (n - 2)$ which implies that $A(\Gamma)$ contains no $3 \times (n - 2)$ block. Proceed inductively to deduce that $A(\Gamma)$ contains no zero $p \times (n + 1 - p)$ block. Hence, $A(\Gamma)$ is fully indecomposable. □

Theorem 3.13. *Let $A \in M_n(0, 1)$, $n \geq 3$, be a fully indecomposable nonconvertible matrix with $\nu(A) = 2n + 3$. Let $\Gamma = \Gamma(V, W, E)$ be a simple bipartite graph with A as its biadjacency matrix. Then up to renumbering of vertices, Γ has the following three properties.*

(1) Vertices v_i, w_j, where $i, j \in \{1, 2, 3\}$, have valency 3, and every other vertex has valency 2.

(2) If $i, j \in \{1, 2, 3\}$ and $\{v_i, w_j\} \notin E$, then there is a unique path connecting v_i to w_j whose intermediate vertices are all of valency 2.

(3) The graph is connected.

Remark 3.14. The disjoint union of a complete bipartite graph and an even cycle $K_{3,3} + C_{2n-6}$ satisfies all the assumptions of Theorem 3.13 except the third item. This graph is not a biadjacency graph of fully indecomposable n-by-n matrix with $2n + 3$ units.

Proof. By Lemma 3.6 there is a sequence of $n - 3$ convolutions to obtain matrix J_3 from A. Matrix J_3 is a biadjacency matrix of a complete bipartite graph $K_{3,3}$. This graph fulfills the conditions of the theorem. Let us reverse these convolutions to obtain graph Γ. Note that by Remark 3.11 on each reverse step the resulted graph is bipartite.

By Lemma 3.9 each convolution replaces a chain of three edges by a single edge. Thus the reverse operation will add two vertices with valency 2 and replace a single edge by a chain of three edges, hence the valencies of vertices which were added on the previous steps do not change. Thus Condition (1) of the theorem is satisfied after each reverse operation.

All edges in the graph $K_{3,3}$ can be represented as a chain of length 1 from vertex v_i to vertex w_j. Thus each reverse operation replaces a single edge by a chain of three edges whose both intermediate vertices are of valency 2 in some chain of edges. Obviously this operation preserves chains of edges from v_i to w_j, where $i, j \in \{1, 2, 3\}$, possibly extending a length of one of these chains by 2. Thus Conditions (2) and (3) are satisfied. □

Remark 3.15. With the help of Remark 3.11 and Lemma 3.12 we can formulate Theorem 3.13 also in the following way. A bipartite graph Γ corresponds to a fully indecomposable nonconvertible biadjacency matrix A with $\nu(A) = 2n + 3$ if and only if Γ is obtained from $K_{3,3}$ by subdividing each edge with an even number of vertices (possibly 0).

Recall that if two matrices are the same modulo permutations of rows/columns and transposition, then their biadjacency graphs are isomorphic. Conversely, assume the biadjacency graphs Γ_1 and Γ_2 of two fully indecomposable nonconvertible n-by-n matrices $A_1, A_2 \in M_n(0, 1)$ with $2n + 3$ units are isomorphic. The two graphs are bipartite having two maximum sets of independent vertices V_i and W_i. Their graph isomorphism must either map V_1 bijectively onto V_2 and W_1 bijectively onto W_2, or it maps V_1 bijectively onto W_2 and W_1 bijectively onto V_2. The first case corresponds to permuting rows/columns of matrix A_1 to obtain A_2, while the second case composes this with transposition.

Therefore, the cardinality of the set Ω of equivalent classes of fully indecomposable nonconvertible matrices $A \in M_n(0, 1)$ with $v(A) = 2n+3$, modulo permutations of rows, columns, and transposition, equals the number of pairwise nonisomorphic graphs, obtained from $K_{3,3}$ by subdividing each edge with an even number of vertices (possibly 0) such that in total we place additional $2(n - 3)$ vertices.

Theorem 3.16. *Up to a permutation of rows and columns and up to a transposition, any fully indecomposable nonconvertible matrix $A \in M_n(0, 1)$ with $\nu(A) = 2n + 3$ can be described by a matrix $C \in M_3(\mathbb{Z}_+)$, such that the sum of elements of C is $n - 3$.*

Proof. In the proof of Theorem 3.13 it is was shown that any bipartite graph Γ with a fully indecomposable nonconvertible biadjacency matrix $A \in M_n(0, 1)$, $\nu(A) = 2n + 3$, can be constructed by a sequence of $n - 3$ replacements of a single edge by a chain of three edges. Thus for a full description of Γ we must define lengths of chains from v_i to w_j, where $i, j \in \{1, 2, 3\}$. Each chain has length $2k + 1$, where $k \geq 0$ is a number of times when an edge from this chain was replaced by a chain of three edges. Equivalently, it is a number of convolutions that modified this chain. By Lemma 3.6 total number of convolutions to obtain $K_{3,3}$ from Γ is $n - 3$. It follows that Γ can be described by 9 numbers k_i, $i \in \{1, \dots, 9\}$, such that $\sum_{i=1}^{9} k_i = n - 3$.

Let us arrange these numbers in a matrix $C = (c_{ij}) \in M_3(\mathbb{Z}_+)$ such that c_{ij} is equal to a number of convolutions corresponding to a chain from v_i to w_j. Permutation of rows (columns) is equivalent to renumbering of vertices v_i, $i \in \{1, 2, 3\}$ (w_i, $i \in \{1, 2, 3\}$). Transposition of C is equivalent to a permutation of sets of vertices V and W of a graph Γ. Thus the structure of Γ does not change and the theorem is proved. □

Example 3.17. For $n = 7$ there are 16 not equivalent nonconvertible $(0, 1)$-matrices with $2n + 3$ ones. They are described by the following nonnegative integer matrices with the sum of elements equal to 4.

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix}2&0&0\\0&2&0\\0&0&0\end{pmatrix}\quad\begin{pmatrix}2&1&1\\0&0&0\\0&0&0\end{pmatrix}\quad\begin{pmatrix}2&1&0\\0&0&1\\0&0&0\end{pmatrix}\quad\begin{pmatrix}2&1&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}2&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad\begin{pmatrix}2&0&0\\0&1&1\\0&0&0\end{pmatrix}\quad\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&1&1\\1&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\0&1&1\\0&0&0\end{pmatrix}\quad\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}\quad\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&1&0\\0&0&1\\0&0&1\end{pmatrix}$$

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 153–183
https://doi.org/10.26493/1855-3974.1195.c71
(Also available at http://amc-journal.eu)

Generating polyhedral quadrangulations of the projective plane*

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Received 9 September 2016, accepted 7 June 2019, published online 17 September 2019

Abstract

We determine the 26 families of irreducible polyhedral quadrangulations of the projective plane under three reductions called a face-contraction, a 4-cycle removal and a 2_3-path shrink, which were first given by Batagelj in 1989. Every polyhedral quadrangulation of the projective plane can be obtained from one of them by a sequence of the inverse operations of the reductions.

Keywords: Quadrangulation, projective plane, generating theorem.

Math. Subj. Class.: 05C10

1 Introduction

A *quadrangulation* (resp., *triangulation*) of a closed surface is a simple graph cellularly embedded on the surface so that each face is quadrilateral (resp., triangular); in particular, a 2-path on the sphere is not a quadrangulation in this paper. It is known that every quadrangulation G of any closed surface is 2-connected and hence the minimum degree of G is at least 2. For quadrangulations of closed surfaces, we introduce typical three reductional operations called a *face-contraction*, a 4-*cycle removal* and a 2_3-*path shrink*, which were first given by Batagelj [2]. (See Figure 1. For a formal definition, see the next section.) In this paper, we call the above three operations $\mathcal{P}$-*reductions*, while call the inverse operations $\mathcal{P}$-*expansions*.

A quadrangulation of a closed surface is *irreducible* if no face-contraction is applicable without making a loop or multiple edges. In [20], it was proved that a 4-cycle is the unique irreducible quadrangulation of the sphere, and that there exist precisely two irreducible quadrangulations of the projective plane which are the unique quadrangular embeddings

*This work was supported by JSPS KAKENHI Grant Number 16K05250.
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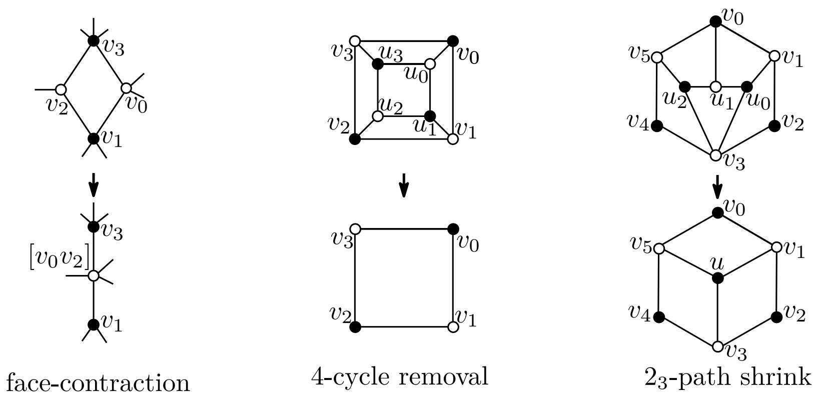

Figure 1: $\mathcal{P}$-reductions.

of K_4 and $K_{3,4}$ on the projective plane, respectively (see Figure 2). The irreducible quadrangulations of the torus and the Klein bottle had also been determined in [15] and [14], respectively.

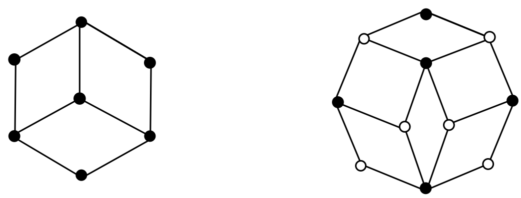

Figure 2: Irreducible quadrangulations of the projective plane where antipodal points of the hexagon and the octagon are identified respectively.

There are some results of quadrangulations of closed surfaces with some conditions. Batagelj [2] proved that any 3-connected quadrangulation on the sphere can be deformed into a cube by a sequence of $\mathcal{P}$-reductions preserving 3-connectedness. However his proof contained a small mistake, and Brinkmann et al. [3] pointed out it and gave a corrected proof. Observe that a 3-connected quadrangulation of the sphere corresponds to a 4-regular 3-connected graph on the same surface by taking its dual. Broersma et al. [4] considered the same problem of the dual version with weaker conditions than Brinkmann et al. [3]. Nakamoto [17] discussed quadrangulations with minimum degree 3 and proved that any quadrangulation of the sphere (resp., the projective plane) with minimum degree 3 can be deformed into a pseudo double wheel (resp., a Möbius wheel or the unique quadrangular embedding of $K_{3,4}$ on the projective plane) by a sequence of face-contractions and 4-cycle removals, preserving the minimum degree at least 3. Brinkmann et al. [3] also proved the same result only on the sphere using a restricted face-contraction. Furthermore, the results in [13] implies that every 3-connected quadrangulation of a closed surface F^2 except the sphere can be reduced into one of irreducible quadrangulations of F^2 by $\mathcal{P}$-reductions, preserving the 3-connectedness. In addition, the recent study [25] discussed another reductional operation defined for 3-connected quadrangulations of closed surfaces.

Let G be a graph embedded on a non-spherical closed surface F^2. The *representativity* of G, denoted by $r(G)$, is the minimum number of intersecting points of G and γ, where γ ranges over all essential simple closed curves on the surface. A graph G embedded on F^2 is r-*representative* if $r(G) \geq r$ (see [22] for the details). A graph G embedded on a closed surface F^2 is *polyhedral* if G is 3-connected and 3-representative. For example, each of two quadrangulations in Figure 2 is 3-connected but not polyhedral since these embeddings have representativity 2. Observe that all facial walks in a polyhedral embedded graph G are cycles, and any two of them are either disjoint, intersect in one vertex, or intersect in one edge. From such a point of view, polyhedral embedded graphs are frequently regarded as "good" embeddings in topological graph theory (see e.g., [8, 9, 10, 11]); note that every simple triangulation of a closed surface is polyhedral, while simple quadrangulations are not necessarily so. Furthermore, it is known that there is one to one correspondence between the set of polyhedral quadrangulations of a nonspherical closed surface F^2 (resp., 3-connected quadrangulations of the sphere) and the set of *optimal* 1-*embeddings* of F^2 (resp., *optimal* 1-*planar graphs* of the sphere, see [5, 6, 12, 21, 23, 24] for definitions and some results).

A face $f = v_0v_1v_2v_3$ of a polyhedral quadrangulation G of F^2 is $\mathcal{P}$-*contractible* (or simply *contractible*) if a face-contraction at either $\{v_0, v_2\}$ or $\{v_1, v_3\}$ results in another polyhedral quadrangulation of the same surface. Similarly, we define "$\mathcal{P}$-*removable* (or simply *removable*)" and "$\mathcal{P}$-*shrinkable* (or simply *shrinkable*)" for a 4-cycle C and a 2-path P, both of which are induced by vertices of degree 3, respectively. A polyhedral quadrangulation G of F^2 is $\mathcal{P}$-*irreducible* if G has none of a contractible face, a removable 4-cycle and a shrinkable 2-path. The following is our main theorem in this paper. In the figures, to obtain the projective plane, identify antipodal pairs of points of each hexagon or octagon.

Theorem 1.1. *There are precisely* 26 *families of* $\mathcal{P}$*-irreducible quadrangulations of the projective plane presented in Figures 8, 11 and 16. Every polyhedral quadrangulation of the projective plane can be obtained from one of them by a sequence of* $\mathcal{P}$*-expansions.*

This paper is organized as follows. In the next section, we define basic terminology and reductional operations for quadrangulations. In Section 3, we show some lemmas to prove Theorem 1.1. In Section 4, we determine inner structures of 2-cell regions bounded by 4, 5 or 6-cycles of $\mathcal{P}$-irreducible quadrangulations. Furthermore in Section 5, we consider ones bounded by several 6 or 8-walks. Before proving the main theorem, we classify $\mathcal{P}$-irreducible quadrangulations with *attached cubes* into five types in Section 6. The last section is devoted to prove Theorem 1.1.

2 Basic definitions

We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. A k-*path* (resp., k-*cycle*) in a graph G means a path (resp., cycle) of length k. (We define the *length* of a path (or cycle) by the number of its edges.) We say that $S \subset V(G)$ is a *cut* of a connected graph G if $G - S$ is disconnected. In particular, S is called a k-*cut* if S is a cut with $|S| = k$. A cycle C of G is *separating* if $V(C)$ is a cut.

Let G be a graph 2-cell embedded on a closed surface F^2. That is, each connected component of $F^2 - G$ is homeomorphic to an open 2-cell (or an open disc), which is called a *face* of G. We denote the face set of G by $F(G)$. A *facial cycle* C of a face f is a cycle bounding f in G; i.e., $C = \partial f$. Furthermore in our argument, we often discuss the interior of a 2-cell region D bounded by a closed walk W of G, i.e., $W = \partial D$, which contains some vertices and edges. (Note that a 2-cell region implies an "open" 2-cell region in this paper.) Then, $\bar{D}$ (resp., $\bar{f}$) denotes a closure of D (resp., f), i.e., $\bar{D} = D \cup \partial D$ (resp., $\bar{f} = f \cup \partial f$). Let $f_1, \dots, f_k$ denote the faces of G incident to $v \in V(G)$ where $\deg(v) = k$. Then, the boundary walk of $\bar{f}_1 \cup \dots \cup \bar{f}_k$ is the *link walk* of v and denoted by $lw(v)$. Clearly, $lw(v)$ bounds a 2-cell region containing a unique vertex v.

A simple closed curve γ on a closed surface F^2 is *trivial* if γ bounds a 2-cell on F^2, and γ is *essential* otherwise. Furthermore, γ is *surface separating* if $F^2 - \gamma$ is disconnected. Clearly, a trivial closed curve on F^2 is always separating, whereas an essential one is either separating or not. We apply these definitions to cycles of graphs embedded in the surface, regarding them as simple closed curves. It is an important property of the projective plane that any two essential simple closed curves are homotopic to each other.

Let G be a quadrangulation of a closed surface F^2 and let f be a face of G bounded by a cycle $v_0v_1v_2v_3$. (For brevity, we also use the notation like $f = v_0v_1v_2v_3$.) The *face-contraction* of f at $\{v_0, v_2\}$ in G is to identify v_0 and v_2, and replace the two pairs of multiple edges $\{v_0v_1, v_2v_1\}$ and $\{v_0v_3, v_2v_3\}$ with two single edges respectively. In the resulting graph, let $[v_0v_2]$ denote the vertex arisen by the identification of v_0 and v_2. See the left-hand side of Figure 1. The inverse operation of a face-contraction is called a *vertex-splitting*. If the graph obtained from G by a face-contraction is not simple, then we do not apply it.

Let G be a quadrangulation of a closed surface F^2, and let f be a face of G bounded by $v_0v_1v_2v_3$. A 4*-cycle addition* to f is to put a 4-cycle $C = u_0u_1u_2u_3$ inside f in G and join v_i and u_i for each $i \in \{0, 1, 2, 3\}$. The inverse operation of a 4-cycle addition is called a 4*-cycle removal* (of C), as shown in the center of Figure 1. We call the subgraph H isomorphic to a cube with eight vertices u_i, v_i for $i \in \{0, 1, 2, 3\}$ an *attached cube*. We denote $\partial(H) = v_0v_1v_2v_3$, and we call C an *attached* 4*-cycle* of H.

As mentioned in the introduction, there exist some results of 3-connected quadrangulations (or quadrangulations with minimum degree 3) of closed surfaces; see [2, 3, 13, 17] for example. In those results, the 4-cycle removal is necessary by the following reason: Let $\tilde{G}$ denote the resulting graph obtained from a 3-connected quadrangulation G of a closed surface by applying 4-cycle additions to all faces of G. Clearly $\tilde{G}$ is 3-connected, however we cannot apply any face-contraction to $\tilde{G}$ without making a vertex of degree 2.

In [3, 17], pseudo double wheels W_{2k} ($k \geq 3$) and a Möbius wheels $\widetilde{W}_{2k-1}$ ($k \geq 2$) are treated as minimal quadrangulations of the sphere and the projective plane, respectively; for their formal definitions, see [17]. However, the following third reduction can reduce a pseudo double wheel W_{2k} ($k \geq 4$) into $W_{2(k-1)}$. That is, W_{2k} can be deformed into a cube by $k - 3$ such reductions.

Assume that a polyhedral quadrangulation G of a closed surface F^2 has a vertex u of degree 3. (Every 3-connected quadrangulation of either the sphere or the projective plane has such a vertex of degree 3, by Euler's formula.) Let $v_0v_1 \cdots v_5$ be a 6-cycle bounding a 2-cell region D on F^2, which contains a unique vertex u and we assume that v_1, v_3 and v_5 are neighbors of u. The 2_3*-vertex splitting* of u is the expansion of G, defined as follows:

(i) Delete u and the three edges incident to u.

(ii) Put a 2-path $u_0u_1u_2$ into the interior of D and add edges u_0v_1, u_0v_3, u_1v_0, u_2v_3 and u_2v_5.

Note that each of u_0, u_1 and u_2 has degree 3 in the resulting graph. The inverse operation of a 2_3-vertex splitting is called a 2_3-*path shrink*, as shown in the right-hand side of Figure 1.

Similarly to the case of 4-cycle removals, it is not difficult to see that 2_3-path shrinks are necessary, when considering $\mathcal{P}$-irreducible quadrangulations; replace an attached cube with a graph having a long path consisting of vertices of degree 3 under some conditions. Now, we have defined all the operations in the paper. Note that all of them preserve the bipartiteness of quadrangulations of closed surfaces.

3 Lemmas

To prove our main theorem, we show some lemmas which state properties of polyhedral ($\mathcal{P}$-irreducible) quadrangulations of closed surfaces. We first give the following proposition which is however clear by the definition of polyhedral quadrangulations.

Proposition 3.1. *A polyhedral quadrangulation has no vertex of degree* 2.

The following holds not only for quadrangulations but also for even embeddings of closed surfaces F^2, that is, a graph on F^2 with each face bounded by a cycle of even length. Taking a dual of an even embedding and using the odd point theorem, it is easy to show the following.

Lemma 3.2. *An even embedding of a closed surface has no separating closed walk of odd length.*

The length of two cycles in an even embedding of a closed surface F^2 have the same parity if they are homotopic to each other on F^2 (see [1, 7, 16]). Furthermore, it is well-known that any two essential closed curves on the projective plane are homotopic to each other, and hence the following holds.

Lemma 3.3. *The length of two essential cycles in an even embedding of the projective plane have the same parity.*

When classifying $\mathcal{P}$-irreducible quadrangulations in the latter half of the paper, we focus on whether such a quadrangulation is bipartite or non-bipartite.

Lemma 3.4. *If a quadrangulation G of the projective plane admits an essential cycle of even (resp., odd) length, then G is bipartite (resp., non-bipartite).*

Proof. If G admits an essential cycle of even length, then every essential cycle of G has even length by the previous lemma. Of course, all trivial cycles of G is separating and hence have even length by Lemma 3.2. Therefore, G is bipartite. □

We denote the set of vertices of a graph G with degree i by $V_i(G)$ (or simply V_i). In this paper, we often focus on the subgraph of G induced by V_3, and denote it by $\langle V_3 \rangle_G$. In [17], the following lemma was proved.

Lemma 3.5. *Let G be a quadrangulation of a closed surface F^2 with minimum degree at least* 3 *and assume that $\langle V_3 \rangle_G$ contains a cycle C of length k. Then $k \geq 3$ and one of the followings holds;*

(i) if $k = 4$, then G is a cube on the sphere or C is an attached 4*-cycle of an attached cube in G,*

(ii) if k is odd, then G is a Möbius wheel $\tilde{W}_k$ on the projective plane,

(iii) if k is even and at least 6, then G is a pseudo double wheel W_k on the sphere.

Let G be a quadrangulation of a closed surface F^2 and let $f = v_0v_1v_2v_3$ be a face of G. Then a pair $\{v_i, v_{i+2}\}$ is called a *diagonal pair* of f in G for each $i \in \{0, 1\}$. A closed curve γ on F^2 is a *diagonal k-curve* for G if γ passes only through distinct k faces $f_0, \dots, f_{k-1}$ and distinct k vertices $x_0, \dots, x_{k-1}$ of G such that for each i, f_i and f_{i+1} share x_i, and that for each i, $\{x_{i-1}, x_i\}$ forms a diagonal pair of f_i of G, where the subscripts are taken modulo k. Furthermore, we call a simple closed curve γ on F^2 a *semi-diagonal k-curve* if in the above definition $\{x_{i-1}, x_i\}$ is not a diagonal pair for exactly one i; note that $x_{i-1}x_i$ is an edge of ∂f_i in this case.

Lemma 3.6. *Let G be a quadrangulation of a closed surface F^2 with a 2-cut $\{x, y\}$. Then there exists a surface separating diagonal 2-curve for G only through x and y.*

Proof. Observe that every quadrangulation of any closed surface F^2 is 2-connected and admits no such closed curve on F^2 crossing G at most once. Thus there exists a surface separating simple closed curve γ on F^2 crossing only x and y, since $\{x, y\}$ is a cut of G.

We shall show that γ is a diagonal 2-curve. Suppose that γ passes through two faces f_1 and f_2 meeting at two vertices x and y. If γ is not a diagonal 2-curve, then x and y are adjacent on ∂f_1 or ∂f_2. Since G has no multiple edges between x and y, and since $\{x, y\}$ is a 2-cut of G, we may suppose that x and y are adjacent in ∂f_1, but not in ∂f_2. Here we can take a separating 3-cycle of G along γ. This contradicts Lemma 3.2. □

Lemma 3.7. *Let G be a 3-connected quadrangulation of a closed surface F^2, and let $f = v_0v_1v_2v_3$ be a face of G. If the face-contraction of f at $\{v_0, v_2\}$ violates the 3-connectedness of the graph but preserves the simplicity, then G has a separating diagonal 3-curve passing through v_0, v_2 and another vertex $x \in V(G) - \{v_0, v_1, v_2, v_3\}$.*

Proof. Let G' be the quadrangulation of F^2 obtained from G by the face-contraction of f at $\{v_0, v_2\}$. Since G' has connectivity 2, G' has a 2-cut. By Lemma 3.6, G' has a separating diagonal 2-curve γ' passing through two vertices of the 2-cut. Clearly, one of the two vertices must be $[v_0v_2]$ of G', which is the image of v_0 and v_2 by the contraction of f; otherwise, G would not be 3-connected, a contradiction. Let x be another vertex of G' on γ' other than $[v_0v_2]$. Note that x is not a neighbor of $[v_0v_2]$ in G'.

Now apply the vertex-splitting of $[v_0v_2]$ to G' to recover G. Then a diagonal 3-curve for G passing through only v_0, v_2 and x arises from γ' for G'. □

Lemma 3.8. *Let G be a 3-representative quadrangulation of a non-spherical closed surface F^2 and let $f = v_0v_1v_2v_3$ be a face of G. If the face-contraction of f at $\{v_0, v_2\}$ yields another quadrangulation with representativity at most 2 but preserves the simplicity, then G has either an essential diagonal 3-curve or an essential semi-diagonal 3-curve, which passes through v_0, v_2 and another vertex $x \in V(G) - \{v_0, v_1, v_2, v_3\}$.*

Proof. Let G' be the quadrangulation of the non-spherical closed surface F^2 obtained from G by a face-contraction of f at $\{v_0, v_2\}$. If the representativity of G' is at most 1, then G would have an essential simple closed curve crossing with G at most twice, contrary to G being 3-representative. Thus G' has representativity 2 and hence G' admits either an essential diagonal 2-curve or an essential semi-diagonal 2-curve. Similarly to Lemma 3.7, one of the two vertices passed by the curve must be $[v_0v_2]$ of G' and G has an essential diagonal (resp., semi-diagonal) 3-curve when the former (resp., the latter) case happens. □

The following lemmas show properties of $\mathcal{P}$-irreducible quadrangulations of non-spherical closed surfaces. To simplify our statements, we suppose that G represents a $\mathcal{P}$-irreducible quadrangulation of a non-spherical closed surface F^2 hereafter in this section.

Lemma 3.9. *If G has a 4-cycle $C = v_0v_1v_2v_3$ bounding a 2-cell region D, then there is no face f of G in D such that one of the diagonal pairs of f is $\{v_0, v_2\}$ or $\{v_1, v_3\}$.*

Proof. Suppose, for a contradiction, that G has a 4-cycle $C = v_0v_1v_2v_3$ bounding a 2-cell region D and a face f bounded by av_1cv_3 in D. We assume that D contains as few vertices of G as possible. We denote the subgraph of G in $\bar{D}$ by H; note that H can be regarded as a quadrangulation of the sphere.

Since C is separating, we have $\partial f \neq C$. Furthermore, G is $\mathcal{P}$-irreducible and hence f is not $\mathcal{P}$-contractible at $\{a, c\}$. If the face-contraction at $\{a, c\}$ breaks the simplicity of the graph, then G has edges $\{ax, cx\}$ for $x \in V(G) - \{v_1, v_3\}$. (Clearly, it does not have a loop.) If $x \in V(G) - V(H)$, we would have $\partial f = C$, contrary to our assumption. Thus, we may assume that x is either v_0 or v_2, now say v_0; observe that $v_0 \neq a, c$ in this case. Now G would have an edge av_0 (or cv_0) and it contradicts Lemma 3.2.

By the above argument, the face-contraction at $\{a, c\}$ does not break the simplicity, hence it breaks the 3-connectedness or the property of representativity at least 3. That is, we find either a surface separating diagonal 3-curve or an essential diagonal 3-curve (or an essential semi-diagonal 3-curve) passing through f and $\{a, c\}$ by Lemmas 3.7 and 3.8. In each case, if $\{a, c\} \cap \{v_0, v_2\} = \emptyset$, then f could not be passed by such a diagonal curve. Therefore we may suppose that $a = v_0$ and $c \neq v_2$.

By Lemma 3.2 again, there is not an edge joining c and v_2. Thus, we can find a face f' of H one of whose diagonal pairs is $\{c, v_2\}$. Let C' be the 4-cycle $v_1v_2v_3c$ of G. Since $\deg(c) \geq 3$, we have $\partial f' \neq C'$. Therefore, C' and f' are a 4-cycle and a face which satisfy the assumption of the lemma, and moreover, C' can cut a strictly smaller graph than H from G. Thus, this contradicts the choice of C. □

Lemma 3.10. *Let $f = v_0v_1v_2v_3$ be a face of G. If the face-contraction of f at $\{v_0, v_2\}$ breaks the simplicity of the graph, then there is a vertex $x \in V(G) - \{v_1, v_3\}$ adjacent to both of v_0 and v_2 such that $v_0v_1v_2x$ is an essential 4-cycle in G. In particular, if F^2 is the projective plane, then G is bipartite.*

Proof. First, assume that the face-contraction yields a loop. Then, we have $v_0v_2 \in E(G)$. By Lemma 3.2, $v_0v_1v_2$ should be an essential 3-cycle. However, we would find an essential simple closed curve intersecting G at only v_0 and v_2, contrary to G being 3-representative.

Therefore, we may assume that the face-contraction yields multiple edges. Under the conditions, there should be a vertex $x \in V(G) - \{v_0, v_1, v_2, v_3\}$ which is adjacent to both of v_0 and v_2. If a 4-cycle $v_0v_1v_2x$ is trivial and bounds a 2-cell region D, then D and f would satisfy the conditions of Lemma 3.9, a contradiction. Therefore $v_0v_1v_2x$ should be essential. If F^2 is the projective plane, then G is bipartite by Lemma 3.4. □

Lemma 3.11. *If G has a trivial diagonal 3-curve γ, then the disc bounded by γ contains the unique vertex, which has degree 3.*

Proof. Suppose that γ passes through three vertices $\{v_0, v_1, v_2\}$ and three faces $\{f_0, f_1, f_2\}$ where each f_i is bounded by $v_i a_i v_{i+1} b_i$ so that the 6-cycle $v_0 b_0 v_1 b_1 v_2 b_2$ bounds a 2-cell region D including f_i and a_i for $i \in \{0, 1, 2\}$ ($v_3 = v_0$). Suppose, for a contradiction, that D contains at least two vertices. That is, this implies that a_0, a_1 and a_2 could not be identified to one vertex. Thus, we can find a vertex $a_i \neq a_{i+1}, a_{i+2}$, now say a_0 ($a_0 \neq a_1, a_2$).

If there is an edge joining a_0 and v_2, then we can find a 2-cell region D' bounded by $a_0 v_2 b_2 v_0$. Since $a_0 \neq a_2$, D' is not a face of G. Furthermore we have $\deg(a_2) \geq 3$ and hence the region bounded by $a_0 v_2 a_2 v_0$ is not a face of G and includes at least one vertex. This means that D' satisfies the conditions of Lemma 3.9, a contradiction. Therefore, we conclude that $a_0 v_2 \notin E(G)$.

Now consider the face-contraction of f_0 at $\{a_0, b_0\}$. Since G is $\mathcal{P}$-irreducible, G should have a diagonal 3-curve or a semi-diagonal 3-curve passing through three vertices $\{a_0, b_0, x\}$ for $x \in V(G) - \{a_0, b_0\}$. (Note that the face-contraction clearly preserves the simplicity of the graph by the above argument, i.e., $a_0 v_2 \notin E(G)$.) Since a_0 is an inner vertex of D, x must be a vertex of ∂D.

However, since $a_0 \neq a_1, a_2$, x must coincide with v_2. Since $a_0 v_2 \notin E(G)$ again, there should be a face whose diagonal pair is $\{a_0, v_2\}$, but it contradicts Lemma 3.2. Hence, we can conclude that D contains exactly one vertex a_0 ($= a_1 = a_2$) and the lemma follows. □

Lemma 3.12. *Let $f = v_0 v_1 v_2 v_3$ be a face of G with* $\deg(v_0), \deg(v_2) \geq 4$.

(i) If F^2 is the projective plane, then a face-contraction of f at $\{v_1, v_3\}$ preserves the 3-connectedness.

(ii) If F^2 is not the projective plane and if a face-contraction of f at $\{v_1, v_3\}$ breaks the 3-connectedness, then G has an essential separating diagonal 3-curve γ passing through v_1, v_3 and another vertex $x \in V(G) - \{v_0, v_1, v_2, v_3\}$.

Proof. The statement (ii) immediately follows from Lemmas 3.7 and 3.11. In the projective-planar case, we cannot take such an essential separating diagonal 3-curve γ. □

Lemma 3.13. *The induced subgraph $\langle V_3 \rangle_G$ has no vertex of degree* 3.

Proof. Suppose, for a contradiction, that G has a vertex v with $\deg(v) = 3$ and each of its three neighbors also has degree 3 (see the left-hand side of Figure 3). Note that the boundary of the hexagon is a cycle of G; otherwise, it would disturb the simplicity of G, Lemma 3.2, Lemma 3.9 or the property of representativity at least 3. We can easily find a trivial separating diagonal 3-curve passing through $\{v_0, v_1, v_2\}$ and that the 3-cut cuts off the four vertices, contrary to Lemma 3.11. □

Suppose that the induced subgraph $\langle V_3 \rangle_G$ of a $\mathcal{P}$-irreducible quadrangulation G has a path $P = u_0 u_1 u_2$ of length 2. Then the configuration around P becomes the center of Figure 3. The following lemma refers to the non-shrinkability of P.

Lemma 3.14. *Let $P = u_0 u_1 u_2$ be a 2-path in G induced by vertices of degree 3 (as shown in the center of Figure 3) and assume that* $\deg(v_4) \geq 4$. *Then, there is an essential diagonal 3-curve or an essential semi-diagonal 3-curve passing through $\{v_0, u_1, v_2\}$.*

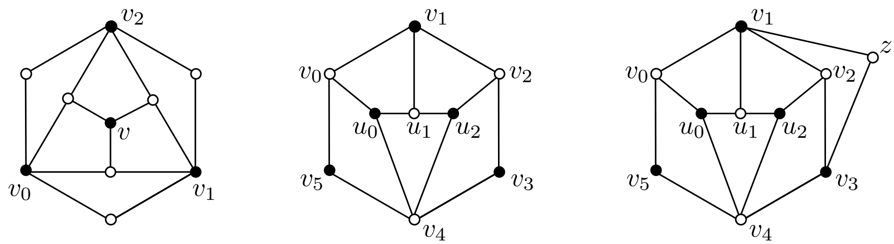

Figure 3: Partial structures of $\mathcal{P}$-irreducible quadrangulations.

Proof. Apply the 2_3-path shrink to P and denote G' be the resulting graph. Let u be a vertex of G' which is the shrunk image of P; note that u is adjacent to v_0, v_2 and v_4. Since G is $\mathcal{P}$-irreducible, G' is not a polyhedral quadrangulation. If G' is not simple, uv_0 and uv_2 must be multiple edges. This implies that $v_0 = v_2$, however this also implies that G is not simple or v_1 has degree 2 in G, a contradiction.

Next, we assume that G' has a 2-cut. By Lemma 3.6, G' has a separating diagonal 2-curve γ' passing through $\{v_0, v_2\}$; otherwise, G would have a 2-cut. Now we can find a separating diagonal 3-curve γ in G corresponding to γ' naturally. Note that γ is not a semi-diagonal 3-curve by Lemma 3.2. Let $f = v_0xv_2y$ be the third face passed by γ, which lies outside of the hexagon bounded by $v_0v_1v_2v_3v_4v_5$. If γ is essential, then we are done. Therefore, we assume that γ is trivial. If neither of x and y corresponds to v_1, then we have got a contradiction by Lemma 3.9. Thus, one of x and y, say x, corresponds to v_1. This means that $\deg(v_1) = 3$, however, it contradicts Lemma 3.13.

Finally, assume that G' has representativity at most 2. Similarly, G' has an essential diagonal 2-curve or an essential semi-diagonal 2-curve passing through $\{v_0, v_2\}$. We can easily find our required essential curve passing through $\{v_0, u_1, v_2\}$ of G. □

Lemma 3.15. *The induced subgraph $\langle V_3 \rangle_G$ has no path of length at least* 3.

Proof. Suppose to the contrary that G has such a path $P = u_0u_1u_2v_2$ (see the right-hand side of Figure 3). By the above lemma, z should coincide with v_0. However, v_1z would become multiple edges, a contradiction. □

Lemma 3.16. *Assume that G has an attached cube H with $\partial(H) = v_0v_1v_2v_3$, an attached* 4*-cycle $C = u_0u_1u_2u_3$ and $u_iv_i \in E(G)$ for each $i \in \{0, 1, 2, 3\}$. Then there is an essential diagonal (or semi-diagonal)* 3*-curve γ passing through $\{v_0, u_1, v_2\}$ or $\{v_1, u_2, v_3\}$.*

Proof. Apply the 4-cycle removal of C to G and let G' denote the resulting graph. It is clear that the 4-cycle removal clearly preserves the simplicity of the graph. Thus, first suppose that G' is not 3-connected. By Lemma 3.6, we can find a separating diagonal 2-curve γ' in G' passing through $\{v_0, v_2\}$ or $\{v_1, v_3\}$. If γ' is trivial, then it contradicts Lemma 3.9. If γ' is essential, we can find our requied diagonal 3-curve γ in G.

Therefore, we may assume that G' has representativity at most 2 and has an essential diagonal (or semi-diagonal) k-curve γ' where k is at most 2. If γ' does not pass through a face $f = v_0v_1v_2v_3$, then G also has representativity at most 2, contrary to our assumption. Thus, γ' passes through f and two vertices $\{v_0, v_2\}$ or $\{v_1, v_3\}$ and we got our conclusion. (Note that γ' does not pass through two neighboring vertices of $v_0v_1v_2v_3$. Otherwise, γ' would be an essential semi-diagonal 2-curve also in G.) □

For an attached cube H with $\partial(H) = v_0v_1v_2v_3$, we call a pair of two vertices $\{v_i, v_{i+2}\}$ a *cube diagonal pair* of H for each $i \in \{0, 1\}$. In particular, a cube diagonal pair is *facing* if they are on a boundary cycle of a face f of G outside the 2-cell region bounded by $\partial(H)$. According to the above argument, an essential diagonal (or semi-diagonal) 3-curve passes through f.

4 Regions bounded by 4-, 6- or 8-cycles

Consider a disk D bounded by a cycle $C = v_0v_1 \cdots v_{2m-1}$ of length $2m$. Put a vertex x into the center of D and join it to v_{2i} for each $i \in \{0, \dots, m-1\}$. Then, the resulting disk quadrangulation is a *pseudo wheel* and denoted by W^-_{2m}.

Lemma 4.1. *Let G be a quadrangulation of a closed surface F^2 and let D be a 2-cell region bounded by a closed walk C of length $4, 6$ or 8 such that*

(i) there is at least one vertex inside D,

(ii) all vertices inside D have degree at least 3 and

(iii) D does not have a unique vertex x of degree 4 such that $lw(x) = C$ (when $|C| = 8$).

Then, there exists a vertex of degree 3 inside D.

Proof. Let H be a graph contained in $\bar{D}$. It suffices to prove the case when C is a cycle. (Even if C is not a cycle, i.e., there exists a vertex appearing twice on C, the analogous proof works.) We use induction on $|V(H)|$. Let $v_0, \dots, v_{m-1}$ be vertices lying on C in this order for some $m \in \{4, 6, 8\}$. The initial step of the induction is the case that $|V(H)| = 7$. In this case, H must be isomorphic to W^-_6 and its center vertex has degree 3. (When the length of C equals 4, it is not difficult to list up all the (disc) quadrangulations with at most 7 vertices, e.g., see [19]. Every such graph has a vertex of degree 2 not lying on any specified outer cycle.) Thus, we suppose that $|V(H)| \geq 8$ in the following argument.

First, assume that there is a diagonal of C. Since at least one of the two regions separated by the diagonal satisfies Conditions (i) – (iii), there is a vertex of degree 3 inside the region by the induction hypothesis. Thus, we suppose that there is no diagonal in D.

Furthermore, suppose that there is a vertex x joining two vertices v_i and v_{i+2}. Then, the 2-path v_ixv_{i+2} separates D into a quadrilateral region D' and the other region D''. If D' contains a vertex, then the induction hypothesis works immediately. Thus, we may assume that D' contains no vertex. Further, if D'' contains at least one vertex and $G \cap \bar{D}''$ is not isomorphic to W^-_8, then we can also apply the induction hypothesis. When the case that $G \cap \bar{D}''$ is isomorphic to W^-_8, the unique inner vertex y of D'' should be adjacent to x, and hence x has degree 3; otherwise, the degree of x would become 2.

Therefore, we suppose that D'' contains no vertex. Under the condition, there should be edges joining x and alternate vertices on C so that H becomes disc quadrangulation since C has no diagonal. Then, H is isomorphic to W^-_8 since $|V(H)| \geq 8$. However, it contradicts (iii).

By the above arguments, we may assume that D contains no diagonal and no 2-path joining v_i and v_{i+2}. This implies that all vertices v_i of C have degree at least 3. When $|C|$ is equal to 6 or 8, add an extra vertex $\hat{x}$ outside D and join it to alternate vertices to obtain a quadrangulation $\hat{H}$ of the sphere; if $|C| = 4$, then we do nothing and let $\hat{H} = H$. Observe that $\hat{H}$ has minimum degree at least 3.

By Euler's formula, we have $|V_3(\hat{H})| \geq 8$. Even if $|C| = 8$, the number of vertices of degree 3 on C is at most 4 by our construction of $\hat{H}$. Therefore, the lemma follows. □

The following lemma is important to determine the inner structures of 2-cell regions of $\mathcal{P}$-irreducible quadrangulations bounded by closed walks of length 4, 6 or 8.

Lemma 4.2. *Let G be a $\mathcal{P}$-irreducible quadrangulation of a non-spherical closed surface and let D be a 2-cell region bounded by a closed walk $W = w_0 w_1 \cdots w_{k-1}$ for some $k \in \{4, 6, 8\}$. Suppose that W does not bound a face of G and that $G \cap \bar{D}$ is not isomorphic to an attached cube. Then $G \cap \bar{D}$ includes;*

(i) a diagonal edge (when $k \in \{6, 8\}$),

(ii) a 2-path $w_i x w_{i+2}$,

(iii) a 2-path $w_i x w_{i+4}$ (when $k = 8$ and $w_i \neq w_{i+4}$),

(iv) a 3-path (or a 3-cycle if $w_i = w_{i+3}$) $w_i x y w_{i+3}$ (when $k \in \{6, 8\}$) or

(v) a 4-cycle $w_i x y z w_{i+4}$ (when $k = 8$ and $w_i = w_{i+4}$),

where x, y and z are distinct inner vertices of D and the indices are taken modulo k.

Proof. In this proof, we call a path (or a cycle) in the statement a *short path* of D. Suppose, for a contradiction, that D includes no short path. By Lemma 4.1, D contains a vertex of degree 3 as an inner vertex; since if D has a unique vertex, then it clearly includes a short path of type (ii). First, assume that D contains a vertex u_i of degree 3 of an attached cube Q; where Q consists of a 4-cycle $C = u_0 u_1 u_2 u_3$ induced by vertices of degree 3 and $\partial(Q) = v_0 v_1 v_2 v_3$ with an edge $u_i v_i$ for each $i \in \{0, 1, 2, 3\}$. We consider the cases depending on the order of $V(C) \cap V(W)$.

Case I. $|V(C) \cap V(W)| = 1$ (assume $w_0 = u_0$): Then u_0 would have a vertex of degree at least 4, contrary to the assumption.

Case II. $|V(C) \cap V(W)| = 2$: If such vertices are diagonal vertices of C, say u_0 and u_2, then we have $\deg(u_0) \geq 4$, as well as the above case. Thus, we suppose that such two vertices are adjacent on both of C and W, say $w_0 = u_0$ and $w_1 = u_1$. Note that u_2 and u_3 are inner vertices in this case. Since $\deg(w_0) = \deg(w_1) = 3$, v_0 (resp., v_1) should coincide with w_{k-1} (resp., w_2). In this case, v_2 and v_3 are inner vertices of D; otherwise D would contain a short path (ii) or (iii). However, $w_2 v_2 v_3 w_{k-1}$ would become (iv) if $k \in \{6, 8\}$; note that if $k = 4$, then $w_{k-1} w_2$ would form multiple edges since $\deg(w_0) = \deg(w_1) = 3$.

Case III. $|V(C) \cap V(W)| = 3$: We can easily exclude this case, since the unique inner vertex of C is adjacent to two vertices of W and it would form either (ii) or (iii).

Case IV. $V(C) \cap V(W) = \emptyset$: By Lemma 3.16, at least one of cube diagonal pairs, say $\{v_0, v_2\}$, should be facing. We further divide this case into the following subcases.

Case IV-a. W is a cycle of G: Then both of v_0 and v_2 should be vertices of W. Note that by Lemma 3.2, $\{v_0, v_2\}$ coincides with $\{w_i, w_{i+2}\}$ or $\{w_i, w_{i+4}\}$. If one of v_1 and v_3 is an inner vertex of D, then D clearly would contain a 2-path of (ii) or (iii) in the lemma. Therefore, they also should be vertices of W. However, if k equals 6 or 8, then D would have a diagonal edge (i), on the other hand, if $k = 4$, then it corresponds to an attached cube, contrary to our assumption.

Case IV-b. W is not a cycle: Note that we only have to consider the case of $k \in \{6, 8\}$. This case is further divided into the following subcases.

Case IV-b-1. $w_i = w_{i+3}$ (say $w_0 = w_3$): Note that G is nonbipartite since it includes an essential cycle of odd length. Now we may suppose that an essential simple closed curve of Lemma 3.16 passes through such a vertex $w_0 = w_3$. We may suppose that $v_0 = w_0$ in this case and there should be the edge v_2w_3 (see (a) in Figure 4). In the figure, we find a hexagonal region bounded by $W' = w_0w_1w_2w_3v_2v_1$. If there is no identification of vertices of W', then we would have a short path $w_0v_1v_2w_3$ of type (iv). Even if there is such an identification, we find either a short path (i) or (ii), a contradiction.

Case IV-b-2. $w_i = w_{i+4}$ (assume $w_0 = w_4$): Similarly to the above arguments, we assume that $v_0 = w_0$ and there is a face bounded by v_2sw_4t in G where $s, t \in V(G)$. If there is no identification of vertices of closed walk $W'' = w_0w_1w_2w_3w_4sv_2v_1$ bounding an octagonal region, there would be a short path of type (v). When there is identification of vertices of W'', we pay attention to the simplicity and the representativity of the whole graph; e.g., if $v_1 = s$, we would have multiple edges w_0v_1. In any case, we find our required short path.

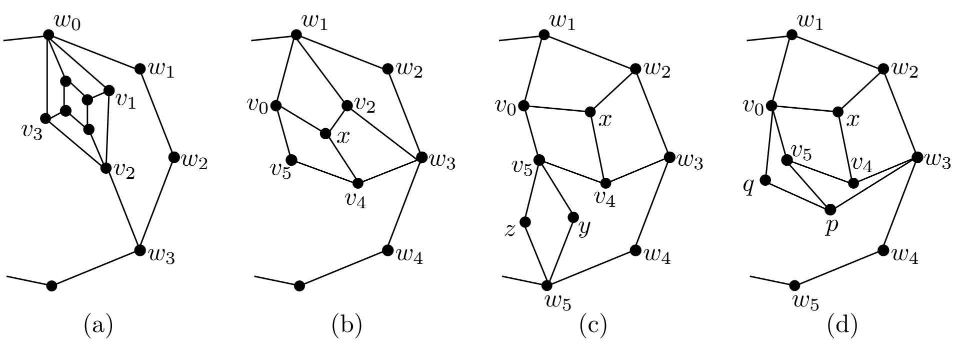

Figure 4: Inside of a region bounded by closed walks of length 4, 6 or 8.

Therefore after this, we may assume that D does not contain a vertex of a 4-cycle induced by vertices of degree 3, that is, each inner vertex of degree 3 is on the path of $\langle V_3 \rangle_G$ with length at most 2, by Lemmas 3.5, 3.13 and 3.15; note that a Möbius wheel in Lemma 3.5 is not polyhedral. We can take an inner vertex x of degree 3 so as to be an endpoint of a path of $\langle V_3 \rangle_G$; otherwise, each path of $\langle V_3 \rangle_G$ would join two vertices of W, contrary to our assumption and Lemmas 3.2 and 3.15.

Let $lw(x) = v_0v_1v_2v_3v_4v_5$ be the link walk of x and assume that v_0, v_2 and v_4 are adjacent to x and that $\deg(v_0), \deg(v_2) \geq 4$. Now we apply the face-contraction of $xv_0v_1v_2$ at $\{x, v_1\}$, and denote the resulting graph by G'.

We first assume that G' is not simple. By Lemma 3.10, there is an edge joining v_1 and v_4 in G such that a cycle $v_1v_2xv_4$ of G is essential. Suppose that the edge v_1v_4 is in D. Clearly, W is not a cycle, and we may assume that $k = 8$ and that $w_0 = w_4 = v_4$. However, it easily follows that there exists a short path passing through x. Also in the case that v_1v_4 runs outside of D, v_1 and v_4 should be vertices of W and hence we can find a short path. Then, assume that G' is simple in the following argument.

Next, we assume that either the representativity or connectivity of G' is at most 2. In each case, G has an essential diagonal (or semi-diagonal) 3-curve γ passing through x and v_1 by Lemmas 3.8 and 3.12. In fact, there are some cases depending on the positions and identifications of vertices in D. However, in each case, the similar argument holds and hence we prove only one substantial case below, for the sake of brevity.

Here, we consider the case that γ passes through $\{v_1, x, v_3\}$ and that $v_1 = w_1$ and $v_3 = w_3$ (see (b) in Figure 4). In this case, if $v_2 \neq w_2$, D would contain a 2-path $w_1v_2w_3$ of type (ii) in the lemma. Therefore we suppose $v_2 = w_2$. Note that each of v_0, v_4 and v_5 is an inner vertex of D and that there is no edge v_lw for $l \in \{0, 4, 5\}$ and $w \in V(W)$; otherwise, there would be a short path.

Next, we assume $\deg(v_4) \geq 4$ and consider the face-contraction of $v_0xv_4v_5$ at $\{v_5, x\}$. By the above argument, v_5 has no adjacent vertex of W and hence we do not have to care about the simplicity of the resulting graph. Thus, similarly to the above argument, we can find a face v_5yw_5z in D by Lemmas 3.8 and 3.12, where either $w_1 = w_5$ or $w_2 = w_5$, i.e., W is not a cycle of G. We assume $w_1 = w_5$ here. (The case when $w_2 = w_5$ can be shown in a similar way.) See (c) in Figure 4. Actually, $k \neq 4$ in this case. Note that y and z are inner vertices of D and further note that $\{y, z\} \cap \{v_0, v_4\} = \emptyset$ by the above argument. It also implies that $\deg(v_5) \geq 4$ and $\deg(w_5) \geq 4$.

By Lemmas 3.8 and 3.12 again and by Lemma 3.2, there should be diagonal 3-curve γ'' passing z, y and $w \in V(W)$; note that semi-diagonal 3-curve is not suitable since each of y and z is not adjacent to a vertex of $V(W)$. In this case, we have $k = 8$ and $w = w_0 = w_4$ since if $w = w_4 = w_6$, w_4w_5 and w_5w_6 become multiple edges. However in this case, we find a short path (iv) of length 3 linking w_0 and w_5 (or a short path (iii) of length 2 linking w_5 and w_7).

Therefore, suppose that $\deg(v_4) = 3$ and there is a face $w_3v_4v_5p$ where p is an inner vertex of D; otherwise we would find a short path. Observe that $\deg(w_3) \geq 4$ in this case. Furthermore, if $\deg(v_5) \geq 4$, then we consider the face-contraction of $w_3v_4v_5p$ at $\{p, v_4\}$. Similarly to the above argument, there must be a face psw_6t where $w_2 = w_6$ since x and v_0 are inner vertices of D and hence there is an essential diagonal 3-curve passing through $\{w_2, v_4, p\}$. However, we find a short 3-path w_3psw_6 in this case.

Hence, we may assume that $\deg(v_5) = 3$ and there is a face v_0v_5pq (see (d) in Figure 4). Then there is a 2-path xv_4v_5 induced by vertices of degree 3. By Lemma 3.14, there should be an essential diagonal (or semi-diagonal) 3-curve passing through $\{w_2, v_4, p\}$. Similarly, we can find a short path around it. (For example, if $w_2 = w_5$ and the edge $pw_5 \in E(G)$ exists, then we find a short path w_3pw_5 of type (ii).) Thus, the lemma follows. □

Figure 5 shows some partial structures of polyhedral quadrangulations of closed surfaces, each of which is bounded by a trivial 4-cycle $v_0v_1v_2v_3$. The center graph in the figure has a 4-cycle $u_0u_1u_2u_3$ induced by vertices of degree 3 and hence this partial structure is an attached cube. Recall that if a polyhedral quadrangulation is $\mathcal{P}$-irreducible and has an attached cube, then one of two cube diagonal pairs is facing by Lemma 3.16. Next, see the right-hand side of Figure 5. For a natural number n, $Q_2^{(n)}$ represents the graph having the following structure: There are $n + 1$ internally vertex-disjoint paths of length 2 between v_0 and v_2, including $v_0v_1v_2$ and $v_0v_3v_2$, so that they divide the region bounded by $v_0v_1v_2v_3$ into n quadrilateral regions each of which has the structure Q_2 having a facing cube diagonal pair $\{v_0, v_2\}$. Note that $Q_2^{(1)}$ corresponds to Q_2.

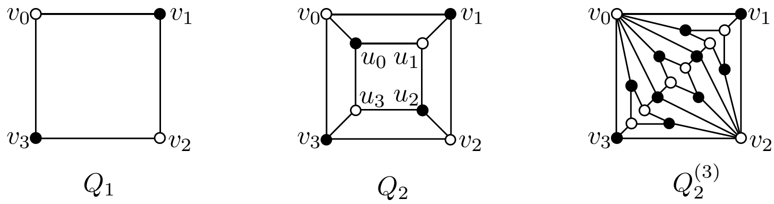

Figure 5: Inside a quadrilateral region.

Lemma 4.3. *Let $C = v_0v_1v_2v_3$ be a cycle of length 4 bounding a 2-cell region D in a $\mathcal{P}$-irreducible quadrangulation G of a non-spherical closed surface. Then, the interior of D has one of the structures Q_1 and $Q_2^{(n)}$ ($n \geq 1$), as shown in Figure 5.*

Proof. Use induction on the number of faces in D, say $F \geq 1$. If $F = 1$, then it is clear that D corresponds to a face of G and it has the structure Q_1. Hence we suppose that $F \geq 2$ below.

If $G \cap \bar{D}$ is not an attached cube Q_2, then there is a vertex x which is adjacent to both v_0 and v_2 (or v_1 and v_3) by Lemma 4.2. By the inductive hypothesis and Lemma 3.9, two quadrilateral regions bounded by $v_0v_1v_2x$ and $v_2v_3v_0x$ are filled with $Q_2^{(l)}$ and $Q_2^{(m)}$ for $n, m \geq 1$. As a result, we obtain $Q_2^{(n)}$ with $n = l + m$ and the induction is completed. □

Note that replacing Q_2 with $Q_2^{(n)}$ having the same facing cube diagonal pair preserves the property being a $\mathcal{P}$-irreducible quadrangulation for any $n \geq 2$. Hence, there exist infinitely many $\mathcal{P}$-irreducible quadrangulations of a non-spherical closed surface F^2 if F^2 admits one with an attached cube. To avoid the complexity in figures, we use simply Q_2 to represent any $Q_2^{(n)}$ after this.

In the following lemmas, we discuss inside structures of regions bounded by 6- and 8-cycles. For brevity, we shall omit routines in the proofs.

Lemma 4.4. *Let $C = v_0v_1v_2v_3v_4v_5$ be a trivial cycle of length 6 bounding a 2-cell region D in a $\mathcal{P}$-irreducible quadrangulation G of a non-spherical closed surface. Then, the interior of D has one of the structures $H_1, H_2, \ldots, H_{17}$, as shown in Figure 6.*

Proof. As well as the previous lemma, we use induction on the number of faces in D, say $F \geq 2$. If $F = 2$, then D has the structure H_1. Hence we suppose that $F \geq 3$. Observe that the existence of a short path of (i), (ii) or (iv) is guaranteed by Lemma 4.2. We fill the divided regions with pieces as follows.

If C has a diagonal, then we apply Lemma 4.3 and obtain H_1, H_6 and H_{10} in Figure 6. Further, if there is an inner vertex x which is adjacent to both v_0 and v_2, then the quadrilateral region bounded by $xv_0v_1v_2$ is filled with Q_1 or $Q_2^{(n)}$ ($n \geq 1$), and the hexagonal region bounded by $v_0xv_2v_3v_4v_5$ is filled with H_i for some $i \in \{1, \ldots, 17\}$ by the inductive hypothesis. Checking the whole cases is a routine, so we omit it, however, most cases are excluded by lemmas in Section 3.

Furthermore, assume that D contains two inner vertices x and y such that 3-path v_0xyv_3 runs across D. Also in this case, we apply the inductive hypothesis to two separated hexagonal regions and obtain H_i's in Figure 6. □

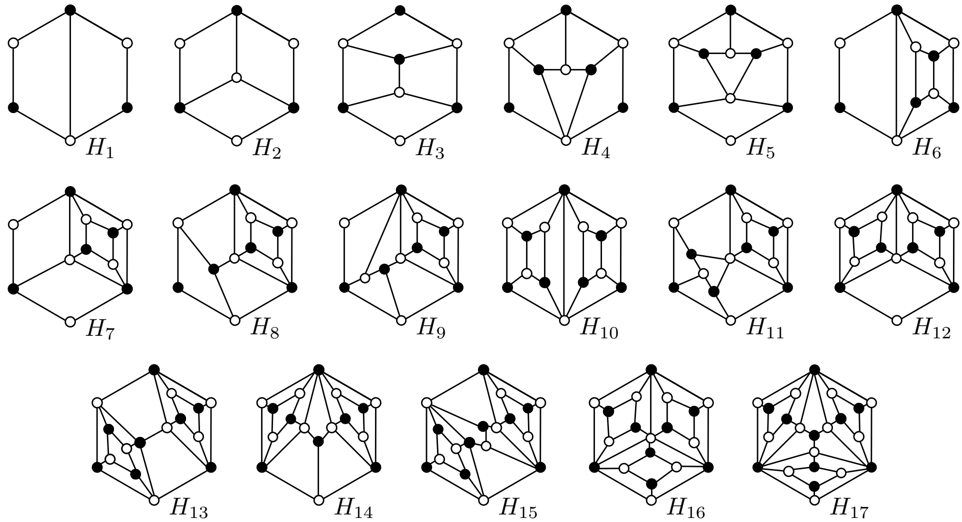

Figure 6: Inside a hexagonal region.

Lemma 4.5. *Let $C = v_0v_1v_2v_3v_4v_5v_6v_7$ be a trivial cycle of length 8 bounding a 2-cell region D in a $\mathcal{P}$-irreducible quadrangulation G of a non-spherical closed surface. If D has no diagonal edge and no attached cube, then the interior of D has one of the structures $O_1, O_2, \ldots, O_8$, as shown in Figure 7.*

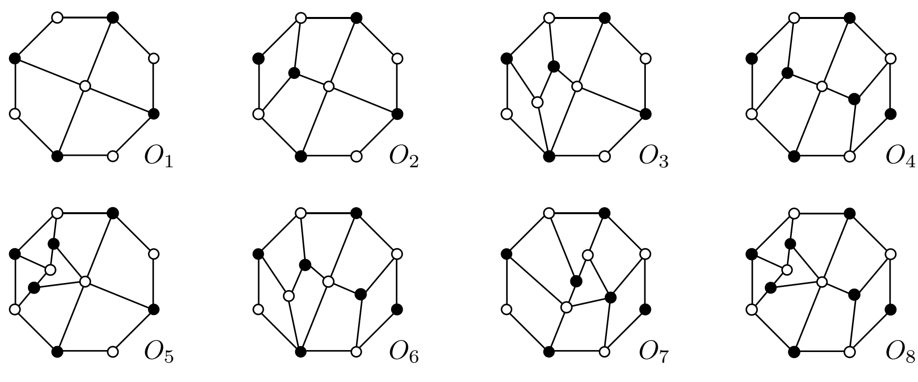

Figure 7: Inside an octagonal region.

Proof. In this proof, all subscripts of vertices are taken modulo 8. We also use induction on the number of faces in D, say F. If F is at most 3, then D has a diagonal, contrary to the assumption of the lemma. If $F = 4$, then D includes a single vertex by Euler's formula and it should be adjacent to v_i, v_{i+2}, v_{i+4} and v_{i+6}; for otherwise, D would contain a diagonal. This is clearly O_1 in Figure 7. Therefore, we assume $F \geq 5$ hereafter. Observe that D contains a short path of type (ii), (iii) or (iv) by Lemma 4.2.

First, we assume that D includes an inner vertex x which is adjacent to both v_0 and v_4. Then there are two hexagonal regions D' and D'' bounded by $xv_0v_1v_2v_3v_4$ and $xv_4v_5v_6v_7v_0$ respectively. Note that each of D' and D'' contains no attached cube. Then by the previous lemma, we fill them with H_1, H_2, H_3, H_4 and H_5 in Figure 6 so that the whole configuration satisfies the condition of this lemma. By considering lemmas in Section 3, most cases are excluded and we obtain $O_1, O_2, O_3, O_4, O_5, O_6$ and O_8 in Figure 7. Therefore, after this, we suppose that D contains no such vertex.

Secondly we assume that there is an inner vertex x in D which is adjacent to both v_0 and v_2. Then, there are a quadrilateral region D' and an octagonal region D'' divided by the 2-path v_0xv_2. By the assumption and Lemma 4.3, D' bounds a face of G. If D'' contains a diagonal edge, then it should be xv_4 or xv_6 by Lemma 3.2. However, in each case, there would be a forbidden 2-path; e.g., v_0xv_4 if the diagonal xv_4 exists. Hence, we may assume that D'' contains no diagonal. Now we apply the inductive hypothesis and fill D'' with $O_1, \dots, O_8$ in Figure 7; note that most cases would contain a contractible face or a shrinkable 2-path by lemmas in Section 3. As a result, we obtain $O_1, \dots, O_8$. Then we also assume that D does not include such a 2-path.

Finally, we assume that D has a short path of type (iv) in Lemma 4.2. Actually, this 3-path divides D into a hexagonal region and an octagonal one. As well as the above case, we use the inductive hypothesis and Lemma 4.4, and obtain our conclusion. □

Lemma 4.6. *Let G be a $\mathcal{P}$-irreducible quadrangulation of the projective plane. If G has a hexagonal 2-cell region D such that $G \cap \bar{D}$ is isomorphic to either H_{13} or H_{15} in Lemma 4.4, then G is one of I_1, I_2 and I_3 shown in Figure 8.*

Proof. Let $C = v_0v_1v_2v_3v_4v_5$ be a 6-cycle bounding a hexagonal region D such that $G \cap \bar{D}$ is isomorphic to either H_{13} or H_{15}. We may assume that each of $v_0v_1v_2x$ and $v_3v_4v_5y$ bounds Q_2 where x and y are distinct inner vertices of D. Now, cube diagonal pairs $\{v_0, v_2\}$ and $\{v_3, v_5\}$ are facing and there are such faces $f_1 = v_0pv_2q$ and $f_2 = v_3sv_5t$ outside of D by Lemma 3.16, where $p, q, s, t \in V(G)$.

However, if $f_1 \neq f_2$, the two essential diagonal (or semi-diagonal) curves in Lemma 3.16 do not exist together on the projective plane. Therefore, we have $f_1 = f_2$, that is, $v_0v_3, v_2v_5 \in E(G)$ and $f_1 = f_2$ is bounded by $v_0v_3v_2v_5$. Under the conditions, the 6-cycle $v_0xv_2v_5yv_3$ bounds a 2-cell region and it should be filled with either H_{13} or H_{15} by Lemma 4.4. Actually we have three ways to take a pair $\{H_i, H_j\}$ for $i, j \in \{13, 15\}$ and the lemma follows; for example, if we fill those hexagonal regions with two H_{13}'s then we obtain I_1. □

5 Regions bounded by 6- or 8-walks

A boundary walk of a hexagonal region of a $\mathcal{P}$-irreducible quadrangulation is not always a cycle, and the same vertex often appears twice along it. Such a hexagonal region can contain the following structure that generates an infinite series of $\mathcal{P}$-irreducible quadrangulations of a non-spherical closed surface.

Let h_1, h_2 and h_3 be three pieces with two terminals x_1 and x_2 shown in the first three configurations of Figure 9, and let $[s_1, \dots, s_m]$ be a given sequence of $1, 2$ and 3 of any length such that each of 2 and 3 does not continue; i.e., we do not permit a sequence like $[\dots, 2, 2, \dots]$. Put h_{s_1} to h_{s_m} in a hexagon $a_1b_1ca_2b_2d$ so that each x_i coincides with a_i for $i \in \{1, 2\}$, and identify paths between x_1 and x_2 in each neighboring pair

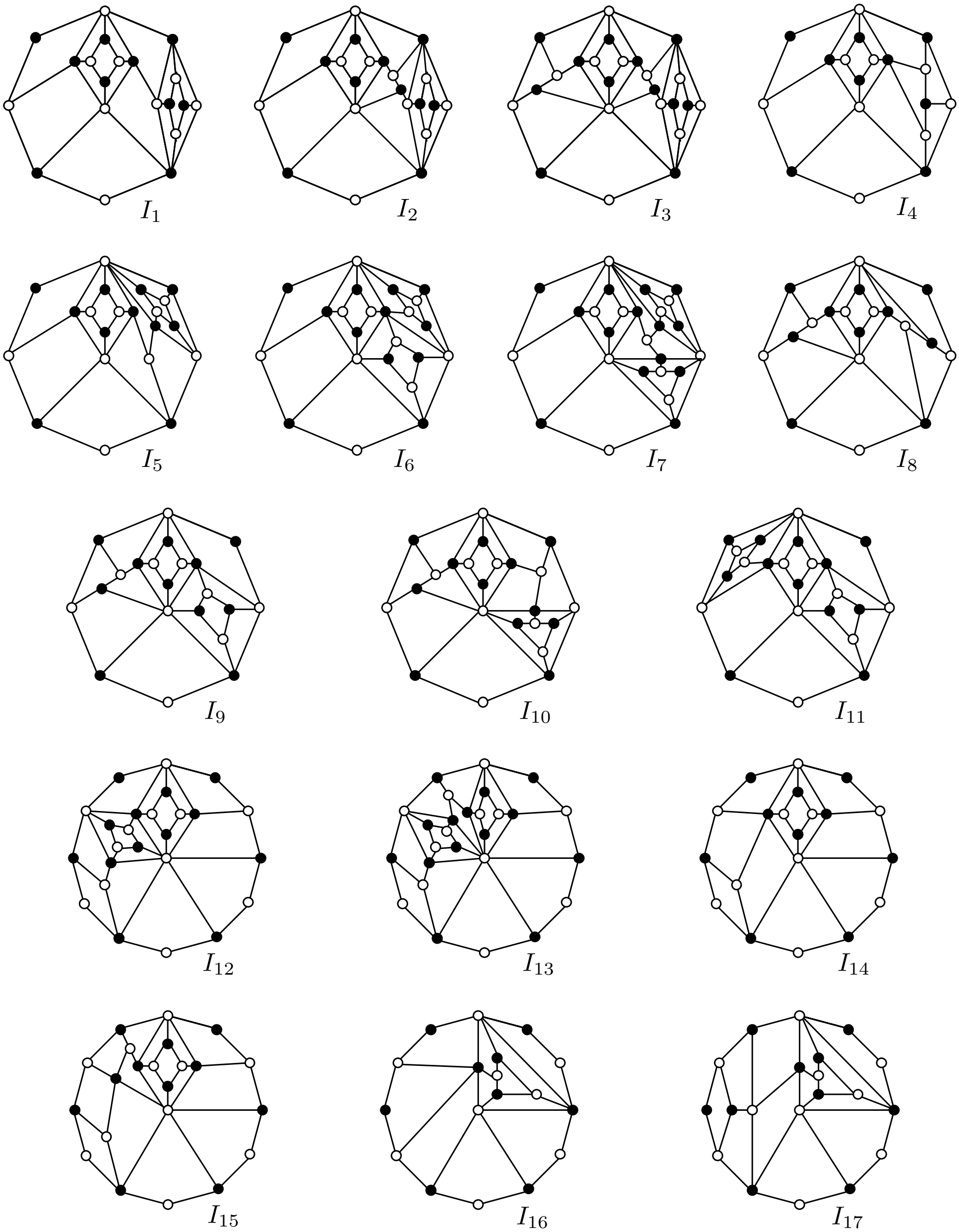

Figure 8: The 17 families of bipartite $\mathcal{P}$-irreducible quadrangulations with attached cubes.

of pieces. (See the rightmost configuration of Figure 9.) We denote the resulting graph by $H_{18}[s_1, \dots, s_m]$; note that we implicitly exclude $H_{18}[2]$ and $H_{18}[3]$ since they cannot fill the hexagonal region solely. If $H_{18}[s_1, \dots, s_m]$ is contained in a $\mathcal{P}$-irreducible quadrangulation G so that $a_1 = a_2$, then each attached cube is not removable and each face is not contractible in this configuration; note that G is nonbipartite. We often denote $H_{18}[s_1, \dots, s_m]$ simply by H_{18}.

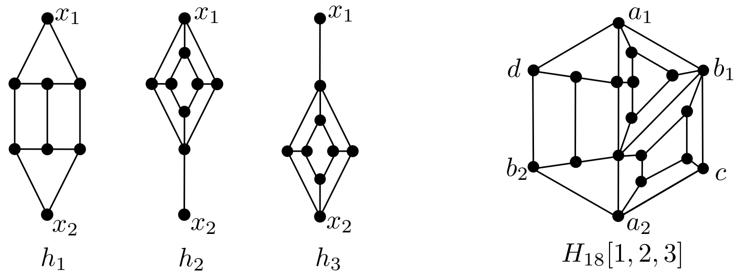

Figure 9: Inside a hexagonal region including an infinite series H_{18}.

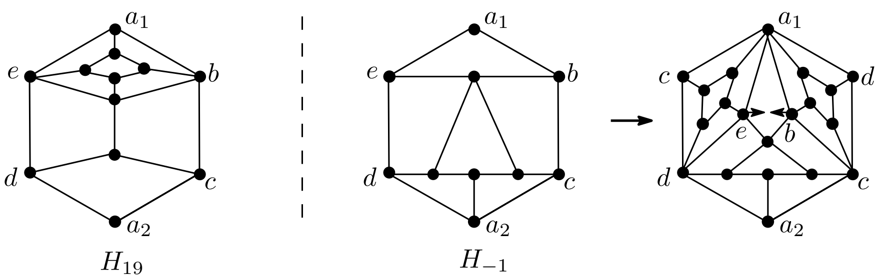

Figure 10: Inside a hexagonal region bounded by a closed walk (1).

See H_{19} in Figure 10. Note that the hexagonal region is bounded by a closed walk $W = a_1 b c a_2 d e$ where $a_1 = a_2$ $(= a)$ and the other four vertices b, c, d and e are distinct. Actually, H_{19} is appeared as a partial structure in $\mathcal{P}$-irreducible quadrangulations of the projective plane. (In Lemma 5.3, it will be mentioned.) However, the following lemma can exclude H_{19} from the later arguments.

In the following three lemmas (Lemmas 5.1, 5.2 and 5.3), we let D be a hexagonal region bounded by a closed walk $W = a_1 b c a_2 d e$ in a $\mathcal{P}$-irreducible quadrangulation G of the projective plane where $a_1 = a_2$ $(= a)$ and the other four vertices b, c, d and e are distinct.

Lemma 5.1. *If $G \cap \bar{D} \cong H_{19}$, then G is isomorphic to I_{20} in Figure 11.*

Proof. Note that G is nonbipartite since G contains an essential cycle of length 3. Therefore, G has an edge be outside of D by Lemma 3.16. Then there are two quadrilateral regions bounded by $abed$ and $aebc$. By Lemma 4.3, each of these regions is filled with either Q_1 or Q_2. However, if Q_1 is used, that is, it corresponds to a face of G, then we can

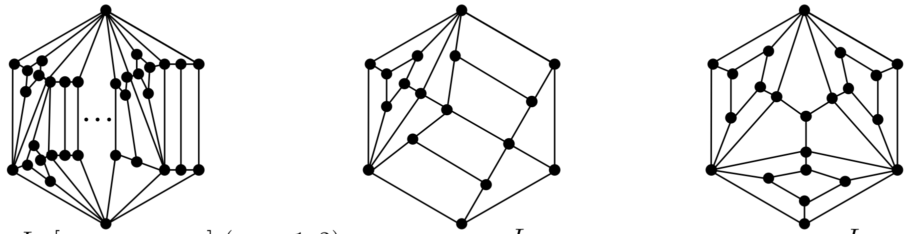

Figure 11: The 3 families of nonbipartite $\mathcal{P}$-irreducible quadrangulations.

easily find an essential simple closed curve intersecting G at only two vertices, a contradiction. Hence, we fill each of those regions with Q_2 and obtain I_{20} in Figure 11. □

Lemma 5.2. *$G \cap \bar{D}$ cannot be isomorphic to H_{-1} in Figure 10.*

Proof. Similarly to Lemma 5.1, G has an edge cd in this case by Lemma 3.14, and both of two quadrilateral regions outside of D are filled with Q_2 (see the right-hand side of Figure 10). However in this case, we would find a contractible face at $\{e, b\}$ by Lemmas 3.8 and 3.12. Therefore the lemma follows. □

Lemma 5.3. *$G \cap \bar{D}$ is isomorphic to either H_{18} or H_{19}.*

Proof. We use induction on the number of faces in D, say F. If F is at most 3, then D includes at most one inner vertex by Euler's formula. In this case, although $\partial(D)$ is not a cycle, $G \cap \bar{D}$ forms a structure like either H_1 or H_2 in Figure 6; we have to identify the top and the bottom vertices of H_i for $i \in \{1, 2\}$. However, G would have representativity at most 2, a contradiction. (Such an essential simple closed curve passes through a.) If $F = 4$, D includes exactly two vertices and we have $G \cap \bar{D} \cong H_{18}[1]$. Therefore, we assume that $F \geq 5$ after this. Similarly to the former lemmas, we discuss inner structures of divided regions by a short path; we have to consider (i), (ii) and (iv) in Lemma 4.2.

First, we assume that D contains a diagonal edge. By Lemma 3.2 and the simplicity of G, it should be ce or bd, now say ce, up to symmetry. Then each of two quadrilateral regions bounded by a_1bce and da_2ce should be filled with Q_2; otherwise at least one of those regions forms a face of G, but we can easily find an essential simple closed curve passing through only two vertices of G. Therefore, we obtain $H_{18}[3, 2]$ from this case. Then, we assume that D contains no diagonal hereafter.

Secondly, we assume that D includes an inner vertex x which is adjacent to a_1 and c. Then the quadrilateral region D' bounded by a_1bcx is filled with either Q_1 or Q_2. If we have the former, that is, a_1bcx bounds a face of G, then we would find an essential simple closed curve intersecting G only at a and c, contrary to the assumption. Therefore, we assume the latter case. In this case, the hexagonal region D'' bounded by a 6-walk a_1xca_2de satisfies the assumption of this lemma. Thus, we use the inductive hypothesis and fill the region with either H_{18} or H_{19}. If we use H_{19}, then the configuration becomes a part of I_{20} by Lemma 5.1. However, this is not the case since b corresponds to d. Hence we fill D'' with H_{18} and obtain our desired conclusion. Then after this, we assume that there is no such inner vertex like x. (We also exclude similar paths a_1xd, a_2xb and a_2xe.)

Thirdly, we assume that there is an inner vertex x which is adjacent to b and e (or c and d). Then, there are a quadrilateral region bounded by a_1bxe and a hexagonal region bounded by a 6-cycle bca_2dex. We fill these regions by using the results of Lemmas 4.3 and 4.4 respectively. Most cases are excluded by lemmas in Section 3, but we obtain $H_{18}[1]$, $H_{18}[3,2,3]$ and H_{19} by filling them with $\{Q_1, H_2\}$, $\{Q_2, H_{10}\}$ and $\{Q_2, H_2\}$, respectively. (For reference, when we use $\{Q_1, H_4\}$, we obtain H_{-1} in Figure 10, but it had already been excluded by Lemma 5.2.)

Next, we consider the existence of an essential 3-cycle a_1xya_2 where x and y are inner vertices of D. In this case, we can apply the inductive hypothesis and fill two hexagonal regions with H_{18}'s and obtain our conclusion; we do not have to consider H_{19} by Lemma 5.1.

Finally we assume that there is a 3-path $bxyd$ (or $cxye$) where both x and y are inner vertices of D. Then the boundary of each hexagonal region divided by the 3-path is a cycle, and we fill them by using Lemma 4.4. We only have to check H_i for $i \in \{3,4,5\}$, since the existence of an attached cube, a diagonal edge and a single vertex of degree 3 clearly yields a short path discussed above. However, there is no pair to satisfy the conditions from this case. Hence, the induction is completed. □

In the following lemma, we discuss a hexagonal region bounded by a 6-walk in which two vertices each appear twice.

Lemma 5.4. *Let D be a hexagonal region bounded by a closed walk $W = a_1b_1ca_2b_2d$ in a $\mathcal{P}$-irreducible quadrangulation G of the projective plane with $a_1 = a_2$ $(= a)$ and $b_1 = b_2$ $(= b)$. Then $G \cap \bar{D}$ is isomorphic to one of H_{18}, H_{20} and H_{21}.*

Proof. Since almost the same argument of the previous proof holds, we omit the proof of this lemma. However, we should pay attention to the following points:

(1) When assuming that there is a 3-path $cxyd$ where x and y are inner vertices of D, we obtain H_{20} in Figure 12; note that such a configuration was excluded in the previous lemma, since at least one of shaded faces in the right-hand side of Figure 12 is contractible by Lemma 3.8.

(2) If there is an essential 3-cycle a_1xya_2 (or b_1xyb_2), then we apply Lemma 5.3 to each of two hexagonal regions divided by the cycle.

(3) Using Q_2 and H_{19}, we can construct H_{21} in Figure 12. □

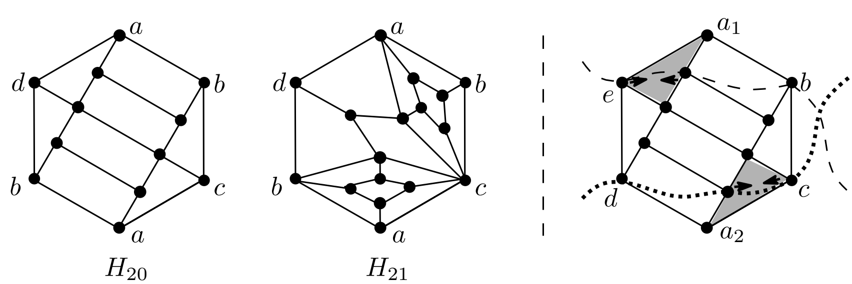

Figure 12: Inside a hexagonal region bounded by a closed walk (2).

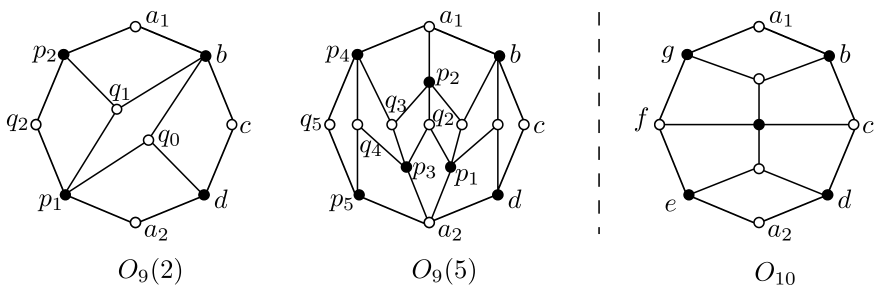

Figure 13: Octagonal structure generating infinite series.

See the graph denoted by $O_9(2)$ shown in the left-hand side of Figure 13. Observe that the octagonal region D is bounded by a closed walk $W = a_1bcda_2p_1q_2p_2$ where $a_1 = a_2$ $(= a)$ and the other vertices are distinct. Now add two vertices p_3 and q_3 so that $q_2p_1a_2p_3$ and $p_2q_2p_3q_3$ are quadrilateral faces. The resulting graph is denoted by $O_9(3)$. We inductively define the general form $O_9(m)$ from $O_9(m-1)$ by adding two vertices p_m and q_m so that $q_{m-1}p_{m-2}a_2p_m$ and $p_{m-1}q_{m-1}p_mq_m$ (resp., $a_1p_{m-2}q_{m-1}p_m$ and $p_mq_{m-1}p_{m-1}q_m$) are quadrilateral faces if m is odd (resp., even); note that we define $O_9(m)$ for $m \geq 2$. This $O_9(m)$ satisfies the followings:

(a) $\deg(q_i) = 3$ for each $i \in \{0, \ldots, m-1\}$, while $\deg(p_i) = 4$ for each $i \in \{1, \ldots, m-2\}$ if $m \geq 3$.

(b) If m is odd, then $\deg_D(b) = 2$, $\deg_D(c) = 0$, $\deg_D(d) = 1$, $\deg_D(p_m) = 1$, $\deg_D(q_m) = 0$ and $\deg_D(p_{m-1}) = 2$.

(c) If m is even, then $\deg_D(b) = 2$, $\deg_D(c) = 0$, $\deg_D(d) = 1$, $\deg_D(p_{m-1}) = 2$, $\deg_D(q_m) = 0$ and $\deg_D(p_m) = 1$.

Lemma 5.5. *Let D be an octagonal region bounded by a closed walk $W = a_1bcda_2efg$ in a $\mathcal{P}$-irreducible quadrangulation G of the projective plane such that $a_1 = a_2$ $(= a)$ and the other vertices are distinct. Suppose the following conditions hold:*

(α) Each of $\deg_D(b), \deg_D(d), \deg_D(e)$ and $\deg_D(g)$ is at least 1.

(β) No two vertices of degree 3 in D are adjacent.

Then $G \cap \bar{D}$ is isomorphic to either $O_9(m)$ or O_{10} in Figure 13.

Proof. First of all, we show that D contains no diagonal edge. Suppose to the contrary that there is a diagonal edge in D, say bf; note that a diagonal edge like a_1d is immediately excluded since it yields multiple edges. Then, there is a quadrilateral region D' bounded by a 4-cycle a_1bfg. By Lemma 4.3 and the condition (β) in the lemma, D' should be filled with Q_1, that is D' corresponds to a face of G. However, it contradicts (α) in the lemma. Therefore, we conclude that D has no diagonal.

Now, we use induction on the number of faces in D, say F as well as previous lemmas. If F is at most 4, then D includes at most one inner vertex x by Euler's formula. Since D has no diagonal, $G \cap \bar{D}$ is a graph obtained from O_1 in Figure 7 by identifying a pair of antipodal vertices. However in this case, we would find an essential simple closed curve intersecting G at only $\{a, x\}$, a contradiction. By careful observation, we have the unique configuration $O_9(2)$ with $F = 5$ faces in D; D includes exactly two inner vertices of degree 3. Therefore, the first step of the induction holds.

Similarly to the former lemmas, we divide the following argument along Lemma 4.2; other than (i) which is already excluded. Note that we shall implicitly exclude a short path already discussed in the former arguments.

Case I. There exists a short 2-path (ii) or (iii): First, such a vertex x adjacent to a_1 and c violates condition (α) in the lemma, since D does not contain an attached cube by (β). (We also exclude such 2-paths a_1xf, a_2xc and a_2xf.) Therefore, we assume that there is such a vertex x adjacent to b and d. Then the 2-path bxd divides D into an octagonal region D' and a quadrilateral region D''; note that D'' corresponds to a face of G. If $\deg_{D'}(b), \deg_{D'}(d) \geq 1$, then we can apply the inductive hypothesis. However, if we use $O_9(m)$, then x would become degree 2. On the other hand, if we fill D' with O_{10}, then the face-contraction of $bcdx$ at $\{c, x\}$ can be applied by Lemma 3.8. If $\deg_{D'}(b) = 0$ and $\deg_{D'}(d) = 0$, then we can easily find an essential simple closed curve intersecting G at only a and x; we can take such a curve along a_1bxda_2.

Therefore, we assume that one of $\deg_{D'}(b)$ and $\deg_{D'}(d)$ is equal to 0 and the other is at least 1. We may assume that $\deg_{D'}(b) = 0$ and $\deg_{D'}(d) \geq 1$ without loss of generality. Under the condition, there is a face of G in D' bounded by a_1bxy for $y \in V(G)$. If y is a vertex of W, then we have either $y = e$ or $y = g$ by Lemma 3.2. If we assume the former, then there would be multiple edges ae, contrary to our assumption. On the other hand, if the latter holds, there is a hexagonal region D''' bounded by a cycle da_2efgx of G. By Lemma 4.4 and the condition (β), D''' is filled only with H_2 and we obtain $O_9(2)$; the unique inner vertex of degree 3 in D''' must have neighbors $\{d, e, g\}$, otherwise, (α) cannot be held.

Therefore we may suppose that y is an inner vertex of D'. In this case, the octagonal region D^* bounded by a_1yxda_2efg satisfies the conditions of this lemma and hence we can apply the inductive hypothesis to D^*; observe that $\deg_{D^*}(y) \geq 1$. Under our assumptions, we fill D^* with $O_9(m)$ so as not to have adjacent vertices of degree 3, and obtain $O_9(m+1)$; O_{10} is inappropriate since it yields two adjacent vertices of degree 3 in D.

Next, we assume that there is an inner vertex x of D adjacent to both of b and g. Let D' be an octagonal region bounded by $bcda_2efgx$; note that the 4-cycle a_1bxg bounds a face of G. If D' has a diagonal edge, then the one end should be x since D admits no diagonal. However, if there is such a diagonal, say xd, then there would be a forbidden 2-path bxd, which was already discussed above. Thus D' has no diagonal edge and we can apply Lemma 4.5 to the region. In fact, most cases are excluded by some conditions but we obtain $O_9(3)$ and O_{10} by using O_4 and O_2, respectively. By the similar argument as above, we obtain $O_9(2)$ (resp., O_{10}) if we assume that there is a 2-path bxe (resp., cxf) for an inner vertex x of D.

Case II. There exists a short 3-path (iii): First, assume that such a short 3-path is a_1xyd where x and y are inner vertices of D. In this case, the hexagonal 2-cell region D' bounded by a_1xydcb should be filled with either H_1 or H_2 by Lemma 4.4 and the condition (β) in this lemma. However in each case, D would contain a diagonal or a forbidden 2-path excluded by the above arguments. By the same reason, we do not have to consider a 3-path like $bxyf$. (Of course, we exclude the paths of the same type, considering the symmetry;

e.g., a_2xyg.)

Case III. There exists a short 4-cycle (iv): We assume that there exists an essential 4-cycle a_1xyza_2 for inner vertices x, y and z of D. Then, D is divided into two octagonal regions D' and D'' and they are bounded by a_1xyza_2dcb and a_1xyza_2efg, respectively. Here, we consider degrees of x and z and first, suppose that $\deg_{D'}(x) = 0$. In this case, D' contains a face bounded by a_1xyw for $w \in V(G)$. (Note that $\deg_{D'}(z) \geq 1$ and $\deg_{D''}(z) \geq 1$; otherwise there would be an essential simple closed curve passing through only a and y. Also, $\deg_{D''}(x)$ is clearly at least 1.) The vertex w is an inner vertex of D' since if not, D would have a diagonal or a forbidden 3-path by the above argument. Let D''' be the octagonal region bounded by a_1wyza_2dcb; note that $\deg_{D'''}(w) \geq 1$. Then both of D'' and D''' satisfy the inductive hypothesis and we fill them with $O_9(m)$ or O_{10} so as not to make two adjacent vertices of degree 3 in D. Under the conditions, we only obtain $O_9(l)$ if D'' and D''' are filled with $O_9(l'')$ and $O_9(l''')$ respectively, where $l = l'' + l''' + 1$.

Therefore, we may assume that each of $\deg_{D'}(x)$, $\deg_{D''}(x)$, $\deg_{D'}(z)$ and $\deg_{D''}(z)$ is at least 1. Then we also use the inductive hypothesis into D' and D''. However, every case is inappropriate, since using O_{10} yields contractible face by Lemmas 3.8 and 3.12 and using two $O_9(m)$'s makes y to have degree 2. Thus, the lemma follows. □

6 Classification by attached cubes

Let G be a $\mathcal{P}$-irreducible quadrangulation of the projective plane. Assume that G has an attached cube H with $\partial(H) = v_0v_1v_2v_3$ and an attached 4-cycle $C = u_0u_1u_2u_3$ such that $u_iv_i \in E(G)$ for each $i \in \{0, 1, 2, 3\}$. Now, observe that any essential cycle of bipartite quadrangulations of the projective plane has even length while that of nonbipartite quadrangulations has odd length. This means that

(I) G has an essential diagonal 3-curve γ if G is bipartite or

(II) G has an essential semi-diagonal 3-curve γ if G is nonbipartite, such that γ passes through $\{v_0, u_1, v_2\}$ by Lemma 3.16.

First, we consider the case (I). In this case, γ is passing through three faces $f_1 = v_0u_0u_1v_1$, $f_2 = v_1u_1u_2v_2$ and $f = v_0av_2b$ for $a, b \in V(G)$. Since G is $\mathcal{P}$-irreducible, applying the face-contraction of f at $\{a, b\}$ breaks the property. However, each of $\deg(v_0)$ and $\deg(v_2)$ is clearly at least four and hence we do not have to consider the 3-connectedness of the graph by Lemma 3.12. Thus, we further divide it into the following two cases:

(I-a) The face-contraction of f disturbs the simplicity of the graph.

(I-b) The face-contraction of f yields a quadrangulation with representativity at most 2.

In (I-a), there exists a vertex x adjacent to both a and b such that the 4-cycle $C' = v_0axb$ is essential on the projective plane by Lemma 3.10. In this case, we cut the projective plane along C' and obtain (A) in Figure 14. In (I-b), G has a diagonal 3-curve passing through $\{a, b, x\}$ and three faces $f, f' = acxd$ and $f'' = bc'xd'$ by Lemma 3.8. Considering the identification of vertices except u_i for $i \in \{1, \ldots, 4\}$, we obtain (B), (C) and (D) in Figure 14 up to symmetry; we have to pay attention to the simplicity, the degree conditions of the graph and Lemma 4.3, further and that it does not have the structure of (I-a). (For example, if $d' = v_2$ in (B), then we have (C). Furthermore, if $x = v_3$ in (B), then d' (resp.,

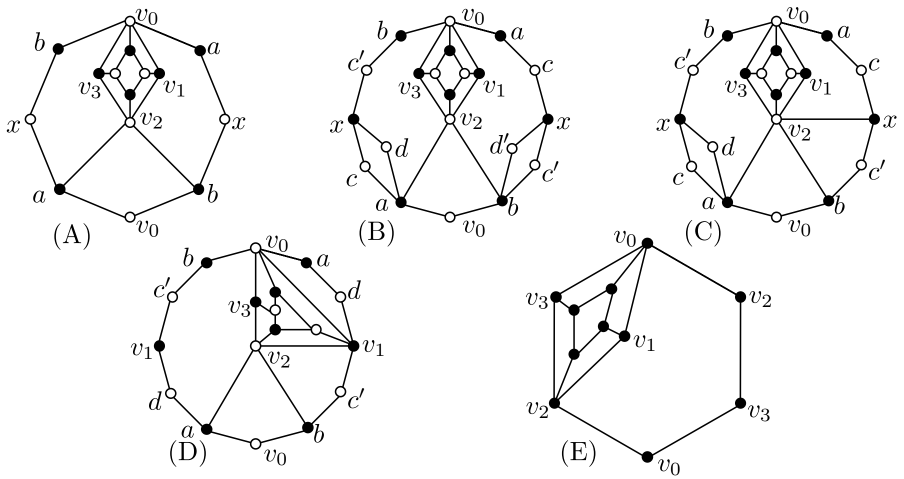

Figure 14: Around an attached cube.

d) must also coincide with v_2 (resp., c) by Lemma 4.3 and hence we obtain (D) in the figure. It is not so difficult to confirm that they are all.)

Secondly, we assume the case (II). In this case, G has an essential semi-diagonal 3-curve passing through $\{v_0, u_1, v_2\}$, that is, there is an edge joining v_0 and v_2. We cut open the projective plane along the essential 3-cycle $v_0v_1v_2$ and obtain (E) in the figure.

In the first half of the next section, we determine $\mathcal{P}$-irreducible quadrangulations of the projective plane with attached cubes by filling each blank non-quadrilateral region of (A) to (E) with results in Sections 4 and 5.

7 Proof of the main theorem

We shall classify $\mathcal{P}$-irreducible quadrangulations of the projective plane in this section to prove Theorem 1.1, using the lemmas proved in the former sections. For our purpose, we divide our main result into the following four theorems, depending on the existence of an attached cube and bipartiteness.

Theorem 7.1. *Let G be a bipartite $\mathcal{P}$-irreducible quadrangulation of the projective plane. If G has an attached cube, then G is one of the graphs shown in Figure 8.*

Proof. By the argument in the previous section, we first fill the two non-quadrilateral regions of (A) shown in Figure 14 with $H_1, \dots, H_{17}$ so as to form a $\mathcal{P}$-irreducible quadrangulation. (However, we implicitly exclude H_{13} and H_{15} by Lemma 4.6.) In fact, we consider the hexagonal regions bounded by $v_0v_1v_2axb$ and $v_0v_3v_2bxa$ and fill them with $H_7, H_8, H_9, H_{11}, H_{12}, H_{14}, H_{16}$ and H_{17} since we have $\{v_1, v_3\} \cap \{a, b\} = \emptyset$; otherwise, G would have multiple edges. When putting a pair of such pieces, we have to check the polyhedrality of G, and the absence of contractible face, removable 4-cycle and shrinkable 2-path, by using Proposition 3.1 and Lemmas 3.8, 3.9, 3.12 – 3.16.

Checking all the cases is a routine, and hence we present two bad examples below. First, see (i) of Figure 15, which is filled with a pair (H_7, H_9). However, it is easy to see that this graph has representativity 2. Secondly, see (ii) in the figure with a pair (H_{11}, H_{12}).

In this case, we can easily find a removable 4-cycle by Lemma 3.16, which is presented as the shaded region in the figure. Similarly to the above two bad cases, we can exclude almost pairs.

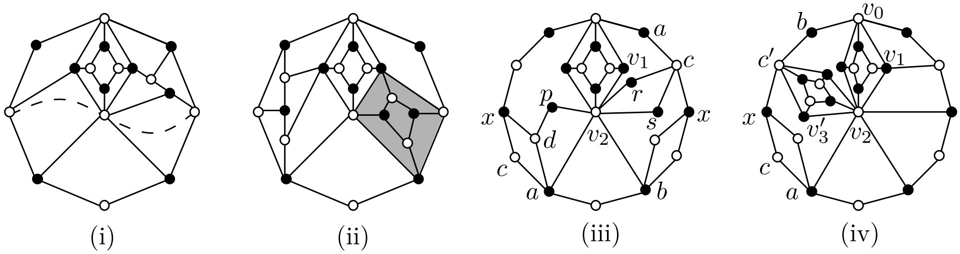

Figure 15: Configurations in the proof of Theorem 7.1.

As a result, 8 pairs (H_7, H_{11}), (H_7, H_{14}), (H_7, H_{16}), (H_7, H_{17}), (H_9, H_9), (H_9, H_{12}), (H_9, H_{14}) and (H_{12}, H_{12}) are available and we obtain $I_4, I_5, I_6, I_7, I_8, I_9, I_{10}$ and I_{11} in Figure 8, respectively.

Next, we consider (B) in Figure 14. Consider the face-contraction of the face bounded by $xdac$ at $\{c, d\}$. Observe that we have no identification of vertices c, d, c' and d' to other white vertices. (It was already done in the previous section.) Thus, we have that $\deg(x), \deg(a) \geq 4$. By Lemma 3.12, the face-contraction breaks the simplicity or the property of representativity at least 3. It is easy to see that the former does not happen and hence we suppose the latter. That is, there is a diagonal 3-curve passing through either $\{c, d, v_2\}$ or $\{c, d, v_0\}$.

Assume that the curve passes the $\{c, d, v_2\}$ and other two faces f_1 and f_2 are bounded by dpv_2q and v_2rcs respectively, for $p, q, r, s \in V(G)$. (Actually, by Lemma 4.3, one of p and q, say q, coincides with a. See (iii) in Figure 15.) If $s = x$ in the figure, then it would yield the configuration (C) in the Figure 14; we discuss (C) next. Further, if $s = b$, then the vertex c is adjacent to both of a and b and hence it would become (A); it was already discussed. Moreover, if $r = a$, we would have multiple edges v_2a. Therefore, we can conclude that the unique possibility of the identification of such vertices is that $r = v_1$; note that we have considered all the possibility around f_2, since G is bipartite and both of r and s should be black vertices. However, regardless of the unique identification, we can apply the face-contraction of f_2 at $\{r, s\}$ since there is no diagonal 3-curve passing through r and s. This is contrary to G being $\mathcal{P}$-irreducible. By the similar argument, we can find a contractible face when assuming that the diagonal 3-curve passes through $\{c, d, v_0\}$. As a result, (B) cannot be extended to any $\mathcal{P}$-irreducible quadrangulation.

As the third case, we consider (C) in Figure 14. By Lemma 3.16, there is no attached cube in the hexagonal region D bounded by $v_0acxv_2v_1$. Therefore, we try to put $H_1, \ldots, H_5$ into D. However, by Proposition 3.1 and Lemmas 3.9, 3.15 and 4.3, it is easy to confirm (but routine) that only H_1 is available and we have edge v_1c in D.

Next, we consider the octagonal region D' bounded by $v_0v_3v_2adxc'b$. Assume that D' contains another attached cube A such that $\partial(A) = v_0'v_1'v_2'v_3'$. Then its one cube diagonal pair, now say $\{v_0', v_2'\}$, coincides with either $\{a, b\}$ or $\{c', v_2\}$ since it should be facing by Lemma 3.16. If the former occurs, then it clearly causes I_1, I_2 or I_3 by Lemmas 4.4 and 4.6. Thus, we suppose the latter (see (iv) of Figure 15). Now we fill the two hexagonal regions

H and H' bounded by $v_0v_1v_2v_3'c'b$ and $v_2acxc'v_1'$, respectively. However, H admits only H_{12} and H_{14}, and H' does only H_8 by considering their partial structures. Therefore, we obtain I_{12} and I_{13} in Figure 8 in this case.

Then, we consider the possibility of existence of diagonal edges in the octagonal region D' and conclude that either v_3c' or v_3d is available by some lemmas and $\mathcal{P}$-irreducibility. If both of diagonals are taken as edges of G, then G becomes I_{14}. If one of two diagonals, say v_3c', is used, then we find the hexagonal region H bounded by $c'v_3v_2acx$; note that H does not contain its diagonal. We put H_2, H_3, H_4 and H_5 into H, however each of the resulting graphs has a contractible face; it is actually reducible into I_{14}. Furthermore, the same argument works when we use the diagonal v_3d.

Therefore we may assume that D' in (C) contains no attached cube and no diagonal, that is, it satisfies the condition of Lemma 4.5. Now, we try to put O_j for $j \in \{1, \ldots, 8\}$ into D'. Considering some lemmas in Section 3, we have only I_{15} from this case by filling it with O_2 in Figure 7.

Similarly to (C), we consider the inside of octagonal region O bounded by cycle $v_0v_3v_2adv_1c'b$ in the case (D). However, the argument is almost the same as the previous one and just a routine and hence we omit it here. (We first discuss the existence of an attached cube and diagonals in O. Next, we put the configurations of Figure 7.) As a result, we obtain I_{16} I_{17} from (D). Therefore, the theorem follows. □

We define $I_{18}[2; s_1, \ldots, s_n]$ as a graph obtained from (E) in Figure 14 by putting $H_{18}[s_1, \ldots, s_n]$ inside the hexagonal region. Recall that we forbid $I_{18}[2; \ldots, 2, 2, \ldots]$, $I_{18}[2; \ldots, 3, 3, \ldots]$, $I_{18}[2; 2, \ldots]$ and $I_{18}[2; \ldots, 3]$, since we make it a rule to unify consecutive Q_2's to one.

Theorem 7.2. *Let G be a nonbipartite $\mathcal{P}$-irreducible quadrangulation of the projective plane. If G has an attached cube, then G is one of $I_{18}[2; s_1, \ldots, s_n]$, I_{19} and I_{20} shown in Figure 11.*

Proof. By the argument in the previous section, we have (E) in Figure 14 in this case. There is the unique blank hexagonal region D which satisfies the conditions of Lemma 5.4. Hence, we fill D with $H_{18}[s_1, \ldots, s_n]$ (resp., H_{20}) and obtain $I_{18}[2; s_1, \ldots, s_n]$ (resp., I_{19}); note that H_{21} was already discussed in Lemma 5.1 and we obtained I_{20}.

In fact, some of $I_{18}[2; s_1, \ldots, s_n]$ with short sequences cannot satisfy the polyhedrality, hence we should exclude such "bad" sequences, which are listed in Table 1. It is not difficult to confirm that if $n \geq 4$, then any $I_{18}[2; s_1, \ldots, s_n]$ satisfying the above rule is acceptable. (Observe that there are different sequences $[s_1, \ldots, s_n] \neq [s_1', \ldots, s_n']$ such that $I_{18}[2; s_1, \ldots, s_n] \cong I_{18}[2; s_1', \ldots, s_n']$; e.g., $I_{18}[2; 1, 1, 2] \cong I_{18}[2; 3, 1, 1]$ in the table.) □

Figure 16 presents six bipartite $\mathcal{P}$-irreducible quadrangulations of the projective plane without attached cubes. In the figure, $I_{26}(2n + 1)$ $(n \geq 2)$ represents an infinite series of such graphs. The center white vertex of $I_{26}(2n + 1)$ has degree $2n + 1$ and each its black neighbors has degree 4. Furthermore, it has $2n + 1$ vertices of degree 3 on the essential simple closed curve drawn by dotted circle. (We obtain the projective plane by identifying all pairs of antipodal points of the dotted circle.) In fact, the figure represents $I_{26}(7)$ with 15 vertices.

Theorem 7.3. *Let G be a bipartite $\mathcal{P}$-irreducible quadrangulation of the projective plane. If G has no attached cube, then G is one of the graphs shown in Figure 16.*

Table 1: Good and bad sequences for $[s_1, \ldots, s_n]$ $(n \leq 3)$.

$[s_1, \ldots, s_n]$ $(n \leq 3)$	
$[1]$	bad (rep. 2)
$[1, 1], [3, 2]$	good
$[1, 2], [3, 1]$	bad (rep. 2)
$[1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 3, 1], [1, 3, 2], [3, 1, 1], [3, 1, 2], [3, 2, 1]$	good

Proof. For brevity, we write only the outline of the proof. We divide the proof into the following three cases by Lemmas 3.5, 3.13 and 3.15. Note that we prove those cases in this order, that is, we implicitly exclude a graph already appeared in the former cases.

Case I. G has a 2-path $u_0u_1u_2$ induced by three vertices of degree 3: See (i) in Figure 17. In the figure, each antipodal pair of points of the dotted circle should be identified to obtain the projective plane. Note that $v_0v_1v_2v_3v_4v_5$ is a cycle of G since if $v_3 = v_5$, then $\deg(v_4) = 3$ and G would contain an attached cube.

The 2-path $u_0u_1u_2$ is not shrinkable and hence we have a face v_0bv_2b' by Lemma 3.14. Furthermore, we consider the face-contraction of the face $v_2v_3v_4u_2$ at $\{v_3, u_2\}$. Since $\deg(v_2), \deg(v_4) \geq 4$, we do not have to pay attention to the connectivity of the resulting graph by Lemma 3.12. Also, since u_1 is an inner vertex of the hexagon, the face-contraction preserves the simplicity of the graph. Hence, by Lemma 3.8, we have a face v_3cv_1c'. By the same way, we find a face v_5av_1a' (see the figure again).

Similarly to the argument in Section 6 and the previous theorem, we consider the possibility of identification of vertices and fill blank non-quadrilateral regions with $H_1, \ldots, H_5$ in Lemma 4.4. As a result, we obtain I_{21}, I_{22} and I_{23} from this case.

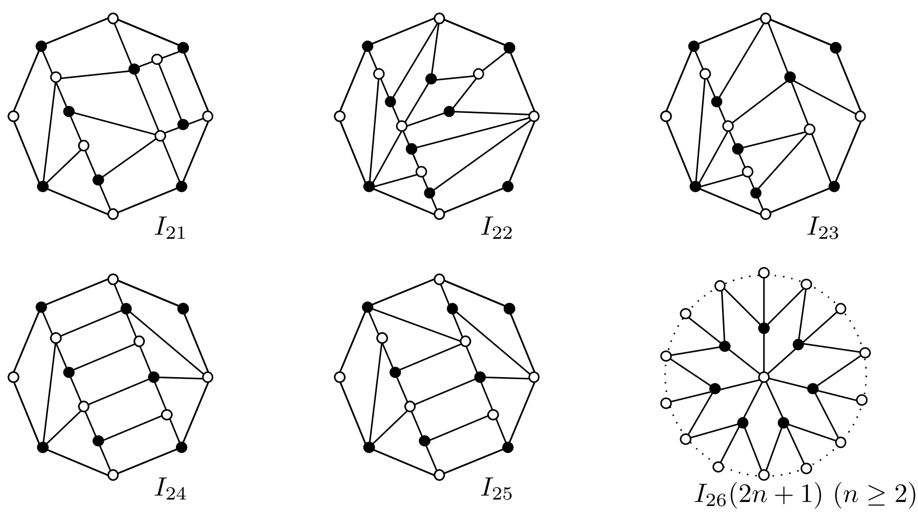

Figure 16: The 6 families of bipartite $\mathcal{P}$-irreducible quadrangulations without an attached cube.

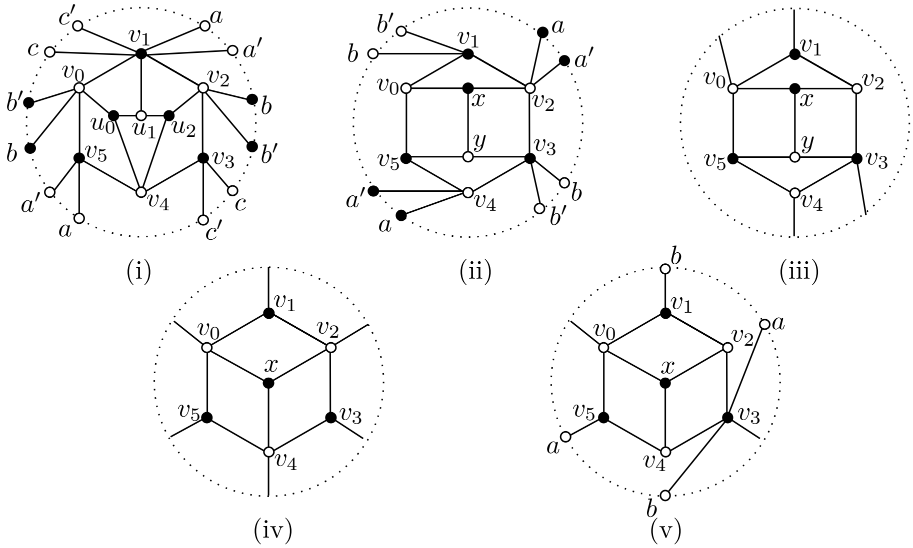

Figure 17: Structures of bipartite $\mathcal{P}$-irreducible quadrangulations with no attached cube.

Case II. G has two adjacent vertices x and y of degree 3: See the inside of the hexagon in (ii) of Figure 17. Note that each of $\deg(v_0), \deg(v_2), \deg(v_3)$ and $\deg(v_5)$ is at least 4 since there is no 2-path induced by vertices of degree 3 by the previous argument. As well as Case I, we consider face-contractions of $v_1v_2xv_0$ at $\{v_1, x\}$ and $v_3v_4v_5y$ at $\{v_4, y\}$. By Lemma 3.8, we have two diagonal 3-curves γ and γ' passing through $\{v_1, x\}$ and $\{v_4, y\}$, respectively. We may assume that γ passes v_3 as the third vertex, up to symmetry.

If γ' passes v_2, then it (resp., γ) goes through v_2av_4a' (resp., v_1bv_3b') in (ii) of Figure 17. We consider the identification of vertices and further fill the blank non-quadrilateral regions, and obtain I_{24} and I_{25}. On the other hand, if γ' passes v_0 as the third vertex, then both of γ and γ' pass a common face $v_0v_1v_4v_3$ (see (iii) in the figure). However, we can fill the unique hexagonal region with neither H_1, H_2 nor H_3 in Figure 6.

Case III. All vertices of degree 3 are independent: Let x be a vertex of degree 3 having neighbors $\{v_0, v_2, v_4\}$ and $v_0v_1v_2v_3v_4v_5$ as its link walk. By the assumption, each of $\deg(v_0), \deg(v_2)$ and $\deg(v_4)$ is at least 4. We consider face-contractions of three faces incident to x and have some cases depending on the forbidden structure of the resulting graph. (For example, if each operation yields multiple edges, we have (iv) in Figure 17, but it is immediately excluded since we can find a simple closed curve intersecting G at only $\{v_0, v_2\}$.) Further, we try to identify vertices as well as the previous cases but most cases are not suited other than the following one case.

See (v) in Figure 17 that has the unique blank octagonal region D bounded by a closed walk $v_1v_2v_3av_5v_4v_3b$. Note that each of $\deg_D(v_2), \deg_D(v_4), \deg_D(a)$ and $\deg_D(b)$ is at least 1, since G has no vertex of degree 2 and no two adjacent vertices of degree 3. Therefore, D satisfies the conditions of Lemma 5.5. However, putting either $O_9(2l + 1)$ ($l \geq 1$) or O_{10} in Figure 13 into D would yield two adjacent vertices of degree 3. Actually, when filling D with $O_9(2l)$ ($l \geq 1$), we obtain $I_{26}(2l + 3) = I_{26}(2(l + 1) + 1)$. Then, we got the conclusion of the theorem. □

As well as $I_{18}[2; s_1, \ldots, s_n]$, we can naturally define $I_{18}[1; s_1, \ldots, s_n]$ by using (iii) of Figure 18 which also has the unique hexagonal region.

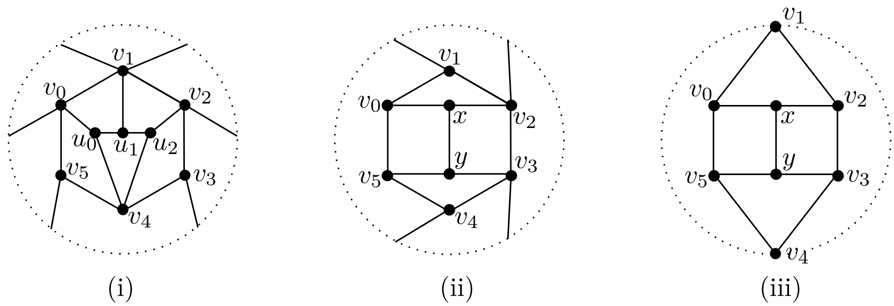

Figure 18: Structures of nonbipartite $\mathcal{P}$-irreducible quadrangulations with no attached cube.

Theorem 7.4. *Let G be a nonbipartite $\mathcal{P}$-irreducible quadrangulation of the projective plane. If G has no attached cube, then G is isomorphic to $I_{18}[1; 1, \ldots, 1]$.*

Proof. As well as the proof of Theorem 7.3, we divide our argument into the following three cases.

Case I. G has a 2-path $u_0u_1u_2$ induced by three vertices of degree 3: See (i) in Figure 18. Since G is nonbipartite, we have three semi-diagonal 3-curves passing through $\{v_0, u_1, v_2\}, \{v_1, u_2, u_3\}$ and $\{v_1, u_0, v_5\}$, respectively. (Consider the 2_3-path shrink $u_0u_1u_2$ and face-contractions of $v_2v_3v_4u_2$ and $v_4v_5v_0u_0$.) Under the conditions, there should be three edges v_0v_2, v_1v_3 and v_1v_5 since $v_0v_1v_2v_3v_4v_5$ forms a cycle of G. By Lemma 4.3, the quadrilateral region $v_1v_2v_0v_5$ corresponds to a face of G. However, there is an essential simple closed curve passing through only v_0 and v_1, a contradiction.

Case II. G has two adjacent vertices x and y of degree 3: See (ii) in Figure 18. Note that each of $\deg(v_0)$, $\deg(v_2)$, $\deg(v_3)$ and $\deg(v_5)$ is at least 4. Suppose that $v_0v_1v_2v_3v_4v_5$ is a cycle of G. We consider the face-contraction of $v_0v_1v_2x$ (resp., $v_3v_4v_5y$) at $\{v_1, x\}$ (resp., $\{v_4, y\}$). Then, there are two semi-diagonal 3-curves and hence we have $v_1v_3, v_2v_4 \in E(G)$, up to symmetry. Clearly, we find an essential simple closed curve intersecting G at only $\{v_2, v_3\}$, a contradiction.

Therefore, we assume that $v_0v_1v_2v_3v_4v_5$ is not a cycle of G. Under the conditions, v_1 and v_4 must coincide and the other vertices of the closed walk are distinct (see (iii) in Figure 18). Then the configuration contains a blank hexagonal region $v_1v_2v_3v_4(= v_1)v_0v_5$ and it satisfies the conditions of Lemma 5.3. Now, we apply the result of the lemma. But, H_{19} is excluded immediately since it contains an attached cube. In this case, $H_{18}[1, \ldots, 1]$ only fits the region. The resulting graph is clearly $I_{18}[1; 1, \ldots, 1]$.

Case III. All vertices of degree 3 are independent: Do the same procedure as in the previous theorem. (Begin with considering face-contractions of three faces incident to a vertex of degree 3.) However, we obtain no $\mathcal{P}$-irreducible quadrangulation from this case; since two adjacent vertices of degree 3 often appear. Therefore, the theorem follows. □

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 185–202
https://doi.org/10.26493/1855-3974.1559.390
(Also available at http://amc-journal.eu)

A diagram associated with the subconstituent algebra of a distance-regular graph

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Received 20 December 2017, accepted 6 June 2019, published online 18 September 2019

Abstract

In this paper we consider a distance-regular graph Γ. Fix a vertex x of Γ and consider the corresponding subconstituent algebra $T = T(x)$. The algebra T is the $\mathbb{C}$-algebra generated by the Bose-Mesner algebra M of Γ and the dual Bose-Mesner algebra M^* of Γ with respect to x. We consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \ldots$ along with their intersections and sums. In our notation, MM^* means $\mathrm{Span}\{RS \mid R \in M, S \in M^*\}$, and so on. We introduce a diagram that describes how these subspaces are related. We describe in detail that part of the diagram up to $MM^* + M^*M$. For each subspace U shown in this part of the diagram, we display an orthogonal basis for U along with the dimension of U. For an edge $U \subseteq W$ from this part of the diagram, we display an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement.

Keywords: Subconstituent algebra, Terwilliger algebra, distance-regular graph.

Math. Subj. Class.: 05E30

1 Introduction

In this paper we consider a distance-regular graph Γ. Fix a vertex x of Γ and consider the corresponding subconstituent algebra (or Terwilliger algebra) $T = T(x)$ [32]. The algebra T is the $\mathbb{C}$-algebra generated by the Bose-Mesner algebra M of Γ and the dual Bose-Mesner algebra M^* of Γ with respect to x. The algebra T is finite-dimensional and semisimple [32]. So it is natural to study the irreducible T-modules. These modules are used in the study of hypercubes [14, 26], dual polar graphs [20, 38], spin models [6, 10],

*The author would like to thank Professor Paul Terwilliger for many valuable ideas and insightful suggestions on my work. This paper was written while the author was an Honorary Fellow at the University of Wisconsin-Madison supported by the Development and Promotion of Science and Technology Talents (DPST) Project, Thailand.

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codes [13, 28], the bipartite property [4, 5, 9, 16, 21, 22, 23, 25, 27], the almost-bipartite property [3, 8, 17], the Q-polynomial property [5, 7, 11, 12, 18, 19, 27, 35], and the thin property [15, 24, 30, 31, 33, 34, 36, 37].

In this paper we discuss the algebra T using a different approach. We consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \ldots$ along with their intersections and sums; see Figure 1. We describe the diagram of Figure 1 up to MM^*+M^*M. For each subspace U shown in this part of the diagram, we display an orthogonal basis for U along with the dimension of U. For an edge $U \subseteq W$ from this part of the diagram, we display an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement. Our main results are summarized in Theorems 6.1 and 6.2. In the last part of the paper we summarize what is known about the part of diagram above $MM^* + M^*M$, and we give some open problems.

2 Preliminaries

In this section we recall some facts about distance-regular graphs. We will use the following notation. Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. For $B \in \mathrm{Mat}_X(\mathbb{C})$ let $\overline{B}$, B^t, and $\mathrm{tr}(B)$ denote the complex conjugate, the transpose, and the trace of B, respectively. We endow $\mathrm{Mat}_X(\mathbb{C})$ with the Hermitean inner product $\langle\ ,\ \rangle$ such that $\langle R, S\rangle = \mathrm{tr}(R^t\overline{S})$ for all $R, S \in \mathrm{Mat}_X(\mathbb{C})$. The inner product $\langle\ ,\ \rangle$ is positive definite. Let U, V denote subspaces of $\mathrm{Mat}_X(\mathbb{C})$ such that $U \subseteq V$. The *orthogonal complement* of U in V is defined by $U^\perp = \{v \in V \mid \langle v, u\rangle = 0 \text{ for all } u \in U\}$.

Let $\Gamma = (X, \mathcal{E})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{E}$. Let ∂ denote the shortest path-length distance function for Γ. Define the diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. For a vertex $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. For notational convenience abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, we say that Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* of Γ. From now on assume that Γ is distance-regular with diameter $D \geq 3$. We abbreviate $k_i := p^0_{ii}$ $(0 \leq i \leq D)$. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \qquad x, y \in X.$$

We call A_i the *i-th distance matrix* of Γ. We call $A = A_1$ the *adjacency matrix* of Γ. Observe that A_i is real and symmetric for $0 \leq i \leq D$. Note that $A_0 = I$ is the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $\sum_{i=0}^{D} A_i = J$, where J is the all-ones matrix in $\mathrm{Mat}_X(\mathbb{C})$. Observe that for $0 \leq i, j \leq D$,

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h. \tag{2.1}$$

For integers h, i, j $(0 \leq h, i, j \leq D)$ we have

$$p^h_{0j} = \delta_{hj}, \tag{2.2}$$
$$p^0_{ij} = \delta_{ij} k_i. \tag{2.3}$$

Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. By [2, p. 44] the matrices $A_0, A_1, \ldots, A_D$ form a basis for M. We call M the *Bose-Mesner algebra of* Γ. By [1, p. 59, 64], M has a basis $E_0, E_1, \ldots, E_D$ such that

(i) $E_0 = |X|^{-1} J$;

(ii) $\sum_{i=0}^{D} E_i = I$;

(iii) $E_i^t = E_i$ $(0 \leq i \leq D)$;

(iv) $\overline{E_i} = E_i$ $(0 \leq i \leq D)$;

(v) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$.

The matrices $E_0, E_1, \ldots, E_D$ are called the *primitive idempotents* of Γ, and E_0 is called the *trivial* idempotent. For $0 \leq i \leq D$ let m_i denote the rank of E_i. For $0 \leq i \leq D$ let θ_i denote an eigenvalue of A associated with E_i. Let λ denote an indeterminate. Define polynomials $\{u_i\}_{i=0}^{D}$ in $\mathbb{C}[\lambda]$ by $u_0 = 1$, $u_1 = \lambda/k$, and

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (1 \leq i \leq D-1).$$

By [2, p. 131, 132],

$$A_j = k_j \sum_{i=0}^{D} u_j(\theta_i) E_i \qquad (0 \leq j \leq D), \tag{2.4}$$
$$E_j = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \qquad (0 \leq j \leq D). \tag{2.5}$$

Since $E_i E_j = \delta_{ij} E_i$ and by (2.4) we have $A_j E_i = k_j u_j(\theta_i) E_i = E_i A_j$ $(0 \leq i, j \leq D)$. By [1, Theorem 3.5] we have the orthogonality relations

$$\sum_{i=0}^{D} u_i(\theta_r) u_i(\theta_s) k_i = \delta_{rs} m_r^{-1} |X| \qquad (0 \leq r, s \leq D), \tag{2.6}$$
$$\sum_{r=0}^{D} u_i(\theta_r) u_j(\theta_r) m_r = \delta_{ij} k_i^{-1} |X| \qquad (0 \leq i, j \leq D). \tag{2.7}$$

We recall the Krein parameters of Γ. Let $\circ$ denote the entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{C})$. Note that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$. So M is closed under $\circ$. By [2, p. 48], there exist scalars $q^h_{ij} \in \mathbb{C}$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \leq i, j \leq D). \tag{2.8}$$

We call the q^h_{ij} the *Krein parameters* of Γ. By [2, Proposition 4.1.5], these parameters are real and nonnegative for $0 \leq h, i, j \leq D$.

We recall the dual Bose-Mesner algebra of Γ. Fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \qquad y \in X.$$

We call E^*_i the *i-th dual idempotent* of Γ with respect to x. Observe that

(i) $\sum_{i=0}^{D} E^*_i = I$;

(ii) $E^{*t}_i = E^*_i$ $(0 \leq i \leq D)$;

(iii) $\overline{E^*_i} = E^*_i$ $(0 \leq i \leq D)$;

(iv) $E^*_i E^*_j = \delta_{ij} E^*_i$ $(0 \leq i, j \leq D)$.

By construction $E^*_0, E^*_1, \ldots, E^*_D$ are linearly independent. Let $M^* = M^*(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ with basis $E^*_0, E^*_1, \ldots, E^*_D$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x.

We now recall the dual distance matrices of Γ. For $0 \leq i \leq D$ let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(A^*_i)_{yy} = |X|(E_i)_{xy} \qquad y \in X. \tag{2.9}$$

We call A^*_i the *dual distance matrix* of Γ with respect to x and E_i. By [32, p. 379], the matrices $A^*_0, A^*_1, \ldots, A^*_D$ form a basis for M^*. Observe that

(i) $A^*_0 = I$;

(ii) $\sum_{i=0}^{D} A^*_i = |X| E^*_0$;

(iii) $A^{*t}_i = A^*_i$ $(0 \leq i \leq D)$;

(iv) $\overline{A^*_i} = A^*_i$ $(0 \leq i \leq D)$;

(v) $A^*_i A^*_j = \sum_{h=0}^{D} q^h_{ij} A^*_h$ $(0 \leq i, j \leq D)$.

From (2.4) and (2.5) we have

$$A^*_j = m_j \sum_{i=0}^{D} u_i(\theta_j) E^*_i \qquad (0 \leq j \leq D), \tag{2.10}$$

$$E^*_j = |X|^{-1} k_j \sum_{i=0}^{D} u_j(\theta_i) A^*_i \qquad (0 \leq j \leq D). \tag{2.11}$$

3 The subconstituent algebra T

In this section we study the subconstituent algebra of a distance-regular graph. For the rest of the paper, fix a distance-regular graph Γ and a vertex x of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. The algebra T is called the *subconstituent algebra* (or *Terwilliger algebra*) [32]. In order to describe T, we consider how M, M^* are related. We will use the following notation. For any two subspaces $\mathcal{R}, \mathcal{S}$ of $\mathrm{Mat}_X(\mathbb{C})$ we define $\mathcal{RS} = \mathrm{Span}\{RS \mid R \in \mathcal{R}, S \in \mathcal{S}\}$. Consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \ldots$ along with their intersections and sums. To describe the inclusions among the resulting subspaces we draw a diagram; see Figure 1. In this diagram, a line segment that goes upward from U to W means that W contains U.

Consider the diagram in Figure 1. For each subspace U shown in the diagram, we seek an orthogonal basis for U and the dimension of U. Also, for each edge $U \subseteq W$ shown in the diagram, we seek an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement. We accomplish these goals for that part of the diagram up to $MM^* + M^*M$. Our main results are summarized in Theorems 6.1 and 6.2. Before we get started, we recall a few inner product formulas.

Lemma 3.1 ([11, Lemma 3.1, Lemma 4.1])**.** *For* $0 \leq h, i, j, r, s, t \leq D$,

(i) $\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir}\delta_{js}\delta_{ht} k_h p_{ij}^h$,

(ii) $\langle E_i A_j^* E_h, E_r A_s^* E_t \rangle = \delta_{ir}\delta_{js}\delta_{ht} m_h q_{ij}^h$.

The following result is well-known.

Lemma 3.2 ([32, Lemma 3.2])**.** *For* $0 \leq h, i, j \leq D$,

(i) $E_i^* A_h E_j^* = 0$ *if and only if* $p_{ij}^h = 0$,

(ii) $E_i A_h^* E_j = 0$ *if and only if* $q_{ij}^h = 0$.

Lemma 3.3 ([29, Lemma 10])**.** *For* $0 \leq h, i, j, r, s, t \leq D$,

$$\langle A_i E_j^* A_h, A_r E_s^* A_t \rangle = \sum_{\ell=0}^{D} k_\ell p_{ir}^\ell p_{js}^\ell p_{ht}^\ell.$$

4 The subspace $M + M^*$

Our goal in this section is to analyze the inclusion diagram up to $M + M^*$. We begin with the trace of elements in M and M^*.

Lemma 4.1. *For* $0 \leq i \leq D$,

(i) $\mathrm{tr}(A_i) = \delta_{0i}|X|$,

(ii) $\mathrm{tr}(E_i) = m_i$,

(iii) $\mathrm{tr}(E_i^*) = k_i$,

(iv) $\mathrm{tr}(A_i^*) = \delta_{0i}|X|$.

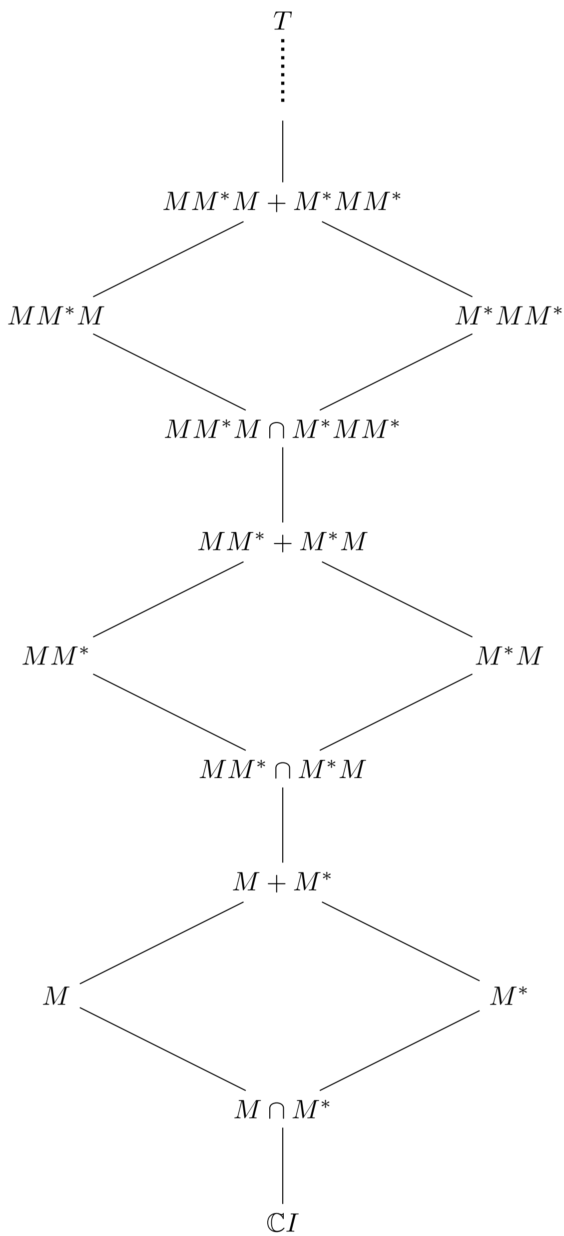

Figure 1: Inclusion diagram.

Proof. (i): Follows from the definition of A_i.
(ii): Since E_i is diagonalizable, we have $\operatorname{tr}(E_i) = \operatorname{rank}(E_i) = m_i$.
(iii): Follows from the definition of E_i^*.
(iv): By (2.5) and since

$$E_0 = |X|^{-1} J = |X|^{-1} \sum_{i=0}^{D} A_i,$$

we have

$$\sum_{i=0}^{D} (1 - u_i(\theta_0)) A_i = 0.$$

Since $\{A_i\}_{i=0}^{D}$ are linearly independent, we obtain $u_i(\theta_0) = 1$ for $0 \leq i \leq D$. By (2.6), (2.10) and (iii), we have

$$\operatorname{tr}(A_i^*) = m_i \sum_{j=0}^{D} u_j(\theta_i) \operatorname{tr}(E_j^*) = m_i \sum_{j=0}^{D} u_j(\theta_i) u_j(\theta_0) k_j = \delta_{0i} |X|. \qquad \square$$

Next we obtain some inner products.

Lemma 4.2. *For* $0 \leq i, j \leq D$*,*

(i) $\langle A_i, A_j \rangle = \delta_{ij} k_i |X|$,

(ii) $\langle E_i, E_j \rangle = \delta_{ij} m_i$,

(iii) $\langle E_i^*, E_j^* \rangle = \delta_{ij} k_i$,

(iv) $\langle A_i^*, A_j^* \rangle = \delta_{ij} m_i |X|$.

Proof. (i): Use (2.1) and Lemma 4.1.
(ii): By Lemma 4.1 and since $E_i E_j = \delta_{ij} E_i$.
(iii): Since $E_i^* E_j^* = \delta_{ij} E_i^*$ ($0 \leq i, j \leq D$) and by Lemma 4.1 (iii).
(iv): By (2.10) and (iii), we obtain

$$\langle A_i^*, A_j^* \rangle = \langle m_i \sum_{h=0}^{D} u_h(\theta_i) E_h^*, m_j \sum_{\ell=0}^{D} u_\ell(\theta_j) E_\ell^* \rangle = m_i m_j \sum_{h=0}^{D} u_h(\theta_i) u_h(\theta_j) k_h.$$

By (2.6), we have $\langle A_i^*, A_j^* \rangle = m_i m_j \delta_{ij} m_j^{-1} |X| = \delta_{ij} m_i |X|$. $\square$

The algebra M has two bases $\{A_i\}_{i=0}^{D}$ and $\{E_i\}_{i=0}^{D}$. The algebra M^* has two bases $\{A_i^*\}_{i=0}^{D}$ and $\{E_i^*\}_{i=0}^{D}$. Next we show that these bases are orthogonal.

Lemma 4.3. *Each of the following is an orthogonal basis for* M*:*

$$\{A_i\}_{i=0}^{D}, \qquad \{E_i\}_{i=0}^{D}.$$

Moreover, each of the following is an orthogonal basis for M^**:*

$$\{A_i^*\}_{i=0}^{D}, \qquad \{E_i^*\}_{i=0}^{D}.$$

Proof. By Lemma 4.2 and the comment below it. □

Recall that $A_0 = I = A_0^*$. Next we compute some inner products between M and M^*.

Lemma 4.4. *For* $0 \leq i, j \leq D$,

$$\langle A_i, A_j^* \rangle = \delta_{i0}\delta_{0j}|X|k_i.$$

Proof. Observe that $\langle A_i, A_j^* \rangle = \langle A_i A_0^* A_0, A_0 A_j^* A_0 \rangle$. By Lemma 3.3 and (2.2), (2.6) and (2.10), the result follows. □

The next results describe orthogonal bases for $M + M^*$ and $M \cap M^*$.

Lemma 4.5. *The following is an orthogonal basis for* $M + M^*$*:*

$$A_D, \dots, A_1, I, A_1^* \dots, A_D^*.$$

Proof. Immediate from Lemmas 4.2 and 4.4. □

Lemma 4.6.

$$\dim(M + M^*) = 2D + 1.$$

Proof. Immediate from Lemma 4.5. □

Lemma 4.7. *We have*

$$M \cap M^* = \mathbb{C}I \quad \textit{and} \quad \dim(M \cap M^*) = 1.$$

Proof. Observe that $I \in M \cap M^*$. By linear algebra, we have

$$\dim(M \cap M^*) = \dim(M) + \dim(M^*) - \dim(M + M^*).$$

By construction $\dim(M) = D + 1$, $\dim(M^*) = D + 1$. By this and Lemma 4.6, $\dim(M \cap M^*) = 1$. The result follows. □

Lemma 4.8. *The following statements hold:*

(i) The matrices $\{A_i\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* $M \cap M^*$ *in* M.

(ii) The matrices $\{A_i^*\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* $M \cap M^*$ *in* M^*.

(iii) The matrices $\{A_i\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* M^* *in* $M + M^*$.

(iv) The matrices $\{A_i^*\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* M *in* $M + M^*$.

Proof. Follows from definitions of M, M^* along with Lemmas 4.5 and 4.7. □

Lemma 4.9. *Each of the following subspaces has dimension* D*:*

$$(M \cap M^*)^\perp \cap M, \qquad (M \cap M^*)^\perp \cap M^*,$$
$$(M^*)^\perp \cap (M + M^*), \qquad M^\perp \cap (M + M^*).$$

Proof. Immediate from Lemma 4.8. □

5 The subspace $MM^* + M^*M$

Our goal in this section is to analyze the inclusion diagram from $M + M^*$ up to $MM^* + M^*M$. We begin with a few inner product formulas.

Lemma 5.1. *For* $0 \leq i, j, r, s \leq D$,

(i) $\langle A_i A_j^*, A_r^* A_s \rangle = \delta_{is}\delta_{jr}|X|k_i m_j u_i(\theta_j)$,

(ii) $\langle A_i A_j^*, A_r A_s^* \rangle = \delta_{ir}\delta_{js}|X|k_i m_j$,

(iii) $\langle A_i^* A_j, A_r^* A_s \rangle = \delta_{ir}\delta_{js}|X|k_i m_j$.

Proof. (i): Since

$$\langle A_i A_j^*, A_r^* A_s \rangle = \operatorname{tr}(A_j^* A_i A_r^* A_s) = \sum_{y \in X} \sum_{z \in X} (A_j^*)_{yy} (A_i)_{yz} (A_r^*)_{zz} (A_s)_{zy}$$

and by (2.9), it follows that

$$\begin{aligned}\langle A_i A_j^*, A_r^* A_s \rangle &= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_j)_{xy} (A_i)_{yz} (E_r)_{xz} (A_s)_{zy} \\ &= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_j)_{xy} (A_i \circ A_s)_{yz} (E_r)_{zx}.\end{aligned}$$

Since $A_i \circ A_s = \delta_{is} A_i$ ($0 \leq i, s \leq D$), we get

$$\langle A_i A_j^*, A_r^* A_s \rangle = |X|^2 \delta_{is} \sum_{y \in X} \sum_{z \in X} (E_j)_{xy} (A_i)_{yz} (E_r)_{zx}.$$

Since

$$\sum_{y \in X} \sum_{z \in X} (E_j)_{xy} (A_i)_{yz} (E_r)_{zx} = |X|^{-1} \operatorname{tr}(E_j A_i E_r),$$

we have

$$\langle A_i A_j^*, A_r^* A_s \rangle = |X| \delta_{is} \operatorname{tr}(E_j A_i E_r) = |X| \delta_{is} \operatorname{tr}(E_r E_j A_i).$$

Since $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$), we obtain

$$\langle A_i A_j^*, A_r^* A_s \rangle = |X| \delta_{is} \delta_{jr} \operatorname{tr}(E_j A_i) = |X| \delta_{is} \delta_{jr} \langle E_j, A_i \rangle.$$

By (2.5) and Lemma 4.2 (i), we get $\langle E_j, A_i \rangle = m_j u_i(\theta_j) k_i$. Hence

$$\langle A_i A_j^*, A_r^* A_s \rangle = \delta_{is} \delta_{jr} |X| k_i m_j u_i(\theta_j).$$

(ii): Since $A_0 = I$, we get $\langle A_i A_j^*, A_r A_s^* \rangle = \langle A_i A_j^* A_0, A_r A_s^* A_0 \rangle$. By (2.10), we obtain

$$\langle A_i A_j^*, A_r A_s^* \rangle = m_j m_s \sum_{h=0}^{D} u_h(\theta_j) \sum_{\ell=0}^{D} u_\ell(\theta_s) \langle A_i E_h^* A_0, A_r E_\ell^* A_0 \rangle.$$

From Lemma 3.3 we have

$$\langle A_i E_h^* A_0, A_r E_\ell^* A_0 \rangle = \sum_{t=0}^{D} k_t p_{ir}^t p_{h\ell}^t p_{00}^t.$$

By (2.2) and (2.3), we obtain

$$\begin{aligned}\langle A_iA_j^*, A_rA_s^*\rangle &= m_jm_s\sum_{h=0}^{D}u_h(\theta_j)\sum_{\ell=0}^{D}u_\ell(\theta_s)k_0p_{ir}^0p_{h\ell}^0\\ &= \delta_{ir}k_im_jm_s\sum_{h=0}^{D}u_h(\theta_j)u_h(\theta_s)k_h.\end{aligned}$$

By (2.6), we get

$$\langle A_iA_j^*, A_rA_s^*\rangle = \delta_{ir}k_im_jm_s\delta_{js}m_s^{-1}|X| = \delta_{ir}\delta_{js}|X|k_im_j.$$

(iii): Since

$$\langle A_i^*A_j, A_r^*A_s\rangle = \operatorname{tr}((A_i^*A_j)^t\overline{(A_r^*A_s)}) = \operatorname{tr}(A_jA_i^*A_r^*A_s) = \operatorname{tr}(A_i^*A_r^*A_sA_j)$$

and

$$A_i^*A_r^* = \sum_{h=0}^{D}q_{ir}^hA_h^*$$

and by (2.1), we get

$$\begin{aligned}\langle A_i^*A_j, A_r^*A_s\rangle &= \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\operatorname{tr}(A_h^*A_\ell) = \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\operatorname{tr}(A_\ell A_h^*)\\ &= \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\operatorname{tr}(A_\ell^t\overline{A_h^*}) = \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\langle A_\ell, A_h^*\rangle.\end{aligned}$$

From Lemma 4.4, we have

$$\sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\langle A_\ell, A_h^*\rangle = |X|\sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\delta_{\ell 0}\delta_{h0}k_\ell = |X|q_{ir}^0p_{js}^0k_0 = |X|q_{ir}^0p_{js}^0.$$

By (2.3) and since $q_{ir}^0 = \delta_{ir}m_i$, we obtain

$$\langle A_i^*A_j, A_r^*A_s\rangle = \delta_{ir}\delta_{js}|X|k_jm_i. \qquad \square$$

Next we obtain orthogonal bases for MM^* and M^*M.

Lemma 5.2. *The following statements hold:*

(i) The matrices $\{A_iA_j^* \mid 0 \le i, j \le D\}$ form an orthogonal basis for MM^*.

(ii) The matrices $\{A_j^*A_i \mid 0 \le i, j \le D\}$ form an orthogonal basis for M^*M.

Proof. Immediate from Lemma 5.1. □

Lemma 5.3. *Each of the following subspaces has dimension $(D+1)^2$:*

$$MM^*, \qquad\qquad M^*M.$$

Proof. Immediate from Lemma 5.2. □

Our next goal is to obtain an orthogonal basis for $MM^* + M^*M$.

Lemma 5.4. *We have*

$$MM^* + M^*M = \sum_{i=0}^{D}\sum_{j=0}^{D} \mathrm{Span}\{A_iA_j^*, A_j^*A_i\} \qquad \text{(orthogonal direct sum)}.$$

Proof. Immediate from Lemma 5.1. □

Corollary 5.5. *We have*

$$\dim(MM^* + M^*M) = \sum_{i=0}^{D}\sum_{j=0}^{D} \dim(\mathrm{Span}\{A_iA_j^*, A_j^*A_i\}).$$

Proof. Immediate from Lemma 5.4. □

Definition 5.6. For $0 \le i, j \le D$ let $H_{i,j}$ denote the 2×2 matrix of inner products for $A_iA_j^*, A_j^*A_i$.

Lemma 5.7. *For $0 \le i, j \le D$,*

$$H_{i,j} = |X|k_im_j \begin{pmatrix} 1 & u_i(\theta_j) \\ u_i(\theta_j) & 1 \end{pmatrix}.$$

Proof. Immediate from Lemma 5.1 and Definition 5.6. □

Lemma 5.8. *For $0 \le i, j \le D$ we have*

$$\det(H_{i,j}) = |X|^2k_i^2m_j^2(1 - (u_i(\theta_j))^2).$$

Proof. Immediate from Lemma 5.7. □

Corollary 5.9. *For $0 \le i, j \le D$, $\det(H_{i,j}) = 0$ if and only if $u_i(\theta_j) = \pm 1$.*

Proof. Immediate from Lemma 5.8. □

Lemma 5.10. *The following elements are orthogonal for $0 \le i, j \le D$:*

$$A_iA_j^* + A_j^*A_i, \qquad\qquad A_iA_j^* - A_j^*A_i.$$

Moreover

$$||A_iA_j^* + A_j^*A_i||^2 = 2|X|k_im_j(1 + u_i(\theta_j)),$$
$$||A_iA_j^* - A_j^*A_i||^2 = 2|X|k_im_j(1 - u_i(\theta_j)).$$

Proof. Immediate from Lemma 5.7. □

Lemma 5.11. *The following statements hold for $0 \le i, j \le D$:*

(i) *Assume $u_i(\theta_j) = 1$. Then $A_iA_j^* = A_j^*A_i$ and this common value is nonzero.*

(ii) Assume $u_i(\theta_j) = -1$. Then $A_i A_j^* = -A_j^* A_i$ and this common value is nonzero.

(iii) Assume $u_i(\theta_j) \neq \pm 1$. Then $A_i A_j^*, A_j^* A_i$ are linearly independent.

Proof. (i), (ii): Immediate from Lemma 5.10.
(iii): Immediate from Lemma 5.8. □

Lemma 5.12. *For $0 \leq i, j \leq D$ we give an orthogonal basis for* $\operatorname{Span}\{A_i A_j^*, A_j^* A_i\}$ *in Table 1.*

Table 1: An orthogonal basis for $\operatorname{Span}\{A_i A_j^*, A_j^* A_i\}$.

Case	Orthogonal basis	Dimension
$u_i(\theta_j) = \pm 1$	$A_i A_j^*$	1
$u_i(\theta_j) \neq \pm 1$	$A_i A_j^* + A_j^* A_i,\ A_i A_j^* - A_j^* A_i$	2

Proof. Follows from Definition 5.6 and Lemmas 5.7 and 5.11. □

Corollary 5.13. *The following is an orthogonal basis for $MM^* + M^*M$:*

$$\{A_i A_j^* + A_j^* A_i,\ A_i A_j^* - A_j^* A_i \mid 0 \leq i, j \leq D,\ u_i(\theta_j) \neq \pm 1\} \\ \cup \{A_i A_j^* \mid 0 \leq i, j \leq D,\ u_i(\theta_j) = \pm 1\}.$$

Proof. Immediate from Lemmas 5.4 and 5.12. □

Our next goal is to find the dimension of $MM^* + M^*M$.

Definition 5.14. Define an integer P as follows:

$$P = |\{(i, j) \mid 1 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}|.$$

Remark 5.15. Recall that $u_0(\theta_j) = 1$ and $u_i(\theta_0) = 1$ for $0 \leq i, j \leq D$. By [2, A.5], the graph Γ is primitive if and only if Γ_i is connected for $1 \leq i \leq D$. From Definition 5.14 and [2, Proposition 4.4.7] we have $P = 0$ if and only if Γ is primitive.

Lemma 5.16.

$$\dim(MM^* + M^*M) = 2D^2 + 2D + 1 - P.$$

Proof. Immediate from Corollary 5.13 and Definition 5.14. □

Our next goal is to obtain an orthogonal basis for $MM^* \cap M^*M$.

Lemma 5.17.

$$\dim(MM^* \cap M^*M) = 2D + 1 + P.$$

Proof. By linear algebra, we have

$$\dim(MM^* \cap M^*M) = \dim(MM^*) + \dim(M^*M) - \dim(MM^* + M^*M).$$

By Lemmas 5.3 and 5.16, the result follows. □

Lemma 5.18. *The following is an orthogonal basis for $MM^* \cap M^*M$:*

$$\{A_iA_j^* \mid 0 \leq i, j \leq D,\, u_i(\theta_j) = \pm 1\}.$$

Proof. Immediate from Lemmas 5.11 and 5.17. □

We now have orthogonal bases for MM^*, M^*M, $MM^* \cap M^*M$ and $MM^* + M^*M$. The next results establish an orthogonal basis for certain orthogonal complements along with the dimension for these orthogonal complements.

Lemma 5.19. *The matrices $\{A_iA_j^* \mid 1 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}$ form an orthogonal basis for the orthogonal complement of $M + M^*$ in $MM^* \cap M^*M$.*

Proof. Follows from Lemmas 4.5 and 5.18. □

Lemma 5.20. *The following statements hold:*

(i) The matrices $\{A_iA_j^* \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$ form an orthogonal basis for the orthogonal complement of $MM^* \cap M^*M$ in MM^*.

(ii) The matrices $\{A_j^*A_i \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$ form an orthogonal basis for the orthogonal complement of $MM^* \cap M^*M$ in M^*M.

Proof. Follows from Lemmas 5.2 and 5.18. □

Lemma 5.21. *The following statements hold:*

(i) The matrices $\{u_i(\theta_j)A_iA_j^* - A_j^*A_i \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$ form an orthogonal basis for the orthogonal complement of MM^* in $MM^* + M^*M$.

(ii) The matrices $\{A_iA_j^* - u_i(\theta_j)A_j^*A_i \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$ form an orthogonal basis for the orthogonal complement of M^*M in $MM^* + M^*M$.

Proof. (i): By Lemma 5.1, for $0 \leq i, j, r, s \leq D$

$$\begin{aligned}
&\langle A_rA_s^* + A_s^*A_r, u_i(\theta_j)A_iA_j^* - A_j^*A_i\rangle \\
&\quad = u_i(\theta_j)\langle A_rA_s^*, A_iA_j^*\rangle - \langle A_rA_s^*, A_j^*A_i\rangle + u_i(\theta_j)\langle A_s^*A_r, A_iA_j^*\rangle - \langle A_s^*A_r, A_j^*A_i\rangle \\
&\quad = \delta_{ir}\delta_{js}|X|k_im_ju_i(\theta_j) - \delta_{ir}\delta_{js}|X|k_im_ju_i(\theta_j) \\
&\qquad\qquad\qquad\qquad + \delta_{ir}\delta_{js}|X|k_im_j(u_i(\theta_j))^2 - \delta_{ir}\delta_{js}|X|k_im_j \\
&\quad = \delta_{ir}\delta_{js}|X|k_im_j((u_i(\theta_j))^2 - 1).
\end{aligned}$$

By similar arguments,

$$\langle A_rA_s^* - A_s^*A_r, u_i(\theta_j)A_iA_j^* - A_j^*A_i\rangle = \delta_{ir}\delta_{js}|X|k_im_j(1 - (u_i(\theta_j))^2)$$

for $0 \leq i, j, r, s \leq D$. By Lemma 5.1, for $0 \leq i, j, r, s \leq D$

$$\begin{aligned}
\langle A_rA_s^*, u_i(\theta_j)A_iA_j^* - A_j^*A_i\rangle &= u_i(\theta_j)\langle A_rA_s^*, A_iA_j^*\rangle - \langle A_rA_s^*, A_j^*A_i\rangle \\
&= \delta_{ir}\delta_{js}|X|k_im_ju_i(\theta_j) - \delta_{ir}\delta_{js}|X|k_im_ju_i(\theta_j) \\
&= 0.
\end{aligned}$$

By Lemma 5.2 and Corollary 5.13, the result follows.

(ii): Similar to the proof of (i). □

Lemma 5.22. *The following subspace has dimension P:*

$$(M + M^*)^\perp \cap (MM^* \cap M^*M).$$

Proof. Immediate from Definition 5.14 and Lemma 5.19. □

Lemma 5.23. *Each of the following subspaces has dimension $D^2 - P$:*

$$(MM^* \cap M^*M)^\perp \cap MM^*, \qquad (MM^* \cap M^*M)^\perp \cap M^*M,$$
$$(MM^*)^\perp \cap (MM^* + M^*M), \qquad (M^*M)^\perp \cap (MM^* + M^*M).$$

Proof. Immediate from Definition 5.14 and Lemmas 5.20 and 5.21. □

6 Summary of main results

In Sections 4 and 5 we obtained an orthogonal basis and the dimension for each subspace U in the diagram of Figure 1 up to $MM^* + M^*M$. Also, for each edge $U \subseteq W$ shown in this part of the diagram of Figure 1, we obtained an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement. The results are summarized in this section.

Theorem 6.1. *In each row of Table 2 we describe a subspace U in the diagram of Figure 1. We give an orthogonal basis for U along with the dimension of U.*

Table 2: An orthogonal basis for each subspace U in the diagram of Figure 1 along with its dimension.

Subspace U	Orthogonal basis for U	Dimension of U
$M \cap M^*$	I	1
M	$\{A_i\}_{i=0}^D$	$D+1$
M^*	$\{A_i^*\}_{i=0}^D$	$D+1$
$M + M^*$	$\{A_D, \ldots, A_1, I, A_1^*, \ldots, A_D^*\}$	$2D+1$
$MM^* \cap M^*M$	$\{A_iA_j^* \mid 0 \le i,j \le D, u_i(\theta_j) = \pm 1\}$	$2D+1+P$
MM^*	$\{A_iA_j^* \mid 0 \le i,j \le D\}$	$(D+1)^2$
M^*M	$\{A_j^*A_i \mid 0 \le i,j \le D\}$	$(D+1)^2$
$MM^* + M^*M$	$\{A_iA_j^* + A_j^*A_i, A_iA_j^* - A_j^*A_i \mid$ $0 \le i,j \le D, u_i(\theta_j) \ne \pm 1\}$ $\cup \{A_iA_j^* \mid 0 \le i,j \le D, u_i(\theta_j) = \pm 1\}$	$2D^2+2D+1-P$

Theorem 6.2. *In each row of Table 3 we describe an edge $U \subseteq W$ from the diagram of Figure 1. We give an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement.*

Table 3: An orthogonal basis for the orthogonal complement of U in W in the diagram of Figure 1 along with the dimension of this orthogonal complement.

U	W	Orthogonal basis for $U^\perp \cap W$	Dimension of $U^\perp \cap W$
$M \cap M^*$	M	$\{A_i\}_{i=1}^D$	D
$M \cap M^*$	M^*	$\{A_i^*\}_{i=1}^D$	D
M	$M + M^*$	$\{A_i^*\}_{i=1}^D$	D
M^*	$M + M^*$	$\{A_i\}_{i=1}^D$	D
$M + M^*$	$MM^* \cap M^*M$	$\{A_i A_j^* \mid 1 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}$	P
$MM^* \cap M^*M$	MM^*	$\{A_i A_j^* \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
$MM^* \cap M^*M$	M^*M	$\{A_j^* A_i \mid 1 \leq i, j \leq D, u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
MM^*	$MM^* + M^*M$	$\{u_i(\theta_j) A_i A_j^* - A_j^* A_i \mid 1 \leq i, j \leq D,\ u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
M^*M	$MM^* + M^*M$	$\{A_i A_j^* - u_i(\theta_j) A_j^* A_i \mid 1 \leq i, j \leq D,\ u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$

7 Open problems

In this section, we give some open problems and suggestions for future research. Earlier in the paper we discussed the diagram of Figure 1. In this discussion we analyzed the diagram up to $MM^* + M^*M$. The remaining part of the diagram is not completely understood. We mention what is known. By Lemma 3.1 the subspace M^*MM^* has an orthogonal basis $\{E_i^* A_j E_h^* \mid 0 \leq h, i, j \leq D,\ p_{ij}^h \neq 0\}$. Similarly, the subspace MM^*M has an orthogonal basis $\{E_i A_j^* E_h \mid 0 \leq h, i, j \leq D,\ q_{ij}^h \neq 0\}$.

Problem 7.1. Find an orthogonal basis for the following subspaces:

(i) $MM^*M \cap M^*MM^*$,

(ii) $MM^*M + M^*MM^*$.

Problem 7.2. In each row of Table 4 we give an edge $U \subseteq W$ from the diagram of Figure 1. Find an orthogonal basis for the orthogonal complement of U in W for the following cases.

Table 4: Subspaces U and W from the diagram of Figure 1.

U	W
$MM^* + M^*M$	$MM^*M \cap M^*MM^*$
$MM^*M \cap M^*MM^*$	MM^*M
$MM^*M \cap M^*MM^*$	M^*MM^*
MM^*M	$MM^*M + M^*MM^*$
M^*MM^*	$MM^*M + M^*MM^*$

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 203–222
https://doi.org/10.26493/1855-3974.1964.297
(Also available at http://amc-journal.eu)

Distance-regular Cayley graphs with small valency*

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Received 28 March 2019, accepted 9 May 2019, published online 20 September 2019

Abstract

We consider the problem of which distance-regular graphs with small valency are Cayley graphs. We determine the distance-regular Cayley graphs with valency at most 4, the Cayley graphs among the distance-regular graphs with known putative intersection arrays for valency 5, and the Cayley graphs among all distance-regular graphs with girth 3 and valency 6 or 7. We obtain that the incidence graphs of Desarguesian affine planes minus a parallel class of lines are Cayley graphs. We show that the incidence graphs of the known generalized hexagons are not Cayley graphs, and neither are some other distance-regular graphs that come from small generalized quadrangles or hexagons. Among some "exceptional" distance-regular graphs with small valency, we find that the Armanios-Wells graph and the Klein graph are Cayley graphs.

Keywords: Cayley graph, distance-regular graph.

Math. Subj. Class.: 05E30

*We thank Sasha Gavrilyuk, who, in the final stages of writing this paper, at a conference in Plzeň, pointed us to [27] for what he called Higman's method applied to generalized polygons.

†The research of Mojtaba Jazaeri was in part supported by a grant from School of Mathematics, Institute for Research in Fundamental Sciences (IPM) (No. 95050039).

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1 Introduction

The classification of distance-regular Cayley graphs is an open problem in the area of algebraic graph theory [28, Problem 71-(ii)]. Partial results have been obtained by Abdollahi and the authors [2], Miklavič and Potočnik [20, 21], and Miklavič and Šparl [22], among others.

Here we focus on distance-regular graphs with small valency. It is known that there are finitely many distance-regular graphs with fixed valency at least 3 [7]. In addition, all distance-regular graphs with valency 3 are known (see [11, Theorem 7.5.1]), as are all intersection arrays for distance-regular graphs with valency 4 [13]. There is however no complete classification of distance-regular graphs with fixed valency at least 5. It is believed though that every distance-regular graph with valency 5 has intersection array as in Table 3. Besides these results, all intersection arrays for distance-regular graphs with girth 3 and valency 6 or 7 have been determined. We therefore study the problem of which of these distance-regular graphs with small valency are Cayley graphs.

After some preliminaries in Section 2, we study several families of distance-regular graphs that have members with small valency. Several of the results in this section are standard. Besides these standard results, we obtain in Proposition 3.2 that the incidence graphs of the Desarguesian affine planes minus a parallel class of lines are Cayley graphs. In Section 3.7, we study generalized polygons. By extending a known method for generalized quadrangles, we are able to prove (among other results) that the incidence graphs of all known generalized hexagons are not Cayley graphs; see Proposition 3.6. Moreover, we show that neither are some other distance-regular graphs that come from small generalized quadrangles or hexagons.

We then determine all distance-regular Cayley graphs with valency 3 and 4 in Sections 4 and 5, respectively. Next, we characterize in Section 6 the Cayley graphs among the distance-regular graphs with valency 5 with one of the known putative intersection arrays. Most of our new results (besides the above mentioned ones) are negative, in the sense that we prove that certain distance-regular graphs are not Cayley graphs. However, we surprisingly do find that the Armanios-Wells graph is a Cayley graph. This gives additional, previously unknown, information about the structure of this distance-transitive graph on 36 vertices, as we remark after Proposition 6.1.

In the final section, we consider distance-regular graphs with girth 3 and valency 6 or 7. Most of these graphs have been discussed in earlier sections. As another exception, we obtain that the Klein graph on 24 vertices is a Cayley graph.

2 Preliminaries

All graphs in this paper are undirected and simple, i.e., there are no loops or multiple edges. A connected graph Γ is called distance-regular with diameter d and intersection array

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

whenever for every pair of vertices x and y at distance i, the number of neighbors of y at distance $i-1$ from x is c_i and the number of neighbors of y at distance $i+1$ from x is b_i, for all $i = 0, \ldots, d$. It follows that a distance-regular graph is regular with valency $k = b_0$. The number of neighbors of y at distance i from x is denoted by a_i, and $a_i = k - b_i - c_i$. The girth of a distance-regular graph follows from the intersection array. The odd-girth (of a non-bipartite graph) equals the smallest i for which $a_i > 0$; the even-girth equals the smallest i for which $c_i > 1$. A distance-regular graph is called antipodal if its distance-d graph is a disjoint union of complete graphs. This property follows from the intersection array.

A distance-regular graph with diameter 2 is called strongly regular. A strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph with n vertices such that every pair of adjacent vertices has λ common neighbors and every pair of non-adjacent vertices has μ common neighbours. Thus, $\lambda = a_1$, $\mu = c_2$, and the intersection array is $\{k, k-1-\lambda; 1, \mu\}$. For more background on distance-regular graphs, we refer to the monograph [11] or the recent survey [28].

Let G be a finite group and S be an inverse-closed subset of G not containing the identity element e of G. Then the (undirected) Cayley graph $\mathrm{Cay}(G, S)$ is a graph with vertex set G such that two vertices a and b are adjacent whenever $ab^{-1} \in S$. Recall that all Cayley graphs are vertex-transitive and a Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if the subgroup generated by S, which is denoted by $\langle S \rangle$, is equal to G. Following Alspach [3], the subset S in $\mathrm{Cay}(G, S)$ is called the connection set. It is well-known that a graph Γ is a Cayley graph if and only if it has a group of automorphisms G that acts regularly on the vertices of Γ.

The commutator of two elements a and b in a group G is denoted by $[a, b]$. Furthermore, the center of G is denoted by $Z(G)$.

2.1 Halved graphs

The following observation is straightforward but very useful. Let Γ be a Cayley graph $\mathrm{Cay}(G, S)$ with diameter d. Define sets S_i recursively by $S_{i+1} = SS_i \setminus (S_i \cup S_{i-1})$ for $i = 2, \ldots, d$, where $S_1 = S$ and $S_0 = \{e\}$. Then the distance-i graph Γ_i of Γ is again a Cayley graph, $\mathrm{Cay}(G, S_i)$. In particular, when Γ is bipartite, then its halved graphs (the components of Γ_2) are Cayley graphs.

Lemma 2.1. *The distance-i graph of a Cayley graph Γ with diameter d is again a Cayley graph, for $i = 2, \ldots, d$. Also the halved graphs of Γ are Cayley graphs.*

Clearly, also the complement $\overline{\Gamma}$ of a Cayley graph Γ is a Cayley graph.

2.2 Large girth

In the later sections we will see many distance-regular graphs with large girth. The following lemmas will then turn out to be useful.

Lemma 2.2. *Let Γ be a Cayley graph $\mathrm{Cay}(G, S)$ with girth g, where $|S| > 2$. If G is abelian, then $g \leq 4$ and Γ contains a — not necessarily induced — 4-cycle.*

Proof. Let a and b be in S such that $a \neq b^{-1}$. Then $e \sim a \sim ba = ab \sim b \sim e$, so Γ contains a 4-cycle, and hence $g \leq 4$. □

Lemma 2.3. *Let Γ be a Cayley graph $\mathrm{Cay}(G, S)$ with girth $g > 4$. Suppose that S contains an element of order m, with $m > 2$. Then $g \leq m$ and the vertices of Γ can be partitioned into induced m-cycles.*

Proof. Suppose $a \in S$ has order $m > 2$. Then $b \sim ab \sim a^2 b \sim \cdots \sim a^{m-1} b \sim b$, for every $b \in G$. Now suppose that this m-cycle is not induced. Then it follows that there is an i, with $1 < i < m-1$, such that $a^i \in S$. But then $b \sim a^i b \sim a^{i+1} b \sim ab \sim b$, which contradicts the assumption that $g > 4$. So every vertex is in an induced m-cycle, and the result follows. □

Note that the above partition of vertices into m-cycles is the same as the partition of G into the right cosets of the cyclic subgroup H generated by a.

In general, if Γ is a Cayley graph $\mathrm{Cay}(G, S)$, and H is a subgroup of G, then the induced subgraph on each of the right cosets of H is regular, and all these subgraphs are isomorphic to each other.

2.3 Normal subgroups and equitable partitions

If Γ is a Cayley graph $\mathrm{Cay}(G, S)$ and H is a normal subgroup of G, then the partition into the (distinct) cosets Hc is equitable, in the sense that each vertex in Hc has the same number of neighbors in Hb, for each c and b. This number is easily shown to be $|S \cap Hcb^{-1}|$. The quotient matrix Q of the equitable partition contains these numbers, i.e. $Q_{Hc,Hb} = |S \cap Hcb^{-1}|$. It is well-known and easy to show (by "blowing up" eigenvectors [12, Lemma 2.3.1]) that each eigenvalue of Q is also an eigenvalue of Γ. We will use this fact in some of the later proofs, for example to show that the Biggs-Smith graph is not a Cayley graph.

Note also that the quotient matrix is in fact the adjacency matrix of a Cayley multigraph on the quotient group G/H, with connection multiset $S/H = \{Hs \mid s \in S\}$. When Γ is an antipodal distance-regular (Cayley) graph with diameter d, then it is easy to show that $N_d = S_d \cup \{e\}$ is a subgroup of G. If this group is normal, then it follows that there is a Cayley graph over the quotient group G/N_d with connection set $\{N_d s \mid s \in S\}$ (cf. [21, Lemma 2.2]). This quotient graph is the folded graph of Γ, and it is well-known to be distance-regular, too.

2.4 Dihedral groups

Miklavič and Potočnik [20, 21] classified the distance-regular Cayley graphs over a cyclic or dihedral group. They already observed in [20] that a primitive distance-regular graph over a dihedral group must be a complete graph. In [21], they moreover showed the following.

Proposition 2.4 ([21]). *A distance-regular Cayley graph over a dihedral group must be a cycle, complete graph, complete multipartite graph, or the bipartite incidence graph of a symmetric design.*

We will see these graphs also in Section 3. More importantly, we will use this classification in some of the results in the later sections.

2.5 Erratum

In [2], we claimed that in the distance-regular line graph Γ of the incidence graph of a generalized d-gon of order (q, q), any induced cycle is either a triangle or a $2d$-cycle. This is not correct however. Instead, every induced cycle in Γ is either a 3-cycle or an even cycle of length at least $2d$. Consequently, Theorem 3.1 in [2] may not be correct. Instead, we have the following result.

Theorem 2.5. *Let $d \geq 2$, let Γ be the line graph of the incidence graph of a generalized d-gon of order (q, q), and suppose that Γ is a Cayley graph $\mathrm{Cay}(G, S)$. Then there exist* *two subgroups H and K of G such that $S = (H \cup K) \setminus \{e\}$, with $|H| = |K| = q + 1$ and $H \cap K = \{e\}$ if and only if $\langle a \rangle \subseteq S \cup \{e\}$ for every element a of order $2i$ in S, with $i \geq d$.*

The correction of the above result has no impact on the validity of the following result in [2, Proposition 3.4]. In fact, by Lemma 2.2, the proof can do without the above theorem.

Proposition 2.6. *The line graph of Tutte's 8-cage is not a Cayley graph.*

Proof. Let Γ be the line graph of Tutte's 8-cage, and suppose that it is a Cayley graph $\mathrm{Cay}(G, S)$. Then $|G| = 45$ and $|S| = 4$. By Lemma 2.2, G cannot be abelian because Γ has no 4-cycles. But all groups of order 45 are abelian, so we have a contradiction. □

3 Some families of distance-regular graphs

It is clear that the cycle C_n is a distance-regular Cayley graph over the cyclic group. Thus, every distance-regular graph with valency 2 is a Cayley graph. Here we mention some other relevant families of distance-regular graphs with members of small valency.

3.1 Complete graphs, complete multipartite graphs, and complete bipartite graphs minus a matching

The complete graph K_n and the regular complete multipartite graph $K_{m\times n}$ are distance-regular Cayley graphs (with diameters 1 and 2, respectively). Indeed, K_n is a Cayley graph over any group of order n, whereas $K_{m\times n}$ (with m parts of size n) is a Cayley graph over the cyclic group $\mathbb{Z}_{mn}$, with connection set $S = \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}$. Note that the complete bipartite graph $K_{2\times n}$ is usually denoted by $K_{n,n}$.

A complete bipartite graph $K_{n,n}$ minus a complete matching, which is denoted by $K^*_{n,n}$, is distance-regular with valency $n - 1$ and diameter 3. Even though it may be clear that this is also a Cayley graph, we will describe it as such explicitly. Indeed, let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$. Then the Cayley graph $\mathrm{Cay}(D_{2n}, S)$, where $S = \{ba^i \mid 1 \leq i \leq n - 1\}$ is the complete bipartite graph $K_{n,n}$ minus a complete matching, with two bipartite parts $\langle a \rangle$ and $b\langle a \rangle$. This graph can also be described as the incidence graph of a symmetric design; see Section 3.5.

3.2 Paley graphs

The Paley graphs are defined as Cayley graphs. Let q be a prime power such that $q \equiv 1 \pmod 4$. Let G be the additive group of $\mathrm{GF}(q)$ and let S be the set of nonzero squares in $\mathrm{GF}(q)$. Then the Paley graph $P(q)$ is defined as the Cayley graph $\mathrm{Cay}(G, S)$. It is distance-regular with diameter 2 and valency $(q - 1)/2$.

3.3 Hamming graphs, cubes, and folded cubes

The Hamming graph $H(d, q)$ is the d-fold Cartesian product of K_q. It can therefore be described as a Cayley graph over (for example) $\mathbb{Z}_q^d$ with the set of vectors of (Hamming) weight one as connection set. It is distance-regular with valency $d(q - 1)$ and diameter d.

The Hamming graph $H(2, q)$ is also known as the lattice graph $L_2(q)$. The Shrikhande graph is a distance-regular graph with the same intersection array as $L_2(4)$, and it is a Cayley graph $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{\pm(0, 1), \pm(1, 0), \pm(1, 1)\})$. A Doob graph is a Cartesian product of Shrikhande graphs and K_4's. These Doob graphs are thereby distance-regular Cayley graphs as well.

The Hamming graph $H(d, 2)$ is also known as the d-dimensional (hyper)cube graph Q_d. The folded d-cube can be obtained from Q_{d-1} by adding a perfect matching connecting its so-called antipodal vertices. This implies that it is a Cayley graph over $\mathbb{Z}_2^{d-1}$ with connection set the set of unit vectors and the all-ones vector. The folded d-cube is distance-regular with valency d and diameter $\lfloor d/2 \rfloor$.

3.4 Odd and doubled Odd graphs

The Odd graph O_n is the Kneser graph $K(2n-1, n-1)$. It is distance-regular with valency n and diameter $n-1$. Godsil [15] determined which Kneser graphs are Cayley graphs, and it follows that the Odd graph is not a Cayley graph.

The doubled Odd graph DO_n is the bipartite double of the Odd graph O_n. It is distance-regular with valency n and diameter $d = 2n - 1$. It is easy to see that if a graph Γ is a Cayley graph $\mathrm{Cay}(G, S)$, then its bipartite double is again a Cayley graph over the group $G \times \mathbb{Z}_2$ with connection set $S = \{(s, 1) \mid s \in S\}$. But the Odd graph is not a Cayley graph, so we cannot apply this argument. Indeed, it turns out that the doubled Odd graph is also not a Cayley graph.

Proposition 3.1. *The doubled Odd graph is not a Cayley graph.*

Proof. The distance-$(d-1)$ graph of a doubled Odd graph DO_n (with diameter $d = 2n-1$) is a disjoint union of two Odd graphs O_n. If this graph is a Cayley graph, then its distance-$(d-1)$ graph is again a Cayley graph, by Lemma 2.1. But an Odd graph is not a Cayley graph [15], so neither is the doubled Odd graph. □

Godsil's results [15] also imply the classification by Sabidussi [25] of Cayley graphs among the triangular graphs $T(n)$; these are Cayley graphs if and only if $n = 2, 3, 4$ or $n \equiv 3 \pmod 4$ and n is a prime power.

3.5 Incidence graphs of symmetric designs

Miklavič and Potočnik [21] showed that there is a correspondence between difference sets and connection sets for the incidence graphs of a symmetric design. Recall that a k-subset D of a group G of order n is called an (n, k, λ) difference set if every nonidentity element $g \in G$ occurs λ times among all possible differences $d_1 d_2^{-1}$ (we prefer to use multiplicative notation) of distinct elements d_1 and d_2 of D. The development $\{Dg \mid g \in G\}$ of such a difference set is a symmetric 2-(n, k, λ) design.

If D is a difference set in an abelian group G, then we can easily construct the incidence graph of its development as a Cayley graph for the group $G \rtimes \mathbb{Z}_2$. The elements of this group can be (identified and) partitioned as $G \cup Gc$, where $c^2 = 1$ and $cgc = g^{-1}$ for all $g \in G$. As a connection set, we take $S = Dc$. It follows that S is inverse closed, and that the corresponding Cayley graph is indeed the incidence graph of the development (a block Dg corresponds to the group element $g^{-1}c$).

Because the Desarguesian projective plane (over $\mathrm{GF}(q)$) is a symmetric 2-$(q^2 + q + 1, q + 1, 1)$ design, and can be obtained from a (Singer) difference set in the cyclic group, it follows that the incidence graph of a Desarguesian projective plane is a Cayley graph. It was shown by Loz et al. [18] that this Cayley graph is 4-arc-transitive. We note that all projective planes of order at most 8 are Desarguesian, and hence all incidence graphs of projective planes with valency at most 9 are Cayley graphs.

We also note that if D is a difference set in G, then the complement $G \setminus D$ is also a difference set in G, and its development is the complementary design of the development of D. This implies that also the incidence graph of the 2-$(7, 4, 2)$ design is a Cayley graph. Also the 2-$(11, 5, 2)$ biplane comes from a difference set (the set of nonzero squares in $\mathbb{Z}_{11}$), so its incidence graph is a Cayley graph. Note that also the (trivial) 2-$(n, n-1, n-2)$ design comes from a difference set ($D = G \setminus \{e\}$), which gives an alternative proof that $K^*_{n,n}$ is a Cayley graph (see Section 3.1).

We denote the incidence graph of a 2-(n, k, λ) design by $IG(n, k, \lambda)$. Such a graph is distance-regular with valency k and diameter 3.

3.6 Incidence graphs of affine planes minus a parallel class of lines

Similar to the case of symmetric designs, there is a correspondence between certain relative difference sets and connection sets for the incidence graph of an affine plane minus a parallel class of lines. A k-subset R of a group G of order mn is called a relative (m, n, k, λ) difference set relative to a subgroup N of order n of G if every element of $G \setminus N$ occurs λ times among all possible differences $r_1 r_2^{-1}$ of elements r_1 and r_2 of R. The development of such a relative difference set is a so-called (m, n, k, λ) divisible design. We will not go into the details of the definition of such a divisible design, but restrict to the remark that an $(n, n, n, 1)$ divisible design is the same as an affine plane of order n minus a parallel class of lines (for details, see [24]). Similar as in Section 3.5, if such a divisible design comes from a relative difference set in an abelian group, then its incidence graph is a Cayley graph.

It is known that all Desarguesian planes correspond to relative difference sets, so the incidence graphs of the Desarguesian affine planes minus a parallel class are all Cayley graphs. These include all such distance-regular graphs with valency at most 8. In particular, for odd prime powers q, the set $\{(x, x^2) \mid x \in \mathrm{GF}(q)\}$ is a relative difference set in $\mathrm{GF}(q)^2$. To include even prime powers, we need a more involved construction of a relative difference set that actually works also for semifields (see [24, Theorem 4.1]). Indeed, if $\mathbb{S}$ is a semifield of order q, then we define a group on $\mathbb{S}^2$ using the addition $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + x_1 y_1)$. In this group, the set $\{(x, x^2) \mid x \in \mathbb{S}\}$ is a relative $(q, q, q, 1)$ difference set. We note that if $\mathbb{S}$ is the field on 2^n vertices, then the constructed group is isomorphic to $\mathbb{Z}_4^n$.

We denote the incidence graph of a the Desarguesian affine plane of order q minus a parallel class of lines (pc) by $IG(AG(2, q) \setminus pc)$. Such a graph is distance-regular with valency q and diameter 4. We conclude the following.

Proposition 3.2. *For every prime power q, the incidence graph of the Desarguesian affine plane of order q minus a parallel class of lines, $IG(AG(2, q) \setminus pc)$, is a Cayley graph.*

3.7 Generalized polygons

The incidence graph of a generalized quadrangle or generalized hexagon of order (q, q) is distance-regular with valency $q + 1$ and girth 8 and 12, respectively. These graphs thus arise in the tables in the following sections. In this section, we will first show, among other results, that for $q \leq 4$, none of these is a Cayley graph. Next to that, we will consider some of the distance-regular line graphs and halved graphs (point graphs) of these graphs.

Indeed, first suppose that the incidence graph Γ of generalized polygon of order (s, s) is a Cayley graph. Then its automorphism group contains a subgroup that acts regularly on the vertices of Γ. It follows that there is an index 2 subgroup G that acts regularly on both the point set and on the line set, as an automorphism group of the generalized polygon. This situation has been studied by Swartz [26] for generalized quadrangles. Using results by Yoshiara [29] (who exploited an idea of Benson [9]; cf. [23, 1.9.1]), Swartz [26] showed that $s+1$ must be coprime to 2 and 3. Consequently, we have the following result.

Proposition 3.3. *If the incidence graph of a generalized quadrangle of order* (s, s) *is a Cayley graph, then* $s+1$ *is not divisible by* 2 *or* 3*.*

In particular, it shows that the incidence graphs of generalized quadrangles of orders $(2, 2)$ and $(3, 3)$ are not Cayley graphs.

We will next derive a similar result for generalized hexagons. The line of proof is the same as for generalized quadrangles. By extracting the main ideas and fine-tuning them, we are able to give a self-contained proof, which in the end even leads to a somewhat stronger result. We note that similar more general techniques and results on generalized hexagons (but not our main results) have also been obtained by Temmermans, Thas, and Van Maldeghem [27].

As in the above, we assume that the generalized hexagon of order (s, s) has an automorphism group G that acts regularly on points as well as on lines. Thus, the order of G is $(s+1)(s^4+s^2+1)$. We start with a lemma.

Lemma 3.4. *Let* $p = 2, 3,$ *or* 5*, and let* $g \in G$ *be of order* p*. Then* $x^g \neq x$ *and* x^g *is not collinear to* x*, for every point* x*.*

Proof. Let x be an arbitrary point. Because G is regular, g fixes no points, and also no lines (otherwise $g = e$) so $x^g \neq x$. In order to show that x^g is not collinear to x, we assume that ℓ is a line through x and x^g, and show that this leads to a contradiction.

If g has order 2, then ℓ^g is a line through x^g and $x^{g^2} = x$, so $\ell^g = \ell$, which is indeed a contradiction.

If g has order 3, then x, x^g, and x^{g^2} are pairwise collinear. Similar as in the previous case (order 2), these three points cannot all be on the line ℓ, and it follows that they "generate" three lines ℓ, ℓ^g, and ℓ^{g^2}. This however gives a 6-cycle in the incidence graph, which is a contradiction, because its girth is 12.

Similarly, if g has order 5, then this gives rise to a 10-cycle in the incidence graph, which is again a contradiction. □

Note that the case $p = 5$ seems specific for generalized hexagons, whereas the cases 2 and 3 clearly also apply to generalized quadrangles, because their incidence graphs have girth "only" 8.

Next, we consider the adjacency matrix A of the point graph of the generalized hexagon, and let $M = A + I$. Note that this matrix could also be used to obtain the results for generalized quadrangles. Our matrix M has eigenvalue s^2+s+1 with multiplicity one (from the constant eigenvector), $2s$, 0, and $-s$. From an automorphism g we make a permutation matrix Q, where $Q_{x,y} = 1$ if $y = x^g$. Because g is an automorphism, we have that $QA = AQ$, and hence that $QM = MQ$. Using the eigenvalues of M, we obtain the following lemma.

Lemma 3.5.

$$\operatorname{tr} QM \equiv 1 \pmod{s}.$$

Proof. If g has order n, then $(QM)^n = Q^n M^n = M^n$. It follows that QM has the same eigenvalues as M, possibly multiplied by a root of unity. It has the same eigenvalue

$s^2 + s + 1$ with multiplicity one (from the constant eigenvector) as M. For each other eigenvalue, also its conjugates are eigenvalues, and the sum of these is a multiple of the "original" eigenvalue θ of M (because the sum of the relevant roots of unity is integer; for details, see the similar proof for generalized quadrangles by Benson [9]). It follows that the sum of all eigenvalues equals $s^2 + s + 1$ plus integer multiples of $2s$, 0, and $-s$. Hence $\operatorname{tr} QM \equiv 1 \pmod{s}$. □

We can now prove the following.

Proposition 3.6. *If the incidence graph of a generalized hexagon of order* (s, s) *is a Cayley graph, then* s *is a multiple of* 6 *and* $s + 1$ *is not divisible by* 5.

Proof. Suppose that the incidence graph is a Cayley graph, and that $(s + 1)(s^4 + s^2 + 1)$ is divisible by 2, 3, or 5. Then the generalized hexagon has a regular group G of automorphisms, acting regularly on both the point set and the line set. Because the order of this group is divisible by 2, 3, or 5, there is an automorphism $g \in G$ of order 2, 3, or 5. By Lemma 3.4, $x^g \neq x$ and x^g is not collinear to x, for every point x. It follows that both Q and QA have zero diagonal, hence $\operatorname{tr} QM = 0$. But this contradicts Lemma 3.5, hence $(s + 1)(s^4 + s^2 + 1)$ is not divisible by 2, 3, or 5, and this implies that s is a multiple of 6 and $s + 1$ is not divisible by 5. □

Because generalized hexagons of order (s, s) are only known for prime powers s, it follows that all the incidence graphs of the known generalized hexagons are not Cayley graphs. Note that automorphisms of a putative generalized hexagon of order $(6, 6)$ have been studied by Belousov [8].

Similarly, generalized quadrangles of order (s, s) are only known for prime powers s. Among these known ones, Proposition 3.3 thus rules out all s except $s = 4^i$ (for $i \in \mathbb{N}$). Among the distance-regular incidence graphs of generalized polygons with valency at most 5, we still need to consider the incidence graph of the generalized quadrangle of order $(4, 4)$. For this, we also consider one of the halved graphs, i.e., the collinearity (or point) graph.

Proposition 3.7. *The incidence graph of the generalized quadrangle* $GQ(4, 4)$ *is not a Cayley graph.*

Proof. Suppose that this bipartite graph Γ is a Cayley graph. By Lemma 2.1, its halved graphs are also Cayley graphs. These halved graphs (one of them being the collinearity graph of the generalized quadrangle) are again distance-regular, with intersection array $\{20, 16; 1, 5\}$ [11, Proposition 4.2.2]. In other words, it is a strongly regular graph with parameters $(85, 20, 3, 5)$. By Sylow's theorem, the only group of order 85 is the cyclic group $\mathbb{Z}_{85}$. Using the properties of a generalized quadrangle and that the cyclic group is abelian, it is easy to show that each line (a 5-clique) through e forms a subgroup of $\mathbb{Z}_{85}$, but there is only one such subgroup, which gives a contradiction, because there are 5 lines through each point. □

We note that this result also follows from more extensive results by Bamberg and Giudici [5, Theorem 1.1] and by Swartz [26, Theorem 1.3]. We remark that also the result that Tutte's 8-cage — the incidence graph of the unique generalized quadrangle of order $(2, 2)$ — is not a Cayley graph, can be obtained using the point graph. The latter is the complement of the triangular graph $T(6)$. Sabidussi [25] determined the Cayley graphs among the triangular graphs (see also Section 3.4), and $T(6)$ is not one of them. Thus, Tutte's 8-cage, also known as the Tutte-Coxeter graph, is not a Cayley graph.

Also Tutte's 12-cage — the unique incidence graph of a generalized hexagon of order $(2,2)$ — is not a Cayley graph for an elementary reason, i.e., because it is not vertex-transitive. Note that there are two generalized hexagons of order $(2,2)$, and these are dual, but not isomorphic, to each other. Thus, there are two orbits of vertices in the incidence graph.

We note that similarly there are precisely two generalized quadrangles of order $(3,3)$, and these are dual to each other. This implies that the corresponding incidence graph is not vertex-transitive, and hence this gives another argument for why this graph is not a Cayley graph.

Another argument for why Tutte's 12-cage is not a Cayley graph is obtained by considering the point graphs of the two generalized hexagons of order $(2,2)$. These distance-regular graphs have intersection array $\{6,4,4;1,1,3\}$ and automorphism group $\mathrm{PSU}(3,3) \rtimes \mathbb{Z}_2$ [4]. If such a graph would be a Cayley graph $\mathrm{Cay}(G,S)$, then G must be a subgroup of order 63 of the above group. Moreover, because the graph has no 4-cycles, the group must be nonabelian by Lemma 2.2. However, we checked with GAP [14] that there are no such subgroups, so we conclude that these graphs are not Cayley graphs. A similar argument applies to the line graph of Tutte's 12-cage, the unique distance-regular graph with intersection array $\{4,2,2,2,2,2;1,1,1,1,1,2\}$. Also this graph has automorphism group $\mathrm{PSU}(3,3) \rtimes \mathbb{Z}_2$ [4] and no 4-cycles. Thus, after having checked that there are no nonabelian subgroups of order 189, we conclude the following.

Proposition 3.8. *The line graph of Tutte's* 12*-cage and the point graphs of the two generalized hexagons of order* $(2,2)$ *are not Cayley graphs.*

Similarly, we can show that the unique distance-regular graph with intersection array $\{6,3,3;1,1,2\}$, the line graph of the incidence graph of the projective plane (generalized 3-gon) of order 3 is not a Cayley graph. Indeed, the automorphism group of the incidence graph (and hence of its line graph) is $\mathrm{PSL}(3,3) \rtimes \mathbb{Z}_2$, and we checked again with GAP [14] that it has no subgroups of order 52. We recall from Section 3.5 that the incidence graph itself is a Cayley graph. We had already observed in [2, Theorem 5.8] that if the line graph of the incidence graph of a projective plane of small odd order is a Cayley graph, then it should come from a group of both collineations and correlations of the projective plane.

Proposition 3.9. *The line graph of the incidence graph of the projective plane of order* 3 *is not a Cayley graph.*

We next consider the line graph of the incidence graph of the generalized quadrangle of order $(3,3)$.

Proposition 3.10. *The line graph of the incidence graph of the generalized quadrangle of order* $(3,3)$ *is not a Cayley graph.*

Proof. Suppose that this graph Γ is a Cayley graph $\mathrm{Cay}(G,S)$. Then G is a subgroup of the automorphism group of the incidence graph of the generalized quadrangle that acts regularly on its 160 flags. It follows that G acts transitively on the point set P and on the line set L. Hence $|G_x| = |G_\ell| = 4$ for every $x \in P$ and $\ell \in L$. This implies that for every point (and similarly, for every line), there is an involution in G that fixes it. On the other hand, it is not hard to show that every involution in G fixes either a point or a line, using Benson's results [9] or the approach as in Lemma 3.5 (see also [6, Lemma 3.4]).

Now let H be a Sylow 2-subgroup of G. We claim that the intersection of $Z(G)$ and H is trivial. To show this, assume that it is not. Then $H \cap Z(G)$ contains an involution σ, say, and suppose without loss of generality that σ fixes a point x, say. Let ℓ be a line through x and let θ be an involution that fixes ℓ. If $y = x^\theta$, then it is easy to see that σ also fixes y, and hence ℓ. But then it fixes a flag (x, ℓ), which is a contradiction.

Because $Z(G)$ is normal in G, it follows that $HZ(G)$ is a subgroup of G such that $|HZ(G)| = |H||Z(G)|$. This implies that $|Z(G)| = 1$ or 5. We checked with GAP [14] that there is no group of order 160 with $|Z(G)| = 5$ and there exists only one group G of order 160 such that $|Z(G)| = 1$; this group is $(\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$.

Now G has a normal subgroup $N = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$ of index 2, and this group does not have any dihedral subgroup, except the ones of order 2 and 4. Moreover, the two cosets of N induce an equitable partition of the graph, with quotient matrix of the form

$$\begin{bmatrix} m & 6-m \\ 6-m & m \end{bmatrix},$$

with $m = |S \cap N|$. This implies that Γ must have an eigenvalue $2m-6$ (besides eigenvalue 6) and because the integer eigenvalues of Γ are 6, 2, and -2, it follows that $m = 2$ or $m = 4$.

By Theorem 2.5 and the fact that G only has elements of orders $1, 2, 4$, and 5, it follows that $S = (K_1 \cup K_2) \setminus \{e\}$, where K_1 and K_2 are subgroups of G of order 4 such that $K_1 \cap K_2 = \{e\}$.

In both the cases $m = 2$ and $m = 4$, it follows that $S \cap N$ contains involutions $s_1 \in K_1$ and $s_2 \in K_2$. These two involutions generate a dihedral subgroup of N, which implies that this must be the dihedral group of order 4. But then s_1 and s_2 commute, and it is clear that e and $s_1 s_2$ have at least two common neighbors, while being at distance 2, and we have a contradiction. □

The last case we will handle in this section is that of the line graph of the incidence graph of a generalized hexagon of order $(3, 3)$. Note that it is currently unknown how many such generalized hexagons there are.

Proposition 3.11. *The line graph of the incidence graph of a generalized hexagon of order* $(3, 3)$ *is not a Cayley graph.*

Proof. Suppose that this graph Γ is a Cayley graph $\mathrm{Cay}(G, S)$. Then by the same approach as in the proof of Proposition 3.10, it follows that $G = (\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7) \times D_{26}$. Again, G has a normal subgroup $N = (\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7) \times \mathbb{Z}_{13}$ of index 2, and from the eigenvalues of Γ, we obtain that $m = 2$ or $m = 4$, where $m = |S \cap N|$.

Observe that N contains seven involutions, which generate an abelian subgroup $\mathbb{Z}_2^3$. Because $S \cap N$ contains an even number of elements, it also contains an even number of involutions. But these involutions commute and there are no induced 4-cycles in Γ, so it easily follows that $S \cap N$ contains no involutions. Because N only has elements of order $1, 2, 7, 13, 26$, and 91, and Γ contains no induces odd-cycles besides triangles, it follows that $S \cap N$ only contains elements of order 26. Thus, the connection set S has at least two elements of order 26.

Next, we consider the normal subgroup $K = \mathbb{Z}_2^3 \times D_{26}$, with quotient group G/K isomorphic to $\mathbb{Z}_7$. Note that all elements of order 26 in G are in K, so it follows that $S \cap K$ contains at least two elements. Because the quotient matrix corresponding to the equitable partition of the cosets of K is symmetric and cyclic, it follows that there are essentially only three options; the first row of the quotient matrix must be

$$\begin{bmatrix}4 & 1 & 0 & 0 & 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}2 & 2 & 0 & 0 & 0 & 0 & 2\end{bmatrix}, \quad \text{or} \quad \begin{bmatrix}2 & 0 & 1 & 1 & 1 & 1 & 0\end{bmatrix}.$$

All three matrices have eigenvalues of degree 3 (related to eigenvalues of the 7-cycle; the roots of $x^3 + x^2 - 2x - 1$). But Γ has no such eigenvalues, so we have a contradiction. □

Finally, we note that Bamberg and Giudici [5] claim that none of the classical generalized hexagons and octagons have a group of automorphisms that acts regularly on the points. This implies that none of the point graphs of the known generalized hexagons and octagons are Cayley graphs.

4 Distance-regular graphs with valency 3

All distance-regular graphs with valency 3 are known; see [11, Theorem 7.5.1]. In Table 1, we give an overview of all possible intersection arrays and corresponding graphs, and indicate which of these is a Cayley graph. The latter will follow from the results in the previous section, and the investigations in the current section, as commented in the table. Note that for each intersection array in Table 1 there is a unique distance-regular graph. By n, d, and g, we denote the number of vertices, diameter, and girth, respectively. The first graph in

Table 1: Distance-regular graphs with valency 3.

Intersection array	n	d	g	Name	Cayley	Comments
$\{3;1\}$	4	1	3	K_4	Yes	Sec. 3.1
$\{3,2;1,3\}$	6	2	4	$K_{3,3}$	Yes	Sec. 3.1
$\{3,2,1;1,2,3\}$	8	3	4	Cube $\sim K_{3,3}^*$	Yes	Sec. 3.1
$\{3,2;1,1\}$	10	2	5	Petersen $\sim O_3$	No	Sec. 3.4
$\{3,2,2;1,1,3\}$	14	3	6	Heawood $\sim IG(7,3,1)$	Yes	Sec. 3.5
$\{3,2,2,1;1,1,2,3\}$	18	4	6	Pappus $\sim IG(AG(2,3) \setminus pc)$	Yes	Prop. 3.2
$\{3,2,2,1,1;1,1,2,2,3\}$	20	5	6	Desargues $\sim DO_3$	No	Prop. 3.1
$\{3,2,1,1,1;1,1,1,2,3\}$	20	5	5	Dodecahedron	No	Folklore
$\{3,2,2,1;1,1,1,2\}$	28	4	7	Coxeter	No	Prop. 4.1
$\{3,2,2,2;1,1,1,3\}$	30	4	8	Tutte's 8-cage $\sim IG(GQ(2,2))$	No	Prop. 3.3
$\{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3\}$	90	8	10	Foster	No	Prop. 4.2
$\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$	102	7	9	Biggs-Smith	No	Prop. 4.4
$\{3,2,2,2,2,2;1,1,1,1,1,3\}$	126	6	12	Tutte's 12-cage $\sim IG(GH(2,2))$	No	Prop. 3.6

the table that does not occur in the previous section is the dodecahedron. It is however well known that this graph is not a Cayley graph; see for example [19], where it is shown that the only fullerene Cayley graph is the football (or buckyball) graph.

Also the fact that the Coxeter graph is not a Cayley graph is folklore. In the literature, e.g., [17], it is mentioned as one of the four non-Hamiltonian vertex-transitive graphs on more than two vertices, and it is noted that none of these four is a Cayley graph. Indeed, the automorphism group of the Coxeter graph is $\mathrm{PGL}(2,7)$, and this group has no subgroups of order 28.

Proposition 4.1. *The Coxeter graph is not a Cayley graph.*

The Foster graph is a bipartite distance-regular graph that can be described as the incidence graph of a partial linear space that can be considered as a 3-cover of the generalized quadrangle of order $(2,2)$. Its halved graphs are distance-regular with intersection array $\{6,4,2,1;1,1,4,6\}$ (e.g., see [11, Proposition 4.2.2]). The halved graph on the points is the collinearity graph of this partial linear space.

Proposition 4.2. *The Foster graph is not a Cayley graph.*

Proof. Suppose that the Foster graph is a Cayley graph. By Lemma 2.1, its halved graphs are also Cayley graphs, and these are distance-regular with intersection array $\{6,4,2,1;1,1,4,6\}$ on 45 vertices. So suppose that this halved graph is a Cayley graph $\mathrm{Cay}(G,S)$, with G of order 45 and S of size 6. By Sylow's theorem, G must be abelian. By Lemma 2.2, it follows that Γ contains a 4-cycle, which contradicts the fact that both the intersection numbers a_1 and c_2 are equal to 1. Thus, a distance-regular graph with intersection array $\{6,4,2,1;1,1,4,6\}$ cannot be a Cayley graph, and hence neither can the Foster graph. □

As a side result, we have thus obtained the following.

Corollary 4.3. *The collinearity graph of the* 3*-cover of the generalized quadrangle* $GQ(2,2)$*, the unique distance-regular graph with intersection array* $\{6,4,2,1;1,1,4,6\}$*, is not a Cayley graph.*

What remains is to consider the Biggs-Smith graph. The eigenvalues of this graph are very exceptional for a distance-regular graph. It has five distinct irrational eigenvalues, and distinct rational eigenvalues $3, 2$, and 0.

Proposition 4.4. *The Biggs-Smith graph is not a Cayley graph.*

Proof. Suppose that the Biggs-Smith graph Γ is a Cayley graph $\mathrm{Cay}(G,S)$. Then $|G| = 102$, so G has a subgroup H of order 51. It follows that the two cosets of H induce an equitable partition for Γ. Because Γ is connected and not bipartite, the quotient matrix is of the form

$$\begin{bmatrix} m & 3-m \\ 3-m & m \end{bmatrix},$$

where $m = 1$ or $m = 2$. This implies that Γ has an eigenvalue -1 or 1, which is a contradiction. □

Now we can conclude this section by the following result.

Theorem 4.5. *Let* Γ *be a distance-regular Cayley graph with valency* 3*. Then* Γ *is isomorphic to one of the following graphs:*

- *the complete graph K_4,*
- *the complete bipartite graph $K_{3,3}$,*
- *the cube Q_3,*
- *the Heawood graph $IG(7,3,1)$,*
- *the Pappus graph $IG(AG(2,3)\setminus pc)$.*

5 Distance-regular graphs with valency 4

The feasible intersection arrays for distance-regular graphs with valency four were determined by Brouwer and Koolen [13]. In Table 2, we give an overview of these intersection arrays and corresponding graphs, and indicate which of these is a Cayley graph, like in the previous section. Note that for each intersection array in the table there is a unique distance-regular graph, except possibly for the last array, which corresponds to the incidence graph of a generalized hexagon of order $(3,3)$.

Table 2: Distance-regular graphs with valency 4.

Intersection array	n	d	g	Name	Cayley	Reference
$\{4;1\}$	5	1	3	K_5	Yes	Sec. 3.1
$\{4,1;1,4\}$	6	2	3	$K_{2,2,2}$	Yes	Sec. 3.1
$\{4,3;1,4\}$	8	2	4	$K_{4,4}$	Yes	Sec. 3.1
$\{4,2;1,2\}$	9	2	3	$P(9)\sim H(2,3)$	Yes	Sec. 3.2
$\{4,3,1;1,3,4\}$	10	3	4	$K_{5,5}^*$	Yes	Sec. 3.1
$\{4,3,2;1,2,4\}$	14	3	4	$IG(7,4,2)$	Yes	Sec. 3.5
$\{4,2,1;1,1,4\}$	15	3	3	L(Petersen)	No	[2, Prop. 5.1]
$\{4,3,2,1;1,2,3,4\}$	16	4	4	Q_4	Yes	Sec. 3.3
$\{4,2,2;1,1,2\}$	21	3	3	L(Heawood)	Yes	[2, Ex. 5.7]
$\{4,3,3;1,1,4\}$	26	3	6	$IG(13,4,1)$	Yes	Sec. 3.5
$\{4,3,3,1;1,1,3,4\}$	32	4	6	$IG(A(2,4)\setminus pc)$	Yes	Prop. 3.2
$\{4,3,3;1,1,2\}$	35	3	6	O_4	No	Sec. 3.4
$\{4,2,2,2;1,1,1,2\}$	45	4	3	L(Tutte's 8-cage)	No	Prop. 2.6
$\{4,3,3,2,2,1,1;$ $1,1,2,2,3,3,4\}$	70	7	6	DO_4	No	Prop. 3.1
$\{4,3,3,3;1,1,1,4\}$	80	4	8	$IG(GQ(3,3))$	No	Prop. 3.3
$\{4,2,2,2,2,2;$ $1,1,1,1,1,2\}$	189	6	3	L(Tutte's 12-cage)	No	Prop. 3.8
$\{4,3,3,3,3,3;$ $1,1,1,1,1,4\}$	728	6	12	$IG(GH(3,3))$	No	Prop. 3.6

In [2], distance-regular Cayley graphs with least eigenvalue -2 were studied. It was, among others, shown that the line graph of the Petersen graph is not a Cayley graph (see [2, Proposition 5.1]), and that the line graph of Tutte's 8-cage is not a Cayley graph (see Section 2.5). On the other hand, it was shown that the line graph of the Heawood graph is a Cayley graph, over $\mathbb{Z}_7\rtimes\mathbb{Z}_3$ (see [2, Example 5.7]). In Proposition 3.8, we obtained that the line graph of Tutte's 12-cage is not a Cayley graph. We can therefore conclude this section with the following result.

Theorem 5.1. *Let* Γ *be a distance-regular Cayley graph with valency* 4*. Then* Γ *is isomorphic to one of the following graphs:*

- *the complete graph* K_5*,*
- *the octahedron graph* $K_{2,2,2}$*,*
- *the complete bipartite graph* $K_{4,4}$*,*
- *the Paley graph* $P(9)$*,*
- *the complete bipartite graph* $K_{5,5}$ *minus a complete matching,*
- *the incidence graph of the* 2-$(7, 4, 2)$ *design,*
- *the cube graph* Q_4*,*
- *the line graph of the Heawood graph,*
- *the incidence graph of the projective plane over* $\mathrm{GF}(3)$*,*
- *the incidence graph of the affine plane over* $\mathrm{GF}(4)$ *minus a parallel class of lines.*

6 Distance-regular graphs with valency 5

In Table 3, we list all known putative intersection arrays for distance-regular graphs with valency 5. We expect that this list is complete, but there is no proof for this. It contains all intersection arrays with diameter at most 7. This can be derived from the tables in [10] and [28]. All of the graphs in the table are unique, given their intersection arrays, except possibly the incidence graph of a generalized hexagon of order $(4, 4)$ (the last case).

It is well-known that the icosahedron is a Cayley graph. By using GAP [14] and similar codes as in [1, p. 3], we checked that we can indeed describe the icosahedron as a Cayley graph over the alternating group $\mathrm{Alt}(4)$, with connection set $S = \{(123), (132), (12)(34), (134), (143)\}$. According to Miklavič and Potočnik [21], the icosahedron is the smallest distance-regular Cayley graph over a non-abelian group, if we exclude cycles and the graphs from Section 3.1.

Also the Armanios-Wells graph is a Cayley graph. As far as we know, this was not known before.

Indeed, let G be the group generated by elements g_i, with $i = 1, 2, 3, 4$, each of order 2, such that $[g_i, g_j]$ is the same element, a say, for all $i \neq j$. This group is isomorphic to $(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$, where Q_8 is the group of quaternions. Now let $S = \{g_1, g_2, g_3, g_4, g_1g_2g_3g_4\}$. Then it is not hard to check that the Cayley graph $\mathrm{Cay}(G, S)$ is distance-regular with the same intersection array as the Armanios-Wells graph Γ, and hence that it must be the latter. In order to indeed check this, it is useful to know that Γ is an antipodal double cover with diameter 4, and that in this case $S_4 = \{a\}$, and consequently $S_3 = Sa$ (see Section 2.3). We double-checked this with GAP [14], and thus we have the following.

Proposition 6.1. *The Armanios-Wells graph is a Cayley graph over* $(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$.

A few more observations that we should make are the following. The center of G equals $\langle a \rangle$, which is of order 2. The quotient $G/\langle a \rangle$ is isomorphic to the elementary abelian 2-group $\mathbb{Z}_2^4$, which leads to the well-known description of the quotient graph — the folded 5-cube — as a Cayley graph (see Section 3.3).

Table 3: Distance-regular graphs with valency 5.

Intersection array	n	d	g	Name	Cayley	Reference
$\{5;1\}$	6	1	3	K_6	Yes	Sec. 3.1
$\{5,4;1,5\}$	10	2	4	$K_{5,5}$	Yes	Sec. 3.1
$\{5,2,1;1,2,5\}$	12	3	3	Icosahedron	Yes	Folklore
$\{5,4,1;1,4,5\}$	12	3	4	$K_{6,6}^*$	Yes	Sec. 3.1
$\{5,4;1,2\}$	16	2	4	Folded 5-cube	Yes	Sec. 3.3
$\{5,4,3;1,2,5\}$	22	3	4	$IG(11,5,2)$	Yes	Sec. 3.5
$\{5,4,3,2,1;1,2,3,4,5\}$	32	5	4	Q_5	Yes	Sec. 3.3
$\{5,4,1,1;1,1,4,5\}$	32	4	5	Armanios-Wells	Yes	Prop. 6.1
$\{5,4,2;1,1,4\}$	36	3	5	Sylvester	No	Prop. 6.2
$\{5,4,4;1,1,5\}$	42	3	6	$IG(21,5,1)$	Yes	Sec. 3.5
$\{5,4,4,1;1,1,4,5\}$	50	4	6	$IG(A(2,5)\setminus pc)$	Yes	Prop. 3.2
$\{5,4,4,3;1,1,2,2\}$	126	4	6	O_5	No	Sec. 3.4
$\{5,4,4,4;1,1,1,5\}$	170	4	8	$IG(GQ(4,4))$	No	Prop. 3.7
$\{5,4,4,3,3,2,2,1,1;$ $1,1,2,2,3,3,4,4,5\}$	252	9	6	DO_5	No	Prop. 3.1
$\{5,4,4,4,4,4;$ $1,1,1,1,1,5\}$	2730	6	12	$IG(GH(4,4))$	No	Prop. 3.6

The group G has a normal subgroup $\langle g_1g_2, g_2g_3, g_3g_1\rangle$, which is isomorphic to Q_8. This gives rise to an equitable partition of Γ into 4 cocliques of size 8.

In addition, the normal subgroup $\langle g_1g_2, g_2g_3, g_3g_1, g_4\rangle$ is isomorphic to $\mathbb{Z}_2\times Q_8$, which gives an equitable partition of Γ into two 1-regular induced subgraphs. Together these form a matching, and removing the edges of this matching results in a bipartite 4-regular graph. This turns out to be the incidence graph of the affine plane of order 4 minus a parallel class (see Section 3.6 and Table 2). Alternatively, we obtain that the latter is isomorphic to the Cayley graph $\mathrm{Cay}(G,\{g_1,g_2,g_3,g_4\})$.

The remaining intersection array in Table 3 is that of the Sylvester graph. This graph has distinct eigenvalues $5,2,-1$, and -3 and full automorphism group $\mathrm{Sym}(6)\rtimes\mathbb{Z}_2$ [11, p. 394].

Proposition 6.2. *The Sylvester graph is not a Cayley graph.*

Proof. Suppose that the Sylvester graph Γ is a Cayley graph $\mathrm{Cay}(G,S)$, then $|G|=36$ and $|S|=5$. Because Γ has girth 5, the group G is non-abelian by Lemma 2.2. It is known that there are 10 non-abelian groups of order 36, of which two do not have a normal subgroup of order 9; these are $\mathbb{Z}_3\times\mathrm{Alt}(4)$ and $(\mathbb{Z}_2\times\mathbb{Z}_2)\rtimes\mathbb{Z}_9$.

If G is the latter group (and contains elements of order 9), then it has automorphisms of order 9. This contradicts the fact that the full automorphism group of Γ equals $\mathrm{Sym}(6)\rtimes\mathbb{Z}_2$.

Next, we will also show that G cannot be $\mathbb{Z}_3\times\mathrm{Alt}(4)$, and hence that G must have a normal subgroup of order 9. Indeed, suppose that G equals $\mathbb{Z}_3\times\mathrm{Alt}(4)$. The center of this group is isomorphic to $\mathbb{Z}_3$, say $Z(G)=\langle c\rangle$, with c of order 3. Moreover, G has a normal subgroup H isomorphic to $\mathrm{Alt}(4)$ (with cosets H, Hc, Hc^2 that form an equitable partition of Γ).

Now suppose that $hc^i \in S$ for some $h \in H$ and $i = 0, 1, 2$. Then the order of h must be 2, for if it were 3 (or 1, the only other options), then $e \sim hc^i \sim (hc^i)^2 \sim (hc^i)^3 = e$, which contradicts the fact that Γ has girth 5. Moreover, if $h \in S$, then hc and $hc^2 = (hc)^{-1}$ are not in S because that would imply that $e \sim hc \sim c \sim hc^2 \sim e$, which again gives a contradiction.

Because $\mathrm{Alt}(4)$ has only three involutions, there are also only three involutions h_1, h_2, and h_3, say, in H. Thus, it follows without loss of generality that $S = \{h_1, h_2c, h_2c^2, h_3c, h_3c^2\}$. However, now $e \sim h_2c \sim h_3h_2 \sim h_2c^2 \sim e$, which gives the final contradiction, and hence G cannot be $\mathbb{Z}_3 \times \mathrm{Alt}(4)$.

Thus, the group G has a normal subgroup N of order 9. The four cosets of N form an equitable partition of Γ with quotient matrix

$$\begin{bmatrix} n_1 & n_2 & n_3 & n_4 \\ n_2 & n_1 & n_4 & n_3 \\ n_3 & n_4 & n_1 & n_2 \\ n_4 & n_3 & n_2 & n_1 \end{bmatrix},$$

for certain n_1, n_2, n_3, n_4 summing to 5, and because Γ is connected, at most one of n_2, n_3, n_4 can be 0. Now the quotient matrix has eigenvalues $n_1+n_2+n_3+n_4$, $n_1+n_2-n_3-n_4$, $n_1 - n_2 + n_3 - n_4$, and $n_1 - n_2 - n_3 + n_4$. Because Γ has no eigenvalues 3 and 1, it follows that $n_1 = 0, n_2 = 1, n_3 = 2$, and $n_4 = 2$, up to reordering of the latter three (we omit the easy but technical details).

So there is one coset that intersects S in $n_2 = 1$ element. Let us call this element a, then clearly $O(a) = 2$, and the subgroup $N\langle a\rangle$ is a normal subgroup (of index 2). Given the quotient matrix, it follows easily that every vertex in the coset Na except a itself is at distance 2 from e.

Now we claim that a is the only involution in $N\langle a\rangle$. Clearly there are no involutions in N because it has order 9. Every other element in Na is at distance 2 from e, and hence can be written as s_1s_2 for some $s_1, s_2 \in S$. Suppose now that $O(s_1s_2) = 2$. Then $e \sim s_2 \sim s_1s_2 \sim s_2s_1s_2 \sim e$, a contradiction since the girth of Γ is 5, and we proved our claim.

Now suppose that $s \in S$, with $s \neq a$. Then $s^{-1}as \in N\langle a\rangle$ since $N\langle a\rangle$ is a normal subgroup. Because $O(s^{-1}as) = 2$, it follows from our above claim that $s^{-1}as = a$. Thus, $sa = as$ and $e \sim a \sim sa = as \sim s \sim e$, which is again a contradiction to the girth of Γ, and which completes the proof. □

Now we can conclude this section with the following proposition.

Proposition 6.3. *Let* Γ *be a distance-regular Cayley graph with valency* 5*, with one of the intersection arrays in Table 3*[1]*. Then* Γ *is isomorphic to one of the following graphs:*

- *the complete graph* K_6*,*
- *the complete bipartite graph* $K_{5,5}$*,*
- *the icosahedron,*
- *the complete bipartite graph* $K_{6,6}$ *minus a complete matching,*
- *the folded* 5*-cube,*

[1] Currently, these are the only known putative intersection arrays for distance-regular graphs with valency 5.

- *the incidence graph of the* 2-$(11, 5, 2)$ *design,*
- *the cube graph* Q_5*,*
- *the Armanios-Wells graph,*
- *the incidence graph of the projective plane over* $\mathrm{GF}(4)$*,*
- *the incidence graph of the affine plane over* $\mathrm{GF}(5)$ *minus a parallel class of lines.*

7 Distance-regular graphs with girth 3 and valency 6 or 7

Hiraki, Nomura, and Suzuki [16] determined the feasible intersection arrays of all distance-regular graphs with valency at most 7 and girth 3 (i.e., with triangles). Besides the ones with valency at most 5 that we have encountered in the previous sections, these are listed in Table 4. For each of the intersection arrays $\{6, 3; 1, 2\}$ and $\{6, 4, 4; 1, 1, 3\}$, there are exactly two distance-regular graphs (as mentioned in the table). For all others, except possibly the last one with valency 6, the graphs in the table are unique, given their intersection arrays. For this last case, it is unknown whether the generalized hexagon of order $(3, 3)$ is unique.

Table 4: Distance-regular graphs with girth 3 and valency 6 or 7.

Intersection array	n	d	g	Name	Cayley	Reference
$\{6; 1\}$	7	1	3	K_7	Yes	Sec. 3.1
$\{6, 1; 1, 6\}$	8	2	3	$K_{2,2,2,2}$	Yes	Sec. 3.1
$\{6, 2; 1, 6\}$	9	2	3	$K_{3,3,3}$	Yes	Sec. 3.1
$\{6, 2; 1, 4\}$	10	2	3	$T(5)$	No	Sec. 3.4
$\{6, 3; 1, 3\}$	13	2	3	$P(13)$	Yes	Sec. 3.2
$\{6, 4; 1, 3\}$	15	2	3	$\overline{T(6)} \sim GQ(2, 2)$	No	Sec. 3.4
$\{6, 3; 1, 2\}$	16	2	3	$L_2(4)$, Shrikhande	Yes	Sec. 3.3
$\{6, 4, 2; 1, 2, 3\}$	27	3	3	$H(3, 3)$	Yes	Sec. 3.3
$\{6, 4, 2, 1; 1, 1, 4, 6\}$	45	4	3	halved Foster	No	Cor. 4.3
$\{6, 3, 3; 1, 1, 2\}$	52	3	3	$L(IG(13, 4, 1))$	No	Prop. 3.9
$\{6, 4, 4; 1, 1, 3\}$	63	4	3	$GH(2, 2)$ $(2\times)$	No	Prop. 3.8
$\{6, 3, 3, 3; 1, 1, 1, 2\}$	160	4	3	$L(IG(GQ(3, 3)))$	No	Prop. 3.10
$\{6, 3, 3, 3, 3, 3;$ $1, 1, 1, 1, 1, 2\}$	1456	6	3	$L(IG(GH(3, 3)))$	No	Prop. 3.11
$\{7; 1\}$	8	1	3	K_8	Yes	Sec. 3.1
$\{7, 4, 1; 1, 2, 7\}$	24	3	3	Klein	Yes	Prop. 7.1

What remains is to consider the Klein graph. We observe that this is a Cayley graph on the symmetric group $\mathrm{Sym}(4)$. Indeed, one can check[2] that with

$$S = \{(123), (132), (12)(34), (13), (14), (1234), (1432)\},$$

the Cayley graph $\mathrm{Cay}(\mathrm{Sym}(4), S)$ is a distance-regular antipodal 3-cover of K_8, and hence it must be the Klein graph. We note that in this case the set $S_3 = \{(124), (142)\}$, and de-

[2] We double-checked this with GAP [14].

spite the fact that $N_3 = S_3 \cup \{e\}$ is not a normal subgroup, its right cosets form an equitable partition (with quotient K_8, of course); cf. Section 2.3. We thus have the following.

Proposition 7.1. *The Klein graph is a Cayley graph over* $\mathrm{Sym}(4)$.

We also note that the normal subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ gives an equitable partition into 6 parts, with each coset inducing a matching (which together gives a perfect matching). More interesting is the (normal) alternating subgroup $\mathrm{Alt}(4)$, which gives an equitable partition into two parts. On each part, the induced subgraph is the truncated tetrahedron, which is thus a Cayley graph $\mathrm{Cay}(\mathrm{Alt}(4), \{(123), (132), (12)(34)\})$. This is also the line graph of a bipartite biregular graph on $4 + \binom{4}{2}$ vertices with valencies 3 and 2, respectively (the Pasch configuration), and a subgraph of the icosahedron; cf. Section 6.

We conclude with the following proposition.

Proposition 7.2. *Let* Γ *be a distance-regular Cayley graph with girth* 3 *and valency* 6 *or* 7*. Then* Γ *is isomorphic to one of the following graphs:*

- *the complete graph* K_7*,*
- *the complete graph* K_8*,*
- *the complete multipartite graph* $K_{2,2,2,2}$*,*
- *the complete multipartite graph* $K_{3,3,3}$*,*
- *the Paley graph* $P(13)$*,*
- *the lattice graph* $L_2(4)$*,*
- *the Shrikhande graph,*
- *the Hamming graphs* $H(3, 3)$*,*
- *the Klein graph.*

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 223–253
https://doi.org/10.26493/1855-3974.1695.144
(Also available at http://amc-journal.eu)

The orientable genus of the join of a cycle and a complete graph

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Received 8 May 2018, accepted 22 April 2019, published online 2 October 2019

Abstract

Let m and n be two integers. In the paper we show that the orientable genus of the join of a cycle C_m and a complete graph K_n is $\lceil \frac{(m-2)(n-2)}{4} \rceil$ if $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$.

Keywords: Surface, orientable genus of a graph, join of two graphs.

Math. Subj. Class.: 05C10

1 Introduction

Let G and H be two disjoint graphs. The *join* of G with H, denoted by $G + H$, is the graph obtained from the union of G and H by adding edges joining every vertex of G to every vertex of H. A cycle with m vertices is denoted by C_m, and a complete graph with n vertices denoted by K_n.

Our investigation of the orientable genus of $C_m + K_n$ is inspired by the problem of the critical graphs on surfaces. A graph G is *k-critical* if $\chi(G) = k$ but $\chi(G') < k$ for every proper subgraph of G, where $\chi(H)$ denotes the chromatic number of a graph H. If G_1 is k-critical and G_2 is l-critical, it is known that $G_1 + G_2$ is $(k + l)$-critical. Since an odd cycle is 3-critical and K_n is n-critical, the join of an odd cycle and K_n is $(n + 3)$-critical. Also, there are only finite many k-critical graphs on a surface if $k \geq 7$ ([4, 6, 7, 13]). So it is an interesting problem to explore the orientable genus of the join of an odd cycle (or a cycle) and K_n.

*Corresponding author. Supported by NNSFC under the granted number 11171114.

†Supported by NNSFC under the granted number 11171114, Science and Technology Commission of Shanghai Municipality under grant No. 13dz2260400.

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Let us look back the history of studying the orientable genus of the join of two graphs. Let $\bar{K}_t$ be the compliment graph of K_t. The complete bipartite graph $K_{m,n}$ and K_n $(n \geq 2)$ can be viewed as $\bar{K}_m + \bar{K}_n$ and $K_1 + K_{n-1}$, respectively. It is cheerful that the orietable genera of K_n and $K_{m,n}$ have been determined ([10, 11]). Upon the orientable genus of $\bar{K}_m + K_n$ there are some results. Craft [3] verified that $\bar{K}_m + K_n$ has the same orientable genus as that of $K_{m,n}$, when n is even and $m \geq 2n - 4$. Ellingham and Stephens [5] determined the orientable genus of $\bar{K}_m + K_n$ if n is even and $m \geq n$, or $n = 2^p + 2$ for $p \geq 3$ and $m \geq n - 1$, or $n = 2^p + 1$ for $p \geq 3$ and $m \geq n + 1$. Korzhik [8] contributed many results on the orientable genus of $\bar{K}_m + K_n$ with $m \leq n - 2$.

Let $m \geq 3$ and $n \geq 1$ be two integers. If $n = 1$, then $C_m + K_n$ is a planar graph. If $n = 2$, then $C_m + K_n$ has a minor isomorphic to K_5. So the orientable genus of $C_m + K_2$ is at least one. Since $C_m + K_2$ can be embedded on the torus, the orientable genus of $C_m + K_2$ is one. If $n = 3$, then K_n is exactly the cycle C_3. Craft [2] has proved that the orientable genus of $C_m + C_3$ is $\lceil \frac{m-2}{4} \rceil$. What is the orientable genus of $C_m + K_n$ if $n \geq 4$? In the paper we shall show that the orientable genus of $C_m + K_n$ is $\lceil \frac{(m-2)(n-2)}{4} \rceil$ if $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$.

Since $K_{m,n}$ is a spanning subgraph of $C_m + K_n$, a lower bound of the oreintable genus of $C_m + K_n$ is that of $K_{m,n}$, which is $\lceil \frac{(m-2)(n-2)}{4} \rceil$. The key to determine the orientable genus of $C_m + K_n$ is the construction of an embedding of $C_m + K_n$ on the orientable surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$. We mainly use two methods of adding tubes to construct an embedding of $C_m + K_n$. Our general strategy of constructing an embedding is as follows. First, we construct an embedding of a spanning subgraph of $C_m + K_n$ which contains C_m, a spanning subgraph of K_n, and some edges between C_m and K_n on some orientable surface. Second, we apply the first method of adding tubes described in Section 2 to attach all the rest edges in K_n and some edges between C_m and K_n. Third, we apply the second method of adding tubes described in Section 2 to attach all the rest edges between C_m and K_n.

The remainder of the section is contributed for some terms. The other undefined terms can be found in [1, 9], or [14].

A *surface* is a compact connected 2-dimensional manifold without boundary. The orientable surface S_g $(g \geq 0)$ can be obtained from a sphere with g handles attached, where g is called the *genus* of S_g. A graph G is able to embed in a surface S if it can be drawn in the surface such that any edge does not pass through any vertex and any two edges do not cross each other. The *orientable genus* of a connected graph G, denoted by $\gamma(G)$, is the smallest nonnegative integer g such that G can be embedded in the orientable surface S_g.

An embedding Π of a connected graph in a surface S is called 2*-cell embedding* if any connected component of $S - \Pi$, called a face, is homeomorphic to an open disc. In a 2-cell embedding of a connected graph G, the boundary of a face in Π is a closed walk of G, which is called the *facial walk*. If a facial walk is a cycle, then it is called a *facial cycle*. Let v be a vertex of a graph G embedded on a surface. A local rotation π_v at the vertex v is a cyclic permutation of the edges incident with v. Suppose that v is incident with edges $vu_1, vu_2, \ldots, vu_n$ in this order. Then π_v can be written by $u_1, u_2, \ldots, u_n$. Furthermore, if $i_1, i_2, \ldots, i_k$ are k continuous numbers in $\{1, 2, \ldots, n\}$, where $2 \leq k \leq n$, then we call $u_{i_1}, u_{i_2}, \ldots, u_{i_k}$ a segment of the local rotation at v.

A graph H is a *supergraph* of G if G is a subgraph of H. If a cycle with n (≥ 3) vertices $v_1, v_2, \ldots, v_n$ in this order, then it is written by $v_1v_2 \ldots v_nv_1$ and it is always oriented by this order.

2 Two methods of constructing embeddings

Let D_1 and D_2 be two facial cycles of a 2-cell embedding on a surface S such that the orientation of D_1 is the reverse of that of D_2. By adding a tube T to the surface S between D_1 and D_2, we mean that we cut two holes Δ_1 and Δ_2 in S such that Δ_i is in the interior of D_i and orient the boundary of Δ_i as that of D_i, then the tube T welds Δ_1 with Δ_2 in such a way that the rim of one of the ends of T coincides with the boundary of Δ_1 and the rim of the other end of T coincides with the boundary of Δ_2.

Lemma 2.1. *Suppose that G is a graph which has a vertex subset*

$$\{w_0, z_1, z_2, \ldots, z_t\} \cup \{x_i \mid i = 1, 2, \ldots, 2t\} \cup \{y_j \mid j = 1, 2, \ldots, 4t\},$$

where $z_1, z_2, \ldots, z_t$ need not be different, and suppose that G contains no edges in the set

$$\begin{aligned} E' = &\{w_0x_i \mid i = 1, 2, \ldots, 2t\} \cup \{x_iy_j \mid i = 1, 2, \ldots, 2t; j = 1, 2, \ldots, 4t\} \\ &\cup (\{x_ix_{i+1}, \ldots, x_ix_{2t} \mid i = 1, 2, \ldots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \ldots, t\}). \end{aligned}$$

Suppose that Π is a 2-cell embedding of G on the orientable surface S_g with the following properties:

(i) For $i = 1, 2, \ldots, t$, $R_{0,i} = w_0y_{4i-3}y_{4i-2}w_0$ and $R'_{0,i} = w_0y_{4i-1}y_{4i}w_0$ are facial cycles of Π.

(ii) For $i = 1, 2, \ldots, t$, $Q_{0,i} = z_ix_{2i-1}x_{2i}z_i$ is a facial cycle of Π such that $Q_{0,i}$ has not any common vertex with each of $R_{0,1}, \ldots, R_{0,t}, R'_{0,1}, \ldots, R'_{0,t}$.

Then there is a supergraph H of G satisfying the following conditions:

(i) E' is an edge subset of $E(H)$.

(ii) H has an embedding on the orientable surface of genus $g + 2t^2$ such that it has a set of t facial 3-cycles $\{Q_{t,i} \mid Q_{t,i} = y_{l_i}x_{2i-1}x_{2i}y_{l_i}, i = 1, 2, \ldots, t\}$, where y_{l_i} is some vertex in $\{y_{4i-3}, y_{4i-2}, y_{4i-1}, y_{4i} \mid i = 1, 2, \ldots, t\}$.

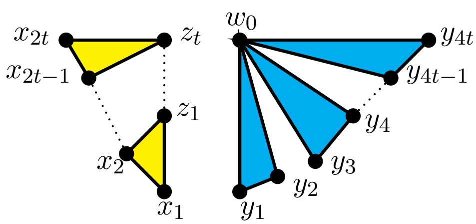

Figure 1: A local structure in Π.

Remark 2.2.

(1) A local structure of Π is shown in Figure 1.

(2) An application of Lemma 2.1 to the construction of an embedding of $C_m + K_n$ is as follows. After an embedding of a spanning subgraph of $C_m + K_n$ on some orientable surface has been constructed, all the rest edges of K_n and some edges between C_m and K_n can be attached by applying Lemma 2.1.

Proof. We shall construct an embedding on the surface of genus $g + 2t^2$ from the embedding Π by applying the operation of adding tubes t times. Every time $2t$ tubes are added to the present surface.

For $i = 1, 2, \dots, t$, the tube $T_{0,i}$ is added between $Q_{0,i}$ and $R_{0,i}$. Next, the five edges $w_0x_{2i}, x_{2i-1}y_{4i-3}, x_{2i-1}y_{4i-2}, x_{2i}y_{4i-3}$ and $x_{2i}y_{4i-2}$ are drawn on $T_{0,i}$ in the way shown in (1) of Figure 2. For $i = 1, 2, \dots, t$, let $Q'_{0,i} = y_{4i-2}x_{2i-1}x_{2i}y_{4i-2}$.

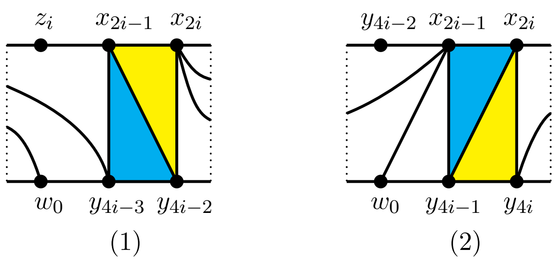

Figure 2: Two drawings of five edges on a tube.

For $i = 1, 2, \dots, t$, the tube $T'_{0,i}$ is added between $Q'_{0,i}$ and $R'_{0,i}$. Next, the five edges w_0x_{2i-1}, $x_{2i-1}y_{4i-1}$, $x_{2i-1}y_{4i}$, $x_{2i}y_{4i-1}$ and $x_{2i}y_{4i}$ are drawn on $T'_{0,i}$ in the way shown in (2) of Figure 2.

Need to say that the rectangle represents a tube and that the two dot curves are identified with each other in Figure 2. In the rest of the paper we always use a rectangle to represent a tube and the two dot curves in the rectangle are always identified with each other.

For the convenience of argument, the way of drawing edges shown in (i) of Figure 2 is called the *drawing of Type-i* for $i = 1, 2$. To help the readers to understand how those $2t$ tubes are added and how five edges are drawn on each tube, we give an example that $t = 5$ which is shown in Figure 3. The diagrams in Figure 3 are partitioned into four columns from left to right. The three rectangles in the first column respectively represent $T_{0,1}, T_{0,2}$ and $T_{0,3}$ from top to bottom, and the two rectangles in the third column respectively represent $T_{0,4}$ and $T_{0,5}$ from top to bottom. Similarly, the three rectangles in the second column respectively represent $T'_{0,1}, T'_{0,2}$ and $T'_{0,3}$, and the two rectangles in the fourth column respectively represent $T'_{0,4}$ and $T'_{0,5}$.

After those $2t$ tubes have been added, there are three sets of facial 3-cycles which are

$$\begin{aligned}
\mathcal{X}_1 &= \{Q_{1,i} \mid Q_{1,i} = y_{4i-1}x_{2i-1}x_{2i}y_{4i-1}, i = 1, 2, \dots, t\},\\
\mathcal{Y}_1 &= \{R_{1,i} \mid R_{1,i} = x_{2i-1}y_{4i-3}y_{4i-2}x_{2i-1}, i = 1, 2, \dots, t\}, \text{ and}\\
\mathcal{Y}'_1 &= \{R'_{1,i} \mid R'_{1,i} = x_{2i}y_{4i-1}y_{4i}x_{2i}, i = 1, 2, \dots, t\}.
\end{aligned}$$

For the convenience of argument, we now define t permutations. For $k = 0, 1, \dots, t-1$, we define the permutation τ_k on the set $\{1, 2, \dots, t\}$ as follows. For $i = 1, 2, \dots, t$,

$$\tau_k(i) \equiv i + (-1)^{k+1}k \pmod t,$$

where $0 \le i + (-1)^{k+1}k \le t - 1$.

Obviously, τ_0 is the identity mapping on $\{1, 2, \dots, t\}$. For $0 \le k \le t-1$, we define

$$\tau'_k(i) \equiv \begin{cases} \tau_k(i) \pmod t, & \text{if } k = 0,\\ \tau_0\tau_1\cdots\tau_k(i) \pmod t, & \text{if } 1 \le k \le t-1,\end{cases}$$

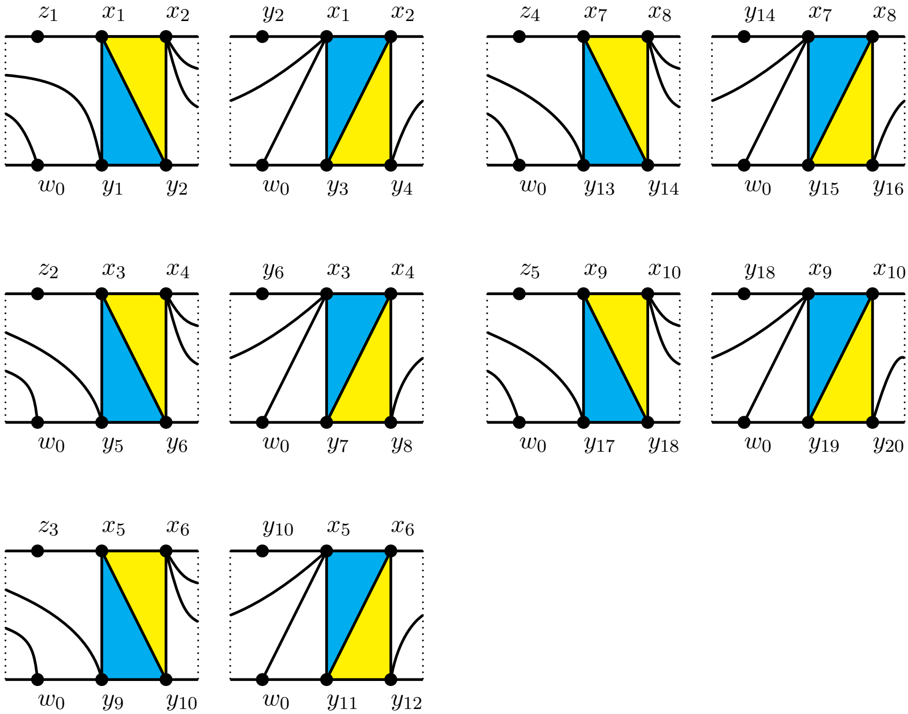

Figure 3: The first operation of adding $2t$ tubes when $t = 5$.

where $0 \le \tau'_k(i) \le t-1$ and $\tau_0\tau_1\cdots\tau_k$ is the product of $\tau_0, \tau_1, \ldots, \tau_k$ in this order. For example, $\tau_0\tau_1(1) = \tau_1(\tau_0(1)) = 2$.

Thus, $Q_{1,i}$, $R_{1,i}$ and $R'_{1,i}$ can be alternately expressed as follows:

$$\begin{aligned} Q_{1,i} &= y_{4\tau'_0(i)-1}x_{2i-1}x_{2i}y_{4\tau'_0(i)-1}, \\ R_{1,i} &= x_{2i-1}y_{4\tau'_0(i)-3}y_{4\tau'_0(i)-2}x_{2i-1}, \text{ and} \\ R'_{1,i} &= x_{2i}y_{4\tau'_0(i)-1}y_{4\tau'_0(i)}x_{2i}. \end{aligned}$$

We continue to add tubes, and consider two cases.

Case 1: $t \equiv 1 \pmod 2$. In this case we firstly add t tubes $T_{1,1}, \ldots, T_{1,t}$ to the present surface such that $T_{1,i}$ is between $Q_{1,i}$ and $R_{1,\tau_1(i)}$. Note that

$$\begin{aligned} R_{1,\tau_1(i)} &= x_{2\tau_1(i)-1}y_{4\tau_0\tau_1(i)-3}y_{4\tau_0\tau_1(i)-2}x_{2\tau_1(i)-1}, \text{ i.e.,} \\ R_{1,\tau_1(i)} &= x_{2\tau_1(i)-1}y_{4\tau'_1(i)-3}y_{4\tau'_1(i)-2}x_{2\tau_1(i)-1}. \end{aligned}$$

For $i = 1, 2, \ldots, t$, the five edges $x_{2i-1}y_{4\tau'_1(i)-3}$, $x_{2i-1}y_{4\tau'_1(i)-2}$, $x_{2i}y_{4\tau'_1(i)-3}$, $x_{2i}y_{4\tau'_1(i)-2}$ and $x_{2i}x_{2\tau_1(i)-1}$ are drawn on $T_{1,i}$ in the way of the drawing of Type-1. Thus, there is a set $\mathcal{X}'_1$ of t facial 3-cycles, where

$$\mathcal{X}'_1 = \{Q'_{1,i} \mid Q'_{1,i} = y_{4\tau'_1(i)-2}x_{2i-1}x_{2i}y_{4\tau'_1(i)-2}, i = 1, 2, \ldots, t\}.$$

Next, the t tubes $T'_{1,1}, \ldots, T'_{1,t}$ are added to the present surface such that $T'_{1,i}$ is between $Q'_{1,i}$ and $R'_{1,\tau_1(i)}$. Then the five edges $x_{2i-1}y_{4\tau'_1(i)-1}$, $x_{2i-1}y_{4\tau'_1(i)}$, $x_{2i}y_{4\tau'_1(i)-1}$, $x_{2i}y_{4\tau'_1(i)}$

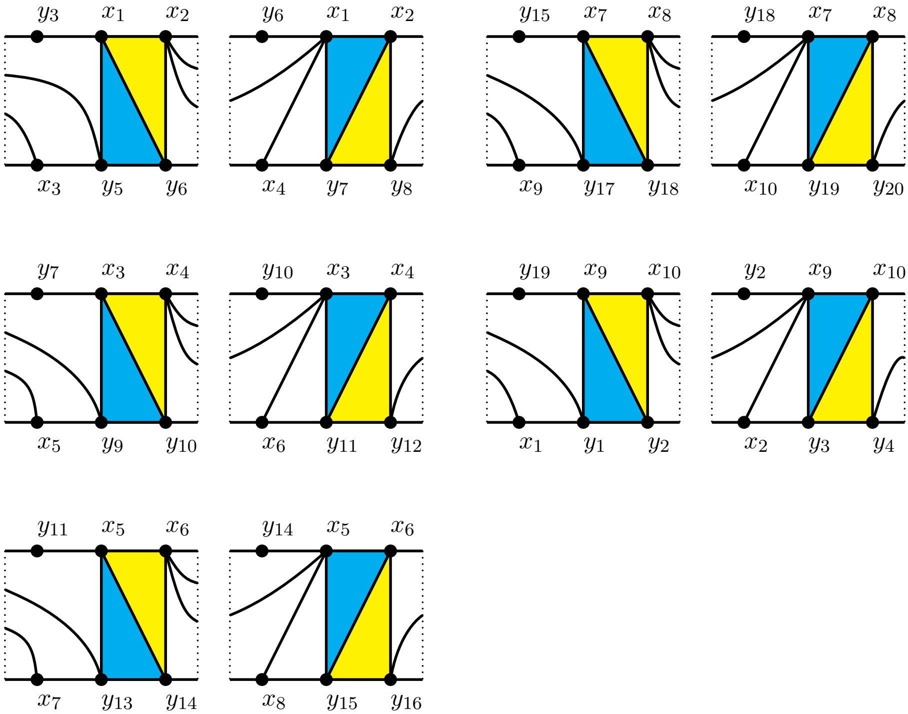

Figure 4: The second operation of adding $2t$ tubes when $t = 5$.

and $x_{2i}x_{2\tau_1(i)}$ are drawn on $T'_{1,i}$ in the way of the drawing of Type-2. For example, if $t = 5$, the above operation of adding $2t$ tubes is shown in Figure 4. The order of diagrams in Figure 4 is as that in Figure 3.

After those $2t$ tubes have been added, there are three sets $\mathcal{X}_2$, $\mathcal{Y}_2$, and $\mathcal{Y}'_2$ of facial 3-cycles which are

$$\begin{aligned}
\mathcal{X}_2 &= \{Q_{2,i} \mid Q_{2,i} = y_{4\tau'_1(i)-1}x_{2i-1}x_{2i}y_{4\tau'_1(i)-1}, i = 1, 2, \dots, t\},\\
\mathcal{Y}_2 &= \{R_{2,i} \mid R_{2,i} = x_{2i-1}y_{4\tau'_1(i)-3}y_{4\tau'_1(i)-2}x_{2i-1}, i = 1, 2, \dots, t\}, \text{ and}\\
\mathcal{Y}'_2 &= \{R'_{2,i} \mid R'_{2,i} = x_{2i}y_{4\tau'_1(i)-1}y_{4\tau'_1(i)}x_{2i}, i = 1, 2, \dots, t\}.
\end{aligned}$$

In general, if the s-th operation ($s \geq 1$) of adding $2t$ tubes has been applied, then there are three sets of facial 3-cycles, i.e.,

$$\mathcal{X}_s = \{Q_{s,i} \mid i = 1, 2, \dots, t\}, \qquad \mathcal{Y}_s = \{R_{s,i} \mid i = 1, 2, \dots, t\}, \text{ and}$$
$$\mathcal{Y}'_s = \{R'_{s,i} \mid i = 1, 2, \dots, t\}.$$

Next, we apply the $(s+1)$-th of adding $2t$ tubes $T_{s,1}, \dots, T_{s,t}, T'_{s,1}, \dots, T'_{s,t}$ to the present surface satisfying the following conditions.

(1) If $1 \leq s \leq \frac{t-1}{2}$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R_{s,\tau_s(i)}$, where $i = 1, 2, \dots, t$. In this case $R_{s,\tau_s(i)} = x_{2\tau_s(i)-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2\tau_s(i)-1}$. Next,

the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3},\quad x_{2i-1}y_{4\tau'_s(i)-2},\quad x_{2i}y_{4\tau'_s(i)-3},\quad x_{2i}y_{4\tau'_s(i)-2},\ \text{and}\ x_{2i}x_{2\tau_s(i)-1}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. After those t tubes have been added, there is a set $\mathcal{X}'_s$ of t facial 3-cycles, where

$$\mathcal{X}'_s = \{Q'_{s,i} \mid Q'_{s,i} = y_{4\tau'_s(i)-2}x_{2i-1}x_{2i}y_{4\tau'_s(i)-2},\ i = 1, 2, \ldots, t\}.$$

For $i = 1, 2, \ldots, t$, the tube $T'_{s,i}$ is added between $Q'_{s,i}$ and $R'_{s,\tau_s(i)}$. Note that $R'_{s,\tau_s(i)} = x_{2\tau_s(i)}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2\tau_s(i)}$. Next, the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1},\quad x_{2i-1}y_{4\tau'_s(i)},\quad x_{2i}y_{4\tau'_s(i)-1},\quad x_{2i}y_{4\tau'_s(i)},\ \text{and}\ x_{2i-1}x_{2\tau_s(i)}$$

are drawn on $T'_{s,i}$ in the way of the drawing of Type-2.

After the $(s+1)$-th operation of adding $2t$ tubes has been applied, there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$, and $\mathcal{Y}'_{s+1}$ of facial 3-cycles which are

$$\mathcal{X}_{s+1} = \{Q_{s+1,i} \mid Q_{s+1,i} = y_{4\tau'_s(i)-1}x_{2i-1}x_{2i}y_{4\tau'_s(i)-1},\ i = 1, 2, \ldots, t\},$$
$$\mathcal{Y}_{s+1} = \{R_{s+1,i} \mid R_{s+1,i} = x_{2i-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i-1},\ i = 1, 2, \ldots, t\},\ \text{and}$$
$$\mathcal{Y}'_{s+1} = \{R'_{s+1,i} \mid R'_{s+1,i} = x_{2i}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i},\ i = 1, 2, \ldots, t\}.$$

(2) If $\frac{t+1}{2} \leq s \leq t-1$, suppose that k and k' are the maximum even and odd numbers which are not more than $\frac{t-1}{2}$, respectively. There are two cases to consider.

If $s = \frac{t+1}{2}, \frac{t+1}{2}+2, \ldots, \frac{t+1}{2}+k$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R'_{s,\tau_s(i)}$. Next, the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1},\quad x_{2i-1}y_{4\tau'_s(i)},\quad x_{2i}y_{4\tau'_s(i)-1},\quad x_{2i}y_{4\tau'_s(i)},\ \text{and}\ x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. After those t tubes have been added, there is a set $\mathcal{X}'_s$ of t facial 3-cycles, where

$$\mathcal{X}'_s = \{Q'_{s,i} \mid Q'_{s,i} = y_{4\tau'_s(i)}x_{2i-1}x_{2i}y_{4\tau'_s(i)},\ i = 1, 2, \ldots, t\}.$$

For $i = 1, 2, \ldots, t$, the tube $T'_{s,i}$ is added between $Q'_{s,i}$ and $R_{s,\tau_s(i)}$. Then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3},\quad x_{2i-1}y_{4\tau'_s(i)-2},\quad x_{2i}y_{4\tau'_s(i)-3},\quad x_{2i}y_{4\tau'_s(i)-2},\ \text{and}\ x_{2i-1}x_{2\tau_s(i)-1}$$

are drawn on $T'_{s,i}$ in the way of the drawing of Type-2. In this case there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$, and $\mathcal{Y}'_{s+1}$ of facial 3-cycles which are

$$\mathcal{X}_{s+1} = \{Q_{s+1,i} \mid Q_{s+1,i} = y_{4\tau'_s(i)-3}x_{2i-1}x_{2i}y_{4\tau'_s(i)-3},\ i = 1, 2, \ldots, t\},$$
$$\mathcal{Y}_{s+1} = \{R_{s+1,i} \mid R_{s+1,i} = x_{2i}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i},\ i = 1, 2, \ldots, t\},\ \text{and}$$
$$\mathcal{Y}'_{s+1} = \{R'_{s+1,i} \mid R_{s+1,i} = x_{2i-1}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i-1},\ i = 1, 2, \ldots, t\}.$$

If $s = \frac{t+1}{2} + 1, \frac{t+1}{2} + 3, \ldots, \frac{t+1}{2} + k'$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R_{s,\tau_s(i)}$. Next, the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3}, \quad x_{2i-1}y_{4\tau'_s(i)-2}, \quad x_{2i}y_{4\tau'_s(i)-3}, \quad x_{2i}y_{4\tau'_s(i)-2}, \text{ and } \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. After those t tubes have been added, there is a set $\mathcal{X}'_s$ of t facial 3-cycles, where $\mathcal{X}'_s$ is the same as in (1). For $i = 1, 2, \ldots, t$, the tube $T'_{s,i}$ is added between $Q'_{s,i}$ and $R'_{s,\tau_s(i)}$. Then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1}, \quad x_{2i-1}y_{4\tau'_s(i)}, \quad x_{2i}y_{4\tau'_s(i)-1}, \quad x_{2i}y_{4\tau'_s(i)}, \text{ and } \quad x_{2i-1}x_{2\tau_s(i)-1}$$

are drawn on $T'_{s,i}$ in the way of the drawing of Type-2. In this case there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$, and $\mathcal{Y}'_{s+1}$ of facial 3-cycles which are the same as in (1), respectively.

Need to say that x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)}$ and $x_{2\tau_s(i)-1}$ in (2), respectively. However, x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)-1}$ and $x_{2\tau_s(i)}$ in (1), respectively.

The above operation of adding $2t$ tubes is not stopped until the t-th operation of adding $2t$ tubes has been applied. Let Π' be the obtained embedding. Then Π' has a set $\mathcal{X}_t$ of t facial 3-cycles, where

$$\mathcal{X}_t = \{Q_{t,i} \mid Q_{t,i} = y_{4\tau'_t(i)-3}x_{2i-1}x_{2i}y_{4\tau'_t(i)-3}, \text{ if } t = \tfrac{t+1}{2} + k, \text{ or } Q_{t,i} = y_{4\tau'_t(i)-1}x_{2i-1}x_{2i}y_{4\tau'_t(i)-1}, \text{ if } t = \tfrac{t+1}{2} + k'\}.$$

Since there are $2t \times t$ $(= 2t^2)$ tubes being used all together, Π' is an embedding on the orientable surface of genus $g + 2t^2$.

Let H be the graph corresponding to Π'. We need to show that H satisfies the demands of the theorem. Before the proof, we give an example that $t = 5$ to illustrate how all 50 tubes are added and how all desired edges are attached. The former two operations of adding 10 tubes are shown in Figure 3 and Figure 4, respectively. The latter three operations of adding 10 tubes are shown in Figure 5. Need to say that the five rectangles in the first column upon (3) respectively represent $T_{2,1}, \ldots, T_{2,5}$, and the five rectangles in the second column upon (3) respectively represent $T'_{2,1}, \ldots, T'_{2,5}$ in Figure 5. Similarly, the first column upon (4) respectively represent $T_{3,1}, \ldots, T_{3,5}$, and the second column upon (4) respectively represent $T'_{3,1}, \ldots, T'_{3,5}$ in Figure 5. The order in (5) in Figure 5 is the same as that in Figure 3.

We now show that H satisfies all demands of the theorem.

Claim 2.3. *w_0 is connected with each of $x_1, x_2, \ldots, x_{2t}$.*

According to the first operation of adding $2t$ tubes, Claim 2.3 is obvious.

Claim 2.4. *For $i = 1, 2, \ldots, 2t$ and $j = 1, 2, \ldots, 4t$, x_i is connected with y_j in H.*

For $i = 1, 2, \ldots, 2t$, each of x_{2i-1} and x_{2i} is connected with $y_{4\tau'_s(i)-3}$, $y_{4\tau'_s(i)-2}$, $y_{4\tau'_s(i)-1}$, and $y_{4\tau'_s(i)}$ after the $(s+1)$-th operation of adding $2t$-tubes has been applied, where $1 \leq s \leq t-1$. Considering that any two of $y_{4\tau'_s(i)-3}$, $y_{4\tau'_s(i)-2}$, $y_{4\tau'_s(i)-1}$, and $y_{4\tau'_s(i)}$ are distinct, it is sufficient to show the following proposition.

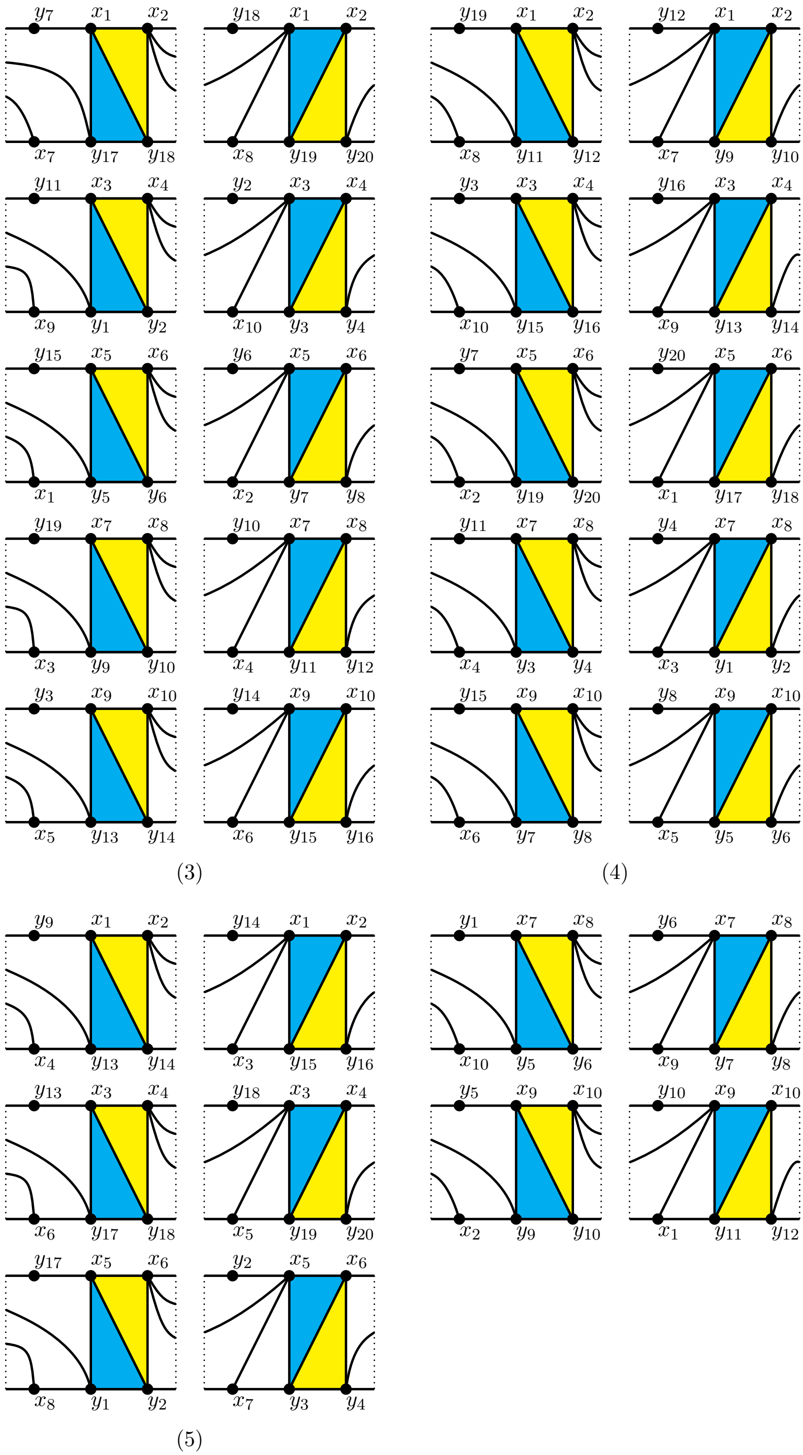

Figure 5: The latter three operations of adding $2t$ tubes when $t = 5$.

Proposition 2.5. *For $i = 1, 2, \dots, t$, $\tau'_{s_1}(i) \neq \tau'_{s_2}(i)$ if $1 \leq s_1, s_2 \leq t-1$ and $s_1 \neq s_2$.*

Assume for the sake of contradiction that there are two distinct number s_1 and s_2 such that $\tau'_{s_1}(i) = \tau'_{s_2}(i)$ for some i. Without loss of generality, suppose that $s_1 > s_2$. Since $\tau'_s(i) \equiv \tau_0\tau_1 \cdots \tau_s(i) \pmod t$ and $\tau_j(i) \equiv i + (-1)^{j+1}j \pmod t$, we have that

$$\tau'_{s_1}(i) \equiv i + \sum\nolimits_{k=0}^{s_1} (-1)^{k+1}k \equiv \tau'_{s_2}(i) \equiv i + \sum\nolimits_{l=0}^{s_2} (-1)^{l+1}l (\mathrm{mod\, t}).$$

Hence

$$\sum\nolimits_{k=0}^{s_1} (-1)^{k+1}k \equiv \sum\nolimits_{l=0}^{s_2} (-1)^{l+1}l \pmod t.$$

Thus,

$$\sum\nolimits_{k=s_2+1}^{s_1} (-1)^{k+1}k \equiv 0 \pmod t.$$

Since $1 \leq s_1 \leq t-1$, we have that

$$\sum\nolimits_{k=s_2+1}^{s_1} (-1)^{k+1}k \not\equiv 0 \pmod t.$$

Then there is a contradiction. Thus, the proposition is verified.

Claim 2.6. *H contains the edge set*

$$\{x_ix_{i+1}, \dots, x_ix_{2t} \mid i = 1, 2, \dots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \dots, t\}.$$

In fact, there are $2t$ edges being added such that each has the form x_kx_j $(k \neq j)$ except for the form $x_{2i-1}x_{2i}$ after the $(s+1)$-th operation of adding $2t$ tubes has been applied, where $1 \leq s \leq t-1$. So there are $2t(t-1)$ edges of the form x_ix_j being added after the t-th operation of adding tubes has been applied. We now show that any two edges in those $2t(t-1)$ edges are different. We need the following proposition.

Proposition 2.7. *Suppose that s_1 and s_2 are two distinct integers such that $1 \leq s_1, s_2 \leq t-1$. If $s_1 + s_2 \equiv 0 \pmod t$, then $\tau_{s_1}(i) = \tau_{s_2}(i)$.*

In fact,

$$\tau_{s_1}(i) \equiv i + (-1)^{s_1+1}s_1 \equiv i + (-1)^{t-s_2+1}(t-s_2) \equiv i + (-1)^{t-s_2}s_2 \pmod t.$$

Since $t \equiv 1 \pmod 2$, $(-1)^{t-s_2} = (-1)^{s_2+1}$. So $\tau_{s_1}(i) \equiv i + (-1)^{s_2+1}s_2 \pmod t$. In other words, $\tau_{s_1}(i) = \tau_{s_2}(i)$.

According to the rule of the $(s+1)$-th operation of adding $2t$ tubes, x_{2i} and x_{2i-1} are respectively connected with $x_{2\tau_s(i)-1}$ and $x_{2\tau_s(i)}$ if $1 \leq s \leq \frac{t-1}{2}$, and x_{2i} and x_{2i-1} are respectively connected with $x_{2\tau_s(i)}$ and $x_{2\tau_s(i)-1}$ if $\frac{t+1}{2} \leq s \leq t-1$. By Proposition 2.7, the pair of vertices connected with the pair of x_{2i-1} and x_{2i} in the s_2-th operation of adding $2t$ tubes is the same as the pair connected with the pair of x_{2i-1} and x_{2i} in the s_1-th operation of adding $2t$ tubes if $s_1 + s_2 \equiv 0 \pmod t$ and $1 \leq s_1, s_2 \leq t-1$. But the methods of two connections are different.

We now view the pair of x_{2i-1} and x_{2i} as a vertex u_i, where $i \in \{1, 2, \dots, t\}$. In order to show Claim 2.6, it is sufficient to show that u_p is connected with u_q, where $p, q \in \{1, 2, \dots, t\}$ and $p \neq q$. For the purpose, it is sufficient to show that there exists some k such that $\tau_k(p) = q$ or $\tau_k(q) = p$. By Proposition 2.7, it is sufficient to show that for any two distinct number $i, j \in \{1, 2, \ldots, t\}$, there exists some $k \in \{1, 2, \ldots, \frac{t-1}{2}\}$ such that $\tau_k(i) \equiv j \pmod t$ or $\tau_k(j) \equiv i \pmod t$.

Without loss of generality, suppose that $j > i$. If $j - i \equiv 1 \pmod 2$, there are two cases to consider. If $j - i \leq \frac{t-1}{2}$, let $k = j - i$. Then

$$\tau_k(i) \equiv i + (-1)^{k+1}k \equiv i + (j - i) \equiv j \pmod t.$$

So $\tau_k(i) = j$. If $j - i > \frac{t+1}{2}$, let $k = t - (j - i)$. Then

$$\tau_k(i) \equiv i + (-1)^{k+1}k \equiv i - t + j - i \equiv j \pmod t.$$

So $\tau_k(i) = j$. If $j - i \equiv 0 \pmod 2$, there are two cases to consider. If $j - i \leq \frac{t-1}{2}$, let $k = j - i$. Then

$$\tau_k(j) \equiv j + (-1)^{k+1}k \equiv j - (j - i) \equiv i \pmod t.$$

Thus, $\tau_k(j) = i$. If $j - i > \frac{t+1}{2}$, let $k = t - (j - i)$. Then

$$\tau_k(j) \equiv j + (-1)^{k+1}k \equiv j + t - j + i \equiv i \pmod t.$$

So $\tau_k(j) = i$.

Therefore, u_p is connected with u_q, where $p \neq q$. Thus, Claim 2.6 has been proved.

Case 2: $t \equiv 0 \pmod 2$. We proceed the similar argument to that in Case 1. Let $\mathcal{X}_s$, $\mathcal{Y}_s$, and $\mathcal{Y}'_s$ be the sets of facial 3-cycles defined in Case 1. When the $(s+1)$-th operation of adding $2t$ tubes $T_{s,1}, \ldots, T_{s,t}, T'_{s,1}, \ldots, T'_{s,t}$ will be applied, it satisfies the following conditions.

(1) If $1 \leq s \leq \frac{t}{2} - 1$, then the ways of adding $2t$ tubes and drawing the five edges are similar to that in (1) of Case 1.

(2) If $s = \frac{t}{2}$, we consider two cases. If $1 \leq i \leq \frac{t}{2}$, then the tube $T_{\frac{t}{2},i}$ is added between $Q_{\frac{t}{2},i}$ and $R_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-3}, \quad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-2}, \quad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3},$$
$$x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-2}, \quad \text{and} \quad x_{2i-1}x_{2\tau_{\frac{t}{2}}(i)-1}$$

are drawn on $T_{\frac{t}{2},i}$ in the way of the drawing of Type-1.

If $\frac{t}{2} + 1 \leq i \leq t$, then the tube $T_{\frac{t}{2},i}$ is added between $Q_{\frac{t}{2},i}$ and $R'_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-1}, \quad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)}, \quad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1},$$
$$x_{2i}y_{4\tau'_{\frac{t}{2}}(i)}, \quad \text{and} \quad x_{2i}x_{2\tau_{\frac{t}{2}}(i)}$$

are drawn on $T_{\frac{t}{2},i}$ in the way of the drawing of Type-1.

After those t tubes have been added, there is a set $\mathcal{X}'_{\frac{t}{2}}$ of t facial 3-cycles, where

$$\mathcal{X}'_{\frac{t}{2}} = \{Q'_{\frac{t}{2},i} \mid Q'_{\frac{t}{2},i} = y_{4\tau'_{\frac{t}{2}}(i)-2}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-2}, \text{ if } i = 1, 2, \ldots, \tfrac{t}{2}, \text{ or}$$
$$Q'_{\frac{t}{2},i} = y_{4\tau'_{\frac{t}{2}}(i)}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)}, \text{ if } i = \tfrac{t}{2}+1, \tfrac{t}{2}+2, \ldots, t-1\}.$$

Next, if $1 \le i \le \frac{t}{2}$, then the tube $T'_{\frac{t}{2},i}$ is added between $Q'_{\frac{t}{2},i}$ and $R'_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-1}, \quad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)}, \quad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1}, \quad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)}, \text{ and } x_{2i-1}x_{2\tau_{\frac{t}{2}}(i)}$$

are drawn on $T'_{\frac{t}{2},i}$ in the way of the drawing of Type-2. If $\frac{t}{2}+1 \le i \le t$, then the tube $T'_{\frac{t}{2},i}$ is added between $Q'_{\frac{t}{2},i}$ and $R_{\frac{t}{2},\tau_{\frac{t}{2}}(i)}$, and the five edges

$$x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-3}, \quad x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-2}, \quad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3}, \quad x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-2}, \text{ and } x_{2i-1}x_{2\tau_{\frac{t}{2}}(i)-1}$$

are drawn on $T'_{\frac{t}{2},i}$ in the way of the drawing of Type-2. There are three sets $\mathcal{X}_{\frac{t}{2}+1}$, $\mathcal{Y}_{\frac{t}{2}+1}$, and $\mathcal{Y}'_{\frac{t}{2}+1}$ of facial 3-cycles, where

$$\begin{aligned}
\mathcal{X}_{\frac{t}{2}+1} = \{Q_{\frac{t}{2}+1,i} \mid {} & Q_{\frac{t}{2}+1,i} = y_{4\tau'_{\frac{t}{2}}(i)-1}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1}, \text{ if } i = 1,\dots,\tfrac{t}{2}, \text{ or} \\
& Q_{\frac{t}{2}+1,i} = y_{4\tau'_{\frac{t}{2}}(i)-3}x_{2i-1}x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3}, \text{ if } i = \tfrac{t}{2}+1,\dots,t\}, \\
\mathcal{Y}_{\frac{t}{2}+1} = \{R_{\frac{t}{2}+1,i} \mid {} & R_{\frac{t}{2}+1,i} = x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-3}y_{4\tau'_{\frac{t}{2}}(i)-2}x_{2i-1}, \text{ if } i = 1,\dots,\tfrac{t}{2}, \text{ or} \\
& R_{\frac{t}{2}+1,i} = x_{2i-1}y_{4\tau'_{\frac{t}{2}}(i)-1}y_{4\tau'_{\frac{t}{2}}(i)}x_{2i-1} \text{ if } i = \tfrac{t}{2}+1,\dots,t\}, \\
\mathcal{Y}'_{\frac{t}{2}+1} = \{R'_{\frac{t}{2}+1,i} \mid {} & R'_{\frac{t}{2}+1,i} = x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-1}y_{4\tau'_{\frac{t}{2}}(i)}x_{2i}, \text{ if } i = 1,\dots,\tfrac{t}{2}, \text{ or} \\
& R'_{\frac{t}{2}+1,i} = x_{2i}y_{4\tau'_{\frac{t}{2}}(i)-3}y_{4\tau'_{\frac{t}{2}}(i)-2}x_{2i} \text{ if } i = \tfrac{t}{2}+1,\dots,t\}.
\end{aligned}$$

(3) If $\frac{t}{2}+1 \le s \le t-1$, then the tube $T_{s,i}$ is added between $Q_{s,i}$ and $R'_{s,\tau_s(i)}$. Since $R'_{s,\tau_s(i)}$ has two forms, we say that

- $R'_{s,\tau_s(i)}$ is of Class 1 if $R'_{s,\tau_s(i)}$ has the form $x_{2i}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i}$, and
- $R'_{s,\tau_s(i)}$ is of Class 2 if $R'_{s,\tau_s(i)}$ has the form $x_{2i}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i}$.

Similarly, we say that

- $R_{s,\tau_s(i)}$ is of Class 1 if $R_{s,\tau_s(i)}$ has the form $x_{2i-1}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i-1}$, and
- $R_{s,\tau_s(i)}$ is of Class 2 if $R_{s,\tau_s(i)}$ has the form $x_{2i-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i-1}$.

If $R'_{s,\tau_s(i)}$ is of Class 1, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1}, \quad x_{2i-1}y_{4\tau'_s(i)}, \quad x_{2i}y_{4\tau'_s(i)-1}, \quad x_{2i}y_{4\tau'_s(i)}, \text{ and } x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. If $R'_{s,\tau_s(i)}$ is of Class 2, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3}, \quad x_{2i-1}y_{4\tau'_s(i)-2}, \quad x_{2i}y_{4\tau'_s(i)-3}, \quad x_{2i}y_{4\tau'_s(i)-2}, \text{ and } x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-1. Then there is a set $\mathcal{X}'_s$ of t facial cycles, where

$$\mathcal{X}'_s = \{Q'_{s,i} \mid Q_{s,i} = y_{4\tau'_s(i)-2}x_{2i-1}x_{2i}y_{4\tau'_s(i)-2}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 1, or}$$
$$Q'_{s,i} = y_{4\tau'_s(i)}x_{2i-1}x_{2i}y_{4\tau'_s(i)}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 2}\}.$$

Next, the tube $T'_{s,i}$ is added between $Q'_{s,i}$ and $R_{s,\tau_s(i)}$. If $R_{s,\tau_s(i)}$ is of Class 1, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-1}, \quad x_{2i-1}y_{4\tau'_s(i)}, \quad x_{2i}y_{4\tau'_s(i)-1},$$
$$x_{2i}y_{4\tau'_s(i)}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T'_{s,i}$ in the way of the drawing of Type-2. If $R_{s,\tau_s(i)}$ is of Class 2, then the five edges

$$x_{2i-1}y_{4\tau'_s(i)-3}, \quad x_{2i-1}y_{4\tau'_s(i)-2}, \quad x_{2i}y_{4\tau'_s(i)-3},$$
$$x_{2i}y_{4\tau'_s(i)-2}, \quad \text{and} \quad x_{2i}x_{2\tau_s(i)}$$

are drawn on $T_{s,i}$ in the way of the drawing of Type-2. Then there are three sets $\mathcal{X}_{s+1}$, $\mathcal{Y}_{s+1}$ and $\mathcal{Y}'_{s+1}$ of t facial cycles, where

$$\mathcal{X}_{s+1} = \{Q_{s+1,i} \mid Q_{s+1,i} = y_{4\tau'_s(i)-2}x_{2i-1}x_{2i}y_{4\tau'_s(i)-2}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 1,}$$
$$\text{or } Q_{s+1,i} = y_{4\tau'_s(i)}x_{2i-1}x_{2i}y_{4\tau'_s(i)}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 2}\},$$
$$\mathcal{Y}_{s+1} = \{R_{s+1,i} \mid R_{s+1,i} = x_{2i-1}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i-1}, \text{ if } R_{s,\tau_s(i)} \text{ is of Class 1,}$$
$$\text{or } R_{s+1,i} = x_{2i-1}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i-1}, \text{ if } R_{s,\tau_s(i)} \text{ is of Class 2}\},$$
$$\mathcal{Y}'_{s+1} = \{R'_{s+1,i} \mid R'_{s+1,i} = x_{2i}y_{4\tau'_s(i)-3}y_{4\tau'_s(i)-2}x_{2i}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 1,}$$
$$\text{or } R'_{s+1,i} = x_{2i}y_{4\tau'_s(i)-1}y_{4\tau'_s(i)}x_{2i}, \text{ if } R'_{s,\tau_s(i)} \text{ is of Class 2}\}.$$

The above operation of adding $2t$ tubes is not stopped until the t-th operation of adding $2t$ tubes has been applied. Let Π' be the obtained embedding and let H the graph corresponding to Π'. Clearly, Π' is an embedding on the orientable surface of genus $g + 2t^2$, and Π' has a set $\mathcal{X}_t$ of t facial 3-cycles in which each has the form $Q_{t,i} = y_{l_i}x_{2i-1}x_{2i}y_{l_i}$, where $y_{l_i} \in \{y_{4j-3}, y_{4j-2}, y_{4j-1}, y_{4j} \mid j = 1, 2, \ldots, t\}$.

In order to help readers to understand the procedure of adding tubes in this case, we give an example that $t = 4$ which is shown in Figure 6. For $i = 1, 2, 3, 4$, the four rectangles in the first column of (i) respectively represent $T_{i,1}, \ldots, T_{i,4}$ from top to bottom, and the four rectangles the second column of (i) respectively represent $T'_{i,1}, \ldots, T'_{i,4}$ from top to bottom.

We need to show that H satisfies the demands of the theorem. Obviously, w_0 is connected with each of $x_1, x_2, \ldots, x_{2t}$ in H. By the similar argument as in Case 1, one can show that for $i = 1, 2, \ldots, 2t$ and $j = 1, 2, \ldots, 4t$, x_i is connected with y_j in H.

Claim 2.8. *H contains the edge set*

$$\{x_i x_{i+1}, \ldots, x_i x_{2t} \mid i = 1, 2, \ldots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \ldots, t\}.$$

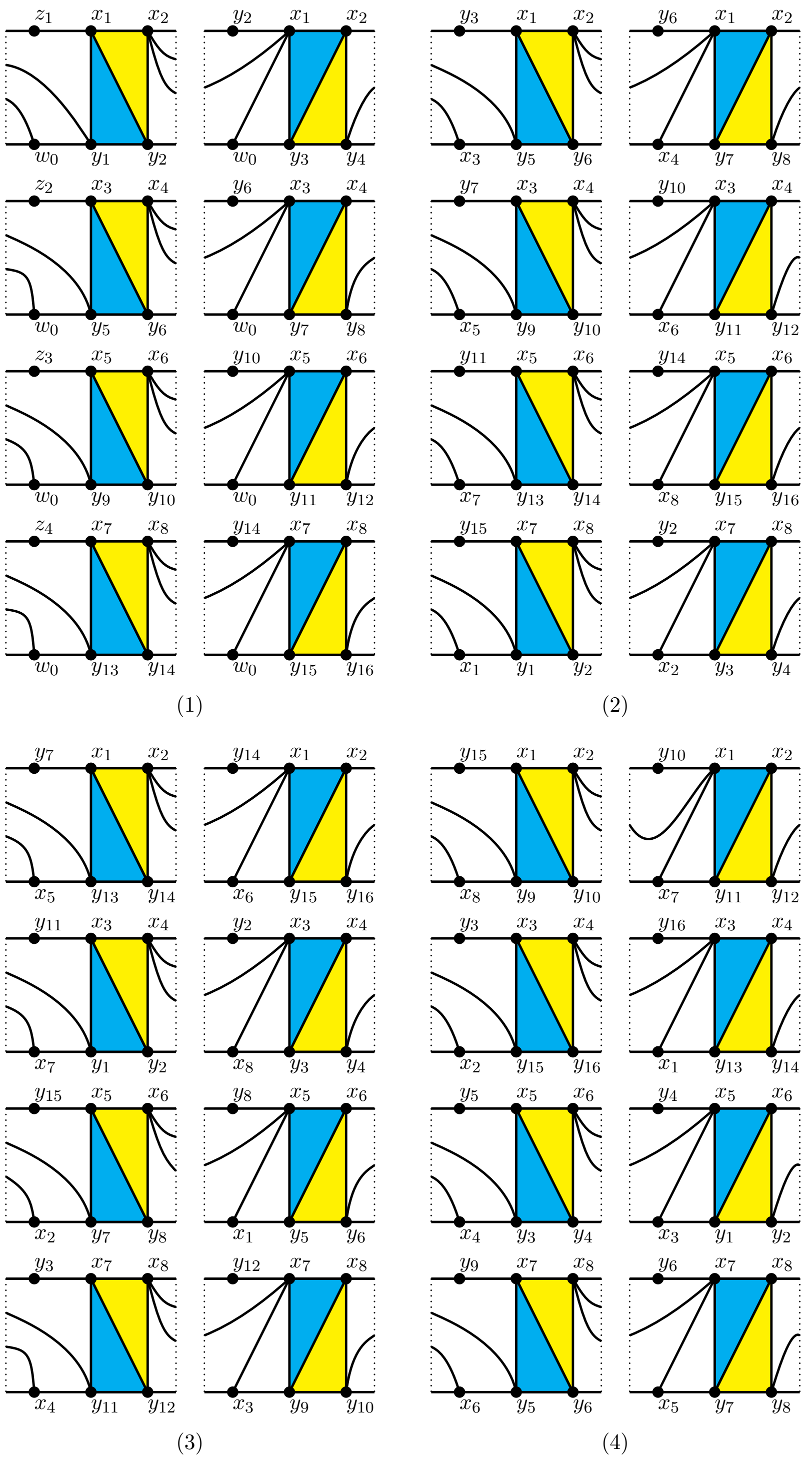

Figure 6: The operations of adding $2t$ tubes when $t = 4$.

We proceed the similar argument to that in Claim 2.6. Obviously, there are $2t(t-1)$ edges of the form x_kx_j $(k \neq j)$ except for the form $x_{2i-1}x_{2i}$ after the t-th operation of adding $2t$ tubes has been applied. According to the rule of the $(s+1)$-th operation of adding $2t$ tubes, x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)-1}$ and $x_{2\tau_s(i)}$, respectively, if $1 \leq s \leq \frac{t}{2}-1$ or $s = \frac{t}{2}$ and $i = 1, 2, \ldots, \frac{t}{2}$, and x_{2i} and x_{2i-1} are connected with $x_{2\tau_s(i)}$ and $x_{2\tau_s(i)-1}$, respectively, if $\frac{t}{2}+1 \leq s \leq t-1$ or $s = \frac{t}{2}$ and $i = \frac{t}{2}+1, \frac{t}{2}+2, \ldots, t$. We now consider the relation between $\tau_{s_1}(i)$ and $\tau_{s_2}(i)$, where $1 \leq s_1, s_2 \leq t-1$ and $s_1 + s_2 \equiv 0 \pmod t$. We have the following proposition.

Proposition 2.9. *Suppose that s_1 and s_2 are two integers such that $1 \leq s_1, s_2 \leq t-1$. If $s_1 + s_2 \equiv 0 \pmod t$, then $\tau_{s_1}(t-i) = t - \tau_{s_2}(i)$ or $\tau_{s_2}(i) = t - \tau_{s_1}(t-i)$.*

In fact,

$$\begin{aligned}\tau_{s_1}(t-i) &\equiv t-i+(-1)^{s_1+1}s_1 \equiv t-i+(-1)^{t-s_2+1}(t-s_2)\\ &\equiv t-i+(-1)^{t-s_2}s_2 \pmod t.\end{aligned}$$

Since $t \equiv 0 \pmod 2$, $(-1)^{t-s_2} = (-1)^{s_2}$. So

$$\tau_{s_1}(t-i) \equiv t-i+(-1)^{s_2}s_2 \equiv t-(i+(-1)^{s_2+1}s_2) \equiv t-\tau_{s_2}(i) \pmod t.$$

In other words, $\tau_{s_1}(t-i) = t - \tau_{s_2}(i)$, or $\tau_{s_2}(i) = t - \tau_{s_1}(t-i)$.

Thus, the pair of vertices of the form $x_{2\tau_{s_2}(i)-1}$ and $x_{2\tau_{s_2}(i)}$ connected with the pair of x_{2i-1} and x_{2i} in the (s_2+1)-th operation of adding $2t$ tubes is the same as the pair of vertices of the form $x_{2(t-\tau_{s_1}(t-i))-1}$ and $x_{2(t-\tau_{s_1}(t-i))}$ connected with the pair of x_{2i-1} and x_{2i} in the (s_1+1)-th operation of adding $2t$ tubes if $0 \leq s_1, s_2 \leq t-1$ and $s_1+s_2 \equiv 0 \pmod t$. But the methods of two connections are different. We now view the pair of x_{2i-1} and x_{2i} as a vertex u_i, where $i \in \{1, 2, \ldots, t\}$. In order to show Claim 2.8, it is sufficient to show that u_p is connected with u_q, where $p, q \in \{1, 2, \ldots, t\}$ and $p \neq q$. For the purpose, it is sufficient to show that there exists some k such that $\tau_k(p) = q$ or $\tau_k(q) = p$. By Proposition 2.9, it is sufficient to show that for any two distinct numbers $i, j \in \{1, 2, \ldots, \frac{t}{2}\}$, there exists some $k \in \{1, 2, \ldots, t\}$ such that $\tau_k(i) = j$ or $\tau_k(j) = i$.

Without loss of generality, suppose that $j > i$. If $j - i \equiv 1 \pmod 2$, let $k = j - i$. Then

$$\tau_k(i) \equiv i + (-1)^{k+1}k \equiv i + (j-i) \equiv j \pmod t.$$

So $\tau_k(i) = j$. If $j - i \equiv 0 \pmod 2$, let $k = j - i$. Then

$$\tau_k(j) \equiv j + (-1)^{k+1}k \equiv j - (j-i) \equiv i \pmod t.$$

So $\tau_k(j) = i$. Hence u_p is connected with u_q for $p \neq q$. Thus, Claim 2.8 has been proved.

Therefore, the obtained embedding is as required. □

In the proof of Lemma 2.1, we apply the operation of adding $2t$ tubes t times starting from $\mathcal{X}_0$, $\mathcal{Y}_0$ and $\mathcal{Y}_0'$ to construct an embedding of H, where $\mathcal{X}_0 = \{Q_{0,i} \mid i = 1, 2, \ldots, t\}$, $\mathcal{Y}_0 = \{R_{0,i} \mid i = 1, 2, \ldots, t\}$, $\mathcal{Y}_0' = \{R_{0,i}' \mid i = 1, 2, \ldots, t\}$. We call the above procedure *the operation of adding $2t^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$ and $\mathcal{Y}_0'$*. Lemma 2.10 below is an analogue of Lemma 2.1. The vertex w_0 in Lemma 2.1 is replaced with two vertices w_0', w_0'' in Lemma 2.10, and the others are not changed. The proof is similar to that in the proof of Lemma 2.1, which is omitted here.

Lemma 2.10. *Suppose that G is a graph which has a vertex subset*

$$\{w_0', w_0'', z_1, z_2, \dots, z_t\} \cup \{x_i \mid i = 1, 2, \dots, 2t\} \cup \{y_j \mid j = 1, 2, \dots, 4t\},$$

where $z_1, z_2, \dots, z_t$ need not be different, and suppose that G contains no edges in the set

$$E' = \{w_0'x_{2i-1}, w_0''x_{2i} \mid i = 1, 2, \dots, t\} \cup \{x_iy_j \mid i = 1, 2, \dots, 2t; j = 1, 2, \dots, 4t\}$$
$$\cup (\{x_ix_{i+1}, \dots, x_ix_{2t} \mid i = 1, 2, \dots, 2t-1\} \setminus \{x_{2i-1}x_{2i} \mid i = 1, 2, \dots, t\}).$$

Suppose that Π is a 2-cell embedding of G on the orientable surface S_g with the following properties:

(i) For $i = 1, 2, \dots, t$, $R_{0,i} = w_0'y_{4i-3}y_{4i-2}w_0$ and $R_{0,i}' = w_0''y_{4i-1}y_{4i}w_0$ are facial cycles of Π.

(ii) For $i = 1, 2, \dots, t$, $Q_{0,i} = z_ix_{2i-1}x_{2i}z_i$ is a facial cycle of Π such that $Q_{0,i}$ has not any common vertex with each of $R_{0,1}, \dots, R_{0,t}, R_{0,1}', \dots, R_{0,t}'$.

Then there is a supergraph H of G satisfying the following conditions:

(i) E' is an edge subset of $E(H)$.

(ii) H has an embedding on the orientable surface of genus $g + 2t^2$ such that it has a set of t facial 3-cycles $\{Q_{t,i} \mid Q_{t,i} = y_{l_i}x_{2i-1}x_{2i}y_{l_i}, i = 1, 2, \dots, t\}$, where y_{l_i} is some vertex in $\{y_{4i-3}, y_{4i-2}, y_{4i-1}, y_{4i} \mid i = 1, 2, \dots, t\}$.

We now introduce another method of constructing an embedding, which is used in the proof of Lemma 2.11.

Lemma 2.11. *Let k and l be two positive integers. Suppose that G has a vertex subset*

$$\{w, z\} \cup \{x_i, y_j \mid i = 1, 2, \dots, 2l,\ j = 1, 2, \dots, 2k\},$$

and suppose that G contains no edges in

$$E' = \{x_iy_j \mid i = 1, 2, \dots, 2l,\ j = 1, 2, \dots, 2k\}.$$

If G has a 2-cell embedding Π on the orientable surface S_g such that $F_i = wx_{2i-1}x_{2i}w$ and $F_j' = zy_{2j-1}y_{2j}z$ are facial cycles in Π for $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, k$, then there is a supergraph H of G with the following properties:

(i) E' is an edge subset of H.

(ii) H has an embedding on the orientable surface of genus $g + kl$ such that it has a set of l facial 3-cycles in which each has the form $y_{h_i}x_{2i-1}x_{2i}y_{h_i}$, where $y_{h_i} \in \{y_1, y_2, \dots, y_{2k}\}$.

Proof. We construct an embedding from Π as follows.

(1) Let $D_{1,1} = F_1$. Then the tube $T_{1,1}$ is added between $D_{1,1}$ and F_1'. Next, the four edges x_1y_1, x_1y_2, x_2y_1 and x_2y_2 are drawn on $T_{1,1}$ in the way shown in Figure 7. Let $D_{1,2} = y_1x_1x_2y_1$, and let $Q_{1,1} = x_2y_1y_2x_2$. The tube $T_{1,2}$ is now added between $D_{1,2}$ and F_2', and the four edges x_1y_3, x_1y_4, x_2y_3 and x_2y_4 are drawn on it in the similar way as in Figure 7. Let $D_{1,3} = y_3x_1x_2y_3$ and $Q_{1,2} = x_2y_3y_4x_2$. Then $D_{1,3}$ and F_3' are dealt with as $D_{1,2}$ and F_2', and so on. The procedure is not stopped until F_k' has been dealt with. Thus, we obtain k facial cycles $Q_{1,1}, \dots, Q_{1,k}$, where $Q_{1,i} = x_2y_{2i-1}y_{2i}x_2$. Moreover, both x_1 and x_2 are connected with each of $y_1, y_2, \dots, y_{2k}$.

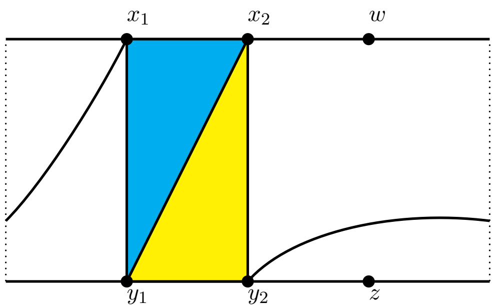

Figure 7: The drawing of the four edges in $T_{1,1}$.

(2) Let $\mathcal{Q}_1 = \{Q_{1,1}, Q_{1,2}, \ldots, Q_{1,k}\}$. Then the tube $T_{2,1}$ is added between F_2 and $Q_{1,1}$, and the four edges x_3y_1, x_3y_2, x_4y_1 and x_4y_2 are drawn on it in the similar way as in Figure 7, and so on. The procedure is stopped till $Q_{1,k}$ has been dealt with. Then we obtain a set of facial walks $\mathcal{Q}_2 = \{Q_{2,1}, Q_{2,2}, \ldots, Q_{2,k}\}$ such that $Q_{2,i} = x_4y_{2i-1}y_{2i}x_4$. Moreover, both x_3 and x_4 are connected with each of $y_1, y_2, \ldots, y_{2k}$.

(3) $\mathcal{Q}_2$ and F_3 are dealt with in the similar way to that of $\mathcal{Q}_1$ and F_2, and so on. The procedure is stopped till F_l has been dealt with. Then x_i is connected with each of $y_1, y_2, \ldots, y_{2k}$ for $i = 1, 2, \ldots, 2l$, and there is a set of l facial 3-cycles in which each has the form $y_{h_i}x_{2i-1}x_{2i}y_{h_i}$. Moreover, there are kl tubes to be added to the primitive surface all together. So the obtained embedding Π' is one on the orientable surface of genus $g + kl$. Let H be the graph corresponding to Π'. It is easy to find that E' is an edge set of H. $\square$

Let $\mathcal{F}_1 = \{F_1, F_2, \ldots, F_l\}$, and let $\mathcal{F}_2 = \{F'_1, F'_2, \ldots, F'_k\}$. We call the procedure of constructing an embedding in the proof of Lemma 2.11 *the operation of adding tubes with respect to* $\mathcal{F}_1$ *and* $\mathcal{F}_2$.

3 An upper bound for $\gamma(C_m + K_n)$ if m is odd

From now on we always suppose that $m \geq 3$ and $n \geq 4$, that $C_m = u_1u_2 \ldots u_mu_1$, and that the vertex set of K_n is $\{v_1, v_2, \ldots, v_n\}$. If no confusion occur, a face and its boundary in an embedding are not distinguished in the rest of the paper.

Lemma 3.1. *Suppose that* $m \equiv 1 \pmod 2$ *and* $n \equiv 0 \pmod 4$. *If* $m \geq 4n - 5$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We shall construct an embedding of $C_m + K_n$ on the oreintable surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$ in the following steps.

(1) In the step we shall construct an embedding on a sphere in which each of v_1 and v_2 is connected with each of $u_1, u_2, \ldots, u_m$, and each of u_1 and u_2 is connected with each of $v_1, v_2, \ldots, v_n$.

First, C_m is placed in the equator of the sphere, and both v_1 and v_2 are situated at the northern pole and the southern pole, respectively. Second, each of v_1 and v_2 joins to each of $u_1, u_2, \ldots, u_m$, and the path $P = v_3v_4 \ldots v_n$ is placed in the interior of the face $v_1u_1u_2v_1$ such that v_3 is near to v_1. Third, v_3 joins to v_1, and each of u_1 and u_2 joins to each of $v_3, v_4, \ldots, v_n$. Thus, we obtain an embedding Π_1 on the sphere, which is shown in Figure 8.

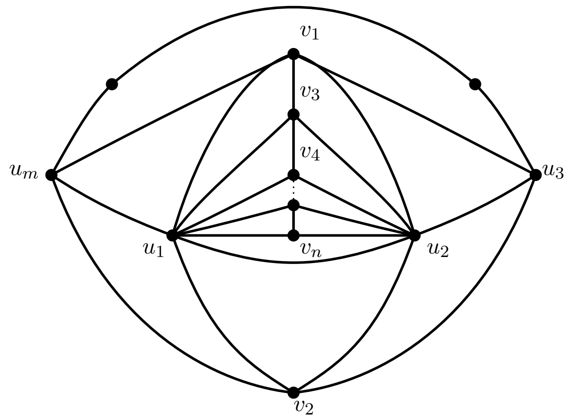

Figure 8: The embedding Π_1.

(2) In the step we shall add $\frac{n}{4}$ tubes to the sphere such that u_3 is connected with each of $v_3, v_4, \ldots, v_n$, and v_1 joins to v_2.

The tube T_1 is now added between the facial cycles $u_2v_3v_4u_2$ and $v_2u_2u_3v_2$. Next, the edge u_2v_3 is redrawn such that it is on T_1 and a segment of local rotation at u_2 in clockwise is that v_4, v_1, u_3, v_3. Then there is a facial walk $W_1 = u_3v_2u_2v_3v_1u_2v_4$ $v_3u_2u_3$. Let $Z_1 = u_3v_2u_2v_3v_1u_2v_4v_3$. Then $W_1 = Z_1u_2u_3$.

The tube T_2 is added between the facial cycle $u_2v_8v_7u_2$ and W_1. Then the two edges u_2v_7 and u_2v_6 are redrawn on T_2 such that a segment of local rotation at u_2 in clockwise is that u_3, v_7, v_6, v_3. Thus, there is a facial walk $W_2 = Z_1u_2v_6v_5u_2v_8v_7u_2u_3$. Let $Z_2 = u_2v_6v_5u_2v_8v_7$. Thus, $W_2 = Z_1Z_2u_2u_3$.

For $i = 3, 4, \ldots, \frac{n}{4}$, the tube T_i is added between the facial cycle $u_2v_{4i}v_{4i-1}u_2$ and W_{i-1}. Next, both edges u_2v_{4i-1} and u_2v_{4i-2} are redrawn on T_i such that a segment of local rotation at u_2 in clockwise is that u_3, v_{4i-1}, v_{4i-2} and v_{4i-5}. Then there is a facial walk $W_i = Z_1Z_2 \ldots Z_{i-1}u_2v_{4i-2}v_{4i-3}u_2v_{4i}v_{4i-1}u_2u_3$. Let $Z_i = u_2v_{4i-2}v_{4i-3}u_2v_{4i}v_{4i-1}$. Thus, $W_i = Z_1Z_2 \ldots Z_iu_2u_3$.

After the tube $T_{\frac{n}{4}}$ has been added, there is a facial walk $W_{\frac{n}{4}} = Z_1Z_2 \ldots Z_{\frac{n}{4}-1}u_2u_3$. For $i = 2, 3, \ldots, \frac{n}{4}$, each of v_{4i-3}, v_{4i-2}, v_{4i-1} and v_{4i} appears in Z_i once, but it does not appear in Z_j if $i \neq j$. Also, v_4 appears in Z_1 once, but it does not appear in Z_j if $j \neq 1$. In the interior of the face $W_{\frac{n}{4}}$, u_3 joins to each of $v_4, v_5, \ldots, v_n$, and v_1 joins to v_2. For example, if $n = 8$, W_2 and all added edges in the interior of W_2 are shown in Figure 9. Let Π_2 be the embedding obtained from Π_1 by the above operation of adding tubes. Then Π_2 is an embedding on the surface of genus $\frac{n}{4}$.

(3) In the step we shall add $2(\frac{n}{2}-1)^2$ tubes to the present surface satisfying the following conditions:

(i) v_1 is connected with each of $v_3, v_4, \ldots, v_n$,

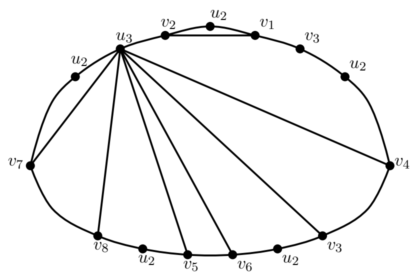

Figure 9: W_2 and all edges added in the interior of W_2.

(ii) for $i = 3, 4, \dots, n$ and $j = 4, 5, \dots, 2n-1$, v_i is connected with u_j, and

(iii) all edges in the set

$$\{v_iv_{i+1}, \dots, v_iv_n \mid i = 3, \dots, n-1\} \setminus \{v_{2i+1}v_{2i+2} \mid i = 1, \dots, \tfrac{n-2}{2}\}$$

are added.

For the above purpose, let

$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_1v_{2i+1}v_{2i+2}u_1,\ i = 1, 2, \dots, \tfrac{n}{2}-1\},$$
$$\mathcal{Y}_0 = \{R_{0,i} \mid R_{0,i} = v_1u_{4i}u_{4i+1}v_1,\ i = 1, 2, \dots, \tfrac{n}{2}-1\}, \text{ and}$$
$$\mathcal{Y}'_0 = \{R'_{0,i} \mid R'_{0,i} = v_1u_{4i+2}u_{4i+3}v_1,\ i = 1, 2, \dots, \tfrac{n}{2}-1\}.$$

Then we apply the operation of adding $2(\frac{n}{2}-1)^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$, and $\mathcal{Y}'_0$. By Lemma 2.1, an embedding Π_3 is obtained which satisfies all the requirements and contains a set $\mathcal{A}_0 = \{A_{0,1}, A_{0,2}, \dots, A_{0,\frac{n}{2}-1}\}$ of facial 3-cycles such that $A_{0,i}$ has the form $u_{k_i}v_{2i+1}v_{2i}u_{k_i}$, where $u_{k_i} \in \{u_j \mid j = 4, 5, \dots, 2n-1\}$.

(4) In the step we shall add $2(\frac{n}{2}-1)^2$ tubes to present surface satisfying the following conditions:

(i) v_2 is connected with $v_3, v_4, \dots, v_n$,

(ii) for $i = 3, 4, \dots, n$ and $j = 2n, 2n+1, \dots, 4n-5$, v_i is connected with u_j.

For the above purpose, let

$$\mathcal{B}_0 = \{B_{0,i} \mid B_{0,i} = v_2u_{2n+4i-4}u_{2n+4i-3}v_2,\ i = 1, 2, \dots, \tfrac{n}{2}-1\}, \text{ and}$$
$$\mathcal{B}'_0 = \{B'_{0,i} \mid B'_{0,i} = v_2u_{2n+4i-2}u_{2n+4i-1}v_2,\ i = 1, 2, \dots, \tfrac{n}{2}-1\}.$$

We now apply the operation of adding $2(\frac{n}{2}-1)^2$ tubes starting from $\mathcal{A}_0$, $\mathcal{B}_0$, and $\mathcal{B}'_0$. By Lemma 2.1, an embedding Π_4 is obtained which satisfies all the requirements and contains a set $\mathcal{F} = \{F_1, F_2, \dots, F_{\frac{n}{2}-1}\}$ of facial 3-cycles such that F_i has the form $u_{l_i}v_{2i+1}v_{2i+2}u_{l_i}$, where $u_{l_i} \in \{u_j \mid j = 2n, 2n+1, \dots, 4n-5\}$. At last, all edges of the form v_iv_j added in the above operations are deleted, since these edges have been existed. Note that the deletion of these edges does not affect each cycle in $\mathcal{F}$.

(5) If $m = 4n-5$, then there is nothing to do. If $m > 4n-5$, then we shall add tubes to the present surface such that v_i is connected with each of $u_{4n-4}, \dots, u_m$ for $i = 3, 4, \dots, n$.

Let

$$\mathcal{D} = \{D_i \mid D_i = v_1 u_{4n+2i-6} u_{4n+2i-5} v_1,\ i = 1, 2, \dots, \tfrac{m-4n+5}{2}\}.$$

We now use the operation of adding tubes respect to $\mathcal{F}$ and $\mathcal{D}$. By Lemma 2.11, there are $\frac{(n-2)(m-4n+5)}{4}$ tubes being used, and v_i is connected with u_j, where $i \in \{3, 4, \dots, n\}$ and $j \in \{4n-4, 4n-3, \dots, m\}$. Let Π_5 be the obtained embedding. Then it is an embedding of $C_m + K_n$ on the surface of genus

$$\frac{n}{4} + \frac{(n-2)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-2)(m-4n+5)}{4}.$$

By simple counting, we have that

$$\frac{n}{4} + \frac{(n-2)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-2)(m-4n+5)}{4} = \frac{n}{4} + \frac{(n-2)(m-3)}{4}.$$

Since $n \equiv 0 \pmod 4$,

$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil = \left\lceil \frac{n-2}{4} \right\rceil + \frac{(n-2)(m-3)}{4} = \frac{n}{4} + \frac{(n-2)(m-3)}{4}.$$

So

$$\frac{n}{4} + \frac{(n-2)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-2)(m-4n+5)}{4} = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Hence, $\gamma(C_m + K_n) \le \lceil \frac{(m-2)(n-2)}{4} \rceil$. □

Lemma 3.2. *Suppose that* $m \equiv 1 \pmod 2$ *and* $n \equiv 2 \pmod 4$. *If* $m \ge 4n-3$, *then*

$$\gamma(C_m + K_n) \le \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We construct an embedding of $C_m + K_n$ in the similar way to that in the proof of Lemma 3.1.

(1) First, place C_m, v_1, and v_2 on a sphere and add edges as (1) in the proof of Lemma 3.1. Let $F_1 = v_1u_1u_2v_1$, $F_2 = v_1u_2u_3v_1$, and $F_3 = v_1u_4u_5v_1$. The path $P = v_7v_8 \dots v_n$ is now placed in the interior of F_1, and each of u_1 and u_2 joins to each of $v_7, v_8, \dots, v_n$. Next, both v_3 and v_5 are placed in the interior of F_2, and they join to each of u_2 and u_3, respectively. Similarly, both v_4 and v_6 are placed in the interior of F_3, and they join to each of u_4 and u_5, respectively. Let Π_1 be the obtained embedding on the sphere, which is shown in Figure 10.

The edge u_3u_4 is now deleted from Π_1. Then the face $v_1u_3u_4v_1$ and the face $v_2u_3u_4v_2$ are merged into a face $F_4 = v_1u_3v_2u_4v_1$. Next, the edge v_1v_2 is drawn in the interior of F_4. Let $F_5 = u_2v_3u_3v_5u_2$ and $F_6 = u_4v_4u_5v_6u_4$. The tube T_1 is added between F_5 and F_6. Then the five edges are drawn on T_1 in the way shown in (1) in Figure 11. Let $F_7 = u_2v_3u_4v_6u_2$ and $F_8 = u_3v_4u_5v_5u_3$. Next, the tube T_2 is

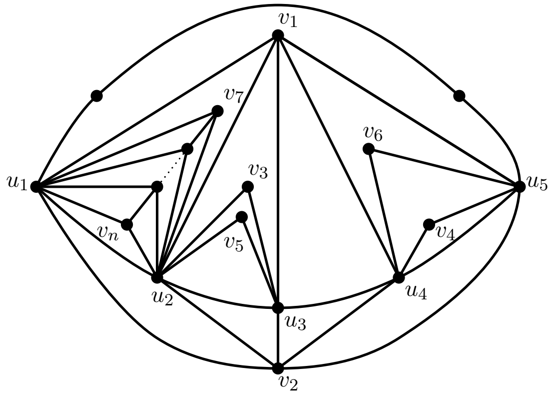

Figure 10: The embedding Π_1.

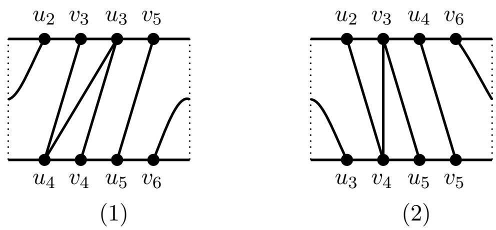

Figure 11: The drawing of edges on T_1 or T_2.

added between F_7 and F_8. Then the five edges are drawn on T_2 in the way shown in (2) in Figure 11.

We observe that the local rotation at u_2 in clockwise is that $u_1, v_n, \ldots, v_1, v_3, v_4, v_6, v_5, u_3, v_2$. Let $F_9 = u_2v_6u_3v_4u_2$, which is a facial cycle (refer to (2) in Figure 11). Let $F_{10} = u_1v_nu_2u_1$ (refer to Figure 10) if $n > 6$, or $F_{10} = u_1v_1u_2u_1$ if $n = 6$. The tube T_3 is now added between F_9 and F_{10}. Then the edges u_2v_5 and u_2v_4 are redrawn on T_3 such that a segment of the local rotation at u_2 is that $u_1, v_6, v_4, v_n, v_3, v_5$. Thus, there is a facial walk $W_1' = u_1u_2v_4v_3u_2v_5u_5v_6u_2v_nu_1$. Next, u_1 joins to each of v_3, v_4, v_5, v_6, and v_5 joins to v_6. Then there are two facial cycles $Q_{0,1} = u_1v_4v_3u_1$ and $Q_{0,2} = u_1v_5v_6u_1$.

(2) If $n = 6$, there is nothing to do. If $n > 6$, then we shall add $\frac{3(n-2)}{4}$ tubes to the present surface such that u_i is connected with each of $v_3, v_4, \ldots, v_n$ for $i = 3, 4, 5$.

Let $F_{11} = v_1u_3v_3u_2v_1$ (refer to Figure 10). For $i = 1, 2, \ldots, \frac{n-6}{4}$, let $F_i' = u_2v_{4i+4}v_{4i+5}u_2$. The tube T_1' is added between F_1' and F_{11}. Then two edges u_2v_{4i+4} and u_2v_{4i+5} are redrawn on T_1'. There is a facial walk $W_1 = u_2v_3u_3v_1u_2v_9v_{10}u_2$ $v_7v_8u_2$. For $i = 2, \ldots, \frac{n-6}{4}$, the tube T_i' is added between F_i' and W_{i-1}, where W_{i-1} is a facial walk which contains $v_7, \ldots, v_{4i+2}$ after T_{i-1}' has added. Next, both u_2v_{4i+4} and u_2v_{4i+5} are redrawn on T_i' and a segment in the local rotation at u_2 in clockwise is that $u_{4(i-1)+5}$, u_{4i+4}, u_{4i+5}, and u_3. After the tube $T'_{\frac{n-6}{4}}$ has been added, there is a facial walk $W_{\frac{n-6}{4}}$ which contains u_3, $v_7, v_8, \ldots, v_n$. Moreover, each of $v_7, v_8, \ldots, v_n$ appears in $W_{\frac{n-6}{4}}$ once. Next, u_3 joins to each of $v_7, v_8, \ldots, v_n$. There are $\frac{n-6}{2}$ facial 3-cycles $D_1, D_2, \ldots, D_{\frac{n-6}{2}}$, where $D_i = u_3v_{2i+5}v_{2i+6}u_3$.

Let $F_{12} = u_4v_4u_5u_4$ (refer to Figure 10). Let $\mathcal{F} = \{F_{12}\}$, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_{\frac{n-6}{2}}\}$. Using the operation of adding tubes with respect to $\mathcal{D}$ and $\mathcal{F}$, each of u_4 and u_5 is connected with each of $v_7, v_8, \ldots, v_n$. By Lemma 2.11, there are $\frac{n-6}{2}$ tubes being used. Also, there are $\frac{n-6}{2}$ facial cycles $Q_{0,3}, \ldots, Q_{0,\frac{n-2}{2}}$ in which $Q_{0,i}$ has the form $u_{l_i}v_{2i+1}v_{2i+2}u_{l_i}$, where $u_{l_i} \in \{u_4, u_5\}$. Let Π_2 be the embedding obtained from Π_1 by the above procedures. Then Π_2 is an embedding on the surface of genus $3 + \frac{n-6}{4} + \frac{n-6}{2}$ $(= \frac{3(n-2)}{4})$. Moreover, u_i is connected with each of $v_1, v_2, \ldots, v_n$ for $i = 1, 2, \ldots, 5$.

(3) For $i = 1, 2, \ldots, \frac{n-6}{2}$, let $R_{0,i} = v_1u_{4i+2}u_{4i+3}v_1$, and let $R'_{0,i} = v_1u_{4i+4}u_{4i+5}v_1$. Let $\mathcal{X}_0 = \{Q_{0,i+2} \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$, $\mathcal{Y}_0 = \{R_{0,i} \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$, and $\mathcal{Y}'_0 = \{R'_{0,i} \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$. Next procedures are similar to that in (4) and (5) in the proof of Lemma 3.1. Note that $\frac{(m-5)(n-2)}{4}$ tubes are added to the present surface such that v_i is connected with u_j for $i = 3, 4, \ldots, n$ and $j = 6, 7, \ldots, m$. Thus, an embedding Π_3 of $C_m + K_n$ on the surface of genus $\frac{3(n-2)}{4} + \frac{(m-5)(n-2)}{4}$ is obtained. Since $n \equiv 2 \pmod 4$, $\lceil \frac{(m-2)(n-2)}{4} \rceil = \frac{3(n-2)}{4} + \frac{(m-5)(n-2)}{4}$. Thus, Π_3 is the desired embedding. Since the operation of adding $n-2$ tubes is used twice, m is at least $5 + 4(n-2)$ $(= 4n-3)$. □

Lemma 3.3. *Suppose that* $m \equiv 1 \pmod 2$ *and* $n \equiv 1 \pmod 2$. *If* $m \geq 6n - 13$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We consider two cases.

Case 1: $m \equiv 1 \pmod 4$. In this case we construct an embedding of $C_m + K_n$ in the following steps.

(1) The path $P_m = u_1u_2 \ldots u_m$ is placed in the equator of a sphere. The edge v_1v_2 is situated in the northern pole and the vertex v_3 placed at the southern pole. Next, each of v_1 and v_3 joins to each of $u_1, u_2, \ldots, u_{\frac{m+1}{2}}$, and each of v_1 and v_2 joins to each of $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m$. Also, v_1 joins to v_3, and v_2 joins to $u_{\frac{m+1}{2}}$. Thus, an embedding Π_1 on the sphere is obtained. For example, the embedding Π_1 is shown in Figure 12 if $m = 17$.

(2) In this step we shall construct an embedding on the surface of genus $\frac{m-1}{4}$ such that v_2 is connected with $u_1, u_2, \ldots, u_{\frac{m-1}{2}}$, v_3 connected with $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m$, and u_1 connected with u_m.

For $i = 1, 2, \ldots, \frac{m-1}{4}$, let $F_i = v_3u_{2i-1}u_{2i}v_3$ and $F'_i = v_2u_{m+1-2i}u_{m+2-2i}v_2$. The tube T_1 is added between F_1 and F'_1, and the five edges are drawn on T_1 in the way shown in (1) in Figure 13. The tube T_2 is added between F_2 and F'_2, and the five edges are drawn on T_1 in the way shown in (2) of Figure 13.

For $i = 3, 4, \ldots, \frac{m-1}{4}$, the tube T_i is added between F_i and F'_i. Then the four edges v_3u_{m+2-2i},v_3u_{m+1-2i}, v_2u_{2i-1}, and v_2u_{2i} are drawn on T_i in the way shown in (2) of Figure 13, but v_2v_3 is not added. Thus, v_3 is connected with each of

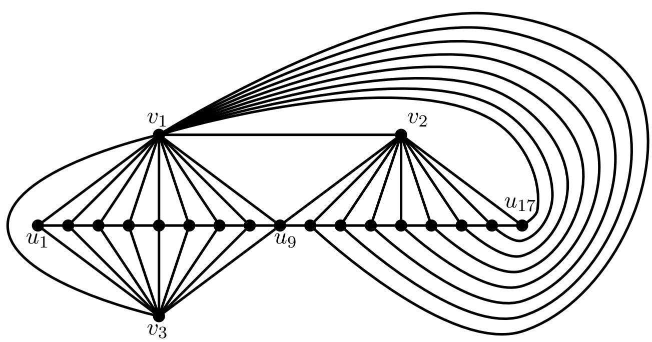

Figure 12: The embedding Π_1.

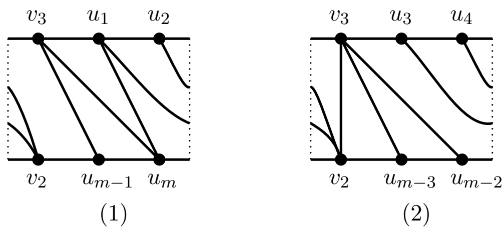

Figure 13: The drawing of edges on T_1 or T_2.

$u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \ldots, u_m, v_2$. Next, v_2 connected with each of $u_1, u_2, \ldots, u_{\frac{m-1}{2}}$. Let Π_2 be the obtained embedding. Note that there are two sets $\mathcal{Z}_0$ and $\mathcal{Z}_0'$ in Π_2, where

$$\mathcal{Z}_0 = \{Z_{0,i} \mid Z_{0,i} = v_2u_{2i-1}u_{2i}v_2,\ i = 1, 2, \ldots, \tfrac{m-1}{4}\} \text{ and}$$
$$\mathcal{Z}_0' = \{Z_{0,i}' \mid Z_{0,i}' = v_3u_{m+1-2i}u_{m+2-2i}v_3,\ i = 1, 2, \ldots, \tfrac{m-1}{4}\}.$$

(3) In this step $\lceil \frac{n-2}{4} \rceil$ tubes will be added to the present surface such that v_i is connected with $u_{\frac{m+1}{2}}, u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}$ for $i = 4, 5, \ldots, n$.

The path $P = v_4v_5 \ldots v_n$ is now placed in the interior of $Z'_{0,\frac{m-1}{4}}$ such that v_4 is near to v_3. Then each of $u_{\frac{m+3}{2}}$ and $u_{\frac{m+5}{2}}$ joins to each of $v_4, v_5, \ldots, v_n$. For $i = 1, 2, \ldots, \lceil \frac{n-1}{4} \rceil$, let $D_i = u_{\frac{m+3}{2}}v_{4i}v_{4i+1}u_{\frac{m+3}{2}}$.

If $n \equiv 1 \pmod 4$, then $\lceil \frac{n-4}{4} \rceil = \frac{n-1}{4}$. The tube T_1' is now added between $D' = v_2u_{\frac{m+1}{2}}u_{\frac{m+3}{2}}v_2$ and D_1. Next, the edge $u_{\frac{m+3}{2}}v_4$ is redrawn on T_1'. Then we obtain a facial walk W_1 which contains $u_{\frac{m+1}{2}}$ and v_4. For $i = 2, 3, \ldots, \frac{n-1}{4}$, the tube T_i' is added between D_i and W_{i-1}, where W_{i-1} is a facial walk which contains $u_{\frac{m+1}{2}}$ and $u_{\frac{m+3}{2}}$ obtained by adding the tube T_{i-1}. Then two edges $u_{\frac{m+3}{2}}v_{4i-1}$ and $u_{\frac{m+3}{2}}v_{4i}$ are redrawn on T_i'. After the tube $T'_{\frac{n-1}{4}}$ has been added, there is a facial walk $W_{\frac{n-1}{4}}$ which contains $u_{\frac{m+1}{2}}, v_4, \ldots, v_n$. Next, $u_{\frac{m+1}{2}}$ joins to v_i if v_i appears once in $W_{\frac{n-1}{4}}$ or a copy of v_i if it appears more than once in $W_{\frac{n-1}{4}}$.

If $n \equiv 3 \pmod 4$, then $\lceil \frac{n-4}{4} \rceil = \frac{n-3}{4}$. We add $\frac{n-3}{4}$ tubes in the similar way to that in the above paragraph. The difference is that two edge $u_{\frac{m+3}{2}}v_{4i+1}$ and $u_{\frac{m+3}{2}}v_{4i+2}$

are redrawn on T_i' for $i = 1, 2, \dots, \frac{n-3}{4}$.

Let Π_3 be the embedding obtained from Π_2 by the above operation of adding tubes. Clearly, $u_{\frac{m+1}{2}}, u_{\frac{m+3}{2}}$, and $u_{\frac{m+5}{2}}$ are connected with each of $v_1, v_2, \dots, v_n$.

(4) In the step we proceed the similar argument as in (3) and (4) of the proof of Lemma 3.1. Let

$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_{\frac{m+5}{2}} v_{2i+2} v_{2i+3} u_{\frac{m+5}{2}},\ i = 1, 2, \dots, \tfrac{n-3}{2}\},$$
$$\mathcal{Y}_0 = \{Z_{0,i} \mid i = 1, 2, \dots, \tfrac{n-3}{2}\}, \text{ and}$$
$$\mathcal{Y}_0' = \{Z_{0,i}' \mid i = 1, 2, \dots, \tfrac{n-3}{2}\}.$$

Then we apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$, and $\mathcal{Y}_0'$. By Lemma 2.10, we have the following results:

(i) v_2 is connected with each of $v_4, v_6, \dots, v_{n-1}$, and v_3 connected with each of $v_5, v_7, \dots, v_n$.

(ii) For $i = 4, 5, \dots, n$ and $j = 1, 2, \dots, \frac{n-3}{2}$, v_i is connected with u_{2j-1}, u_{2j}, u_{m+1-2j}, u_{m+2-2j}.

(iii) There is a set

$$\{v_i v_{i+1}, \dots, v_i v_n \mid i = 1, 2, \dots, n-1\} \setminus \{v_4 v_5, v_6 v_7, \dots, v_{n-1} v_n\}.$$

(iv) There is a set

$$\mathcal{A}_0 = \{A_{0,1}, A_{0,2}, \dots, A_{0,\frac{n-3}{2}}\}$$

of facial cycles such that $A_{0,i}$ has the form $u_{l_i} v_{2i+1} v_{2i} u_{l_i}$, where $u_{l_i} \in \{u_1, \dots, u_{n-3}\} \cup \{u_{m-n+4}, \dots, u_m\}$.

Unfortunately, v_2 is not connected with each of $v_5, v_7, \dots, v_n$ and v_3 is not connected with each of $v_4, v_6, \dots, v_{n-1}$. In order to attach the edges $v_2v_5, \dots, v_2v_n$, $v_3v_4, \dots, v_3v_{n-1}$, we apply the operation of adding $2(\frac{n-3}{2})^2$ tubes again. Let

$$\mathcal{B}_0 = \{B_{0,i} \mid B_{0,i} = v_3 u_{m-n+4-2i} u_{m-n+5-2i} v_3,\ i = 1, 2, \dots, \tfrac{n-3}{2}\} \text{ and}$$
$$\mathcal{B}_0' = \{B_{0,i}' \mid B_{0,i}' = v_2 u_{n-4+2i} u_{n-3+2i} v_2,\ i = 1, 2, \dots, \tfrac{n-3}{2}\}.$$

We now apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{B}_0'$. By Lemma 2.10, we have the following results:

(i) v_2 is connected with each of $v_5, v_7, \dots, v_n$, and v_3 connected with each of $v_4, v_6, \dots, v_{n-1}$.

(ii) For $i = 4, 5, \dots, n$ and $j = 1, 2, \dots, \frac{n-3}{2}$, v_i is connected with u_{n-4+2j}, $u_{n-3+2j}, u_{m-n+4-2j}, u_{m-n+5-2j}$.

(iii) There is a set

$$\mathcal{L}_0 = \{L_{0,1}, L_{0,2}, \dots, L_{0,\frac{n-3}{2}}\}$$

of $\frac{n-3}{2}$ facial cycles such that $L_{0,i}$ has the form $u_{h_i} v_{2i+1} v_{2i} u_{h_i}$, where $u_{h_i} \in \{u_{n-4+2j}, u_{n-3+2j}, u_{m-n+6-2j}, u_{m-n+5-2j} \mid j = 1, \dots, \frac{n-3}{2}\}$.

Need to say that all edges of the form $v_k v_l$ added in the above operations are deleted, since they have been existed.

For $i = 1, 2, \dots, \frac{n-3}{2}$, let $F_{0,i} = v_1 u_{2n-7+2i} u_{2n-6+2i} v_1$ and $F'_{0,i} = v_1 u_{m-2n+7-2i} u_{m-2n+8-2i} v_1$. Let $\mathcal{F}_0 = \{F_{0,i} \mid i = 1, 2, \dots, \frac{n-3}{2}\}$, and let $\mathcal{F}'_0 = \{F'_{0,i} \mid i = 1, 2, \dots, \frac{n-3}{2}\}$. We apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{L}_0$, $\mathcal{F}_0$, and $\mathcal{F}'_0$. By Lemma 2.1, v_1 is connected with each of $v_4, v_5, \dots, v_n$, and there is a set $\mathcal{N}_0 = \{N_{0,1}, N_{0,2}, \dots, N_{0,\frac{n-3}{2}}\}$ of $\frac{n-3}{2}$ facial cycles such that $N_{0,i}$ has the form $u_{k_i} v_{2i+1} v_{2i} u_{k_i}$, where $u_{k_i} \in \{u_{2n-7+2j}, u_{2n-6+2j} u_{m-2n+7-2j}, u_{m-2n+8-2j} \mid j = 1, \dots, \frac{n-3}{2}\}$. Next, all added edges of the form $v_i v_j$ $(i, j \neq 1)$ are deleted, since they have been existed.

(5) In this step we proceed the similar argument to (5) in the proof of Lemma 3.1. For $i = 1, \dots, \frac{1}{2}(\frac{m-1}{2} - 3n + 9)$, let $M_i = v_1 u_{3n-10+2i} u_{3n-9+2i} v_1$, and $M'_i = v_1 u_{m-3n+10-2i} u_{m-3n+11+2i} v_1$. Clearly, $M'_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}$ is exactly the cycle $v_1 u_{\frac{m+3}{2}} u_{\frac{m+5}{2}} v_1$. Since $u_{\frac{m+3}{2}}$ and $u_{\frac{m+5}{2}}$ are connected with each of $v_1, \dots, v_n$, $M'_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}$ should be neglected. Let

$$\mathcal{M} = \{M_i, M'_i \mid i = 1, \dots, \tfrac{1}{2}(\tfrac{m-1}{2} - 3n + 9)\} \setminus \{M'_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}\}.$$

Next, we apply the operation of adding tubes with respect to $\mathcal{M}$ and $\mathcal{N}_0$. There are $\frac{[m-6(n-3)-3](n-3)}{4}$ tubes being added to the present surface. Since $m \equiv 1 \pmod 2$ and $n \equiv 1 \pmod 4$, we have that

$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil = \frac{(m-3)(n-3)}{4} + \frac{m-1}{4} + \left\lceil \frac{n-4}{4} \right\rceil$$

and

$$\frac{[m-6(n-3)-3](n-3)}{4} + \frac{m-1}{4} + \left\lceil \frac{n-4}{4} \right\rceil + 6\left(\frac{n-3}{2}\right)^2 = \frac{(m-3)(n-3)}{4} + \frac{m-1}{4} + \left\lceil \frac{n-4}{4} \right\rceil.$$

Hence an embedding of $C_m + K_n$ on the surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$ is obtained.

Need to say that the operations of adding $2(\frac{n-3}{2})^2$ tubes are used three times, m is at least $6(n-3)$ $(= 6n-18)$. If $u_{\frac{m+1}{2}}, u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}$ and $M_{\frac{1}{2}(\frac{m-1}{2}-3n+9)}$ are considered, m is at least $6n - 18 + 5$ $(= 6n - 13)$.

Case 2: $m \equiv 3 \pmod 4$. In this case we shall construct an embedding of $C_m + K_n$ in the similar way to that in Case 1.

(1) P_m, v_1, v_2, and v_3 are placed in a sphere as in Case 1. Next, each of v_1 and v_3 is connected with each of $u_1, u_2, \dots, u_{\frac{m+1}{2}}$, and each of v_1 and v_2 is connected with each of $u_{\frac{m+3}{2}}, u_{\frac{m+5}{2}}, \dots, u_m$. Also, v_2 is connected with $u_{\frac{m+1}{2}}$, and v_3 is connected with $u_{\frac{m+3}{2}}$. Then we obtain an embedding Π_1 on the sphere. For example, Π_1 is shown in Figure 14 if $m = 15$.

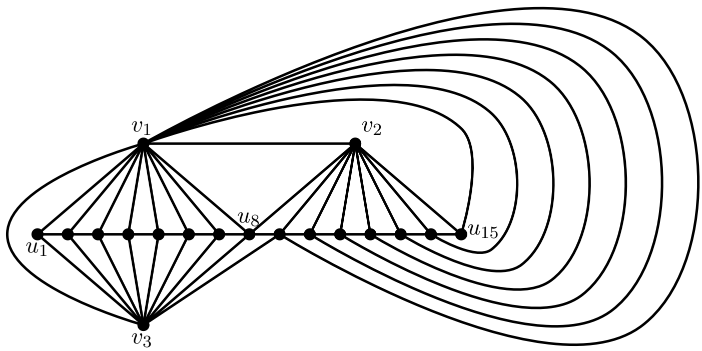

Figure 14: The embedding Π_1.

(2) As in (2) in Case 1, $\frac{m-3}{4}$ tubes are added to the sphere satisfying the following conditions:

  (i) u_1 is connected with u_m,
  (ii) v_2 is connected with each of $u_1, u_2, \ldots, u_{\frac{m-3}{2}}$,
  (iii) v_3 is connected with each of $u_{\frac{m+5}{2}}, u_{\frac{m+7}{2}}, \ldots, u_m$.

Let Π_2 be the obtained embedding. Then it is an embedding on the surface of the genus $\frac{m-3}{4}$.

(3) The path $P = v_4v_5 \ldots v_n$ is now placed in the interior of $v_2u_{\frac{m+1}{2}}u_{\frac{m+3}{2}}v_2$. Then each of $u_{\frac{m+1}{2}}$ and $u_{\frac{m+3}{2}}$ joins to each of $v_4, v_5, \ldots, v_n$. For $j = 1, 2, \ldots, \lceil \frac{n-2}{4} \rceil$, let $D_j = u_{\frac{m+1}{2}}v_{4i}v_{4i+1}u_{\frac{m+1}{2}}$. If $n \equiv 1 \pmod 4$, then $\frac{n-1}{4}$ $(= \lceil \frac{n-2}{4} \rceil)$ tubes $T_1', T_2', \ldots, T_{\frac{n-1}{4}}'$ are added to the present surface one by one such that $u_{\frac{m+1}{2}}v_5$ is redrawn on T_1', and $u_{\frac{m+1}{2}}v_{4i}$ and $u_{\frac{m+1}{2}}v_{4i+1}$ are redrawn on T_i' for $i = 2, 3, \ldots, \frac{n-1}{4}$. If $n \equiv 3 \pmod 4$, then $\frac{n+1}{4}$ $(= \lceil \frac{n-2}{4} \rceil)$ tubes $T_1', T_2', \ldots, T_{\frac{n+1}{4}}'$ are added to the present surface one by one such that $u_{\frac{m+1}{2}}v_4$ is drawn on T_1', and $u_{\frac{m+1}{2}}v_{4i+3}$ and $u_{\frac{m+1}{2}}v_{4i}$ are redrawn on T_i' for $i = 2, 3, \ldots, \frac{n+1}{4}$. As in Case 1, there is a facial walk $W_{\lceil \frac{n-2}{4} \rceil}$ which contains $u_{\frac{m-1}{2}}, v_4, \ldots, v_n$ and v_2. Next, $u_{\frac{m-1}{2}}$ joins to v_j if it appears once in $W_{\lceil \frac{n-2}{4} \rceil}$ or a copy of v_j if it appears more than once in $W_{\lceil \frac{n-2}{4} \rceil}$, where v_j is a vertex in $v_4, v_5, \ldots, v_n$ and v_2. Let Π_3 be the obtained embedding. Then it is an embedding on the surface of the genus $\frac{m-3}{4} + \lceil \frac{n-2}{4} \rceil$.

(4) In this step we proceed the similar argument as in (4) and (5) in Case 1. There are $\frac{(m-3)(n-3)}{4}$ tubes being added to the present surface. The detail is omitted here. Let Π_4 be the obtained embedding. Then it is an embedding of $C_m + K_n$ on the surface of genus $\frac{m-3}{4} + \lceil \frac{n-2}{4} \rceil + \frac{(m-3)(n-3)}{4}$. Need to say that for the purpose that each of v_1, v_2 and v_3 is connected with $v_4, \ldots, v_n$, we need add at least $6(\frac{n-3}{2})^2$ tubes. Since each of $u_{\frac{m-1}{2}}$, $u_{\frac{m+1}{2}}$ and $u_{\frac{m+3}{2}}$ has been connected with each of $v_4, \ldots, v_n$, m is at least $3 + 6(n-3)$ $(= 6n - 15)$.

Since $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 2$, we have that $\lceil \frac{(m-2)(n-2)}{4} \rceil = \frac{m-3}{4} + \lceil \frac{n-2}{4} \rceil + \frac{(m-3)(n-3)}{4}$. So Π_4 is an embedding of $C_m + K_n$ on the surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$. □

4 An upper bound for $\gamma(C_m + K_n)$ if m is even

In the section we shall study the orientable genus of $C_m + K_n$ if m is even.

Lemma 4.1. *Suppose that* $m \equiv 0 \pmod 2$. *If* $m \geq 8$, *then*

$$\gamma(C_m + K_4) \leq \left\lceil \frac{m-2}{2} \right\rceil .$$

Proof. We firstly construct an embedding on a sphere. C_m, v_1, and v_2 are placed in the sphere as in the proof of Lemma 3.1, and each of v_1 and v_2 joins to $u_1, u_2, \ldots, u_n$. Let $F_1 = v_1u_1u_2v_1$ and $F_2 = v_2u_3u_4v_2$. Next, the vertex v_3 is placed in the interior of F_1 and is connected with to u_1, u_2, and v_1, and the vertex v_4 is placed in the interior of F_2 and is connected with u_3, u_4, and v_2. At last, the tube T_1 is added between the facial cycle $v_3u_1u_2v_3$ and the facial cycle $v_4u_3u_4v_4$. Then six edges are drawn on T_1 in the way shown in (1) of Figure 15.

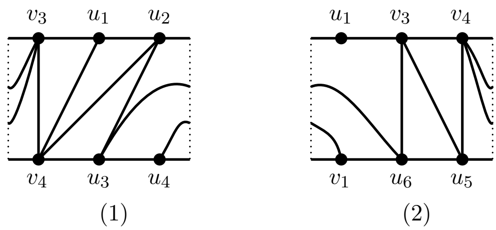

Figure 15: Two drawings of edges on T_1 or T_2.

Note that there are two edges connecting u_2 and u_3. Let $F_3 = v_1u_2u_3v_1$ and $F_4 = v_2u_2u_3v_2$. We now delete the edge u_2u_3 which is a common edge of F_3 and F_4. Then F_3 and F_4 are merged into a facial cycle $F_5 = v_1u_2v_2u_3v_1$. Next, the edge v_1v_2 is drawn in the interior of F_5.

Let $F_6 = u_1v_3v_4u_1$ (refer to (1) of Figure 15), and let $F_7 = v_1u_5u_6v_1$. The tube T_2 is now added between F_6 and F_7. Then the five edges are drawn on T_2 in the way shown in (2) in Figure 15. Let $F_8 = u_5v_3v_4u_5$ (refer to (2) of Figure 15), and let $F_9 = v_2u_8u_7v_2$. Then the tube T_3 is added between F_8 and F_9. Next, the five edges v_3u_8, v_3u_7, v_4u_7, v_4u_8 and v_4v_2 are drawn on T_3 in the similar way to that in (2) in Figure 15. Thus, v_i is connected with v_j if $i \neq j$. If $m = 8$, there is nothing to do. If $m > 8$, let $\mathcal{F} = \{F' \mid F' = u_7v_3v_4u_7\}$, and let $\mathcal{Q} = \{Q_i \mid Q_i = v_1u_{7+2i}u_{8+2i}v_1, i = 1, 2, \ldots, \frac{m-8}{2}\}$. We apply the operation of adding $\frac{m-8}{2}$ tubes with respect to $\mathcal{F}$ and $\mathcal{Q}$ to realize an embedding of $C_m + K_4$. Thus, there are $\frac{m-8}{2} + 3$ $(= \frac{m-2}{2})$ tubes being used. Hence, $\gamma(C_m + K_4) \leq \lceil \frac{m-2}{2} \rceil$. □

Lemma 4.2. *Suppose that* $m \equiv 0 \pmod 2$ *and* $n \equiv 0 \pmod 2$. *If* $n \geq 6$ *and* $m \geq 4n - 4$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

Proof. We construct an embedding of $C_m + K_n$ in the following steps.

(1) The cycle C_m and vertices v_1,v_2 are placed in a sphere as in the proof of Lemma 3.1. Next, each of v_1 and v_2 joins to $u_1, u_2, \ldots, u_m$. Let $F_1 = v_1u_1u_2v_1$ and $F_2 = v_1u_3u_4v_1$. The two vertices v_4 and v_6 are placed in the interior of F_1, and each of u_1 and u_2 joins to each of v_4 and v_6 such that there are two facial 4-cycles $F_1' = u_1v_4u_2v_6u_1$ and $F_2' = v_1u_1v_6u_2v_1$. The two vertices v_3 and v_5 are placed in the interior of F_2, and each of u_3 and u_4 joins to each of v_3 and v_5 such that there are two facial 4-cycle $F_3' = u_3v_3u_4v_5u_3$ and $D_1' = u_3u_4v_5u_3$. The path $P = v_7v_8 \ldots v_n$ is placed in the interior of F_2' such that v_7 is near to v_6. Next, each of u_1 and u_2 joins to each of $v_7, v_8, \ldots, v_n$. The obtained embedding is denoted by Π_1.

(2) In the step each of u_1, u_2, u_3 and u_4 will be connected with each of $v_3, v_4, \ldots, v_n$, and v_1 is connected with v_2. For the above purpose, the tube T_1 is firstly added between F_1' and F_3', and the five edges u_1v_5, u_2v_3, u_3v_4, u_4v_6 and u_2u_3 are drawn on T_1 in the way shown in (1) of Figure 16. Thus, there are two edges connecting u_2 and u_3. The edge u_2u_3 which is the common edge of facial cycles $v_1u_2u_3v_1$ and $v_2u_2u_3v_2$ is deleted. Then there is a facial cycle $F_3 = v_1u_2v_2u_3v_1$. Next, v_1 joins to v_2 in the interior of F_3. The tube T_2 is now added between the facial cycles $u_1v_4u_3v_5u_1$ and $u_2v_3u_4v_6u_2$ (refer to (1) in Figure 16), and the six edges u_1v_3, u_2v_5, u_3v_6, u_4v_4, v_3v_4 and v_5v_6 are drawn on T_2 in the way shown in (2) of Figure 16.

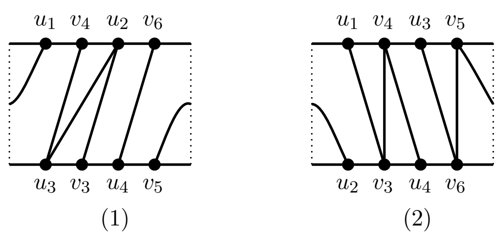

Figure 16: Two drawings of edges on T_1 or T_2.

For $i = 1, 2, \ldots, \frac{n-6}{2}$, let $D_i = u_2v_{2i+5}v_{2i+6}u_2$. Let $\mathcal{D} = \{D_i \mid i = 1, 2, \ldots, \frac{n-6}{2}\}$ and $\mathcal{D}' = \{D_1'\}$. We apply the operation of adding tubes with respect to $\mathcal{D}$ and $\mathcal{D}'$ such that both u_3 and u_4 are connected with each of $v_7, v_8, \ldots, v_n$. By Lemma 2.11, there are $\frac{n-6}{2}$ tubes being used. Let Π_2 be the obtained embedding.

(3) We proceed a similar argument to that in (3) in the proof of Lemma 3.2. We shall add $\frac{(m-4)(n-2)}{4}$ tubes to the present surface to realize an embedding Π_3 of $C_m + K_n$. The detail is omitted here. For the purpose that each of v_1 and v_2 joins to each of $v_3, \ldots, v_n$, $2(\frac{n-2}{2})^2$ tubes will be used by Lemma 2.1. So m is at least $4 + 4 \times \frac{n-2}{2}$ $(= 4n - 4)$.

Obviously, Π_3 is an embedding of $C_m + K_n$ on the surface of genus $2 + \frac{n-6}{2} + \frac{(m-4)(n-2)}{4}$. Since $m \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, we have that

$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil = 2 + \frac{n-6}{2} + \frac{(m-4)(n-2)}{4}.$$

So $\gamma(C_m + K_n) \leq \lceil \frac{(m-2)(n-2)}{4} \rceil$. □

Lemma 4.3. *Suppose that* $m \equiv 0 \pmod 2$ *and* $n \equiv 1 \pmod 2$. *If* $m \geq 6n - 14$ *and* $n \geq 5$, *then*

$$\gamma(C_m + K_n) \leq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Proof. We proceed a similar argument to that in the proof of Lemma 3.3.

(1) Let $P_m = u_1 u_2 \dots u_m$. Then P_m, v_1, v_2, and v_3 are placed in a sphere as in (1) in the proof of Lemma 3.3. If $m \equiv 0 \pmod 4$, then each of v_1 and v_3 joins to each of $u_1, u_2, \dots, u_{\frac{m}{2}}$ such that $v_1 u_i$ and $v_3 u_i$ are in the upper side and lower side of P_m, respectively. Next, each of v_2 and v_1 joins to each of $u_{\frac{m+2}{2}}, u_{\frac{m+4}{2}}, \dots, u_m$ such that $v_2 u_i$ and $v_1 u_i$ are in the upper side and lower side of P_m, respectively. Also, v_1 joins to v_3. If $m \equiv 2 \pmod 4$, then each of v_1 and v_3 joins to each of $u_1, u_2, \dots, u_{\frac{m}{2}}$ such that $v_1 u_i$ and $v_3 u_i$ are in the upper side and lower side of P_m, respectively. Next, each of v_2 and v_1 joins to each of $u_{\frac{m+2}{2}}, u_{\frac{m+4}{2}}, \dots, u_m$ such that $v_2 u_i$ and $v_1 u_i$ are in the upper side and lower side of P_m, respectively. Also, v_1 joins to v_3, v_2 joins to $u_{\frac{m}{2}}$, and v_3 joins to $u_{\frac{m+2}{2}}$. Let Π_1 be the obtained embedding on the sphere.

(2) As in (2) in the proof of Lemma 3.3, there are $\frac{m}{4}$ tubes being added to the sphere if $m \equiv 0 \pmod 4$, or there are $\frac{m-2}{4}$ tubes being added to the sphere if $m \equiv 2 \pmod 4$, such that each of v_2 and v_3 is connected with all rest vertices in $u_1, u_2, \dots, u_m$. Also, u_1 is connected with u_m, and v_2 is connected with v_3. Need to say that $\lceil \frac{m-2}{4} \rceil = \frac{m}{4}$ if $m \equiv 0 \pmod 4$, or $\lceil \frac{m-2}{4} \rceil = \frac{m-2}{4}$ if $m \equiv 2 \pmod 4$. Thus, there are $\lceil \frac{m-2}{4} \rceil$ tubes being used in the above procedure.

(3) Let $P' = v_4 v_5 \dots v_n$. If $m \equiv 0 \pmod 4$, then P' is placed in the facial cycle $v_1 u_1 u_2 v_1$, and each of u_1 and u_2 is connected with $v_4, v_5, \dots, v_n$. If $m \equiv 2 \pmod 4$, then P' is placed in the facial cycle $v_1 u_{\frac{m}{2}} u_{\frac{m}{2}+1} v_1$, and each of $u_{\frac{m}{2}}$ and $u_{\frac{m}{2}+1}$ is connected with $v_4, v_5, \dots, v_n$.

Let

$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_2 v_{2i+2} v_{2i+3} u_2,\ i = 1, 2, \dots, \tfrac{n-3}{2}\} \text{ if } m \equiv 0 \pmod 4, \text{ or}$$
$$\mathcal{X}_0 = \{Q_{0,i} \mid Q_{0,i} = u_{\frac{m}{2}} v_{2i+2} v_{2i+3} u_{\frac{m}{2}},\ i = 1, 2, \dots, \tfrac{n-3}{2}\} \text{ if } m \equiv 2 \pmod 4.$$

Let

$$\mathcal{Y}_0 = \{R_{0,i} \mid R_{0,i} = v_2 u_{2i+1} u_{2i} v_2,\ i = 1, 2, \dots, \tfrac{n-3}{2}\}, \text{ and}$$
$$\mathcal{Y}'_0 = \{R'_{0,i} \mid R'_{0,i} = v_3 u_{m+1-2i} u_{m+2-2i} v_3,\ i = 1, 2, \dots, \tfrac{n-3}{2}\}.$$

We apply the operation of adding $2(\frac{n-3}{2})^2$ tubes starting from $\mathcal{X}_0$, $\mathcal{Y}_0$ and $\mathcal{Y}'_0$. Next procedures are similar to that in (4) in the proof of Lemma 3.3. Eventually, we obtain an embedding of $C_m + K_n$ by adding $\frac{(m-2)(n-3)}{4}$ tubes. Note that for the purpose that each of v_1, v_2 and v_3 is connected with each of $v_4, v_5, \dots, v_n$, we need to add at least $3 \times 2 \times \frac{n-3}{2}$ tubes by Lemma 2.10. Thus, $m \geq 6(n-3) + 2 + 2 = 6n - 14$ if $m \equiv 0 \pmod 4$, or $m \geq 6(n-3) + 2 = 6n - 16$ if $m \equiv 2 \pmod 4$.

Since $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, $\lceil \frac{(m-2)(n-2)}{4} \rceil = \frac{(m-2)(n-3)}{4} + \lceil \frac{m-2}{4} \rceil$. Since $\frac{m}{4} = \lceil \frac{m-2}{4} \rceil$ if $m \equiv 0 \pmod 4$, or $\frac{m-2}{4} = \lceil \frac{m-2}{4} \rceil$ if $m \equiv 2 \pmod 4$, the obtained embedding is an embedding of $C_m + K_n$ on the surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$. □

5 Conclusions

Lemma 5.1 ([10]). *If $m \geq 2$ and $n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Considering that $K_{m,n}$ is a subgraph of $C_m + K_n$, Theorem 5.2 follows from Lemmas 3.1, 3.2, and 3.3, Lemmas 4.1, 4.2, and 4.3, and Lemma 5.1.

Theorem 5.2. *Suppose that m and n are two integers. Then*

$$\gamma(C_m + K_n) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$$

if $n \geq 4$ and m, n satisfy one of the following conditions:

(1) $m \equiv 1 \pmod 2$, $n \equiv 0 \pmod 2$, *and* $m \geq 4n - 5$,

(2) $m \equiv 1 \pmod 2$, $n \equiv 1 \pmod 2$, *and* $m \geq 6n - 13$,

(3) $m \equiv 0 \pmod 2$, $n \equiv 0 \pmod 2$, *and* $m \geq 4n - 4$,

(4) $m \equiv 0 \pmod 2$, $n \equiv 1 \pmod 2$, *and* $m \geq 6n - 14$.

Obviously, the maximal value in $4n - 5, 4n - 4, 6n - 13$ and $6n - 14$ is 12 if $n = 4$, or $6n - 13$ if $n \geq 5$. The result below follows from Lemma 5.1 and Theorem 5.2 directly.

Corollary 5.3. *Suppose that m and n are two integers. Let G_1 be a spanning subgraph of C_m, and let G_2 be a spanning subgraph of K_n. If $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$, then*

$$\gamma(G_1 + G_2) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Since $K_{r,s,t}$ ($r \geq s \geq t \geq 3$) is a spanning subgraph of $C_r + K_{s+t}$, we have the following result by Theorem 5.2.

Corollary 5.4. *If $r \geq s \geq t \geq 3$ and $r \geq 6(s + t) - 13$, then*

$$\gamma(K_{r,s,t}) = \left\lceil \frac{(r-2)(s+t-2)}{4} \right\rceil.$$

Therefore, Stahl and White's conjecture ([12]) on the orientable genus of the complete tripartite graph $K_{r,s,t}$ holds if $r \geq s \geq t \geq 3$ and $r \geq 6(s + t) - 13$.

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ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 255–270
https://doi.org/10.26493/1855-3974.1560.a43
(Also available at http://amc-journal.eu)

Tetrahedral and pentahedral cages for discs*

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Received 21 December 2017, accepted 21 April 2019, published online 14 October 2019

Abstract

This paper is about cages for compact convex sets. A cage is the 1-skeleton of a convex polytope in $\mathbb{R}^3$. A cage is said to hold a set if the set cannot be continuously moved to a distant location, remaining congruent to itself and disjoint from the cage.

In how many "truly different" positions can (compact 2-dimensional) discs be held by a cage? We completely answer this question for all tetrahedra. Moreover, we present pentahedral cages holding discs in a large number (57) of positions.

Keywords: Tetrahedral cages, pentahedral cages, discs.

Math. Subj. Class.: 52B10

1 Introduction

A *cage* is the 1-skeleton of a (convex) polytope in $\mathbb{R}^3$. If P is the polytope, the cage is denoted by $\mathrm{cage}(P)$. A cage G is said to *hold* a compact set K with $G \cap \mathrm{int}\, K = \emptyset$, if no rigid continuous motion can bring K in a position far away without $\mathrm{int}\, K$ meeting G on its way. (Here, $\mathrm{int}\, K$ means the interior of K in its affine hull.) A compact 2-dimensional ball in $\mathbb{R}^3$ will be called a *disc*.

*This work is supported by NSF of China (No. 11871192, No. 11471095).
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Not that long ago, the subject of holding (3-dimensional) balls in cages has been treated by Coxeter [6], Besicovitch [4], Aberth [1] and Valette [12].

In this paper we hold discs instead of balls. The question we ask is about the number of positions of the discs held.

We investigate the capability of the 1-skeleton of the regular tetrahedron as a cage to hold discs. Then, we consider the capability of the 1-skeleton of an arbitrary tetrahedron to hold discs, and discuss in detail the dependence on the shape of the tetrahedron. Finally, we also consider the two combinatorial types of pentahedral cages.

The related phenomenon of holding a convex body using a circle was investigated in [2, 3, 13]. For other related results, see [9, 10, 14, 15].

For distinct $x, y \in \mathbb{R}^3$, let $\overline{xy}$ be the line through x, y and xy the line-segment from x to y. We denote by Π_{xy} the plane through x orthogonal to $\overline{xy}$, and by Π^+_{xy} the closed half-space not containing y, determined by Π_{xy}.

For $M \subset \mathbb{R}^3$, $\overline{M}$ denotes its affine hull, $\operatorname{int} M$ and $\operatorname{bd} M$ denote its interior and boundary in the topology of $\overline{M}$, and $\operatorname{diam} M = \sup_{x,y\in M} \|x - y\|$. A line-segment xy with $\{x, y\} \subset M$ and $\|x - y\| = \operatorname{diam} M$ is called a *diameter* of M. Also, $\operatorname{conv} M$ denotes the intersection of all convex sets including M.

For $x_1, x_2, \dots, x_k \in \mathbb{R}^3$, $x_1x_2 \cdots x_k$ means $\operatorname{conv}\{x_1, x_2, \dots, x_k\}$. For non-collinear elements $x, y, z \in \mathbb{R}^3$, let $C(xyz) \subset \overline{xyz}$ be the circle passing through x, y, z, and let o_{xyz} be its centre. Put $D(xyz) = \operatorname{conv} C(xyz)$. We denote by $\widehat{xyz}$ the angle of xyz at y, and by $\angle xyz$ its measure.

A *face* of a cage G is a 2-dimensional face of the polytope $\operatorname{conv} G$.

The d-dimensional compact unit ball (centred at $\mathbf{0}$) is B_d, and $\operatorname{bd} B_d = S_{d-1}$ ($d \geq 2$).

Also, we denote by λ the 1-dimensional Hausdorff measure (length).

Problem 1.1. Let $\mathcal{G}(K)$ be the space of all cages in $\mathbb{R}^3$ holding the compact set K. Determine

$$L(K) = \inf_{G\in\mathcal{G}(K)} \lambda G,$$

for various sets K.

This problem, in line with the work of Coxeter, Besicovitch, Aberth and Valette, will not be addressed in this paper, but in [8].

For any cage G, let $\mathcal{D}(G)$ be the space of all discs held by G, endowed with the Pompeiu-Hausdorff metric.

Let $\mathcal{D}_r(G)$ be the set of all discs in $\mathcal{D}(G)$ of radius at least r. (Notice that the term "radius" is used for both the distance and the line-segment from the centre to a point of the relative boundary.) Assume that, for some component $\mathcal{E}$ of $\mathcal{D}_r(G)$ and any number $s > r$, $\mathcal{D}_s(G) \cap \mathcal{E}$ is connected or empty. We call such a component $\mathcal{E}$ an *end-component* of $\mathcal{D}(G)$. If n is the maximal number of pairwise disjoint end-components of $\mathcal{D}(G)$, we say that G *holds n discs.*

In fact, intuitively, G does not hold n pairwise disjoint discs simultaneously; merely there are n different positions at which, separately, a disc can be held.

Let the component $\mathcal{E}$ of $\mathcal{D}_r(G)$ be an end-component of $\mathcal{D}(G)$. Put $\sigma(\mathcal{E}) = \sup\{s : \mathcal{D}_s(G) \cap \mathcal{E} \neq \emptyset\}$. Choose an increasing sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers satisfying $s_n > r$ and $\lim_{n\to\infty} s_n = \sigma(\mathcal{E})$. Consider a disc $D_n \in \mathcal{D}_{s_n}(G)$ for each n.

If $\{D_n\}_{n=1}^{\infty}$ converges to some disc $D(\mathcal{E})$ independent of the choice of the numbers s_n and discs D_n, we call $D(\mathcal{E})$ the *limit disc* of $\mathcal{E}$. Several end-components may have the same limit disc.

If the limit disc of an end-component $\mathcal{E}$ lies in the plane of a face F of $\operatorname{conv} G$, we say that *G holds a disc at the face F*. For each end-component, we have a disc held, even if the limit discs coincide. So, a cage may hold several discs at the same face. Also, if a face F is not triangular, several distinct limit discs can be coplanar with F.

Inspired by an earlier version of the present paper, Montejano and Zamfirescu [11] raised the following questions.

Problem 1.2. Does a cage holding 7 discs exist?

Problem 1.3. How many discs can be held by a pentahedral cage?

We give here an affirmative answer to Problem 1.2, establish the precise minimum and find a lower bound for the maximum number of discs that a pentahedral cage can hold.

For a cage which is not tetrahedral it is possible that a disc is held, but not at a face. Such a case we shall meet for a pentahedral cage admitting a limit disc (of some end-component) circumscribed to a triangle which is not a face of the pentahedron, but has vertices among those of the cage. For arbitrary polyhedral cages even the following is possible.

Proposition 1.4. *There exist cages G admitting a limit disc not coplanar with any vertex of* $\operatorname{conv} G$.

Proof. Consider a regular icosagon $\Delta = a_1 a_2 \cdots a_{20} \subset H$ inscribed in S_1, where $H = \{(x, y, z) : z = 0\}$ and S_1 is the unit circle in H. Let $\varepsilon > 0$ and $\tau = (0, 0, \varepsilon)$. Let $\nu > 0$.

Put

$$b_i = \begin{cases} (1+\nu)a_i + \tau & \text{for } i \not\equiv 3 \pmod 4 \\ (1-\nu)a_i + \tau & \text{for } i \equiv 3 \pmod 4 \end{cases}$$

and

$$c_i = \begin{cases} (1+\nu)a_i - \tau & \text{for } i \not\equiv 1 \pmod 4 \\ (1-\nu)a_i - \tau & \text{for } i \equiv 1 \pmod 4. \end{cases}$$

For ν small enough, $\Delta_b = b_1 b_2 \cdots b_{20}$ and $\Delta_c = c_1 c_2 \cdots c_{20}$ are convex icosagons. The polytope $P = \operatorname{conv}(\Delta_b \cup \Delta_c)$ has 42 faces including Δ_b and Δ_c. We claim that B_2 is a limit disc of $\operatorname{cage}(P)$.

Indeed, note that the circle S_1 meets $\operatorname{cage}(P)$ at the vertices $a_1, a_3, \ldots, a_{19}$ of Δ only. Assume that some unit disc D distinct from B_2 but close to it satisfies $\operatorname{cage}(P) \cap \operatorname{int} D = \emptyset$. Let the ellipse E be the orthogonal projection of D onto H and let xy be the long axis of E (or any diameter if E is a circle). Since $\|x - y\| = 2$, one of these end-points, say x, is on S_1 or outside B_2. Let x', y' be the points of D with projections x, y, respectively. Since $\overline{x'y'}$ is parallel to H, it is included in (at least) one of the half-spaces

$$H^+ = \{(x, y, z) : z \geq 0\}, \qquad H^- = \{(x, y, z) : z \leq 0\}.$$

Suppose without loss of generality that $\overline{x'y'} \subset H^-$. Then, at least one of the half-discs of D determined by $x'y'$, say D', entirely lies in H^-.

The intersection $\{x^*\} = \mathbf{0}x \cap S_1$ lies on S_1 between two consecutive vertices of the regular pentagon $a_1 a_5 a_9 a_{13} a_{17}$, or coincides with one of them, say $x^* \in \widehat{a_1 a_5}$. Therefore, since $D \neq B_2$, D' cuts either $a_1 c_1$ or $a_5 c_5$, which yields $\operatorname{int} D \cap \operatorname{cage}(P) \neq \emptyset$, and this contradicts our assumption. $\square$

2 Tetrahedral cages

Consider a regular tetrahedron. If its edge-length is 1, then the circle circumscribed to a face has radius $1/\sqrt{3}$. So, a slightly enlarged tetrahedral cage T will hold the disc $(1/\sqrt{3})B_2$. Clearly, at each face there is such a disc.

In fact there are many discs close to $(1/\sqrt{3})B_2$, held by T, lying in the same component of $\mathcal{D}_{(1/\sqrt{3})}(T)$. The space $\mathcal{D}_{(1/\sqrt{3})}(T)$ has 4 components analogous to the component of $(1/\sqrt{3})B_2$, one corresponding to each face of T. The limit disc of each component is the disc circumscribed to the respective face.

The following lemma is easily verified by the reader.

Lemma 2.1. *If a polytopal cage holds a disc at some triangular face, then that triangle is acute.*

A face being an acute triangle is, however, no guarantee that the disc described above (lying over the face) is held there. Whether it can move away from that face or not, obviously depends on the angle between the edges of the polytope adjacent but not belonging to that face and the corresponding radii of the circumscribed circle of the face.

Lemma 2.2. *If a face of a tetrahedral cage is an acute triangle, then at least one disc is held at that face.*

Proof. Let abc be the given acute face, and o the centre of $C(abc)$. Consider the half-spaces Π^+_{ao}, Π^+_{bo}, Π^+_{co}. As the intersection of these half-spaces is void, there is no point $x \in \mathbb{R}^3$ for which all angles $\widehat{xao}$, $\widehat{xbo}$, $\widehat{xco}$ are non-acute. Assume $\angle dao < \pi/2$. Now take a disc (slightly smaller than $D(abc)$) over ab and ac, but below bc (see Figure 1). This disc is held by the cage. □

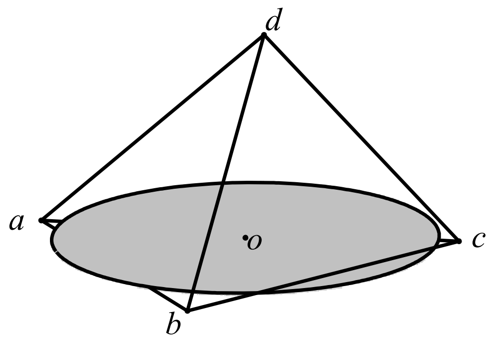

Figure 1: Cage holding a disc.

Lemma 2.3. *If a tetrahedral cage has an acute face, then it has one, two, or four discs held at that face.*

Proof. Keep the notation of the preceding proof. The kind of disc held by the cage in the previous proof requires an angle like $\widehat{dao}$ to be acute. The existence of a second such angle, say $\widehat{dbo}$, provides a second such disc. If at least one such angle, say $\widehat{dco}$, is not acute, then any disc lying over the face abc can move away from the face. If all three angles $\widehat{dao}$, $\widehat{dbo}$, $\widehat{dco}$ are acute, then not only the three discs partly lying below some edge of abc are held, but also the disc lying completely over the face abc, whence the conclusion of the lemma. □

Theorem 2.4. *The regular tetrahedral cage holds* 16 *discs.*

Proof. The last case of the proof of Lemma 2.3 applies at all faces. By Proposition 2.5 below, there is no other disc held by the cage. □

Tetrahedral cages cannot display the situation in Proposition 1.4.

Proposition 2.5 (Fruchard [7])**.** *In any tetrahedral cage, each limit disc is at some face.*

With the author's permission, we reproduce here his proof, for the reader's convenience.

Proof. Let $abcd$ be a non-degenerate tetrahedron, $G = \text{cage}(abcd)$, and assume D is a limit disc which is not at a face. To fix ideas, we assume that D is the unit disc B_2 in the *horizontal plane* $H = \{(x, y, z) : z = 0\}$ of $\mathbb{R}^3$.

It is an easy task to exclude that some vertex of G lies in the plane of D. Furthermore, it is easily seen that D meets four edges of G, say ab, bc, cd, and da, with a and c above D, and b and d below D. Two of these edges have to pass above D and two below, and they must alternate, say ab and cd above, bc and da below. Let $e \in ab \cap D$, $f \in bc \cap D$, $g \in cd \cap D$, and $h \in da \cap D$, see Figure 2.

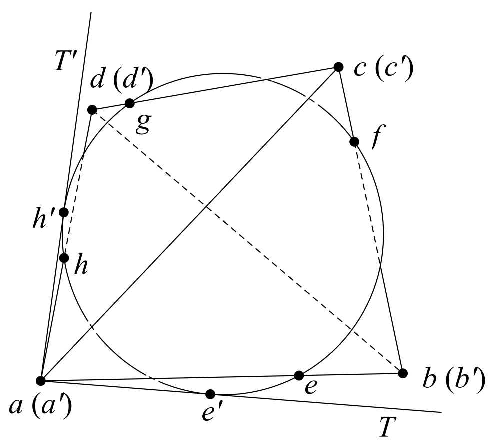

Figure 2: Proof of Proposition 2.5.

Let a', b', c' and d' be the orthogonal projections of a, b, c, d on H. Then, we have

$$\frac{\|a' - e\|}{\|b' - e\|} = \frac{\|a - e\|}{\|b - e\|} = \frac{z_a}{|z_b|},$$

where z_a is the third coordinate of a. Using the analogous formulae for the other three sides of the quadrilateral $a'b'c'd'$, we obtain

$$\frac{\|a' - e\|}{\|b' - e\|} \frac{\|b' - f\|}{\|c' - f\|} \frac{\|c' - g\|}{\|d' - g\|} \frac{\|d' - h\|}{\|a' - h\|} = \frac{z_a}{|z_b|} \frac{|z_b|}{z_c} \frac{z_c}{|z_d|} \frac{|z_d|}{z_a} = 1. \tag{2.1}$$

As we show below, this is impossible. From a', draw the two tangent lines to D, T toward b and T' toward d. Let $e' \in T \cap D$ (hence on the same side as e) and $h' \in T' \cap D$. Because ab is above D, we have $\|a - e\| > \|a' - e'\|$; in the same manner, da is below D, hence $\|a' - h\| < \|a' - h'\|$. Then, $\|a' - e'\| = \|a' - h'\|$ implies $\frac{\|a'-e\|}{\|a'-h\|} > 1$. Similarly, one has $\frac{\|b'-f\|}{\|b'-e\|}$, $\frac{\|c'-g\|}{\|c'-f\|}$ and $\frac{\|d'-h\|}{\|d'-g\|}$ all larger than 1, contradicting equation (2.1). □

Lemma 2.6. *If, for $a, b, c, x, o \in \mathbb{R}^3$, $\angle axb \leq \pi/2$, $\angle cxa < \pi/2$ and o lies in the relative interior of bxc, then $\angle axo < \pi/2$.*

The proof (using for example the basic properties of the scalar product) is left to the reader.

Theorem 2.7. *There are tetrahedral cages holding exactly n discs, for every $n \leq 16$ except for $n \in \{7, 9, 11, 13, 14, 15\}$, and there is no such cage for any other n.*

Proof. Separately, every number n of held discs can be realized at a face, if $n \in \{0, 1, 2, 4\}$, by Lemma 2.3. We have to show that a global realization is possible, for each of the n's from the statement. Moreover, we must show the impossibility of a realization in all other cases.

We keep in mind that limit discs can only be at faces, by Proposition 2.5.

Throughout this proof, o will denote the centre of $C(abc)$.

Case $n = 0$: Take the face abc to have an obtuse angle at a, take a point d' in the relative interior of its height at a, and consider a point d close to d' and having d' as orthogonal projection on $\overline{abc}$. Then the tetrahedral cage $\text{cage}(abcd)$ has all faces obtuse. Now use Lemma 2.1.

Case $n = 1$: Take now the face abc to be an acute triangle and consider o. For any point

$$d \in \Pi^+_{ao} \cap \Pi^+_{ba} \setminus \overline{abc},$$

the triangles abd, bcd, cad are obtuse or right. See Figure 3.

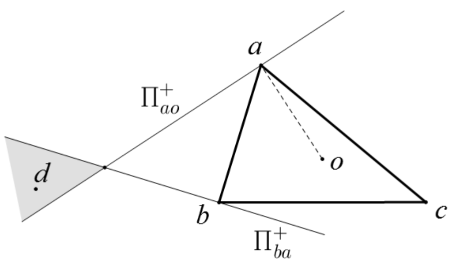

Figure 3: Case $n = 1$.

Moreover, only one of the angles $\widehat{oad}$, $\widehat{obd}$, $\widehat{ocd}$ is acute, namely the latter. Thus, $\text{cage}(abcd)$ holds exactly one disc (at the face abc), as described in the proof of Lemma 2.2.

Case $n = 2$: Let again abc be acute, and choose

$$d \in \Pi^+_{ac} \cap \Pi^+_{ba} \setminus (\Pi^+_{ao} \cup \overline{abc}).$$

In this way abd, bcd, cad are still non-acute, but now precisely two of the angles $\widehat{oad}$, $\widehat{obd}$, $\widehat{ocd}$ are acute, namely the first and the last (see Figure 4). Thus, two discs are held, both at the face abc.

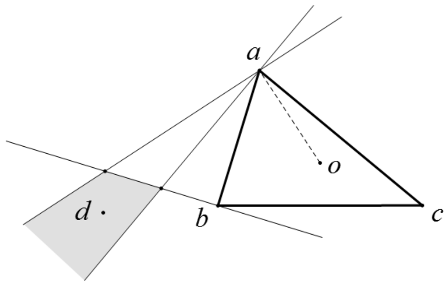

Figure 4: Case $n = 2$.

Case $n = 3$: Take abc acute, as before. Choose

$$d \in \Pi_{ac}^{+} \cap \Pi_{bo}^{+} \setminus (\Pi_{ba}^{+} \cup \Pi_{ao}^{+} \cup \overline{abc}).$$

Now, the triangles bcd and cad are non-acute, while the triangles abc and abd are acute. See Figure 5.

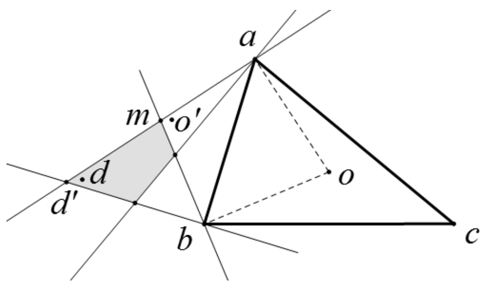

Figure 5: Case $n = 3$.

Regarding abc, $\angle oad < \pi/2$, $\angle obd \geq \pi/2$, $\angle ocd < \pi/2$, whence two discs are held at abc.

Regarding abd, let $\{d'\} = \Pi_{ao} \cap \Pi_{ba} \cap \overline{abc}$, and denote by m the midpoint of ad'. Then $\angle oam = \angle obm = \pi/2$. Hence, $\angle cam > \pi/2$ and $\angle cbm > \pi/2$. If d is chosen close to d' (and in the already assigned region), then the centre o' of $C(abd)$ is close to m, and we also have $\angle cao' > \pi/2$ and $\angle cbo' > \pi/2$. Doubtlessly $\angle cdo' < \pi/2$, whence there is precisely one disc held by cage$(abcd)$ at abd.

Case $n = 4$: Let the face abc be an equilateral triangle of centre o. Choose $d \notin \overline{abc}$ close to o. Thus, the triangles dab, dbc and dca are obtuse. See Figure 6. By Lemma 2.1, no disc is held at any of the faces dab, dbc, dca.

Since $\angle oad$, $\angle obd$ and $\angle ocd$ are close to 0, cage$(abcd)$ holds exactly 4 discs at abc (see the proof of Lemma 2.3).

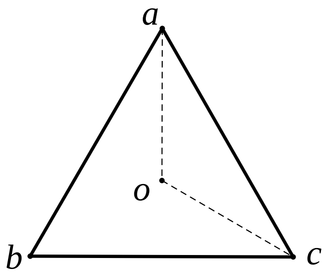

Figure 6: Case $n = 4$.

Case $n = 5$: Let $a'bec$ be a square. Choose $a, d' \in \overline{a'e}$ such that a, a', e, d' lie in this order on their line, with $\|a - a'\|$ small and $\|e - d'\| = \|b - e\|$. See Figure 7. Then

$$\angle abe = \angle ace > \pi/2 \quad \text{and} \quad \angle obd' = \angle ocd' < \pi/2.$$

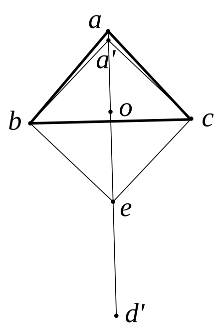

Figure 7: Case $n = 5$.

Rotate slightly d' about $\overline{bc}$ up to a new position d. Then still

$$\angle abo' = \angle aco' > \pi/2 \quad \text{and} \quad \angle obd = \angle ocd < \pi/2,$$

where o' is the centre of $C(bcd)$.

Also, notice that $\angle ado'$ and $\angle oad$ are small.

The triangles abc and bcd are acute, abd and acd obtuse. The inequalities above imply that one disc is held by $\text{cage}(abcd)$ at bcd, and four discs at abc.

Case $n = 6$: Take an equilateral triangle abc, and choose $a' \in ao$ such that $\angle ba'c < \pi/2$. Let $d \in \mathbb{R}^3 \setminus \overline{abc}$ be close to a', such that a' is its orthogonal projection on $\overline{abc}$. Then 4 discs are held at abc and 2 discs at bcd (see the proof of Lemma 2.3).

Case $n \in \{7, 9, 11, 13\}$: By Lemma 2.3, in order to obtain exactly 7 discs held by $\text{cage}(abcd)$, there are 3 possibilities for the number of discs held at each face: $2, 2, 2, 1$, or $4, 1, 1, 1$, or $4, 2, 1, 0$.

To obtain exactly 9 discs held by $\mathrm{cage}(abcd)$, there are 2 possibilities for the number of discs held at each face: $4, 2, 2, 1$, or $4, 4, 1, 0$.

To obtain exactly 11 discs held, there is just one possibility for the number of discs held at each face: $4, 4, 2, 1$.

Similarly for 13 discs held: $4, 4, 4, 1$.

In each of these 7 scenarios, there exists a face at which exactly one disc is held and at most one face at which no disc is held. We prove this to be impossible to realize.

Suppose it is realized. Then at most one of the 12 angles (of the 4 triangles), say $\widehat{acd}$, is non-acute. Consequently, the triangles abc, bcd and abd are acute, and all angles at a, b, d are acute, too.

By Lemma 2.6, $\angle oad < \pi/2$ and $\angle obd < \pi/2$; thus, at least two discs are held at abc. Similarly, at least two discs are held at bcd. At abd exactly four discs are held, as all cage angles at a, b, d are acute.

Now, if acd is not acute, no disc is held there. If acd is acute, then each face behaves like abd, i.e. 4 discs are held at each face. Hence, at no face exactly one disc is held.

Case $n = 8$: Take two coplanar equilateral triangles abc and bcd', and then slightly rotate the latter about $\overline{bc}$ to reach a new position bcd. Then the angles oad, obd and ocd are acute, whence $\mathrm{cage}(abcd)$ holds 4 discs at abc. By symmetry, it also holds 4 discs at bcd. As abd and acd are obtuse triangles, there are no further discs held by $\mathrm{cage}(abcd)$.

Case $n = 10$: Let the triangle abc be equilateral, and d' be close to a, such that $\|a - c\| = \|c - d'\|$ and $ac \cap od' \neq \emptyset$. Let o' be the centre of $C(bcd')$, and o'' the centre of $C(acd')$. (See Figure 8.)

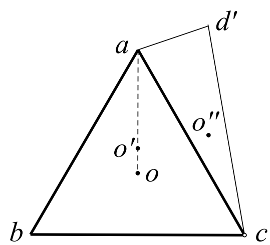

Figure 8: Case $n = 10$.

Clearly,

$$\angle d'ao > \pi/2, \qquad \angle d'bo < \pi/2, \qquad \angle d'co < \pi/2.$$

Also,

$$\angle ad'o' < \pi/2, \qquad \angle abo' < \pi/2, \qquad \angle aco' < \pi/2$$

and

$$\angle bao'' < \pi/2, \qquad \angle bco'' < \pi/2, \qquad \angle bd'o'' < \pi/2.$$

By rotating a little d' about $\overline{ac}$, the above angles don't change much, and the inequalities remain valid. Let d be the new position of d'. So, there are 4 discs held at bcd, 4 at acd, just 2 at abc, and none at abd, as $\angle bad > \pi/2$.

Case $n = 12$: Let $\text{cage}(abcd)$ have three acute triangular faces and a right triangle abc as fourth face, with $\angle bac = \pi/2$. By Lemma 2.6,

$$\angle bao < \pi/2, \qquad \angle bco < \pi/2, \qquad \angle bdo < \pi/2,$$

whence 4 discs are held at cda. Analogously, $abcd$ holds 4 discs at each of the faces dab, bcd. Of course, no disc is held at abc.

Case $n = 14$: Suppose $\text{cage}(abcd)$ holds $4, 4, 4, 2$ discs at the four faces, which is the only possibility of reaching the total number of 14. Then all triangles are acute. By Lemma 2.6, $\angle oad < \pi/2$, $\angle obd < \pi/2$, $\angle ocd < \pi/2$, whence there are 4 discs held at abc. This applies to every face. Hence, at no face the number of discs held is 2.

Case $n = 15$: Impossible as sum of four integers from $\{0, 1, 2, 4\}$.

Case $n = 16$: The regular tetrahedron realizes this, see Theorem 2.4. □

If we briefly say that the cage G *holds* n *unit discs*, this means that G holds n discs, i.e. the maximal number of pairwise disjoint end-components is n, and $\sigma(\mathcal{E})$ does not depend on the chosen end-component $\mathcal{E}$.

One may ask the question: how many unit discs can a tetrahedral cage hold? We shall not deepen this question here, only make some remarks.

Trivially, by Theorem 2.7, there is a cage holding 1 unit disc.

In the proof for $n = 2$, both discs held by the cage were at the same face, so they had the same size. Similarly, Theorem 2.4 shows that the regular tetrahedral cage holds 16 unit discs.

The proof for $n = 3$ provides two discs of same size, and a third disc of a possibly different size. A more concrete construction is needed. We do this here, using the notation from the proof of Theorem 2.7, case $n = 3$.

The acute triangle abc will be taken such that $\angle acb = \frac{\pi}{4}$, which implies $\angle oab = \angle oba = \pi/4$. Now, the two circles $C(abc)$ and $C(abd')$ are congruent.

Let Θ be the torus obtained by rotating $C(abd')$ about $\overline{ab}$. By choosing $d \in \Theta \setminus (\Pi_{ba}^+ \cup \Pi_{ao}^+ \cup \overline{abc})$, still close to d', we get $C(abd)$ and $C(abc)$ congruent.

For the regular tetrahedral cage T of unit side-length, any disc held has radius at least $1/2$.

Altogether T holds 16 discs, by Theorem 2.7. In fact, for any $r \in [3\sqrt{2}/8, \sqrt{3}/3]$, $\mathcal{D}_r(T)$ has 16 components. What happens for smaller r?

Theorem 2.8. *Let T be the regular tetrahedral cage of unit side-length. For any $r \in [1/2, 3\sqrt{2}/8]$, $\mathcal{D}_r(T)$ has 4 components.*

Proof. A disc D in $\mathcal{D}_r(T)$ above abc can be rotated about an axis parallel and close to ab without meeting cd until it reaches a position close to abd, above ad and bd, but below ab (seeing now abd as horizontal, with T above it).

The rotation of the disc D can also be performed about an axis close to bc, or bd, and so we obtain a third and a fourth disc in the same component as D. This means that a group of 4 discs held by T among the 16 analogous to those mentioned in Theorem 2.4 belong to the same component of $\mathcal{D}_r(T)$. As we have 4 such groups, the conclusion of the theorem follows. □

Theorem 2.8 provides illuminating examples of components which are not end-components of $\mathcal{D}(T)$.

3 Pentahedral cages

The convex pentahedra are of two combinatorial types: the pyramid over a quadrilateral and the triangular prism. We do not aim at finding all possible numbers of discs which can be held by pentahedral cages, as we did for tetrahedra. We restrict the otherwise lengthy analysis to the most interesting problem about the maximal number of discs which can be held.

We start with the question: How many discs can a pentahedral cage hold at a face? We know the answer if the face is triangular by adapting the analysis from the tetrahedral case to this new situation: 0, 1, 2, or 4. This is seen like in Lemma 2.3, with the difference that the case of 0 discs may now occur, even if the triangle is acute. For our pentahedra we need the answer for quadrilateral faces, too.

Let $Q = abcd$ be a quadrilateral (bottom) face of a polytope P, and assume that each vertex of Q has degree 3 in P. (This is so in pentahedra.) Each diagonal of Q divides it into two triangles. These four triangles cannot all be acute, at least one must be non-acute. Let a' be the vertex of P, neighbour of a, different from b, d. Also, consider the analogous vertices b', c', d'. (Some of these vertices may coincide.)

An exhaustive investigation would have to consider several cases. But this is not our intention. As an example, we treat the case when a, b, c, d are cocyclic. Assume abc and abd are acute. Obviously, both $d \in D(abc)$, $c \in D(abd)$. Moreover, the inequalities $\angle dao_{abc} < \pi/2$, $\angle dbo_{abc} < \pi/2$ and $\angle dco_{abc} < \pi/2$ are satisfied. Thus, if all inequalities $\angle a'ao_{abc} < \pi/2$, $\angle b'bo_{abc} < \pi/2$, $\angle c'co_{abc} < \pi/2$ are valid, then a disc can be held over ab, bc, cd, and da, or over 3 of them and under the fourth, or over ab, bc and under cd, da, or over da, ab and under bc, cd, which gives 7 possibilities in total.

In case a, b, c, d are not cocyclic, more discs can be held at Q.

Lemma 3.1. *If the triangles abc, abd, bcd and the angles*

$$\widehat{ado_{bcd}}, \quad \widehat{cdo_{abd}}, \quad \widehat{a'ao_{abd}}, \quad \widehat{a'ao_{abc}}, \quad \widehat{b'bo_{abd}},$$
$$\widehat{b'bo_{bcd}}, \quad \widehat{c'co_{abc}}, \quad \widehat{c'co_{bcd}}, \quad \widehat{d'do_{abd}}, \quad \widehat{d'do_{bcd}},$$

are all acute, then 13 *discs are held at* $Q = abcd$.

Proof. First of all, by Lemma 2.6, $\angle b'bo_{abd} < \pi/2$ and $\angle b'bo_{bcd} < \pi/2$ imply $\angle b'bo_{abc} < \pi/2$.

Now, considering abd, a disc is held above all four edges, another one is held under ab and above the other three, yet another disc under ad and above all others, a fourth disc under bc and cd and above ab and da, a fifth under bc and above all others, and a sixth under cd and above the remaining edges.

Analogously, considering bcd, we find other six discs held.

Moreover, considering abc, one more disc is held, namely under cd and da and above ab and bc. □

Lemma 3.2. *There are maximally* 13 *discs held at abcd.*

Proof. It is quickly seen that, in all other cases concerning the angles mentioned at Lemma 3.1, the number of discs held is smaller than 13. □

In conclusion, at any quadrilateral face of a polytopal cage, at most 13 discs can be held, and this only if several angle inequalities are satisfied. If the polytope is a prism, the following holds.

Lemma 3.3. *Let $abca^*b^*c^*$ be a prism. If abb^*a^* has three acute angles close to $\pi/2$, and if, moreover, the angles $\widehat{cao_{aba^*}}$, $\widehat{cbo_{aba^*}}$, $c^*\widehat{a^*o_{aba^*}}$, regarding aba^*, are acute, and all analogous angles regarding bb^*a^*, aa^*b^*, abb^*, are also acute, then the prism holds 13 discs at abb^*a^*.*

Proof. Indeed, all angle conditions required in Lemma 3.1 are satisfied. The condition that the angles of abb^*a^* be close to $\pi/2$ is needed since it implies that abb^*a^* is close to a rectangle, from which $\angle b^*bo_{abb^*} < \pi/2$ and all other analogous inequalities follow. □

A pentahedral cage, in contrast to a tetrahedral one, can hold discs not only at faces.

Consider the pyramid $P = abcde$ with apex e and a quadrilateral face $abcd$. If the triangle ace is acute, the capability of $\mathrm{cage}(P)$ to hold a disc there depends on the angles $\angle bao_{ace}$, $\angle dao_{ace}$, $\angle bco_{ace}$, $\angle dco_{ace}$, $\angle beo_{ace}$, $\angle deo_{ace}$. If all of them are smaller than $\pi/2$, then the pyramid holds 4 discs at ace, one on each side of $\overline{ace}$, and two crossing $\overline{ace}$. Here, holding a disc at ace means, in analogy to holding a disc at a face, that a certain limit disc lies in $\overline{ace}$ (and is, in fact, circumscribed to ace). If ace is not acute, $\mathrm{cage}(P)$ cannot hold any disc there.

Adding the at most 4 discs held at bde, we obtain a maximum of 8 held discs, which traverse the pyramid.

A (combinatorial) prism $abca^*b^*c^*$ with faces abc, $a^*b^*c^*$, abb^*a^*, bcc^*b^*, caa^*c^*, may also hold discs at abc^* and at the other 5 analogous triangles. In order to hold any disc at abc^*, we must have $\angle cc^*o_{abc^*} < \pi/2$ and at least one of the inequalities $\angle a^*co_{ab^*} < \pi/2$, $\angle b^*co_{abc^*} < \pi/2$. Now, if this happens, we have a held disc "separating" ab from c if $\angle cao_{abc^*} < \pi/2$ and $\angle cbo_{abc^*} < \pi/2$, and a similar held disc "separating" ab from a^*b^* if $\angle a^*ao_{abc^*} < \pi/2$ and $\angle b^*bo_{abc^*} < \pi/2$. This amounts to a maximum of 2 discs held at abc^*.

In particular, the following holds.

Lemma 3.4. *If the prism P is close to a long right regular one, then* $\mathrm{cage}(P)$ *holds 2 discs at abc^* and at each of the other 5 analogous places.*

Moreover, Proposition 1.4 warns that there might exist limit discs not coplanar with any three vertices of the cage. Consequently, let us say that a cage G *holds n standard discs* if all corresponding end-components have limit discs coplanar with at least three vertices of $\mathrm{conv}\, G$.

Thus, if P is a pyramid, the total number of standard discs held by $\mathrm{cage}(P)$ would become at most 37, and if it is a prism at most 59. Can these numbers be realized? Is it 59 the true maximum for all pentahedra?

But, first, let us solve Problem 1.2.

Theorem 3.5. *There exists a pentahedral cage holding exactly 7 discs.*

Proof. Let $Q = abcd$ be a rectangle, of centre o, such that the triangles abo and cdo be equilateral. Let m be the centre of abo. Close to m choose a point $e \notin \overline{abc}$, whose orthogonal projection on $\overline{abc}$ is m. Put $o' = o_{cde}$. See Figure 9.

We show that, for the pyramid $P = eabcd$, $\mathrm{cage}(P)$ holds 7 discs.

Indeed, notice that the triangles abm, bcm, and dam are obtuse. So, besides the rectangle Q, P has four triangular faces, of which only cde is acute. Since $\angle aeo' > \pi/2$, $\angle beo' > \pi/2$, $\angle bco' < \pi/2$, $\angle ado' < \pi/2$, P holds 2 discs at cde.

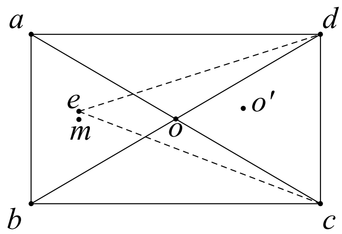

Figure 9: Cage holding 7 discs.

For the face Q, the relevant angles satisfy $\angle eao = \angle ebo < \pi/2$ and $\angle eco = \angle edo < \pi/2$. Hence, above all edges of Q our cage holds 1 disc, while above any three of its edges and under the fourth it also holds a disc. Above any two consecutive edges of Q, but under the remaining two, $\mathrm{cage}(P)$ holds no disc. Hence, it holds 5 discs at Q.

The two triangles eac and ebd traversing P are both obtuse, so no disc can be held at any of them. Clearly, there are no non-standard discs held.

In conclusion, altogether $\mathrm{cage}(P)$ holds 7 discs, as stated. □

We now establish the exact minimum for the number of discs and the exact maximum for the number of standard discs that a pentahedral cage can hold.

Three parallel lines in $\mathbb{R}^3$ determine an unbounded closed prism P having 3 strips as sides. If a triangle $\Delta \subset \mathbb{R}^3$ has its vertices on the sides of P, we say that P is *associated with* Δ.

We shall make use of the following simple, but powerful, result.

Proposition 3.6 (Chevallier, Fruchard [5]). *For any (bounded) combinatorial prism with triangular faces Δ and Δ', it is impossible that Δ lies in the interior of a prism associated with Δ', and Δ' lies in the interior of a prism associated with Δ.*

For the reader's convenience, we give here a short proof.

Proof. Assume that $\Delta = abc$ lies in the interior of a prism P associated with $\Delta' = a'b'c'$. As $\Delta \cap \Delta' = \emptyset$, the triangle Δ entirely lies in one component P^+ of $P \setminus \overline{a'b'c'}$. Thus, $\overline{aa'}$, $\overline{bb'}$, $\overline{cc'}$ meet in some point $z \in P^+$. This determines the order z, a, a' on $\overline{aa'}$. Analogously, the assumption that Δ' lies in the interior of a prism associated with Δ implies the order z, a', a on $\overline{aa'}$. But both orders cannot coexist. □

Lemma 3.7. *For no prism P, $\mathrm{cage}(P)$ can hold more than 6 discs at its triangular faces together.*

Proof. Take the prism $P = abca^*b^*c^*$. We use Lemma 2.3 and its proof. We have $\angle a^*ao_{abc} < \pi/2$ if and only if $a^* \notin H^+_{ao_{abc}}$, Hence, $\widehat{a^*ao_{abc}}$, $\widehat{b^*bo_{abc}}$, $\widehat{c^*co_{abc}}$ are all acute if and only if a^* belongs to the complement of $H^+_{ao_{abc}} \cup H^+_{bo_{abc}} \cup H^+_{co_{abc}}$, which is the interior of a certain prism associated with abc. In order for $\mathrm{cage}(abca^*b^*c^*)$ to hold 4 discs at each of its two triangular faces, all vertices of each of them must lie in the interior of a prism associated with the other. But this is forbidden by Proposition 3.6. So, by Lemma 2.3 (adapted to our needs), $\mathrm{cage}(P)$ cannot hold more than 6 discs at its triangular faces together. □

Theorem 3.8. *A pentahedral cage can hold at least* 0 *and at most* 57 *standard discs. Both bounds are attained.*

Proof. To prove that a pentahedral cage may hold no standard disc, take a trapezoid having all four triangles determined by their diagonals obtuse. A prism with such trapezoids as quadrilateral faces and with two obtuse triangles as remaining faces holds no disc, see Figure 10.

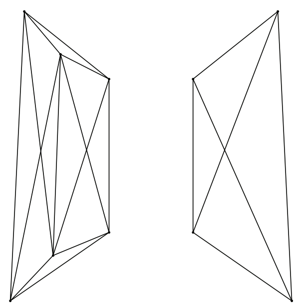

Figure 10: Cage holding no discs.

We now build a prism the cage of which holds 57 standard discs. Consider a long right regular prism $abca^*b^*c^*$ (with aa^*, bb^*, cc^* parallel).

Choose $a_1 \in aa^*$ close to a and $c_1 \in cc^*$ close to c, satisfying

$$2\|a - a_1\| < \|c - c_1\|.$$

Choose a_1^* close to a^*, b_1^* close to b^* and c_1^* close to c^*, such that $a^* \in a_1^* o_{a^*b^*c^*}$, $b^* \in b_1^* o_{a^*b^*c^*}$, $c^* \in c_1^* o_{a^*b^*c^*}$, and

$$\|a^* - a_1^*\| = \|b^* - b_1^*\| = \|c^* - c_1^*\| = \varepsilon.$$

See Figure 11. Also, put $\{a'\} = aa_1^* \cap \overline{a_1bc_1}$ and $\{c'\} = cc_1^* \cap \overline{a_1bc_1}$.

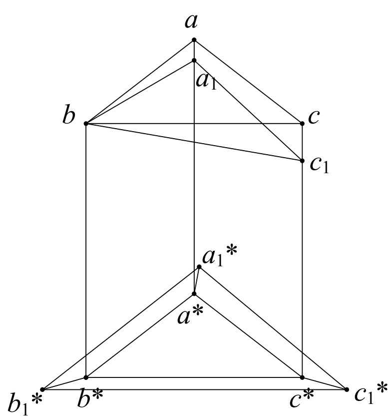

Figure 11: Cage holding 57 discs.

If ε is small enough, then the three quadrilateral faces $a'bb_1^*a_1^*$, $bc'c_1^*b_1^*$, $c'a'a_1^*c_1^*$, have obtuse angles at a', c', c', respectively, and acute angles at all other vertices.

All angles analogous to $\widehat{abo_{bb^*c^*}}$ are acute, so they remain acute after the small changes done to $abca^*b^*c^*$. Thus, by Lemma 3.3, there are 13 discs held at each quadrilateral face.

Passing now to the two triangular faces, we immediately see that all angles $a'\widehat{a_1^*o_{a_1^*b_1^*c_1^*}}$, $b\widehat{b_1^*o_{a_1^*b_1^*c_1^*}}$, $c'\widehat{c_1^*o_{a_1^*b_1^*c_1^*}}$, are acute.

Concerning $a'bc'$, $\angle b^*ba_1 < \pi/2$ and $\angle b^*bc_1 < \pi/2$ imply $\angle b^*bo_{a_1bc_1} < \pi/2$. Therefore, the next moves being gentle enough, $\angle b_1^*bo_{a'bc'} < \pi/2$ too.

The inequality $2\|a - a_1\| < \|c - c_1\|$ yields $\angle a^*a_1o_{a_1bc_1} < \pi/2$. Again, this can be preserved, and $\angle a_1^*a'o_{a'bc'} < \pi/2$. Now, adapting part of the proof of Lemma 2.3, we see that at least two discs are held at $a'bc'$. Hence, by Lemma 2.3 (see its proof) and by Lemma 3.7, our cage holds 13 discs at each of its quadrilateral faces, 4 discs at $a_1^*b_1^*c_1^*$, and 2 discs at $a'bc'$.

Concerning the discs traversing the prism, the maximum number (of 12) is reached, by Lemma 3.4.

Thus, our cage holds 57 standard discs. By Lemmas 3.2 and 3.7, it cannot hold more than these 57. The proof is finished. □

Theorem 3.8 does not prove that 57 is the maximal number of discs that a pentahedral cage can hold. We miss an analogue of Proposition 2.5 for pentahedra. Examples that the referee kindly provided suggest that such an analogue may not exist. Thus, we remain with the following.

Problem 3.9. What is the maximal number of discs that a pentahedral cage can hold?

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ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 271–275
https://doi.org/10.26493/1855-3974.1901.987
(Also available at http://amc-journal.eu)

Logarithms of a binomial series: A Stirling number approach

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Received 8 January 2019, accepted 21 April 2019, published online 15 October 2019

Abstract

The p-th power of the logarithm of the Catalan generating function is computed using the Stirling cycle numbers. Instead of Stirling numbers, one may write this generating function in terms of higher order harmonic numbers.

Keywords: Catalan numbers, logarithm, generating function, Stirling number.

Math. Subj. Class.: 05A15, 05A10

1 Introduction

Knuth [6, 7] proposed the exciting formula

$$(\log C(z))^2 = \sum_{n\geq 1} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n},$$

where

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\geq 0} \frac{1}{n+1} \binom{2n}{n} z^n \tag{1.1}$$

and

$$H_n = \sum_{1\leq k\leq n} \frac{1}{k}$$

with the generating function of Catalan numbers and harmonic numbers.

This formula was recently extended by Chu [1] to general exponents p. Chu's approach is based on the use of (exponential) Bell polynomials. Note that Knuth talked about the exponent 1 in his Christmas lecture from 2014 [5].

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We present here a very simple approach to this question using Stirling cycle numbers; recall [3] that they transform falling powers into ordinary powers viz.

$$x^{\underline{n}} = \sum_{0\le k\le n} {n \brack k} (-1)^{n-k} x^k.$$

For the readers' convenience it is mentioned that the numbers ${n \brack k}(-1)^{n-k}$ appear often in the older literature as $s(n,k)$ and are then denoted as Stirling numbers of the first kind.

2 The expansion of the p-th power

The substitution $z = \frac{u}{(1+u)^2}$ was presented in [2] and it is extremely useful when dealing with Catalan numbers and Catalan statistics. Using it in (1.1), we get $C(z) = 1 + u$, and, by the Lagrange inversion formula [8],

$$u^m = \sum_{n\ge m} \frac{m}{n} \binom{2n}{n-m} z^n$$

for $m \ge 1$. For $m = 0$ the formula is still true when taking a limit. We now consider the bivariate generating function

$$\begin{aligned} F(z,\alpha) &= \sum_{p\ge 0} \frac{\alpha^p}{p!} (\log C(z))^p = \exp(\alpha \log C(z)) \\ &= C^\alpha(z) = (1+u)^\alpha = \sum_{m\ge 0} \binom{\alpha}{m} u^m. \end{aligned}$$

But

$$\binom{\alpha}{m} = \frac{1}{m!}\alpha^{\underline{m}} = \frac{1}{m!} \sum_{0\le k\le m} (-1)^{m-k} {m \brack k} \alpha^k.$$

Therefore

$$F(z,\alpha) = \sum_{0\le k\le m\le n} \frac{1}{m!}(-1)^{m-k} {m \brack k} \alpha^k \frac{m}{n} \binom{2n}{n-m} z^n.$$

The desired formula follows from reading off coefficients of α^p:

$$(\log C(z))^p = p![\alpha^p]F(z,\alpha) = \sum_{p\le m\le n} \frac{p!}{m!}(-1)^{m-p} {m \brack p} \frac{m}{n} \binom{2n}{n-m} z^n. \tag{2.1}$$

3 Special cases

For $p = 1$ in equation (2.1), we get the instance of the Christmas lecture:

$$\log C(z) = [\alpha^1]F(z,\alpha) = \sum_{1\le m\le n} \frac{1}{m!}(-1)^{m-1} {m \brack 1} \frac{m}{n} \binom{2n}{n-m} z^n.$$

Since ${m \brack 1} = (m-1)!$, this leads to

$$\log C(z) = [\alpha^1]F(z,\alpha) = \frac{1}{2} \sum_{n\ge 1} \frac{1}{n} \binom{2n}{n} z^n.$$

Now we turn to the instance $p = 2$ from [6, 7]. (Note that $\left[{m \atop 2}\right] = (m-1)!H_{m-1}$.) Equation (2.1) leads to

$$\begin{aligned}
2[\alpha^2]F(z,\alpha) &= \sum_{2\le m\le n} \frac{2}{m!}(-1)^m \left[{m \atop 2}\right] \frac{m}{n}\binom{2n}{n-m} z^n \\
&= 2\sum_{2\le m\le n} H_{m-1}(-1)^m \frac{1}{n}\binom{2n}{n-m} z^n \\
&= 2\sum_{1\le j<m\le n} \frac{1}{j}(-1)^m \frac{1}{n}\binom{2n}{n-m} z^n \\
&= 2\sum_{1\le j<n} \frac{1}{j}(-1)^{j-1} \frac{1}{n}\binom{2n-1}{n-j-1} z^n.
\end{aligned}$$

In the last step we used the formula

$$\sum_{j<m\le n} (-1)^m \binom{2n}{n-m} = (-1)^{j-1}\binom{2n-1}{n-j-1},$$

which is a standard summation for binomial coefficients [3].

To obtain the form proposed by Knuth, we still need to prove that

$$\binom{2n}{n}(H_{2n-1} - H_n) = 2\sum_{1\le j<n} \frac{(-1)^{j-1}}{j}\binom{2n-1}{n-j-1}.$$

Modern computer algebra systems readily simplify the difference of these two sides to 0, as expected.

4 Connection with harmonic numbers — the general case

In [4], there is the general formula

$$\frac{1}{n!}\left[{n+1 \atop r+1}\right] = (-1)^r \sum_{\{r\}} \prod_{j=1}^{l} \frac{(-1)^{i_j}}{i_j!}\left(\frac{H_n^{(r_j)}}{r_j}\right)^{i_j}.$$

Here, the sum is over all partitions of r:

$$r = i_1 r_1 + \cdots + i_l r_l,$$

with parts $r_1 > \cdots > r_l \ge 1$ and positive integers $i_1, \ldots, i_l$. As an example, the partitions of $r = 4$ are 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$, written alternatively as $1\cdot 4$, $1\cdot 3 + 1\cdot 1$, $2\cdot 2$, $1\cdot 2 + 2\cdot 1$, $4\cdot 1$.

There appear higher order harmonic numbers as well:

$$H_n^{(i)} = \sum_{1\le k\le n} \frac{1}{k^i}.$$

Here are the first few instances:

$$\begin{aligned}
\frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= H_n, \\
\frac{1}{n!}\begin{bmatrix} n+1 \\ 3 \end{bmatrix} &= -\frac{1}{2}H_n^{(2)} + \frac{1}{2}H_n^2, \\
\frac{1}{n!}\begin{bmatrix} n+1 \\ 4 \end{bmatrix} &= \frac{1}{3}H_n^{(3)} - \frac{1}{2}H_n^{(2)}H_n + \frac{1}{6}H_n^3, \\
\frac{1}{n!}\begin{bmatrix} n+1 \\ 5 \end{bmatrix} &= -\frac{1}{4}H_n^{(4)} + \frac{1}{3}H_n^{(3)}H_n + \frac{1}{8}\big(H_n^{(2)}\big)^2 - \frac{1}{4}H_n^{(2)}H_n^2 + \frac{1}{24}H_n^4.
\end{aligned}$$

This allows to replace $\frac{1}{(m-1)!}\begin{bmatrix} m \\ p \end{bmatrix}$ in

$$(\log C(z))^p = \sum_{p\le m\le n} \frac{1}{(m-1)!}\begin{bmatrix} m \\ p \end{bmatrix}(-1)^{m-p}\frac{p!}{n}\binom{2n}{n-m}z^n$$

by an expression involving $H_{m-1}^{(1)}, \dots, H_{m-1}^{(p-1)}$.

5 Extension

If instead of $u = z(1+u)^2$ we work with $u = z(1+u)^\lambda$, then we deal with the generating function of extended (generalized) Catalan numbers

$$C_\lambda(z) = \sum_{n\ge 0}\binom{1+n\lambda}{n}\frac{z^n}{1+n\lambda}.$$

From [3], we infer that

$$u^m = \sum_{n\ge m}\binom{\lambda n+m}{n}\frac{m}{\lambda n+m}z^n.$$

So

$$\begin{aligned}
F(z,\alpha) &= \sum_{p\ge 0}\frac{\alpha^p}{p!}(\log C_\lambda(z))^p = \exp(\alpha\log C_\lambda(z)) = C_\lambda^\alpha(z) \\
&= (1+u)^\alpha = \sum_{m\ge 0}\binom{\alpha}{m}u^m \\
&= \sum_{0\le k\le m\le n}\frac{1}{m!}(-1)^{m-k}\begin{bmatrix} m \\ k \end{bmatrix}\alpha^k\binom{\lambda n+m}{n}\frac{m}{\lambda n+m}z^n.
\end{aligned}$$

The desired formula follows from reading off coefficients of α^p:

$$(\log C_\lambda(z))^p = p![\alpha^p]F(z,\alpha) = \sum_{p\le m\le n}\frac{p!}{m!}(-1)^{m-p}\begin{bmatrix} m \\ p \end{bmatrix}\binom{\lambda n+m}{n}\frac{m}{\lambda n+m}z^n.$$

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 277–290
https://doi.org/10.26493/1855-3974.1782.127
(Also available at http://amc-journal.eu)

On identities of Watson type

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Received 24 August 2018, accepted 23 April 2019, published online 16 October 2019

Abstract

We prove several identities of the type $\alpha(n) = \sum_{k=0}^{\infty} \beta\left(\frac{n-k(k+1)/2}{2}\right)$. Here, the functions $\alpha(n)$ and $\beta(n)$ count partitions with certain restrictions or the number of parts in certain partitions. Since Watson proved the identity for $\alpha(n) = Q(n)$, the number of partitions of n into distinct parts, and $\beta(n) = p(n)$, Euler's partition function, we refer to these identities as Watson type identities. Our work is motivated by results of G. E. Andrews and the second author who recently discovered and proved new Euler type identities. We provide analytic proofs and explain how one could construct bijective proofs of our results.

Keywords: Partitions, combinatorial identities, bijective combinatorics.

Math. Subj. Class.: 05C15, 05A17, 11P81, 11P84

1 Introduction

Any positive integer n can be written as a sum of one or more positive integers, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k. \tag{1.1}$$

When the order of integers λ_i does not matter, this representation is known as an integer partition [1] and can be rewritten as

$$n = m_1 + 2m_2 + \cdots + nm_n,$$

where each positive integer i appears m_i times. If the order of integers λ_i is important, then the representation (1.1) is known as a composition. For

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k,$$

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we have a descending composition. In the literature, partitions are often defined as descending compositions and this is also the convention used in this paper. We refer to $\lambda_1, \lambda_2, \ldots, \lambda_k$ as the parts of λ and use the notation $\lambda \vdash n$ to denote a partition of n, i.e., a partition whose parts add up to n. We denote by $\ell(\lambda)$ the number of parts of λ, i.e.,

$$\ell(\lambda) = k \quad \text{or} \quad \ell(\lambda) = \sum_{j=1}^{n} m_j.$$

As usual, for a positive integer n, we denote by $p(n)$ the number of partitions of n and we set $p(0) = 1$.

In 1936, Watson [24] computed tables of the number of partitions of n into distinct parts $Q(n)$ and the number of partitions of n into distinct odd parts $Q_{odd}(n)$ up to $n = 400$. He notes that his "computations were considerably simplified by the use of certain formulae of elliptic functions in conjunction with the existing table of values of $p(n)$, the number of unrestricted partitions of n, up to $n = 200$ which was constructed by MacMahon and published by Hardy and Ramanujan" [13] in 1918. Watson [24, p. 551] stated two identities whose developments lead to

$$Q(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{2}\right) \tag{1.2}$$

and

$$Q_{odd}(n) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{4}\right), \tag{1.3}$$

where $p(x) = 0$ when x is not a nonnegative integer.

In 2016, the second author [17, Theorem 1] considered the identity (1.2) and obtained a method to compute the values of the partition function $p(n)$ that requires only the values of $p(k)$ with $k \leqslant n/2$, namely

$$p(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\infty} p(k) p\left(\frac{n - j(j+1)/2}{2} - k\right). \tag{1.4}$$

One year later, the identity (1.2) was used by the authors [6, Theorem 2.7] to prove the following parity result related to sums of partition numbers and squares in arithmetic progressions. For $n \geqslant 0$,

$$\sum_{16k+1 \text{ square}} p(n-k) \equiv 1 \pmod 2$$

if and only if $48n + 1$ is a square.

Recently, Fu and Tang [10] generalized Vandervelde's bijection [23] and gave a combinatorial proof of the identity (1.2). A combinatorial proof of (1.3) can be found in [25] where the author uses abacus displays which were first introduced in [14]. We remark that [25] also refers to [16, Proposition 5.2] for a combinatorial proof of (1.2).

In this paper, motivated by these results, we investigate other identities of Watson type (1.2). To begin, we consider a recent paper [2] in which Andrews solved a problem of Beck and provided the following result: For all $n \geqslant 1$,

$$a_1(n) = b_1(n) = c_1(n),$$

where:

- $a_1(n)$ is the number of partitions of n in which the set of even parts has only one element;
- $b_1(n)$ is the difference between the number of parts in all partitions of n into odd parts and the number of parts in all partitions of n into distinct parts;
- $c_1(n)$ is the number of partitions of n in which exactly one part is repeated.

Shortly after that, inspired by Andrews's proof of this result, the second author [19] discovered and proved analytically an analogue of the identity (1.2) involving the number of parts in partitions.

Theorem 1.1. *For $n \geqslant 0$,*

$$b_1(n) = \sum_{k=0}^{\infty} S\left(\frac{n-k(k+1)/2}{2}\right), \tag{1.5}$$

where $S(n)$ denotes the total number of parts in all partitions of n, with $S(x) = 0$ if x is not a positive integer.

We remark that combinatorial proofs of $a_1(n) = b_1(n)$ and $c_1(n) = b_1(n)$ are given in [4] and, as a result of a generalization, in [26]. A combinatorial proof of a generalization of $a_1(n) = c_1(n)$ was initially given in [11]. Thus, a purely combinatorial proof of Theorem 1.1 follows from the combinatorial proof of either of the next two theorems which we present in Section 3.

Theorem 1.2. *Let $\alpha_j(n)$ denote the number of partitions of n whose set of even parts consists of the single element $2j$ and let $S_j(n)$ be the number of parts equal to j in all partitions of n. Then, for $n \geqslant 0$, we have*

$$\alpha_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n-k(k+1)/2}{2}\right). \tag{1.6}$$

Theorem 1.3. *Let $\gamma_j(n)$ denote the number of partitions of n in which exactly one part is repeated and the repeated part is j. Then, for $n \geqslant 0$, we have*

$$\gamma_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n-k(k+1)/2}{2}\right). \tag{1.7}$$

Very recently, Andrews and the second author [3] proved that for all $n \geqslant 1$,

$$a_2(n) = (-1)^n b_2(n) = c_2(n),$$

where:

- $a_2(n)$ is the number of even parts in all partitions of n into distinct parts;
- $b_2(n)$ is the difference between the number of partitions of n into an odd number of parts in which the set of even parts has only one element and the number of partitions of n into an even number of parts in which the set of even parts has only one element;
- $c_2(n)$ is the difference between the number of partitions of n in which exactly one part is repeated and this part is odd and the number of partitions of n in which exactly one part is repeated and this part is even.

Combinatorial proofs of $a_2(n) = c_2(n)$ and $(-1)^n b_2(n) = c_2(n)$ are given by the authors in [5]. We obtain a new analogue of the identity (1.2) which we prove both analytically and combinatorially in Section 4.

Theorem 1.4. *For $n \geqslant 0$,*

$$a_2(n) = \sum_{k=0}^{\infty} S_{o-e}\left(\frac{n-k(k+1)/2}{2}\right), \tag{1.8}$$

where $S_{o-e}(n)$ denotes the difference between the number of odd parts and the number of even parts in all partitions of n, with $S_{o-e}(x) = 0$ if x is not a positive integer.

Let $S'(n)$ be the number of parts that appear at least once in a given partition of n, summed over all partitions of n, i.e., $S'(n)$ equals the number of different parts in all partitions of n. For example, $S'(5) = 12$ since the number of different parts in $(5), (4,1)$, $(3,2), (3,1,1), (2,2,1), (2,1,1,1)$ and $(1,1,1,1,1)$ is $1+2+2+2+2+2+1 = 12$. The following result in partition theory has been widely attributed to Richard Stanley, although it is a particular case of a more general result that had been established by Nathan Fine fifteen years earlier [12]: The number of parts equal to 1 in the partitions of n is equal to $S'(n)$. Recently, the second author and Schmidt [20] provided a new identity for the number of parts equal to 1 in the partitions of n involving a well-known object in multiplicative number theory: Euler's totient $\phi(n)$. We have the following analogue of the identity (1.2) which we prove both analytically and combinatorially in Section 5.

Theorem 1.5. *For $n \geqslant 0$,*

$$E(n) = \sum_{k=0}^{\infty} S'\left(\frac{n-k(k+1)/2}{2}\right),$$

where $E(n)$ counts the partitions of n with exactly one even part and $S'(x) = 0$ if x is not a positive integer.

Related to Theorem 1.5, we have the following result which we prove combinatorially in Section 6.

Theorem 1.6. *For $n \geqslant 0$,*

$$E(n) = \sum_{\lambda \in \mathcal{O}(n)} l_2(\lambda),$$

where $\mathcal{O}(n)$ is the set of all integer partitions of n into odd parts and

$$l_2(\lambda) = \sum_{\substack{k=1 \\ m_k > 0}}^{n} \lfloor \log_2(m_k) \rfloor.$$

Let $S_2'(n)$ be the number of parts equal to 2 in all partitions of n that do not contain 1 as a part. We have the following analogue of the identity (1.2) which we prove both analytically and combinatorially in Section 7.

Theorem 1.7. *For $n \geqslant 5$,*

$$Q_2(n-4) = \sum_{k=0}^{\infty} S_2'\left(\frac{n-k(k+1)/2}{2}\right),$$

where $Q_2(n)$ is the number of partitions of n into distinct parts, none being 2 and $S_2'(x) = 0$ if x is not a positive integer.

2 Review of a combinatorial proof of (1.2)

The combinatorial proof of (1.2) is key to the combinatorial proofs of all our statements. In [10], Fu and Tang give a beautiful bijective proof of (1.2). In this section we reformulate their bijection in a way that is much shorter and easier to convey.

Recall that Dyson [9] defined the rank of a partition λ by $r(\lambda) = \lambda_1 - \ell(\lambda)$. The BG-rank of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$, denoted by $r_{bg}(\lambda)$, is defined in [7] as the excess in the number of odd-indexed odd parts over the number of even-indexed odd parts of λ, i.e.,

$$r_{bg}(\lambda) = \sum_{j=1}^{\ell(\lambda)} (-1)^{j+1} \operatorname{par}(\lambda_j),$$

where $\operatorname{par}(m) = 1$ if m is odd and 0, otherwise.

Start with a partition λ with distinct parts and consider the shifted Young diagram of λ, i.e., the Young diagram in which row i is shifted i boxes to the right, $i = 1, 2, \dots, \ell(\lambda)$. Remove the first $\ell(\lambda)$ columns of the shifted diagram and denote the conjugate of the resulting partition by ν. We have $\ell(\nu) = r(\lambda)$. Suppose $r_{bg}(\lambda) = j \in \mathbb{Z}$. Recall [8] that the 2-core of a partition λ is the partition whose Young diagram is obtained from the Young diagram of λ by repeatedly removing removing pairs of adjacent squares. At each step, the resulting diagram must be a valid Young diagram. Then the 2-core of λ is the staircase partition of size $j(2j-1)$.

Let a equal the height of the 2-core. It is equal to $2j-1$ if $j > 0$ and to $-2j$ if $j \leq 0$. Let $b = \ell(\lambda) - a$. Define a partition μ via its Young diagram as follows.

(i) If $b = 0$, all parts of ν have even multiplicity. Then μ is the partition obtained from ν by removing half the parts of each size.

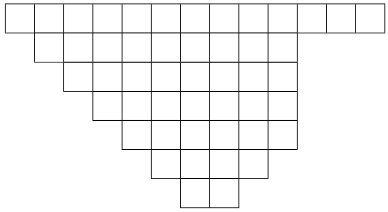
(ii) If $b \neq 0$, set

$$d_0 = \begin{cases} \frac{b}{2} & \text{if } b \text{ is even} \\ a + \lceil \frac{b}{2} \rceil & \text{if } b \text{ is odd} \end{cases}$$

and define recursively $d_i = \nu_i - d_{i-1}$ for $i = 1, 2, \dots, r(\lambda)$. To obtain the Young diagram of μ, begin with a rectangle of size $\lceil \frac{b}{2} \rceil \times (a + \lceil \frac{b}{2} \rceil)$ (i.e., $\lceil \frac{b}{2} \rceil$ rows and $a + \lceil \frac{b}{2} \rceil$ columns). If b is odd (respectively, even), for $i = 1, 2, \dots$ append columns of length d_{2i-1} (respectively, $d_{2(i-1)}$) to the right of the rectangle and rows of length d_{2i} (respectively, d_{2i-1}) below the rectangle. In [10], it is shown that this is a bijection from the set of partitions with distinct parts and BG-rank j to the set of partitions of $\frac{n-j(2j-1)}{2}$. Summing over all $j \in \mathbb{Z}$ gives (1.2).

Example 2.1. Let $\lambda = (13, 9, 8, 7, 6, 4, 2) \vdash 49$. We have $\lambda_1 = 13$ and $\ell(\lambda) = 7$. Then $r(\lambda) = 13 - 7 = 6$. Since the odd parts are the first, second and fourth parts, we have

$r_{bg}(\lambda) = -1$ and $a = 2$. Then, $b = \ell(\lambda) - 2 = 7 - 2 = 5$. The shifted Young diagram of λ is given below.



After removing the first $\ell(\lambda) = 7$ columns and conjugating, we obtain the partition $\nu = (7, 6, 5, 1, 1, 1)$. Since $b = 5$ is odd, $d_0 = a + \lceil \frac{b}{2} \rceil = 5$. We calculate recursively

$$\begin{aligned} d_1 &= \nu_1 - d_0 = 7 - 5 = 2, \\ d_2 &= \nu_2 - d_1 = 6 - 2 = 4, \\ d_3 &= \nu_3 - d_2 = 5 - 4 = 1, \\ d_4 &= \nu_4 - d_3 = 1 - 1 = 0, \\ d_5 &= \nu_5 - d_4 = 1 - 0 = 1, \\ d_6 &= \nu_6 - d_5 = 1 - 1 = 0. \end{aligned}$$

We start with a rectangle of size $\lceil \frac{b}{2} \rceil \times (a + \lceil \frac{b}{2} \rceil) = 3 \times 5$ and append columns of size d_1, d_3, and d_5 (i.e., columns of size $2, 1$, and 1) to the right of the rectangle and rows of size d_2, d_4, and d_6 (i.e., rows of size $4, 0$, and 0) below the rectangle to obtain the Young diagram of the partition $\mu = (8, 6, 5, 4) \vdash \frac{49-(-1)(-2-1)}{2} = 23$.

3 Combinatorial proofs of Theorems 1.2 and 1.3

In this section we use the combinatorial proof of (1.2) reviewed in the previous section to derive combinatorial proofs of Theorems 1.2 and 1.3. Then, summing over $j \geqslant 1$ and using the combinatorial proofs of $b_1(n) = a_1(n)$ and $b_1(n) = c_1(n)$, we obtain two slightly different combinatorial proofs of Theorem 1.1. For the combinatorial proofs of $b_1(n) = a_1(n)$ and $b_1(n) = c_1(n)$, which are fairly straight forward, we refer the reader to [4] or [26]. We do not repeat the argument here.

First, we introduce some notation. For any partition λ and any positive integer j we denote by m_j the multiplicity of j in λ. We denote by $p(n, j, t)$ the number of partitions of n such that $m_j \geqslant t$. Removing t parts equal to j from a partition of n with $m_j \geqslant t$ gives a partition of $n - jt$. Conversely, adding t parts equal to j to a partition of $n - jt$ gives a partition of n with $m_j \geqslant t$, Thus,

$$p(n, j, t) = p(n - jt).$$

As noted in the introduction, we denote by $S_j(n)$ the number of parts equal to j in all partitions of n. Then

$$S(n) = \sum_{j \geqslant 1} S_j(n).$$

Let $\mathcal{A}(n)$ be the set of partitions of n such that the set of even parts has exactly one element and let $\mathcal{C}(n)$ be the set of partitions of n in which exactly one part is repeated.

Proof of Theorem 1.2. Recall that $\alpha_j(n)$ denotes the number of partitions in $\mathcal{A}(n)$ in which the even part is $2j$. Let $\alpha_j^{(t)}(n)$ be the number of partitions in $\mathcal{A}(n)$ with $m_{2j} = t$. The above argument using removing/adding t parts equal to $2j$ shows that $\alpha_j^{(t)}(n) = Q(n-2jt)$. Therefore,

$$\alpha_j(n) = \sum_{t\geqslant 1} \alpha_j^{(t)}(n) = \sum_{t\geqslant 1} Q(n-2jt).$$

From (1.2), we have

$$Q(n-2jt) = \sum_{k=0}^{\infty} p\left(\frac{n-k(k+1)/2}{2} - jt\right).$$

For any $n \geqslant 0$, to determine $S_j(n)$ we count, in order, the first appearance of j in all partitions of n, then the second appearance of j in all partitions of n, and so on. The number of the t^{th} appearance of j in all partitions of n equals $p(n,j,t)$. Thus,

$$S_j(n) = \sum_{t\geqslant 1} p(n,j,t) = \sum_{t\geqslant 1} p(n-jt). \tag{3.1}$$

Then,

$$\alpha_j(n) = \sum_{t\geqslant 1} Q(n-2jt) = \sum_{t\geqslant 1}\sum_{k=0}^{\infty} p\left(\frac{n-k(k+1)/2}{2} - jt\right)$$

and thus

$$\alpha_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n-k(k+1)/2}{2}\right). \qquad \square$$

Summing (1.6) for $j \geqslant 1$, we obtain

$$a_1(n) = \sum_{k=0}^{\infty} S\left(\frac{n-k(k+1)/2}{2}\right).$$

Since there are purely combinatorial proofs of (1.2) and $a_1(n) = b_1(n)$, this gives a combinatorial proof of Theorem 1.1.

Proof of Theorem 1.3. Recall that $\gamma_j(n)$ denotes the number of partitions in $\mathcal{C}(n)$ in which the repeated part is j and, for $t \geqslant 1$ we denote by $\gamma_j^{(t)}(n)$ the number of partitions in $\mathcal{C}(n)$ such that $m_j = t$. Then, $\gamma_j^{(t)}(n)$ equals the number of partitions of $n-tj$ into distinct parts such that j does not appear as a part. To any partition of $n-(2t+1)j$ into distinct parts such that j does not appear as a part, add a part equal to j to obtain a partition of $n-2tj$ into distinct parts such that j appears as a part. Therefore,

$$\gamma_j^{(2t)}(n) + \gamma_j^{(2t+1)}(n) = Q(n-2tj)$$

and

$$\gamma_j(n) = \sum_{t\geqslant 1} Q(n-2tj).$$

Then, the proof of Theorem 1.2 gives a combinatorial argument for

$$\gamma_j(n) = \sum_{k=0}^{\infty} S_j\left(\frac{n-k(k+1)/2}{2}\right). \qquad \square$$

Summing (1.7) over $j \geqslant 1$, we have

$$c_1(n) = \sum_{k=0}^{\infty} S\left(\frac{n-k(k+1)/2}{2}\right).$$

Using the combinatorial proof for $c_1(n) = b_1(n)$ in [4], this gives a second combinatorial proof of Theorem 1.1.

4 Proofs of Theorem 1.4

4.1 An analytic proof

We consider the following factorization for a special case of Lambert series [18]:

$$\sum_{n=1}^{\infty} \frac{q^n}{1+q^n} = (q;q)_\infty \sum_{n=1}^{\infty} S_{o-e}(n) q^n.$$

According to [3], we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_2(n) q^n &= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \\
&= (-q;q)_\infty (q^2;q^2)_\infty \sum_{n=1}^{\infty} S_{o-e}(n) q^{2n} \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} S_{o-e}(n) q^{2n}.
\end{aligned}$$

Considering the theta identity [1, p. 23, Eq. (2.2.13)]

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

the proof follows by equating the coefficients of q^n in

$$\sum_{n=0}^{\infty} a_2(n) q^n = \left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)\left(\sum_{n=1}^{\infty} S_{o-e}(n) q^{2n}\right).$$

4.2 A combinatorial proof

Recall that [5] provides a combinatorial proof for $a_2(n) = c_2(n)$. Using the notation of Theorem 1.3, we have

$$c_2(n) = \sum_{j \geqslant 1} (\gamma_{2j-1}(n) - \gamma_{2j}(n))$$

and the proof of Theorem 1.3 provides a combinatorial argument for

$$\begin{aligned} c_2(n) &= \sum_{j\geqslant 1}\sum_{k=0}^{\infty}\left(S_{2j-1}\left(\frac{n-k(k+1)/2}{2}\right)-S_{2j}\left(\frac{n-k(k+1)/2}{2}\right)\right) \\ &= \sum_{k=0}^{\infty} S_{o-e}\left(\frac{n-k(k+1)/2}{2}\right). \end{aligned}$$

Using the combinatorial proof for $c_2(n) = a_2(n)$ in [5], this gives a combinatorial proof of Theorem 1.4.

5 Proofs of Theorem 1.5

5.1 An analytic proof

We remark that the sequence $E(n)$ is known as sequence A038348 [21] and can be found in the On-Line Encyclopedia of Integer Sequence [22]. The generating function function for $E(n)$ is given by

$$\sum_{n=0}^{\infty} E(n)q^n = \frac{q^2}{1-q^2}\cdot\frac{1}{(q;q^2)_\infty}.$$

On the other hand, according to [20], the generating function for $S'(n)$ is given by

$$\sum_{n=0}^{\infty} S'(n)q^n = \frac{q}{1-q}\cdot\frac{1}{(q;q)_\infty}.$$

Thus we can write

$$\begin{aligned} \sum_{n=0}^{\infty} E(n)q^n &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\cdot\frac{q^2}{1-q^2}\cdot\frac{1}{(q^2;q^2)_\infty} \\ &= \left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)\left(\sum_{n=0}^{\infty} S'(n)q^{2n}\right) \end{aligned}$$

and the proof of the theorem follows by equating the coefficients of q^n.

5.2 A combinatorial proof

We first follow [11] to prove the following Euler type identity.

Proposition 5.1. *Let $n \geqslant 1$. Then, the number of partitions with exactly one even part equals the number of partitions in which exactly one part is repeated with multiplicity 2 or 3.*

Before we prove the proposition, we introduce some notation. Recall that we denote by $\mathcal{O}(n)$ the set of partitions of n into odd parts. We denote by $\mathcal{D}(n)$ the set of partitions of n into distinct parts. In Section 3 we defined $\mathcal{C}(n)$ to be the set of partitions of n in which exactly one part is repeated. Let $\mathcal{T}(n)$ be the subset of $\mathcal{C}(n)$ consisting of partitions of n in which the repeated part has multiplicity 2 and let $\mathcal{T}'(n)$ be the subset of $\mathcal{C}(n)$ consisting of partitions of n in which the repeated part has multiplicity 3. Let $c_3(n) = |\mathcal{T}(n)|$ and $c_4(n) = |\mathcal{T}'(n)|$. Moreover, let $\mathcal{E}(n)$ be the set of partitions of n with exactly one even part.

Proof of Proposition 5.1. Consider the following transformation

$$\psi\colon \mathcal{E}(n) \to \mathcal{T}(n) \cup \mathcal{T}'(n).$$

Let $\mu \in \mathcal{E}(n)$ and suppose the even part is $2^k m$ with $k \geqslant 1$ and m odd. Denote by $\overline{\mu}$ the partition consisting of the single part $2^k m$ and by $\tilde{\mu}$ the partition consisting of the remaining parts of μ. Thus $\tilde{\mu}$ is a partition into odd parts. Let $\overline{\lambda} = (2^{k-1}m, 2^{k-1}m)$ and $\tilde{\lambda}$ be the partition with distinct parts obtained from $\tilde{\mu}$ after applying Glaisher's bijection (i.e., after merging equal parts repeatedly). Define $\psi(\mu) = \overline{\lambda} \cup \tilde{\lambda}$, the partition obtained by listing the parts of $\overline{\lambda}$ and $\tilde{\lambda}$ in non-increasing order. Then, in $\psi(\mu)$, the part $2^{k-1}m$ is the only repeated part and its multiplicity is 2 or 3. Thus, $\psi(\mu) \in \mathcal{T}(n) \cup \mathcal{T}'(n)$.

Conversely, if $\lambda \in \mathcal{T}(n) \cup \mathcal{T}'(n)$ suppose the repeated part is t. Then the multiplicity of t in λ is 2 or 3. Let $\overline{\lambda} = (t, t)$ and $\tilde{\lambda}$ be the partition consisting of the remaining parts of λ (one of which could be t). Let $\overline{\mu} = (2t)$, a partition consisting of a single even part, and $\tilde{\mu}$ be the partition obtained from $\tilde{\lambda}$ after applying the inverse of Glaisher's bijection (i.e., split even parts repeatedly until all parts are odd). Then, $\psi^{-1}(\lambda) = \overline{\mu} \cup \tilde{\mu}$ is a partition in $\mathcal{E}(n)$.

Thus, ψ is a bijection and $E(n) = c_3(n) + c_4(n)$. □

Next we complete the proof of Theorem 1.5.

Combinatorial Proof of Theorem 1.5. Let $d_j(n)$ denote the number of partitions in $\mathcal{T}(n) \cup \mathcal{T}'(n)$ with $m_j > 1$. Then $m_j = 2$ or 3. We have $c_3(n) + c_4(n) = \sum_{j \geqslant 1} d_j(n)$.

From the proof of Theorem 1.3, we have

$$d_j(n) = Q(n - 2j) = \sum_{k=0}^{\infty} p\left(\frac{n - k(k+1)/2}{2} - j\right).$$

Recall that $p\left(\frac{n-k(k+1)/2)}{2} - j\right)$ counts the number of first appearances of j in all partition of $\frac{n-k(k+1)/2}{2}$. Since $E(n) = c_3(n) + c_4(n)$, summing over $j \geqslant 1$, gives a combinatorial proof of the theorem when $S'(n)$ equals the number of different parts in all partitions of n.

On the other hand, from (3.1), we have that the number of parts equal to 1 in all partitions of n is $S_1(n) = \sum_{t \geqslant 1} p(n - t)$. This gives the combinatorial proof of the theorem when $S'(n)$ is viewed as the number of parts equal to 1 in all partitions of n. □

6 Combinatorial proof of Theorem 1.6

Let $b_3(n)$ be the difference between the total number of parts in the partitions of n into distinct parts and the total number of *different* parts in the partitions of n into odd parts. Thus, $b_3(n)$ is the difference between the number of parts in all partitions in $\mathcal{D}(n)$ and the number of different parts in all partitions in $\mathcal{O}(n)$ (i.e., parts counted without multiplicity).

Definition 6.1. Given a partition $\lambda \in \mathcal{O}(n)$, suppose the multiplicity of i in λ is m_i. If i appears in λ, we define the *binary order of magnitude* of the multiplicity of i in λ, denoted $\mathrm{bomm}_\lambda(i)$, to be the number of digits in the binary representation of m_i.

Note that, if $m_i > 0$, then $\mathrm{bomm}_\lambda(i) = \lfloor \log_2(m_i) \rfloor + 1$.

Example 6.2. If $\lambda = (5, 3, 3, 3, 3, 3, 1) \vdash 21$, we have $m_3(\lambda) = 5$. Since the binary representation of 5 is 101, we have $\mathrm{bomm}_\lambda(3) = 3$.

Let $b_4(n)$ denote the difference between the number of parts in all partitions in $\mathcal{O}(n)$, each counted as many times as its bomm, and the number of parts in all partitions in $\mathcal{D}(n)$. Since the number of parts in all partitions in $\mathcal{D}(n)$ equals the number of 1 in all binary representations of all multiplicities in all partitions of $\mathcal{O}(n)$, it follows that $b_4(n)$ equals the number of 0 in all binary representations of all multiplicities in all partitions of $\mathcal{O}(n)$.

Example 6.3. Let $n = 7$. We have $\mathcal{D}(7) = \{(7), (6,1), (5,2), (4,3), (4,2,1)\}$ and the number of parts in $\mathcal{D}(7)$ equals 10. Denote by $z_i(\lambda)$ the number of 0 in the binary representation of $m_i(\lambda)$. In Table 1 we list the partitions in $\mathcal{O}(n)$ with the relevant data (omitting the subscript λ).

Table 1: Partitions in $\mathcal{O}(7)$ and their multiplicity statistics.

λ	$m_i(\lambda)$ in binary	$\mathrm{bomm}_\lambda(i)$	$z_i(\lambda)$
(7)	$m_7 = 1$	$\mathrm{bomm}(7) = 1$	$z_7 = 0$
$(5,1,1)$	$m_5 = 1$, $m_1 = 10$	$\mathrm{bomm}(5) = 1$, $\mathrm{bomm}(1) = 2$	$z_5 = 0$, $z_1 = 1$
$(3,3,1)$	$m_3 = 10$, $m_1 = 1$	$\mathrm{bomm}(3) = 2$, $\mathrm{bomm}(1) = 1$	$z_3 = 1$, $z_1 = 0$
$(3,1,1,1,1)$	$m_3 = 1$, $m_1 = 100$	$\mathrm{bomm}(3) = 1$, $\mathrm{bomm}(1) = 3$	$z_3 = 0$, $z_1 = 2$
$(1,1,1,1,1,1,1)$	$m_1 = 111$	$\mathrm{bomm}(1) = 3$	$z_1 = 0$

Thus $b_4(7) = 1+1+2+2+1+1+3+3-10 = 4$, which equals the sum of z in the right column of the table above.

As shown in [4] combinatorially, we have $c_3(n) = b_3(n)$ and $c_4(n) = b_4(n)$. Together with the combinatorial proof of Theorem 1.5, this gives a combinatorial argument for the identity

$$b_3(n) + b_4(n) = \sum_{k=0}^{\infty} S'\left(\frac{n - k(k+1)/2}{2}\right), \quad \forall n \geqslant 0. \tag{6.1}$$

It follows directly from the definition of $b_3(n)$ and $b_4(n)$ that $b_3(n) + b_4(n)$ equals the number of parts in all partitions in $\mathcal{O}(n)$, where each part i is counted with multiplicity $\mathrm{bomm}_\lambda(i) - 1 = \lfloor \log_2(m_i) \rfloor$ in each partition λ in which it appears.

Example 6.4. The total number of distinct parts in all partitions in $\mathcal{O}(7)$ equals 8. Then $b_3(7) = 10-8 = 2$ and $b_3(7)+b_4(7) = 2+4 = 6$ which equals $0+0+1+1+0+0+2+2$, the number of parts in all partitions in $\mathcal{O}(7)$, where each part i is counted with multiplicity $\mathrm{bomm}_\lambda(i) - 1$ in each partition λ in which it appears.

Therefore, we have a combinatorial proof of Theorem 1.6.

7 Proofs of Theorem 1.7

7.1 An analytic proof

The sequence $Q_2(n)$ is known as sequence A015744 [15] and can be found in the On-Line Encyclopedia of Integer Sequences [22]. Since $(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$, the generating function for $Q_2(n)$ can be written as

$$\sum_{n=0}^{\infty} Q_2(n) q^n = \frac{1}{1+q^2} \cdot \frac{1}{(q;q^2)_\infty}.$$

On the other hand, according to [20], the generating function for $S_2'(n)$ is given by

$$\sum_{n=0}^{\infty} S_2'(n)q^n = \frac{q^2}{1-q^2} \cdot \frac{1}{(q^2;q)_\infty}.$$

We can write

$$\begin{aligned}\sum_{n=0}^{\infty} Q_2(n-4)q^n &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \cdot \frac{q^4}{1+q^2} \cdot \frac{1}{(q^2;q^2)_\infty} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \cdot \frac{q^4}{1-q^4} \cdot \frac{1}{(q^4;q^2)_\infty} \\ &= \left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)\left(\sum_{n=0}^{\infty} S_2'(n)q^{2n}\right)\end{aligned}$$

and the proof follows by equating the coefficients of q^n.

7.2 A combinatorial proof

Let $Q_2'(n)$ denote the number of partitions of n into distinct parts containing 2 as a part. If $\lambda \in \mathcal{D}(n)$ has 2 as a part, removing 2 we obtain a partition counted by $Q_2(n-2)$. Conversely, if $\mu \in \mathcal{D}(n-2)$ does not have 2 as a part, adding a part equal to 2 we obtain a partition counted by $Q_2'(n)$. Thus, $Q_2'(n) = Q_2(n-2)$. Since $Q(n) = Q_2(n) + Q_2'(n)$, it follows that $Q_2(n) = Q(n) - Q_2(n-2)$. Recursively, we have

$$Q_2(n) = \sum_{j\geqslant 0} (-1)^j Q(n-2j). \tag{7.1}$$

Here, $Q(x) = 0$ if x is negative. We rewrite (7.1) as

$$Q_2(n-4) = \sum_{t\geqslant 1} Q(n-4t) - \sum_{t\geqslant 1} Q(n-2-4t).$$

From the proof of Theorem 1.2, we have

$$\begin{aligned}Q_2(n-4) &= \alpha_2(n) - \alpha_2(n-2) \\ &= \sum_{k=0}^{\infty} S_2\left(\frac{n-k(k+1)/2}{2}\right) - \sum_{k=0}^{\infty} S_2\left(\frac{n-k(k+1)/2}{2} - 1\right).\end{aligned}$$

If $\lambda \vdash m-1$, adding a part equal to 1, we obtain a partition μ of m containing 1. The number of parts equal to 2 is the same in λ and in μ. Therefore, $S_2(m) - S_2(m-1) = S_2'(m)$. This completes the proof of the theorem.

8 Concluding remarks

We presented several Watson type identities of the same shape as identity (1.2)

$$Q(n) = \sum_{k=0}^{\infty} p\left(\frac{n-k(k+1)/2}{2}\right)$$

and provided both analytic and combinatorial proofs for our results. Since the identity above has the companion identity (1.3) given by

$$Q_{odd}(n) = \sum_{k=0}^{\infty} p\left(\frac{n-k(k+1)/2}{4}\right)$$

it would be interesting to find Watson type identities of this shape. Because there is a combinatorial proof for identity (1.3), there is hope that such new identities can be proved combinatorially.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 291–310
https://doi.org/10.26493/1855-3974.1953.c53
(Also available at http://amc-journal.eu)

String C-group representations of alternating groups*

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Received 15 March 2019, accepted 15 July 2019, published online 22 October 2019

Abstract

We prove that for any integer $n \geq 12$, and for every r in the interval $[3, \ldots, \lfloor \frac{n-1}{2} \rfloor]$, the group A_n has a string C-group representation of rank r, and hence that the only alternating group whose set of such ranks is not an interval is A_{11}.

Keywords: Abstract regular polytopes, Coxeter groups, alternating groups, string C-groups.

Math. Subj. Class.: 52B11, 20D06

1 Introduction

String C-group representations have gained much attention in recent years as they are in one-to-one correspondence with abstract regular polytopes. More precisely, given an abstract regular polytope and a base flag of the polytope, one can construct a string C-group representation whose group G is the automorphism group of the polytope that is generated by the set of involutory automorphisms sending the base flag to its adjacent flags [32, Section 2E]. Hence the study of string C-group representations has interest not only for group theory, but also for geometry.

*The authors thank Mark Mixer for observing that there was a mistake somewhere in the case $n \equiv 3 \pmod 4$ in a previous version of this paper. They also thank two anonymous referees for numerous comments that improved a previous version of this paper. This research was supported by the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), through CIDMA – Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019 (CIDMA).

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Classifications of string C-group representations received a big impetus thanks to experimental work of Leemans and Vauthier [31] and also Hartley [20]. These were pushed further for instance in [11, 15, 21, 27]. The results obtained in [31] quickly led to the determination of the highest rank of a string C-group representation of Suzuki groups [26]. Other families of almost simple groups were then investigated: the almost simple groups with socle $\mathrm{PSL}(2,q)$ [14, 28, 29], groups $\mathrm{PSL}(3,q)$ and $\mathrm{PGL}(3,q)$ [5], groups $\mathrm{PSL}(4,q)$ [3], small Ree groups [30], orthogonal and symplectic groups in characteristic 2, and finally, symmetric groups [16] and alternating groups [17, 18]. In particular, only the last four families gave rise to string C-group representations of arbitrary large rank. In [2], it is shown that, for all integers $m \geq 2$, and all integers $k \geq 2$, the orthogonal groups $\mathrm{O}^{\pm}(2m, \mathbb{F}_{2^k})$ act on abstract regular polytopes of rank $2m$, and the symplectic groups $\mathrm{Sp}(2m, \mathbb{F}_{2^k})$ act on abstract regular polytopes of rank $2m+1$. A symmetric group S_n is known to have string C-group representations of highest rank $n-1$ [6] and an alternating group A_n is known to have string C-group representations of highest rank $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$ [8]. It is worth noting that not only almost simple groups have been investigated. For instance, Cameron, Fernandes, Leemans and Mixer determined the maximal rank of a string C-group representation of a transitive permutation group in [7]. Conder determined in [9] the smallest string C-group representations of rank r. It turns out that when r is at least 9, all such groups are 2-groups. Further studies on string C-group representations of 2-groups are available for instance in [23, 24].

The authors looked at the symmetric groups in [16] and proved three important facts. Firstly, when $n \geq 5$, the $(n-1)$-simplex is, up to isomorphism, the unique string C-group representation of S_n with rank $n-1$. Secondly, they showed that when $n \geq 7$, there is also, up to isomorphism, a unique string C-group representation of rank $n-2$. And finally, they showed that for every $n \geq 4$, and for every integer r in the interval $[3, \ldots, n-1]$, a symmetric group S_n has at least one string C-group representation of rank r. Therefore, the symmetric groups have no gaps in their set of ranks. The first and second theorems have been extended in [19] where the authors of this paper, together with Mark Mixer, classified string C-group representations of rank $n-3$ (for $n \geq 9$) and $n-4$ (for $n \geq 11$) of the symmetric group S_n.

Also with Mixer, the authors produced in [17, 18] string C-group representations of rank $\lfloor (n-1)/2 \rfloor$ of the alternating groups, with $n \geq 12$. In the process of obtaining these results, they computed all string C-group representations of A_n with $n \leq 12$. They found that the set of ranks for the alternating groups of small degree were as given in Table 1. The

Table 1: Set of ranks for small alternating groups.

Group	Set of ranks
A_5	$\{3\}$
A_6	$\emptyset$
A_7	$\emptyset$
A_8	$\emptyset$
A_9	$\{3,4\}$
A_{10}	$\{3,4,5\}$
A_{11}	$\{3,6\}$
A_{12}	$\{3,4,5\}$

case $n = 11$ turned out to be special in the sense that it was the only example encountered so far of a group whose set of ranks presented gaps. In this paper, we prove a similar result as the third theorem of [16]. Our main result is stated as follows.

Theorem 1.1. *For $n \geq 12$ and for every $3 \leq r \leq \lfloor (n-1)/2 \rfloor$, the group A_n has at least one string C-group representation of rank r.*

This theorem shows indeed that the case $n = 11$ is special among the alternating groups. The main tool in the proof of our main theorem is to find good permutation representation graphs that turn out to be CPR graphs, for every rank $3 \leq r \leq \lfloor (n-1)/2 \rfloor$ once n is fixed. We use a proof similar to that of the third theorem of [16] to tackle most cases and are just left dealing with finding string C-group representations of ranks four and five for A_n when n is even, and ranks four, five and six, when $n \equiv 3 \pmod 4$.

The paper is organised as follows. In Section 2, we recall the basic definitions about string C-groups. In Section 3, we recall the definitions of permutation representation graphs and CPR-graphs and give some results that will be useful in proving Theorem 1.1. In Section 4, we prove Theorem 1.1. In Section 5, we give some final remarks.

As to notation for groups, we denote a cyclic group of order n by C_n, a dihedral group of degree n and order $2n$ by D_n, and by p^n an elementary abelian group of order p^n. Also, if G is a permutation group, the group G^+ is the subgroup of G generated by the even permutations in G, and if $G^+ = G$ (so that all elements of G are even) then we call G an *even permutation group*.

2 String C-groups

An abstract polytope is a combinatorial object which generalizes a classical convex polytope in Euclidean space. When the automorphism group of an abstract polytope acts regularly on its set of flags, the polytope is called *regular*, and in that case, its automorphism group admits a string C-group representation. Additionally, each abstract regular polytope can be constructed from a string C-group representation, and thus abstract regular polytopes and string C-groups representations are basically the same objects. For more details on the subject see [32, Section 2E].

A *Coxeter group* is a group with generators $\rho_0, \dots, \rho_{r-1}$ and presentation

$$\langle \rho_i \mid (\rho_i \rho_j)^{m_{i,j}} = \varepsilon \text{ for all } i, j \in \{0, \dots, r-1\} \rangle$$

where ε is the identity element of the group, each $m_{i,j}$ is a positive integer or infinity, $m_{i,i} = 1$, and $m_{i,j} = m_{j,i} > 1$ for $i \neq j$. It follows from the definition, that a Coxeter group satisfies the next condition called the *intersection property*.

$$\forall J, K \subseteq \{0, \dots, r-1\},\ \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$$

A Coxeter group G can be represented by a *Coxeter diagram* $\mathcal{D}$. This Coxeter diagram $\mathcal{D}$ is a labelled graph which represents the set of relations of G. More precisely, the vertices of the graph correspond to the generators ρ_i of G, and for each i and j, an edge with label $m_{i,j}$ joins the ith and the jth vertices; conventionally, edges of label 2 are omitted. By a *string (Coxeter) diagram* we mean a Coxeter diagram with each connected component linear. A Coxeter group with a string diagram is called a *string Coxeter group*.

More generally, we define a *string group generated by involutions*, or *sggi* for short, as a pair (G, S) where G is a group, $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a finite set of involutions of G that generate G and that satisfy the following property, called the *commuting property*.

$$\forall i, j \in \{0, \ldots, r-1\},\ |i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

Finally, a *string C-group representation* of a group G is a pair (G, S) that is a sggi and that satisfies the intersection property. In this case the underlying "Coxeter" diagram for (G, S) is a string diagram. The *(Schläfli) type* of (G, S) is $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$, $i \in \{1, \ldots, r-1\}$, and the *rank* of a string C-group representation (or of a sggi) (G, S) is the size of S. When the context is clear, we sometimes do not specify the set of generators S and we talk about a string C-group G instead of a string C-group representation (G, S).

The *set of ranks* of a group G is the largest set of integers I such that for each $r \in I$, there exists at least one string C-group representation of G with rank r.

Let $\Gamma := (G, S)$ be a sggi with $S := \{\rho_0, \ldots, \rho_{r-1}\}$. We denote by G_I with $I \subseteq \{0, \ldots, r-1\}$ the subgroup of G generated by the involutions with indices that are not in I and let $\Gamma_I := (G_I, \{\rho_j : j \notin I\})$; it follows from the definition that if Γ is a string C-group representation of G, each Γ_I is itself a string C-group representation of G_I. Also, for $i, j \in \{0, \ldots, r-1\}$, we denote $G_i = \langle \rho_j \mid j \neq i \rangle$ and $G_{i,j} := (G_i)_j$. The following two results show that when Γ_0 and Γ_{r-1} are string C-group representations, the intersection property for (G, S) is verified by checking only one condition.

Proposition 2.1 ([32, Proposition 2E16]). *Let $\Gamma := (G, S)$ be a sggi with $S := \{\rho_0, \ldots, \rho_{r-1}\}$. Suppose that Γ_0 and Γ_{r-1} are string C-group representations. If $G_0 \cap G_{r-1} = G_{0,r-1}$, then Γ is a string C-group representation of G.*

We point out that the inclusion $G_0 \cap G_{r-1} \geq G_{0,r-1}$ is immediate, and thus we only need to check that $G_0 \cap G_{r-1} \leq G_{0,r-1}$. The following proposition makes it even simpler to check if a pair (G, S) is a string C-group representation when $G_{0,r-1}$ is a maximal subgroup of either G_0 or G_{r-1} (or both).

Proposition 2.2 ([18, Lemma 2.2]). *Let $\Gamma = (G, S)$ be a sggi with $S := \{\rho_0, \ldots, \rho_{r-1}\}$ and $G := \langle S \rangle$. Suppose that Γ_0 and Γ_{r-1}, are string C-group representations of G_0 and G_{r-1} respectively. If $\rho_{r-1} \notin G_{r-1}$ and $G_{0,r-1}$ is maximal in G_0, then Γ is a string C-group representation of G.*

3 Permutation representation graphs and CPR graphs

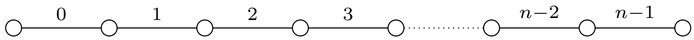
Let G be a group of permutations acting on a set $\{1, \ldots, n\}$. Let $S := \{\rho_0, \ldots, \rho_{r-1}\}$ be a set of r involutions of G that generate G. We define the *permutation representation graph* $\mathcal{G}$ of G, as the r-edge-labeled multigraph with n vertices and with an i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$.

The pair (G, S) is a sggi if and only if $\mathcal{G}$ satisfies the following properties:

1. The graph induced by edges of label i is a matching;
2. Each connected component of the graph induced by edges of labels i and j, for $|i-j| \geq 2$, is a single vertex, a single edge, a double edge, or a square with alternating labels.

When (G, S) is a string C-group representation, the permutation representation graph $\mathcal{G}$ is called a *CPR graph*, as defined in [33]. In rank 3, there are a couple of known results to determine if a 3-edge-labeled multigraph is a CPR graph. For higher ranks, no such arguments were accomplished.

One simple example of a CPR graph is the one corresponding to the $(n-1)$-simplex as follows:

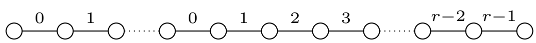
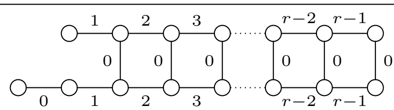
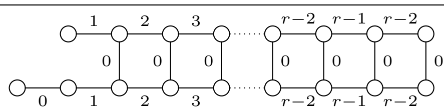


In [16], for each rank $3 \le r \le n-2$, a string C-group representation of rank r of S_n was found. In [18], the authors constructed a string C-group representation of rank $r \ge 4$ of A_n for some n. This is summarized in the following two theorems, and the associated CPR graphs are given in Table 2.

Theorem 3.1 ([16, Theorem 3]). *For $n \ge 5$ and $3 \le r \le n-2$, there is a string C-group representation of rank r and type $\{n-r+2, 6, 3^{r-3}\}$ of S_n.*

Theorem 3.2 ([18, Theorem 1.1]). *For each rank $k \ge 3$, there is a string C-group representation of rank k of A_n for some n. In particular, for each even rank $r \ge 4$, there is a string C-group representation of A_{2r+1} of type $\{10, 3^{r-2}\}$, and for each odd rank $q \ge 5$, there is a string C-group representation of A_{2q+3} of type $\{10, 3^{q-4}, 6, 4\}$.*

Table 2: String C-group representations of S_n and A_n.

Group	Schläfli type	CPR graph
S_n $(3 \le r \le n-2)$	$\{n-r+2, 6, 3^{r-3}\}$	
A_{2r+1} (r even and ≥ 4)	$\{10, 3^{r-2}\}$	
A_{2r+3} (r odd and ≥ 5)	$\{10, 3^{r-4}, 6, 4\}$	

Permutation representation graphs are a very useful tool for the construction of string groups generated by involutions. We will use them in the proof of our main theorem.

The term sesqui-extension was first introduced in [18]. Let us recall its meaning. Let $\Phi = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle$ be a sggi, and let τ be an involution in a supergroup of Φ such that $\tau \notin \Phi$ and τ centralizes Φ. For fixed k, we define the group $\Phi^* = \langle \alpha_i \tau^{\eta_i} \mid i \in \{0, \ldots, d-1\} \rangle$ where $\eta_i = 1$ if $i = k$ and 0 otherwise, and call this the *sesqui-extension* of Φ with respect to α_k and τ. In particular, a permutation representation graph having two connected components, one of which is a single k-edge and the other contains at least one k-edge, represents a sesqui-extention of a group (the group corresponding to the biggest component) with respect to the generator k.

Proposition 3.3 ([17, Proposition 5.4]). *If* $\Phi = \langle \alpha_i \mid i = 0, \dots, d-1\rangle$ *and*

$$\Phi^* = \langle \alpha_i \tau^{\eta_i} \mid i \in \{0, \dots, d-1\}\rangle$$

is a sesqui-extension of Φ *with respect to* α_k*, then* $(\Phi, \{\alpha_i \mid i = 0, \dots, d-1\})$ *is a string C-group representation if and only if* $(\Phi^*, \{\alpha_i\tau^{\eta_i} \mid i \in \{0, \dots, d-1\}\})$ *is a string C-group representation. Moreover one of the following situations occur.*

(1) $\tau \in \Phi^*$*, in which case* Φ^* *is isomorphic to* $\Phi \times \langle\tau\rangle \cong \Phi \times C_2$*; or*

(2) $\tau \notin \Phi^*$*, in which case* Φ^* *is isomorphic to* Φ*.*

Sesqui-extensions will be used later to check the intersection condition on the permutation representations of the groups of our main theorem.

We also apply the techniques used in the proof of Theorem 3.1 based on a construction of Hartley and Leemans available in [22]. The key of the proof of Theorem 3.1 was to start from the CPR graph of the $(n-1)$-simplex with generators $\rho_1, \dots, \rho_{n-1}$ where ρ_i is the transposition $(i, i+1)$ in S_n. Let $d = n - 1$. At each step, we start with a string C-group representation of rank d and generators $\rho_1, \dots, \rho_d$. We replace ρ_{d-2} by $\rho_{d-2}\rho_d$ and we drop ρ_d. As proved in [16], we get in this way a new string C-group representation with generators $\rho_1, \dots, \rho_{d-1}$. We can repeat this until $d = 3$. We give in Table 3 an example of this process for S_7.

Table 3: The induction process used on S_7.

Generators	CPR graph	Schläfli type
$(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)$	○ 1 ○ 2 ○ 3 ○ 4 ○ 5 ○ 6 ○	$\{3,3,3,3,3\}$
$(1,2),(2,3),(3,4),(4,5)(6,7),(5,6)$	○ 1 ○ 2 ○ 3 ○ 4 ○ 5 ○ 4 ○	$\{3,3,6,4\}$
$(1,2),(2,3),(3,4)(5,6),(4,5)(6,7)$	○ 1 ○ 2 ○ 3 ○ 4 ○ 3 ○ 4 ○	$\{3,6,5\}$
$(1,2),(2,3)(4,5)(6,7),(3,4)(5,6)$	○ 1 ○ 2 ○ 3 ○ 2 ○ 3 ○ 2 ○	$\{6,6\}$

In order to prove that the permutation groups of our main theorem are isomorphic to alternating groups we use the following results.

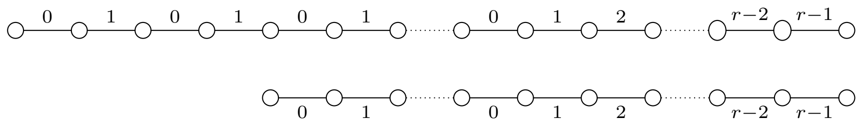
Theorem 3.4 ([25]). *Let* G *be a primitive permutation group of finite degree* n*, containing a cycle of prime length fixing at least three points. Then* $G \geq A_n$*.*

Proposition 3.5 ([17, Proposition 3.3]). *Let* $G = \langle \rho_0, \dots, \rho_{r-1}\rangle$ *be a transitive permutation group acting on the points* $\{1, \dots, n\}$ *with* $n \geq 5$*, and let* $G^* = \langle \rho_0, \dots, \rho_{r-1}, \rho_r, \rho_{r+1}\rangle$*, where*

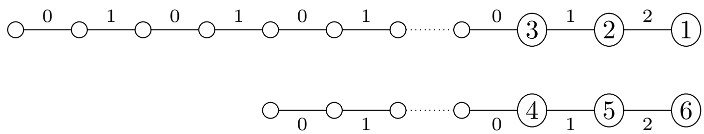
$$\rho_r = (i, n+1)(n+2, n+3) \text{ for some } i \in \{1, \dots, n\}$$
$$\rho_{r+1} = (n+1, n+2)(n+3, n+4).$$

Then $G^* = A_{n+4}$ *or* S_{n+4}*, depending on whether or not* G *is even.*

Proposition 3.6. *The following graph, with $n \geq 8$ vertices, n even and $r \in \{3, \ldots, \frac{n-2}{2}\}$, is a CPR graph for $\left(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}}\right)^+$.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. Let us first consider $r = 3$.



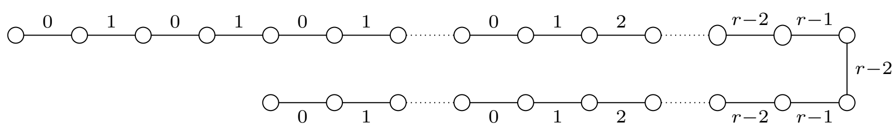
We see that Γ_0 and Γ_2 are string C-group representations and as $G_0 \cap G_2 = G_{0,2} \cong C_2$, Γ is itself a string C-group representation by Proposition 2.1.

Let us prove that G is isomorphic to $\left(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}}\right)^+$. We first prove that G contains the 3-cycles $(1,2,3)$ and $(4,5,6)$ (the vertices of the above graph on the right). Let l be the least integer such that $(\rho_0\rho_1)^l$ fixes all the vertices of the component of the graph on the bottom. We see that $(\rho_1\rho_2)^2 = (1,2,3)(4,5,6)$. The latter element conjugated by $(\rho_0\rho_1)^l$ is equal to $\alpha = (a,b,c)(4,5,6)$ with $\{a,b,c\} \cap \{1,2,3\} = \{1\}$. Hence $(\alpha(\rho_1\rho_2)^2)^5 = (4,6,5)$ and $(1,2,3) = (4,6,5)(\rho_1\rho_2)^2$.

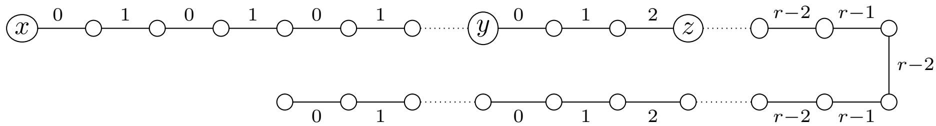
Now by transitivity in each of the two components of the graph we find that G has a subgroup isomorphic to $A_{\frac{n-4}{2}} \times A_{\frac{n+4}{2}}$. As in addition $\rho_2 \notin A_{\frac{n-4}{2}} \times A_{\frac{n+4}{2}}$ and G is a group of even permutations, the group G is isomorphic to $\left(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}}\right)^+$.

Now let $r > 3$. We may assume by induction that Γ_{r-1} is a string C-group representation and G_{r-1} is isomorphic to $\left(S_{\frac{n-6}{2}} \times S_{\frac{n+2}{2}}\right)^+$. In addition Γ_0 is a string C-group representation with group G_0 isomorphic to S_{r-1}. By the intersection of the orbits of G_0 and G_{r-1} we conclude that $G_0 \cap G_{r-1}$ and $G_{0,r-1}$ are both isomorphic to S_{r-2}. Therefore Γ is a string C-group representation of G. Moreover it is clear that G is isomorphic to $\left(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}}\right)^+$. □

Proposition 3.7. *The following graph, with $n \geq 10$ vertices, n even and $r \in \{5, \ldots, \frac{n-2}{2}\}$, is a CPR graph for S_n.*

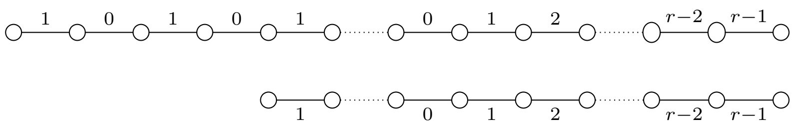


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. The permutation representation graph is connected, hence G is transitive. Let x be the first point on the left of the graph. The stabilizer of x has at most the same orbits as G_0. Consider the vertices y and z as in the following graph.



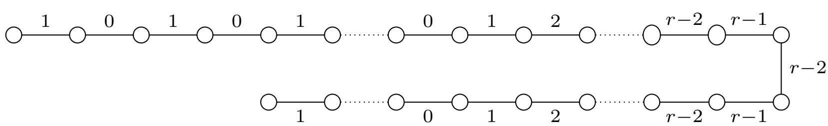
We see that $y\rho_2^{\rho_1\rho_0} = z$ and $\rho_2^{\rho_1\rho_0}$ fixes x. More generally the appropriate conjugations of ρ_2 by powers of $\rho_0\rho_1$ fuse the orbits of G_0 while fixing x. Hence G is 2-transitive and therefore primitive. Moreover, it contains a 3-cycle (explicitly given in the proof of Proposition 3.6) and an odd permutation. Hence, by Theorem 3.4, it is isomorphic to S_{n-1}. By Proposition 3.3 and [17, Table 2] we may conclude that Γ_0 is a string C-group representation of the group $C_2 \times (C_2 \wr S_{r-1})$. By Proposition 3.6, the sggi Γ_{r-1} is a string C-group representation of $\left(S_{\frac{n-6}{2}} \times S_{\frac{n+2}{2}}\right)^+$. From the intersection of the orbits of G_0 and G_{r-1} we also conclude that $G_0 \cap G_{r-1} = G_{0,r-1} \cong C_2 \times \left(S_{\frac{n-7}{2}} \times S_{\frac{n+1}{2}}\right)^+$. Hence Γ is a string C-group representation. □

Proposition 3.8. *The following graph, with $n \geq 10$ vertices, n even and $r \in \{3, \dots, \frac{n-2}{2}\}$, is a CPR graph for $\left(S_{\frac{n-4}{2}} \times S_{\frac{n+4}{2}}\right)^+$.*



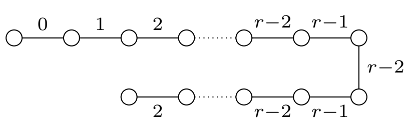
Proof. Similar to that of Proposition 3.6. □

Proposition 3.9. *The following graph, with $n \geq 12$ vertices, n even and $r \in \{5, \dots, \frac{n-2}{2}\}$, is a CPR graph for S_n.*

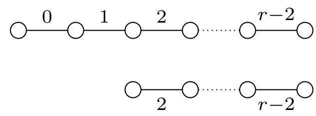


Proof. Similar to that of Proposition 3.7. □

Proposition 3.10. *The following graph, with $n \geq 8$ vertices, n even and $r = n/2$, is a CPR graph for S_n.*

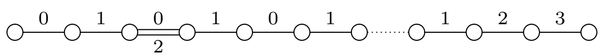


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. Removing the 0-edge from the graph we get a CPR graph for a symmetric group of degree $n-1$ (see Table 2 of [17]). Hence Γ_0 is a string C-group representation. Now consider the sggi $\Phi := (H, T)$ with the following permutation representation graph.



For $r = 4$, Φ is a string C-group representation with H isomorphic to $C_2 \times S_4$. Assume by induction that Φ_{r-2} is a string C-group representation with H_{r-2} isomorphic to $S_{r-1} \times S_{r-3}$. As Φ_0 is a string C-group representation and $H_0 \cap H_{r-2} \leq S_{r-2} \times S_{r-3} \cong H_{0,r-2}$, Φ is a string C-group representation. Moreover H is isomorphic to $S_{r-1} \times S_{r-3}$. Now by Proposition 3.3 the sggi Γ_{r-1} is a string C-group representation and G_{r-1} is isomorphic to $C_2 \times S_{r-1} \times S_{r-3}$. By the intersection of the orbits of G_0 and G_{r-1} we find that $G_0 \cap G_{r-1} = G_{0,r-1}$ Hence Γ is a string C-group representation. As G_0 is isomorphic to S_{n-1} and stabilizes the first vertex on the left, we conclude that G is isomorphic to S_n. □

Proposition 3.11. *The following graph with n vertices, $n \equiv 3 \pmod 4$ and $n \geq 11$, is a CPR graph for S_n.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation given by the graph of this proposition. The group G_3 is an even transitive group containing a 3-cycle, namely $(\rho_1\rho_2)^4$, and the stabilizer of a point in G_3 is transitive on the remaining points. Hence by Theorem 3.4 the group G_3 is isomorphic to A_{n-1}. Consequently G is isomorphic to S_n. Moreover as G_3 is a simple group generated by three independent involution, the sggi Γ_3 is string C-group representation. It is also easy to check that Γ_0 is string C-group representation and that $G_3 \cap G_0 = G_{0,3}$, as it is sufficient to consider the case $n = 11$. Hence Γ is a string C-group representation and G is isomorphic to S_n as wanted. □

4 Proof of Theorem 1.1

For each $n \geq 12$, the group A_n has at least one string C-group representation of rank three. Indeed, we can rely on [12, 13] which covers all but a small number of small cases that can be easily dealt with MAGMA [1], or [34]. Hence we have to construct examples of rank 4 and above. Also, the case where $n = 12$ is done in [18], hence we may assume $n > 12$.

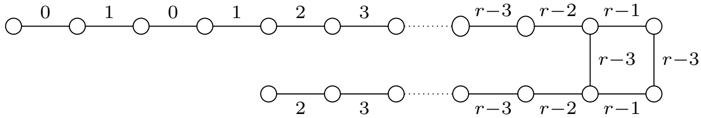
We divide the rest of the proof is a series of theorems depending on the values of n and r as described in Table 4. Theorem 4.1 comes from [17], and we use it in Theorem 4.2 to construct string C-group representations of rank $6 \leq r \leq (n-2)/2$ for n even.

4.1 The even case

We will construct a family of CPR graphs of even ranks "reducing" the rank of a CPR graph having highest possible rank. Let us consider the graph given in the following theorem.

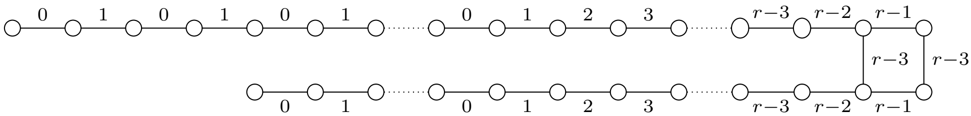
Theorem 4.1 ([17]). *If $n \geq 14$ is even and $r = \frac{n-2}{2} \geq 6$, then the following graph is a*

CPR graph for A_n.

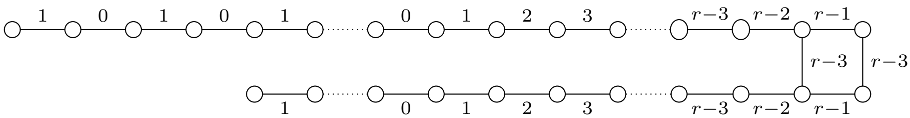


Moreover the corresponding string C-group representation has type $\{5, 6, 3^{r-6}, 6, 6, 3\}$.

Theorem 4.2. *If n is an even integer, $n \geq 14$ and $6 \leq r \leq \frac{n-2}{2}$, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{\mathrm{lcm}(4+i, i), 6, 3^{r-6}, 6, 6, 3\}$ where $i = (n-2)/2 - r + 1$, and with the following CPR graph*



for ($n \equiv 2 \pmod 4$ and $n - r$ even) or ($n \equiv 0 \pmod 4$ and $n - r$ odd) and the following CPR-graph



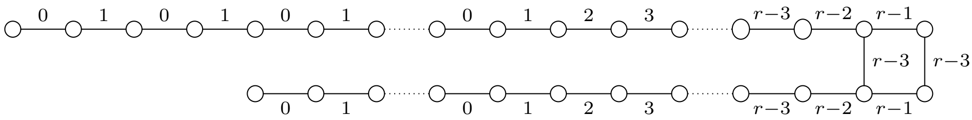
for ($n \equiv 2 \pmod 4$ and $n - r$ odd) or ($n \equiv 0 \pmod 4$ and $n - r$ even).

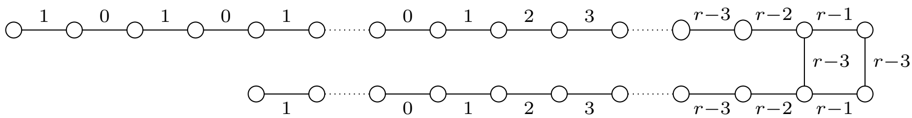
Proof. From the graph of Theorem 4.1 we construct a family of graphs with n vertices and $r \in \{6, \dots, \frac{n-2}{2}\}$ adding, on the top and on the bottom of the graph, two sequences

Table 4: The structure of the proof depending on n and r.

n	r	Reference
n even	$6 \leq r \leq (n-2)/2$	Theorem 4.2
$n \equiv 0 \pmod 4$	$r = 5$	Theorem 4.6
	$r = 4$	Theorem 4.5
$n \equiv 2 \pmod 4$	$r = 5$	Theorem 4.4
	$r = 4$	Theorem 4.3
$n \equiv 1 \pmod 4$	$4 \leq r \leq (n-1)/2$	Theorem 4.7
$n \equiv 3 \pmod 4$	$r = (n-1)/2$	Theorem 4.8
	$7 \leq r < (n-1)/2$ and r odd	Theorem 4.9
	$r = (n-1)/2 - 1$	Theorem 4.10
	$8 \leq r < (n-1)/2$ and r even	Theorem 4.11
	$r = 4$	Theorem 4.12
	$r = 5$	Theorems 4.13 and 4.15
	$r = 6$	Theorem 4.14

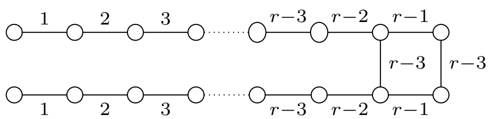
of edges, of the same size, with alternate labels 0 and 1. So we have the following two possibilities.





Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The statement holds for $n = 14$ and $r = 6$ by Theorem 4.1. Assume $n > 14$.

The involution ρ_1 can be decomposed as $\rho_1 = \tau\alpha_1$ where α_1 is the restriction of ρ_1 to the biggest G_0-orbit and τ is the restriction of ρ_1 to the union of G_0-orbits of size 2. The following CPR graph has group isomorphic to $(2^r : S_r)^+$ as shown in [17, Lemma 6.6]. It is exactly the graph we obtain by replacing ρ_1 by α_1 and forgetting about the points fixed by G_0.



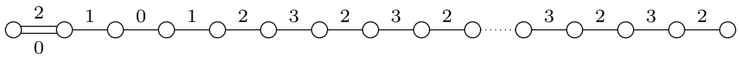
We find that $\alpha_1 = \rho_2\rho_1\rho_2\rho_1\rho_2 \in G_0$, then also $\tau \in G_0$ and therefore by Proposition 3.3, G_0 is a sesqui-extension of the group $(2^r : S_r)^+$ and G_0 is isomorphic to $C_2 \times (2^r : S_r)^+ \cong 2^r : S_r$ as $\tau \in G_0$. Moreover, Γ_0 is a string C-group representation.

We use a similar argument to prove that Γ_{r-1} is a string C-group, starting from the CPR graph given in Proposition 3.7 when ($n \equiv 2 \pmod 4$ and $n - r$ even) or ($n \equiv 0 \pmod 4$ and $n-r$ odd), and from the CPR graph given in Proposition 3.9 when ($n \equiv 2 \pmod 4$ and $n - r$ odd) or ($n \equiv 0 \pmod 4$ and $n - r$ even). In that case, however, since the restriction of $\rho_{r-2}\rho_{r-3}$ to the biggest orbit of G_{r-1} is an element of even order, $G_{r-1} \cong S_{n-2}$. Since A_n acts primitively on the set of unordered pairs of points, the stabilizer in A_n of a fixed pair is maximal in A_n, and such stabilizers have precisely the structure of G_{r-1}. As G_{r-1} is a maximal subgroup of A_n and $\rho_{r-1} \notin G_{r-1}$, it follows that G is isomorphic to A_n. Let us now prove that $G_{0,r-1} = G_0 \cap G_{r-1}$. The orbits of $G_0 \cap G_{r-1}$ have to be suborbits of G_0 and of G_{r-1}, hence $G_0 \cap G_{r-1} \leq (C_2 \times (2^{r-1} : S_{r-1}) \times C_2)^+ \cong G_{0,r-1}$. Hence, by Proposition 2.1, Γ is a string C-group representation of A_n.

Let $i = (n-2)/2 - r + 1$. Then it is easy to see from the CPR-graph that the Schläfli type of the string C-group representation of A_n of rank r obtained by this construction is $\{\mathrm{lcm}(4+i, i), 6, 3^{r-6}, 6, 6, 3\}$. The first entry of the symbol comes from the fact that there are 0-1-components on the upper side of the graph and on the lower side of the graph and the upper one has 4 more vertices than the lower one. □

It remains to construct examples in rank 4 and 5 for n even. We split the discussion in two cases, namely the case where $n \equiv 0 \pmod 4$ and the case where $n \equiv 2 \pmod 4$.

Theorem 4.3. *If* $n \equiv 2 \pmod 4$ *with* $n \geq 10$*, then the group* A_n *admits a string C-group representation of rank* 4*, with Schläfli type* $\{5, 6, n-4\}$*, with the following CPR-graph.*

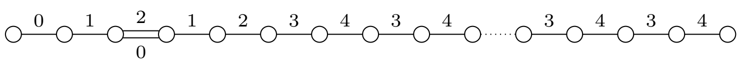


(F_1)

Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case G_3 is a sesqui-extension of a string C-group representation of A_5, hence by Proposition 3.3, $G_3 \cong C_2 \times A_5$ and Γ_3 is a string C-group representation of rank 3. Moreover, $G_{0,3}$ is isomorphic to $C_2 \times D_3 \cong D_6$ and therefore $G_{0,3}$ is maximal in G_3. So, by Proposition 2.2, it remains to prove that Γ_0 is also a string C-group representation. Now, $\Gamma_{0,3}$ and $\Gamma_{0,1}$ are obviously string C-group representations of dihedral groups. The group $G_{0,1,3}$ is a cyclic group of order 2 and the subgroups $G_{0,3}$ and $G_{0,1}$ will have the same intersection no matter what the value of n is. We can thus assume $n = 10$ and check by hand or using MAGMA that $G_0 \cap G_3 = G_{0,3}$. Hence Γ_0 is a string C-group representation. This concludes the proof that a sggi with permutation representation graph (F_1) is a string C-group representation. It remains to show that the four generators generate A_n. The element $\rho_0\rho_1$ is a 5-cycle and G is primitive, as for instance ρ_0 cannot preserve any block system. Hence, by Theorem 3.4, G is isomorphic to A_n.

The Schläfli type is obvious from the permutation representation graph. □

Theorem 4.4. *If* $n \equiv 2 \pmod 4$ *with* $n \geq 10$*, then the group* A_n *admits a string C-group representation of rank* 5*, with Schläfli type* $\{5, 5, 6, n-5\}$*, with the following CPR-graph.*

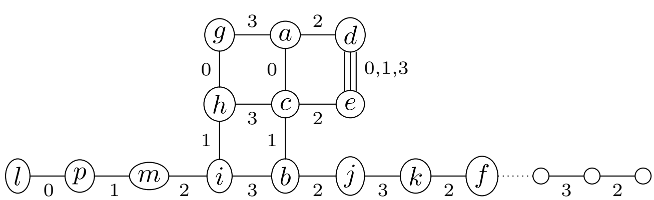


(F_2)

Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case, G_4 is a sesqui-extension of a group isomorphic to $(S_7 \times C_2)^+ \cong S_7$ whose CPR graph is given in Table 2 of [17]. Hence Γ_4 is a string C-group representation. By Proposition 3.5 the group G_0 is isomorphic to A_{n-1}. The subgroup $G_{0,4}$ is isomorphic to S_6, in addition $G_{0,1,4} \cong D_6$ and $G_{0,1} \cong S_{n-4}$. Increasing n will not change the intersection between $G_{0,1}$ and $G_{0,4}$. Hence we can check with MAGMA that $G_{0,1} \cap G_{0,4} = G_{0,1,4}$ for $n = 10$. Thus $\Gamma_{0,1}$ is a string C-group representation and so is Γ_0 and so is Γ, as $G_0 \cong A_{n-1}$ and G is transitive. Moreover G is isomorphic to A_n since it is transitive on n points and the stabilizer of a point in G contains $G_0 \cong A_{n-1}$.

The Schläfli type is obvious from the permutation representation graph. □

Theorem 4.5. *If* $n \equiv 0 \pmod 4$ *with* $n \geq 16$*, then the group* A_n *admits a string C-group representation of rank* 4*, with Schläfli type* $\{3, 12, \text{lcm}(n-8, 6)\}$*, with the following CPR-graph.*

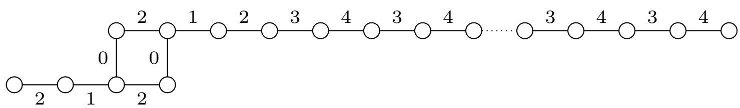


(F_3)

Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case, G_3 is isomorphic to $2^2 : S_3 \times S_3$ and $G_{0,3}$ is isomorphic to D_{12} no matter what the value of n is, thanks to the shape of the graph. Observe that the left connected component of the graph, obtained when removing the 3-edges, gives the CPR graph of the octahedron. Thus it can easily be checked with MAGMA that Γ_3 is a string C-group representation with type $\{3, 12\}$. The group G_0 is transitive on $n-1$ points, namely all vertices of the graph except l. Moreover, the stabilizer of l and p in G has at most two more orbits thanks to the connected components of the permutation representation graph obtained by removing edges labelled 0 and 1. The element $(\rho_1\rho_2\rho_3\rho_2)^3$ moves point i to point d while fixing both l and p. Hence G_0 is 2-transitive on $n-1$ vertices (all but l). Therefore G_0 is primitive on these points. Now the element $(\rho_1\rho_2\rho_3\rho_2) = (l)(p, j, m)(i, e, g, d, h)(a, c, f, b)\ldots$ has the property that the cycles we did not write are transpositions. Indeed, ρ_1 does not do anything on these points and so the action on these points is given by $\rho_2\rho_3\rho_2 = \rho_3^{\rho_2}$ which is an involution. Hence $(\rho_1\rho_2\rho_3\rho_2)^{12} \in G_0$ is a 5-cycle fixing more than three points. By Theorem 3.4, we can therefore conclude that G_0 is isomorphic to A_{n-1}. As G_0 is a simple group, since it is generated by three involutions (namely ρ_1, ρ_2, ρ_3), two of which commute, Γ_0 is a string C-group representation by [10, Theorem 4.1]. It remains to check that $G_{0,3} = G_0 \cap G_3$ to prove that these graphs give indeed string C-group representations. This can be checked with MAGMA for $n = 12$ and the result can be extended for any n.

The Schläfli type is obvious from the permutation representation graph. □

Theorem 4.6. *If $n \equiv 0 \pmod 4$ with $n \geq 12$, then the group A_n admits a string C-group representation of rank 5, with Schläfli type $\{3, 4, 6, n-7\}$, with the following CPR-graph.*



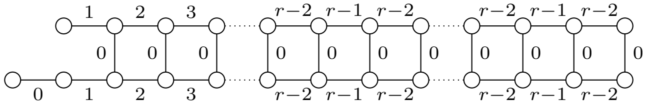
(F_4)

Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. In this case, G_4 is a sesqui-extension of the group of a string C-group representation of S_9, that can be found for instance in the atlas [31]. The sggi $\Gamma_{0,1}$ is a string C-group representation of S_{n-6} and $G_{0,4}$ is isomorphic to $S_5 \times D_4$. Now $\rho_2\rho_3$ has order 6, so $G_{0,1,4}$ is isomorphic to D_6 and it is obvious from the permutation representation graph that $G_{0,4} \cap G_{0,1} = G_{0,1,4}$ and $G_{0,4} \cap G_{1,4} = G_{0,1,4}$. Hence Γ_0 and Γ_4 are string C-group representations by Proposition 2.1. As $G_0 \cap G_4$ must have orbits that are suborbits of those of G_0 and of those of G_4, we readily see that $G_0 \cap G_4 = G_{0,4}$. This concludes the proof that every graph of shape (F_4) gives a string C-group representation. As G is a primitive group generated by even permutations and $(\rho_2\rho_3)^2$ is a 3-cycle, we see that G is isomorphic to A_n by Theorem 3.4.

The Schläfli type is obvious from the permutation representation graph. □

4.2 The odd case

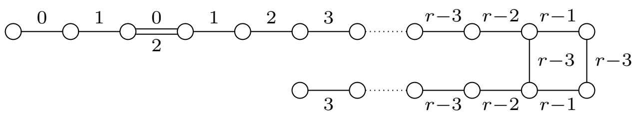
Theorem 4.7. *If n and r are integers with $n \geq 13$, $n \equiv 1 \pmod 4$ and $4 \leq r \leq (n-1)/2$, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{10, 3^{\frac{n-1}{2}-2}\}$ when $r = \frac{n-1}{2}$ and $\{10, 3^{r-4}, 6, \frac{n-1}{2} - r + 3\}$ when $r < \frac{n-1}{2}$, and with the* *following CPR graph.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. Clearly G is a group of even permutations and it must be primitive as ρ_0 cannot preserve a non-trivial block system. Let us prove that G is isomorphic to A_n. We see that $(\rho_0\rho_1)^2$ is a 5-cycle, hence by Theorem 3.4, the group G is isomorphic to A_n. It remains to prove that Γ satisfies the intersection property. We know that for $n = 13$, the sggi Γ is a string C-group representation of rank 6 and Schläfli type $\{10, 3, 3, 3, 3\}$. It can be checked with MAGMA that Γ is also a string C-group representation for $n = 13$ and $r \in \{4, 5\}$. By induction we may assume that G_{r-1} is a sesqui-extension of the group of a string C-group representation. Hence by Proposition 3.3, the sggi Γ_{r-1} satisfies the intersection property. By the first line of Table 2, it is easy to see that Γ_0 is a string C-group representation. Finally, $G_{0,r-1} = G_0 \cap G_{r-1} \cong S_{r-1} \times C_2$. By Proposition 2.1, we conclude that Γ is a string C-group representation. Using this technique, we have just constructed string C-group representations of rank r for every $4 \leq r \leq \frac{n-1}{2}$. Their Schläfli types are $\{10, 3^{\frac{n-1}{2}-2}\}$ when $r = \frac{n-1}{2}$ and $\{10, 3^{r-4}, 6, \frac{n-1}{2} - r + 3\}$ when $r < \frac{n-1}{2}$. □

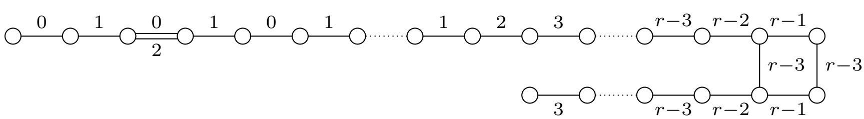
The following theorem gives the string C-group representations of rank $r = (n-1)/2$ in the case where $n \equiv 3 \pmod 4$.

Theorem 4.8 ([17]). *If n and r are integers with $n \geq 15$, $n \equiv 3 \pmod 4$ and $r = (n-1)/2$, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{5, 5, 6, 3^{r-7}, 6, 6, 3\}$, and with the following CPR graph.*



From these examples, we construct examples of the same rank but for groups of degree $n + 4k$ where k is an integer, by adding a sequence of alternating 0- and 1-edges of length $4k$ between the first and the second 2-edge (counting from the left).

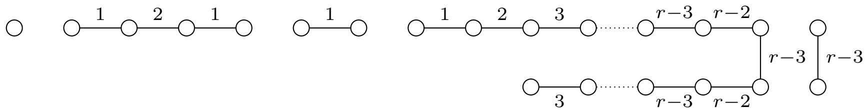
Theorem 4.9. *If n and r are integers with $n \geq 15$, $n \equiv 3 \pmod 4$ and $7 \leq r < (n-1)/2$, r odd, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{n - 2(r-2), 12, 6, 3^{r-7}, 6, 6, 3\}$, and with the following CPR graph.*



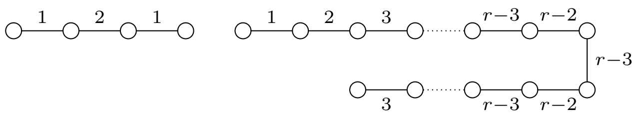
Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is acting as $S_{2(r-1)}$ on the orbit of size $2(r-1)$ and as D_4 on the orbit of size 4, making G_0 isomorphic to $A_{2(r-1)} : D_4$. Observe that G_0 has a structure that only depends on the rank, not on the degree of G.

The group $G_{0,r-1}$ is isomorphic to $S_{2(r-2)} : D_4$. It is a maximal subgroup of G_0. Hence $G_0 \cap G_{r-1} = G_{0,r-1}$.

Let us now prove that Γ_0 and Γ_{r-1} are string C-group representations. We start with Γ_0. The group $G_{0,1}$ is the same (up to removing the fixed points) as the one of Theorem 4.8. Hence Γ_0 is a string C-group representation. The sggi $\Gamma_{0,r-1}$ has the following permutation representation graph, where there might be more than one 1-edge disconnected from the rest of the graph.



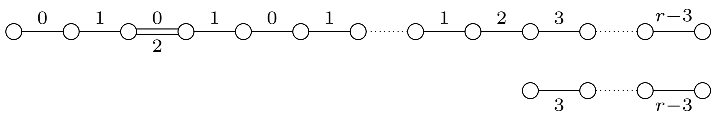
If we prove that the sggi corresponding to the following permutation representation graph is a string C-group representation, we may then apply Proposition 3.3 in order to show that $\Gamma_{0,r-1}$ is also a string C-group representation.



Let us call $\Phi := (H, T)$ the sggi having this permutation representation graph. By Proposition 3.10 the connected component on the right of the graph above gives a string C-group representation. By Proposition 3.3 the graph that we obtain from the graph pictured above by removing the 2-edge on the left is a CPR graph. Since removing the 2-edge on the left does not change the order of the group H_1, by [32, Proposition 2E17] we find that Φ is a string C-group representation. Hence Γ_0 is a string C-group representation.

Let us now prove that Γ_{r-1} is a string C-group representation.

The group $G_{r-2,r-1}$ is a sesqui-extension of the group K of the sggi $\Psi := (K, U)$ having the following permutation representation graph.



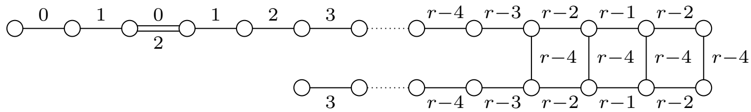
Let a and b be the sizes of the connected components of the graph above. For $r = 6$, K is a sesqui-extension of the group of the string C-group representation of Proposition 3.11, hence by Proposition 3.3, K is isomorphic to $S_a \cong (S_a \times 2)^+$. By induction we may assume that Ψ_{r-3} is a string C-group representation and K_{r-3} is isomorphic to $(S_{a-1} \times S_{b-1})^+$. As Ψ_0 is a string C-group representation and $K_0 \cap K_{r-3} = K_{0,r-3}$ we find that Ψ is itself a string C-group representation. Moreover K is clearly isomorphic to $(S_a \times S_b)^+$. With this, using Proposition 3.3, we see that $\Gamma_{r-2,r-1}$ is a string C-group representation. Finally $G_{0,r-1} \cap G_{r-2,r-1} \leq (D_4 \times S_{2(r-3)} \times 2)^+ \cong G_{0,r-2,r-1}$.

Hence we have proved that Γ_{r-1} is a string C-group representation and therefore G itself is a string C-group.

It is easy to see from the permutation representation graph in the theorem that the Schläfli type of the string C-group representation of rank r of A_n obtained by this construction is $\{n-2(r-2), 12, 6, 3^{r-7}, 6, 6, 3\}$. □

The previous two theorems enable us to construct examples of all possible odd ranks at least 7 for A_n with $n \equiv 3 \pmod 4$ and $n \geq 15$. We now construct an example of rank $(n-3)/2$ for A_n from the example of rank $(n-1)/2$, that we will use to construct all examples of even rank at least 8.

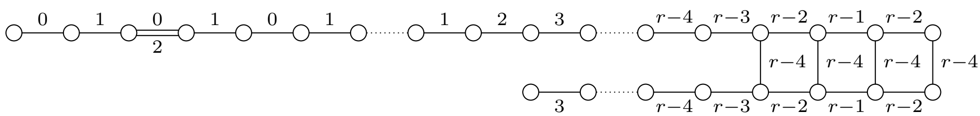
Theorem 4.10. *If n and r are integers are such that $n \geq 19$, $n \equiv 3 \pmod 4$ and $r = (n-1)/2 - 1$, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{5, 5, 6, 3^{r-8}, 6, 6, 6, 4\}$, and with the following CPR graph.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_{r-1} is a sesqui-extension of the group given in Theorem 4.8. Hence Γ_{r-1} is a string C-group representation. The sggi Γ_0 can be proved to be a string C-group representation using similar techniques to those the proof of the previous theorem. The fact that $G_0 \cap G_{r-1} = G_{0,r-1}$ follows from the fact that G_{r-1} is a sesqui-extension of the group given in Theorem 4.8 and the orbits of the respective subgroups. □

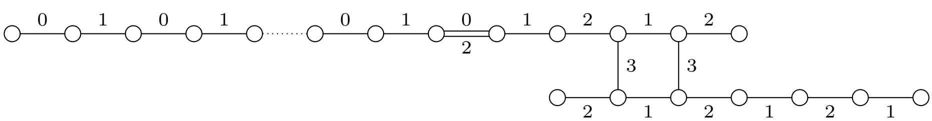
As in the case of odd ranks, from these examples we construct examples of the same rank but for groups of degree $n + 4k$ where k is an integer, by adding a sequence of alternating 0- and 1-edges of length $4k$ between the 1-edge and the second 2-edge (counting from the left).

Theorem 4.11. *If n and r are integers such that $n \equiv 3 \pmod 4$, $n \geq 19$ and $8 \leq r < (n-1)/2 - 1$, r even, then the group A_n admits a string C-group representation of rank r, with Schläfli type $\{n - 2(r-1), 12, 6, 3^{r-8}, 6, 6, 6, 4\}$, and with the following CPR graph.*



There are two ways to prove this theorem, either by a proof similar to that of Theorem 4.9 or by a proof similar to that of Theorem 4.10. We leave the details to the interested reader.

Theorem 4.12. *If $n \equiv 3 \pmod 4$ with $n \geq 15$, then the group A_n admits a string C-group representation of rank 4, with Schläfli type $\{10, 7, 4\}$ for $n = 15$ and $\{2(n-10), 14, 4\}$ for $n > 15$, with the following CPR-graph.*

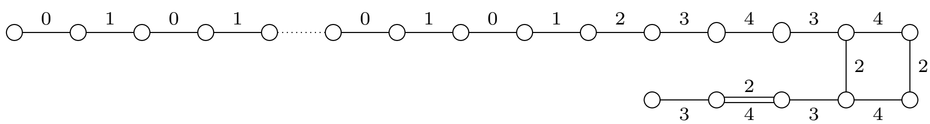


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is isomorphic to $2^6 : A_7 : C_2$ for $n = 15$ and $2^6 : A_7 : C_2 \times C_2$ for $n \geq 19$, no matter how big n is. It can easily be checked with MAGMA that Γ_0 is a string C-group representation for $n = 15$ and $n = 19$ and since adding more points to the graph will not change the structure of G_0, we can conclude that Γ_0 is a string C-group representation for every $n \geq 15$. The group G_3 acts as S_{n-7} on the vertices of the top of the graph and acts as D_7 on the remaining vertices, and is a subgroup of $(A_{n-7} \times D_7)^+$. We can thus conclude that G_3 is $A_{n-7} \times D_7$. The group $G_{0,3}$ is isomorphic to D_7 for $n = 15$ and $C_2 \times D_7$ when $n \geq 19$ (as there are extra 1-edges in the graph). The group $G_{2,3}$ is isomorphic to $D_{(n-10)}$. It is obvious from the permutation representation graph that $G_{0,3} \cap G_{2,3}$ is isomorphic to C_2. Hence, by Proposition 2.1, the sggi Γ_3 is a string C-group representation. Now, the intersection $G_0 \cap G_3 = G_{0,3}$ need only to be checked in the cases $n \in \{15, 19\}$, which can be done with MAGMA. Hence, again, by Proposition 2.1, we see that Γ is a string C-group representation.

It remains to show that G is isomorphic to A_n. The structure of G_3 shows that the action of G_3 on the $(n-7)$ vertices at the top of the graph is A_{n-7}. Hence there exists a cycle of order 3 in G_0 acting on those vertices. This cycle necessarily fixes the 7 other vertices, so it is a cycle of G. Moreover, that action is $(n-9)$-transitive on the top vertices. Hence the stabilizer, in G, of the leftmost vertex of the graph must be transitive on the remaining vertices and G is 2-transitive, therefore primitive. Then, by Theorem 3.4, we can conclude that $G \geq A_n$. Since all generators of G are even permutations, we conclude that G is isomorphic to A_n.

The Schläfli type follows immediately from the permutation representation graph. □

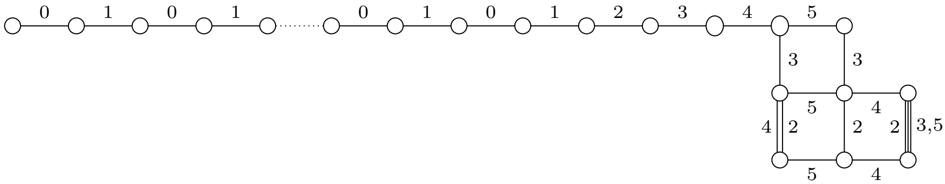
Theorem 4.13. *If* $n \equiv 3 \pmod 4$ *with* $n \geq 15$*, then the group* A_n *admits a string C-group representation of rank* 5*, with Schläfli type* $\{n-10, 6, 6, 5\}$*, with the following CPR-graph.*



Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is isomorphic to S_{12} no matter how large n is. One can easily check with MAGMA that the permutation representation graph corresponding to Γ_0 is a CPR graph. The group $G_{0,4}$ is isomorphic to $2^3 : S_3 \times S_3$ no matter how large n is. $G_{3,4}$ is isomorphic to S_{n-9} by Theorem 3.4, as it contains a cycle of length 3, namely $(\rho_1\rho_2)^2$ and is obviously 2-transitive on $n-9$ vertices. Moreover, by [10, Theorem 4.1], $\Gamma_{3,4}$ is a string C-group representation as it is generated by three involutions, two of which commute. The group $G_{0,3,4}$ is isomorphic to D_6. Looking at the respective orbits of $G_{0,4}$ and $G_{3,4}$ we can conclude that $G_{0,4} \cap G_{3,4} = G_{034}$ and therefore Γ_4 is a string C-group representation. Moreover, one can check that the group G_4 is isomorphic to $A_{n-8} \times C_2 : S_3$ but this is not needed to finish the proof. Now, it is easy to check with MAGMA that $G_0 \cap G_4 = G_{0,4}$ for $n = 15$ and this intersection does not depend on the degree of G. Therefore, by Proposition 2.1, we may conclude that Γ is a string C-group representation with the given permutation representation graph. A similar argument as in the proof of Theorem 4.12 shows that G is isomorphic to A_n. The Schläfli type follows immediately from the permutation representation graph. □

Theorem 4.14. *If* $n \equiv 3 \pmod 4$ *with* $n \geq 15$*, then the group* A_n *admits a string C-group representation of rank* 6*, with Schläfli type* $\{n-10, 6, 3, 5, 3\}$*, with the following*

CPR-graph.

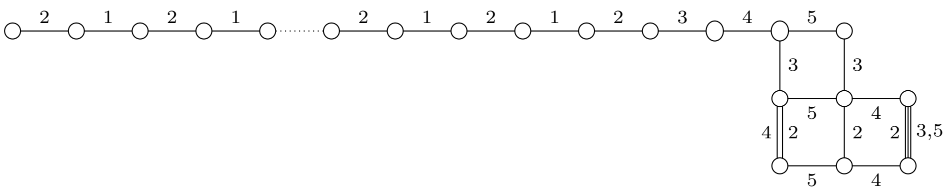


Proof. Let $\Gamma := (G, S)$ be the sggi having the permutation representation graph above. The group G_0 is isomorphic to S_{12} no matter how big n is. One can easily check with MAGMA that the permutation representation graph corresponding to Γ_0 is a CPR graph. We have $G_{0,5} \cong S_7 \times A_5$ no matter how big n is. Here $G_{3,4,5} \cong S_{n-9}$ as proven in the previous theorem (for G_{34} in the previous theorem is the same group as $G_{3,4,5}$ here). Similarly, we have $G_{0,4,5} \cong 2^2 : S_3 \times S_3$. As $G_{3,4,5} \cap G_{0,4,5} = G_{0,3,4,5}$ independently on how big n is, we can conclude by Proposition 2.1 that $\Gamma_{4,5}$ is a string C-group representation. Similarly, as $G_{0,5} \cap G_{4,5} = G_{0,4,5}$ no matter how big n is, we can conclude by Proposition 2.1 that Γ_5 is a string C-group representation. Finally, as $G_0 \cap G_5 = G_{0,5}$ no matter how big n is, we conclude that Γ is a string C-group representation.

It remains to show that G is isomorphic to A_n. Similar arguments as in the proof of the previous two theorems lead to that conclusion. The Schläfli type follows immediately from the permutation representation graph. □

Observe that this last family of string C-group representations of rank 6 gives, using the same general construction we used in Theorems 4.2 and 4.7, a family of string C-groups of rank 5 with Schläfli type $\{n - 10, 6, 5, 3\}$.

Theorem 4.15. *If $n \equiv 3 \pmod 4$ with $n \geq 15$, then the group A_n admits a string C-group representation of rank 5, with Schläfli type $\{n - 9, 6, 5, 3\}$, with the following CPR-graph.*



We leave the proof of this last theorem to the interested reader as it is very similar to the previous proofs.

5 Concluding remarks

Mark Mixer mentioned a similar result in 2015 at the AMS Fall Eastern Sectional Meeting in Rutgers (talk 1115-20-283).

The techniques we developed in this paper inspired Brooksbank and the second author to develop a general rank reduction technique, now available in [4].

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 311–318
https://doi.org/10.26493/1855-3974.1435.c71
(Also available at http://amc-journal.eu)

Vertex transitive graphs G with $\chi_D(G) > \chi(G)$ and small automorphism group*

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Received 24 June 2017, accepted 1 October 2019, published online 29 October 2019

Abstract

For a graph G and a positive integer k, a vertex labelling $f\colon V(G) \to \{1, 2, \ldots, k\}$ is said to be k-distinguishing if no non-trivial automorphism of G preserves the sets $f^{-1}(i)$ for each $i \in \{1, \ldots, k\}$. The distinguishing chromatic number of a graph G, denoted $\chi_D(G)$, is defined as the minimum k such that there is a k-distinguishing labelling of $V(G)$ which is also a proper coloring of the vertices of G. In this paper, we prove the following theorem: Given $k \in \mathbb{N}$, there exists an infinite sequence of vertex-transitive graphs $G_i = (V_i, E_i)$ such that

1. $\chi_D(G_i) > \chi(G_i) > k$,
2. $|\operatorname{Aut}(G_i)| < 2k|V_i|$, where $\operatorname{Aut}(G_i)$ denotes the full automorphism group of G_i.

In particular, this answers a question posed by the first and second authors of this paper.

Keywords: Distinguishing chromatic number, vertex transitive graphs, Cayley graphs.

Math. Subj. Class.: 05C15, 05D40, 20B25, 05E18

*The first and second authors would like to thank Ted Dobson for useful discussions.
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1 Introduction

Let G be a graph. An automorphism of G is a permutation φ of the vertex set $V(G)$ of G such that, for any $x, y \in V(G)$, $\varphi(x), \varphi(y)$ are adjacent if and only x, y are adjacent. The automorphism group of a graph G, denoted by $\mathrm{Aut}(G)$, is the group of all automorphisms of G. A graph G is said to be vertex transitive if, for any $u, v \in V(G)$, there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(u) = v$.

Given a positive integer r, an r-coloring of G is a map $f\colon V(G) \to \{1, 2, \dots, r\}$ and the sets $f^{-1}(i)$, for $i \in \{1, 2 \dots, r\}$, are the color classes of f. An automorphism $\varphi \in \mathrm{Aut}(G)$ is said to fix a color class C of f if $\varphi(C) = C$, where $\varphi(C) = \{\varphi(v) : v \in C\}$. A coloring of G, with the property that no non-trivial automorphism of G fixes every color class, is called a distinguishing coloring of G.

Collins and Trenk in [5] introduced the notion of the distinguishing chromatic number of a graph G, which is defined as the minimum number of colors needed to color the vertices of G so that the coloring is both proper and distinguishing. Thus, the distinguishing chromatic number of G is the least integer r such that the vertex set can be partitioned into sets $V_1, V_2, \dots, V_r$ such that each V_i is independent in G, and for every non-trivial $\varphi \in \mathrm{Aut}(G)$ there exists some color class V_i with $\varphi(V_i) \neq V_i$. The distinguishing chromatic number of a graph G, denoted by $\chi_D(G)$, has been the topic of considerable interest recently (see, for instance, [1, 2, 3, 4]).

One of the many questions of interest regarding the distinguishing chromatic number concerns the contrast between $\chi_D(G)$ and the cardinality of $\mathrm{Aut}(G)$. For instance, the Kneser graphs $K(n, r)$ have very large automorphism groups and yet, $\chi_D(K(n, r)) = \chi(K(n, r))$ for $n \geq 2r + 1$, and $r \geq 3$ (see [2]). The converse question is compelling: Are there infinitely many graphs G_n with 'small' automorphism groups and satisfying $\chi_D(G_n) > \chi(G_n)$?

The question as posed above is not actually interesting for two reasons. First, for all even n, $\chi_D(C_n) > \chi(C_n) = 2$ and $|\mathrm{Aut}(C_n)| = 2n$, where C_n is the cycle of length n. Second, if one stipulates that G also has arbitrarily large chromatic number, then here is a construction for such a graph. Start with a rigid graph G with a leaf vertex x and having large chromatic number (one can obtain this by minor modifications to a random graph, for instance); then, blow up the leaf vertex x to a new disjoint set X whose neighbor in the new graph $\widetilde{G}$ is the same as the neighbor of x in G. In fact one can arrange for $\chi_D(\widetilde{G}) - \chi(\widetilde{G})$ to be as large as one desires. Furthermore, since $|\mathrm{Aut}(\widetilde{G})| = |X|!$, this provides examples of graphs for which the automorphism groups are relatively 'small' in terms of the order of the graph.

In the example above, the fact that $\chi_D(G)$ is larger than $\chi(G)$ is accounted for by a 'local' reason, and that is what makes the problem stated above not very interesting. However, if one further stipulates that the graph is vertex-transitive, then the same question is highly non-trivial. In [1], the first and second authors constructed families of vertex-transitive graphs with $\chi_D(G) > \chi(G) > k$ and $|\mathrm{Aut}(G)| = O(|V(G)|^{3/2})$, for any given k. In this paper, we improve upon that result:

Theorem 1.1. *Given $k \in \mathbb{N}$, there exists an infinite family of graphs $G_n = (V_n, E_n)$ satisfying:*

1. $\chi_D(G_n) > \chi(G_n) > k$,
2. *G_n is vertex transitive and* $|\mathrm{Aut}(G_n)| < 2k|V_n|$.

Our family of graphs consists of Cayley graphs. To recall the definition, let A be a group and let S be an inverse-closed subset of A, i.e., $S = S^{-1}$, where $S^{-1} := \{s^{-1} : s \in S\}$. The Cayley graph $\mathrm{Cay}(A, S)$ is the graph with vertex set A and the vertices u and v are adjacent in $\mathrm{Cay}(A, S)$ if and only if $uv^{-1} \in S$.

We start with a brief description of the graphs of our construction. For q, an odd prime, let $\mathbb{F}_q^n$ denote the n-dimensional vector space over $\mathbb{F}_q$. Our graphs shall be Cayley graphs $\mathrm{Cay}(\mathbb{F}_q^n, S)$ for some suitable inverse-closed set $S \subset \mathbb{F}_q^n$ which is obtained by taking a union of a certain collection of lines in $\mathbb{F}_q^n$ and then deleting the zero element of $\mathbb{F}_q^n$. More precisely, let $\mathcal{H}_0 := \{(x_1, x_2, \dots, x_{n-1}, 0) : x_i \in \mathbb{F}_q, 1 \le i \le n-1\}$ and let $\mathbf{0}$ denote the element $(0, \dots, 0) \in \mathbb{F}_q^n$. For each line (1-dimensional subspace of $\mathbb{F}_q^n$) $\ell \subset \mathbb{F}_q^n$ satisfying $\ell \cap \mathcal{H}_0 = \{\mathbf{0}\}$, pick ℓ independently with probability $1/2$ to form the random set $\widetilde{S}$. Our connection set S for the Cayley graph $\mathrm{Cay}(\mathbb{F}_q^n, S)$ is defined by $S := \{v \in \mathbb{F}_q^n : v \in \ell \text{ for some } \ell \in \widetilde{S}\} \setminus \{\mathbf{0}\}$. Our main theorem states that with high probability, $G_{n,S} := \mathrm{Cay}(\mathbb{F}_q^n, S)$ satisfies the conditions of Theorem 1.1.

To show that these graphs have 'small' automorphism groups, we prove a stronger version of Theorem 4.3 of [6] in this particular context, which is also a result of independent interest.

Theorem 1.2. *Let q be a prime power, let n be a positive integer with $n \ge 2$ and let G be the additive group of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ of cardinality q, and let $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$ be the multiplicative group of the field $\mathbb{F}_q$ with its natural group action on G by scalar multiplication, and write $K := \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$. If S is an inverse-closed subset of G with $K \le \mathrm{Aut}(\mathrm{Cay}(G, S))$, then either*

(i) $\mathrm{Aut}(\mathrm{Cay}(G, S)) = K$, *or*

(ii) there exists $\varphi \in \mathrm{Aut}(\mathrm{Cay}(G, S)) \setminus K$ *with* φ *normalizing* G.

Remark 1.3. Theorem 1.2 is valid even though the connection set S is not inverse-closed. Since we deal with Cayley graphs the phrase inverse-closed subset is used in the statement of the theorem.

The rest of the paper is organized as follows. We start with some preliminaries in Section 2 and then include the proofs of Theorems 1.1 and 1.2 in the next section. We conclude with some remarks and some open questions.

2 Preliminaries

We begin with a few definitions from finite geometry. For more details, one may see [13, 14]. By $\mathrm{PG}(n, q)$ we mean the Desarguesian projective space obtained from the affine space $\mathrm{AG}(n+1, q)$.

Definition 2.1. A cone with vertex $A \subset \mathrm{PG}(k, q)$ and base $B \subset \mathrm{PG}(n-k-1, q)$, where $\mathrm{PG}(k, q) \cap \mathrm{PG}(n-k-1, q) = \emptyset$, is the set of points lying on the lines connecting points of A and B.

Definition 2.2. Let V be an $(n+1)$-dimensional vector space over a finite field $\mathbb{F}$. A subset S of $\mathrm{PG}(V)$ is called an $\mathbb{F}_q$-linear set if there exists a subset U of V that forms an $\mathbb{F}_q$-vector space, for some $\mathbb{F}_q \subset \mathbb{F}$, such that $S = \mathcal{B}(U)$, where

$$\mathcal{B}(U) := \{\langle u \rangle_{\mathbb{F}} : u \in U \setminus \{\mathbf{0}\}\}$$

and where $\langle u\rangle_{\mathbb{F}}$ denotes the projective point of $\mathrm{PG}(V)$, corresponding to the vector u of $U \subset V$.

Further details about $\mathbb{F}_q$-linear sets can be found in [14], for instance.

The projective space $\mathrm{PG}(n, q)$ can be partitioned into an affine space $\mathrm{AG}(n, q)$ and a hyperplane at infinity, denoted by H_∞.

Definition 2.3. Following [13], we say that a set of points $U \subset \mathrm{AG}(n, q)$ determines the direction $d \in H_\infty$, if there is an affine line through d meeting U in at least two points.

We now state the main theorem of [13] which will be relevant in our setting.

Theorem 2.4. *Let $U \subset \mathrm{AG}(n, \mathbb{F}_q)$, $n \geq 3$, $|U| = q^k$. Suppose that U determines at most $\frac{q+3}{2}q^{k-1} + q^{k-2} + \cdots + q^2 + q$ directions and suppose that U is an $\mathbb{F}_p$-linear set of points, where $q = p^h$, $p > 3$ prime. If $n - 1 \geq (n - k)h$, then U is a cone with an $(n-1-h(n-k))$-dimensional vertex at H_∞ and with base a $\mathbb{F}_q$-linear point set $U_{(n-k)h}$ of size $q^{(n-k)(h-1)}$, contained in some affine $(n-k)h$-dimensional subspace of $\mathrm{AG}(n, q)$.*

We end this section by recalling another result that appears in [6] as Theorem 4.2.

Theorem 2.5. *Let G be a permutation group on Ω with a proper self-normalizing abelian regular subgroup. Then $|\Omega|$ is not a prime power.*

3 Proofs of the Theorems

In this section we prove Theorems 1.1 and 1.2 starting with the proof of Theorem 1.2. We believe that this result is only the tip of an iceberg: its current statement has been tailored to the context of our setting, and uses some ideas that appear in [6, Section 3] and [9].

Proof of Theorem 1.2. We suppose that (i) does not hold, that is, K is a proper subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$; we show that (ii) holds. Write $\Gamma := \mathrm{Cay}(G, S)$.

Let B be a subgroup of $\mathrm{Aut}(\Gamma)$ with $K < B$ and with K maximal in B. Suppose that $K \lhd B$. As G is characteristic in K, we get $G \lhd B$. In particular, every element φ in $B \setminus K$ satisfies (ii).

Suppose then that K is not normal in B. Since K is maximal in B and $G \lhd K$, we have $\mathbf{N}_B(G) = K$. Suppose that there exists $b \in B \setminus K$ such that $L := \langle G, G^b\rangle$ (the smallest subgroup of B containing G and G^b) satisfies $L \cap K = G$. We claim that we are now in the position to apply Theorem 2.5 (and implicitly some ideas from [9]). Indeed, as $\mathbf{N}_L(G) = \mathbf{N}_B(G) \cap L = K \cap L = G$, L is a transitive permutation group on the vertices of Γ with a proper regular self-normalizing abelian subgroup G. (Observe that G is a proper subgroup of L because $b \notin \mathbf{N}_B(G) = K$.) By Theorem 2.5, $|G|$ is not a prime power, which is a contradiction because $|G| = q^n$. This proves that, for every $b \in B \setminus K$, we have $\langle G, G^b\rangle \cap K > G$.

Fix $b \in B \setminus K$. Now, G and G^b are abelian and hence $G \cap G^b$ is centralized by $\langle G, G^b\rangle$. From the preceding paragraph, there exists $k \in \langle G, G^b\rangle \cap K$ with $k \notin G$. Observe now that $K = \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ is a Frobenius group with kernel $G = \mathbb{F}_q^n$ and complement $\mathbb{F}_q^*$. Therefore, k acts by conjugation fixed-point-freely on $G \setminus \{\mathbf{0}\}$. As k centralizes $G \cap G^b$, we deduce $|G \cap G^b| = 1$.

Let $C := \bigcap_{x\in B} K^x$ be the core of K in B. As $G \cap G^b = 1$ for all $b \in B \setminus K$, $K \cap K^b$ has no non-identity q-elements. Therefore $C \cap G = 1$. As $C \lhd B$ and $C \leq K$, C is a normal subgroup of the Frobenius group K intersecting its kernel on the identity. This yields $C = 1$.

Let Ω be the set of right cosets of K in B. From the paragraph above, B acts faithfully on Ω. Moreover, as K is maximal in B, the action of B on Ω is primitive. Therefore B is a finite primitive group with a solvable point stabilizer K. In [11], Li and Zhang have explicitly determined such primitive groups: these are classified in [11, Theorem 1.1] and [11, Tables I–VII]. Now, using the terminology in [11], a careful (but not very difficult) case-by-case analysis on the tables in [11] shows that B is a primitive group of affine type, that is, B contains an elementary abelian normal r-subgroup V, for some prime r. For this analysis it is important to keep in mind that the stabilizer K is a Frobenius group with kernel the elementary abelian group $G \cong \mathbb{F}_q^n$ and $n \geq 2$.

Let $|V| = r^t$. Now, the action of B on Ω is permutation equivalent to the natural action of $B = V \rtimes K$ on V, with V acting via its regular representation and with K acting by conjugation. Observe that $q \neq r$, because K acts faithfully and irreducibly as a linear group on V and hence K contains no non-identity normal r-subgroups. Observe further that $|B| = |V||K| = r^t \cdot q^n \cdot (q-1)$.

We are finally ready to reach a contradiction and to do so, we go back studying the action of B on the vertices of Γ. Observe that B is solvable because V is solvable and so is $B/V \cong K$. We write $B_{\mathbf{0}}$ for the stabilizer in B of the vertex $\mathbf{0}$ of Γ. As G acts regularly on the vertices of Γ, we obtain $B = B_{\mathbf{0}}G$ and $B_{\mathbf{0}} \cap G = 1$. In particular, $|B_{\mathbf{0}}| = r^t \cdot (q-1)$. Observe that $B_{\mathbf{0}}$ is a Hall Π-subgroup of the solvable group B, where Π is the set of all the prime divisors of $q-1$ together with the prime r. As V is a Π-subgroup, from the theory of Hall subgroups (see for instance [7], Theorem 3.3), V has a conjugate contained in $B_{\mathbf{0}}$. Since $V \lhd B$, we have $V \leq B_{\mathbf{0}}$. This is clearly a contradiction because V is normal in B, but $B_{\mathbf{0}}$ is core-free in B, being the stabilizer of a point in a transitive permutation group. □

For the next lemma, recall that

$$\mathcal{H}_0 := \{(x_1, x_2, \ldots, x_{n-1}, 0) : x_i \in \mathbb{F}_q, 1 \leq i \leq n-1\}.$$

In what follows, $G_{n,S}$ will denote the Cayley graph $\mathrm{Cay}(\mathbb{F}_q^n, S)$ and $S = \widetilde{S} \setminus \{\mathbf{0}\}$ for some set $\widetilde{S} = \bigcup_{\ell \in \mathcal{L}} \ell$, where $\mathcal{L}$ is a collection of lines in $\mathbb{F}_q^n$ with each $\ell \in \mathcal{L}$ satisfying $\ell \cap \mathcal{H}_0 = \{\mathbf{0}\}$.

Lemma 3.1. *If $\mathcal{L} \neq \emptyset$, then $\chi(G_{n,S}) = q$.*

Proof. Observe that each line that belongs to the set S gives rise to a clique of size q in the graph $G_{n,S}$. Therefore $\chi(G_{n,S}) \geq q$. On the other hand, for a fixed $v \in S$, the partition $(C_\lambda)_{\lambda \in \mathbb{F}_q}$, where $C_\lambda := \{w + \lambda v : w \in \mathcal{H}_0\}$, of the vertex set $\mathbb{F}_q^n$ is a proper coloring of the graph $G_{n,S}$. Indeed, for any distinct $x = w_1 + \lambda v$, $y = w_2 + \lambda v$ in C_λ, we have $x - y = w_1 - w_2 \notin S$ because $w_1 - w_2 \in \mathcal{H}_0$ and $S \cap \mathcal{H}_0 = \emptyset$. Therefore the sets C_λ are independent in $G_{n,S}$ for each $\lambda \in \mathbb{F}_q$. □

Lemma 3.2. *Assume that q is prime. Let $\widetilde{S}$ be the random set corresponding to a union of lines ℓ in $\mathbb{F}_q^n$ with $\ell \cap \mathcal{H}_0 = \{\mathbf{0}\}$ and where each $\ell \in \mathbb{F}_q^n$ is chosen independently with probability $\frac{1}{2}$; and let $S = \widetilde{S} \setminus \{\mathbf{0}\}$. Then*

$$\mathbb{P}\left(\chi_D(G_{n,S}) > q\right) \geq 1 - \exp\left(-\frac{q^{n-3}}{4}\right).$$

Proof. First, note that $\mathbb{E}(|S|) = \frac{q^{n-1}}{2}$, so taking $\delta = \frac{1}{q}$ and $\mu = \mathbb{E}(|S|)$ in the Chernoff bound (see (2.6) on page 26 of [10]) we obtain

$$\mathbb{P}\left(|S| < \frac{q^{n-1} - q^{n-2}}{2}\right) \leq \exp\left(-\frac{q^{n-3}}{4}\right).$$

In particular, with probability at least $1 - \exp(-q^{n-3}/4)$, we have $|S| > \frac{q^{n-1}-q^{n-2}}{2}$. We may thus assume $|S| > \frac{q^{n-1}-q^{n-2}}{2}$ in what follows.

We claim that every color class in a proper q-coloring of $G_{n,S}$ is an affine hyperplane of $\mathbb{F}_q^n$. To see why, let $C_1, \dots, C_q$ be independent sets in $G_{n,S}$ witnessing a proper q-coloring of $G_{n,S}$. Fix $v \in S$ and consider the line $\ell_v := \{\lambda v : \lambda \in \mathbb{F}_q\}$ along with its translates $\ell_v + w := \{\lambda v + w : \lambda \in \mathbb{F}_q\}$, for $w \in \mathcal{H}_0$. Each set $\ell_v + w$ is a clique of size q in $G_{n,S}$, and these cliques partition the vertex set of $G_{n,S}$, so in particular each C_i contains at most one vertex from each of these translates $\ell_v + w$. Consequently, $|C_i| \leq q^{n-1}$ for all $i \in \{1, \dots, q\}$. By size considerations, it follows that $|C_i| = q^{n-1}$ for each $i \in \{1, \dots, q\}$.

Consider a color class C. Suppose C determines at least $\frac{q+3}{2}q^{n-2} + q^{n-3} + \dots + q^2 + q + 1$ directions. Then if $\langle C \rangle$ denotes the set of all vertices in the affine lines intersecting at least two points in C, we have $|\langle C \rangle| + |S| > 1 + q + \dots + q^{n-1}$, so $\langle C \rangle \cap S \neq \emptyset$. However, this contradicts the assumption that C is an independent set in $G_{n,S}$. Therefore C determines at most $\frac{q+3}{2}q^{n-2} + q^{n-3} + \dots + q^2 + q$ directions. Since q is prime, by Corollary 10 in [13], it follows that C is an $\mathbb{F}_q$-linear set. Hence, by Theorem 2.4, the color class C is a cone with an $n-2$ (projective) dimensional vertex $\mathcal{V}$ at H_∞ and an affine point u_1 as base. In particular, the affine plane corresponding to the $\mathbb{F}_q$-subspace spanned by $\mathcal{V}$ passing through the affine point u_1 is contained in C. Since $|C| = q^{n-1}$, it follows that C is this affine hyperplane, and this proves the claim.

To complete the proof, observe that for each $\lambda \in \mathbb{F}_q^* \setminus \{1\}$, the map $\varphi_\lambda(x) = \lambda x$, $x \in \mathbb{F}_q^n$ fixes each color class. Moreover, φ_λ fixes the set S and $\varphi_\lambda(u) - \varphi_\lambda(v) = \varphi_\lambda(u - v)$, so φ_λ is a non-trivial automorphism which fixes each color class. Therefore $\chi_D(G_{n,S}) > q$. □

Lemma 3.3. *If $n \geq 6$ and $q \geq 5$ is prime, then $\operatorname{Aut}(G_{n,S}) \cong \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ with probability at least*

$$1 - 2^{-\frac{q^{n-1}}{3}}.$$

Proof. Since $G_{n,S}$ is a Cayley graph on the additive group $G = \mathbb{F}_q^n$, by Theorem 1.2, either $\operatorname{Aut}(G_{n,S}) = K \cong \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ or there exists $\varphi \in \operatorname{Aut}(G_{n,S}) \setminus K$ with φ normalizing $G = \mathbb{F}_q^n$. We show that with probability at least $1 - 2^{-\frac{q^{n-1}}{3}}$, there is no φ satisfying the latter condition.

Suppose $\varphi \in \operatorname{Aut}(G_{n,S})$ normalizes $\mathbb{F}_q^n$. If $a = \varphi(\mathbf{0})$ and $\lambda_a : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is the right translation via a, then $\lambda_a^{-1}\varphi$ is an automorphism of $G_{n,S}$ normalizing $\mathbb{F}_q^n$ and with $(\lambda_a^{-1}\varphi)(\mathbf{0}) = (\lambda_a^{-1})(\varphi(\mathbf{0})) = (\lambda_a^{-1})(a) = a - a = \mathbf{0}$. Therefore, without loss of generality, we may assume that $\varphi(\mathbf{0}) = \mathbf{0}$. Since S is the neighbourhood of $\mathbf{0}$ in $G_{n,S}$, we get $\varphi(S) = S$. Moreover, since φ acts as a group automorphism on $\mathbb{F}_q^n$, we have $\varphi \in \mathrm{GL}_n(q)$.

Now, for $\varphi \in \mathrm{GL}_n(q)$, let E_φ denote the event $\varphi(S) = S$. Let $\mathcal{L}$ denote the set of all lines ℓ with $\ell \cap \mathcal{H}_0 = \emptyset$. Also, let $\operatorname{Orb}_\varphi(\ell) = \{\ell, \varphi(\ell), \varphi^2(\ell), \dots, \varphi^k(\ell)\}$ where $\varphi^{k+1}(\ell) = \ell$. Then

$$\mathbb{P}(E_\varphi) \leq \prod_{i=1}^{N_\varphi} 2^{1-|\operatorname{Orb}_\varphi(\ell_i)|} = 2^{N_\varphi - |\mathcal{L}|},$$

where N_φ denotes the number of distinct orbits of φ in $\mathcal{L}$. Setting $\mathcal{G} = \mathrm{GL}(n,q) \setminus \{\lambda I : \lambda \in \mathbb{F}_q^*\}$, we have

$$\mathbb{P}\left(\bigcup_{\varphi\in\mathcal{G}} E_\varphi\right) \le \sum_{\varphi\in\mathcal{G}} \mathbb{P}(E_\varphi) \le 2^{-|\mathcal{L}|} \sum_{\varphi\in\mathcal{G}} 2^{N_\varphi}. \tag{3.1}$$

Let $F_\varphi := |\{\ell \in \mathcal{L} : \varphi(\ell) = \ell\}|$ and $F := \max_{\varphi\in\mathcal{G}} F_\varphi$. Now $N_\varphi \le F + \frac{|\mathcal{L}|-F}{2} = \frac{F+|\mathcal{L}|}{2}$. Thus, it suffices to give a suitable upper bound for F. Towards that end, we note that, if $F_\varphi = F$ for $\varphi \in \mathcal{G}$, then every line ℓ fixed by φ corresponds to an eigenvector of φ. If $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$ denote the eigenspaces of φ for some distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then

$$F_\varphi \le \sum_{i=1}^{k} \left(\binom{\dim \mathcal{E}_i}{1}_q - \binom{\dim(\mathcal{E}_i \cap \mathcal{H}_0)}{1}_q \right) \le q^{n-2} + 1.$$

Similarly, we have $|\mathcal{L}| = \binom{n}{1}_q - \binom{n-1}{1}_q = q^{n-1}$, and so by (3.1), we have

$$\mathbb{P}\left(\bigcup_{\varphi\in\mathcal{G}} E_\varphi\right) \le |\mathcal{G}| 2^{\frac{F-|\mathcal{L}|}{2}} < q^{n^2} 2^{-\frac{q^{n-1}-q^{n-2}-1}{2}} < 2^{-\frac{q^{n-1}}{3}},$$

for $q \ge 5$, $n \ge 6$. □

Computations and estimates similar to the ones presented in the proof of Lemma 3.3 have been proved useful in a variety of problems, see for instance [1, 8] and [12, Section 6.4].

Proof of Theorem 1.1. Given $k \in \mathbb{N}$ with $k \ge 4$, pick a prime number q with $k < q < 2k$. For $n \ge 6$, consider the random graph $G_{n,S}$ of the group $\mathbb{F}_q^n$ as constructed above. By Lemmas 3.1, 3.2 and 3.3, with positive probability, the graph $G_{n,S}$ satisfies the statements of the lemmas, and hence satisfies the conclusions of Theorem 1.1. □

4 Concluding remarks

- We observe that, for S chosen randomly as in the proof of our result, the distinguishing chromatic number of $G_{n,S}$ is $q+1$ with high probability. Indeed, consider the q-coloring C described in Lemma 3.1. Re-color the vertex $\mathbf{0}$ using an additional color. Then the coloring described by the partition $C' = C \cup \{\mathbf{0}\}$ is a proper, distinguishing coloring of $G_{n,S}$ with $q+1$ colors. In fact, C' is clearly proper, and to show that it is distinguishing, consider $\varphi \in \mathrm{Aut}(G_{n,S}) = \mathbb{F}_q^n \rtimes \mathbb{F}_q^*$ (by Lemma 3.3) that fixes every color class. Write $\varphi(x) = \lambda x + b$ with $\lambda \in \mathbb{F}_q^*, b \in \mathbb{F}_q^n$. Since φ fixes the color class containing $\mathbf{0}$, we have $b = \mathbf{0}$. Also, x and λx cannot be in same color class unless $\lambda = 1$. Therefore φ is the identity automorphism.

  It is interesting to determine if one can obtain families of vertex-transitive graphs with $\chi_D(G) > \chi(G) + 1$, with 'small' automorphism groups and with $\chi(G)$ being arbitrarily large. In fact, for $k \in \mathbb{N}$, there is no known family of vertex-transitive graphs for which $\chi_D(G) > \chi(G) + 1 > k$ and $|\mathrm{Aut}(G)| = O(|V(G)|^{O(1)})$. It is plausible that Cayley graphs over certain groups may provide the correct constructions.

- Theorem 1.1 establishes, for any fixed k, the existence of vertex-transitive graphs $G_n = (V_n, E_n)$ with $\chi_D(G_n) > \chi(G_n) > k$ and with $|\operatorname{Aut}(G_n)| < 2k|V_n|$. It would be interesting to obtain a similar family of graphs that satisfy with $\chi_D(G_n) > \chi(G_n) > k$ and with $|\operatorname{Aut}(G_n)| \leq C|V_n|$, for some absolute constant C.

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ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 319–347
https://doi.org/10.26493/1855-3974.1516.58f
(Also available at http://amc-journal.eu)

On graphs with exactly two positive eigenvalues*

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Received 27 October 2017, accepted 28 May 2019, published online 30 October 2019

Abstract

The inertia of a graph G is defined to be the triplet $\mathrm{In}(G) = (p(G), n(G), \eta(G))$, where $p(G)$, $n(G)$ and $\eta(G)$ are the numbers of positive, negative and zero eigenvalues (including multiplicities) of the adjacency matrix $A(G)$, respectively. Traditionally $p(G)$ (resp. $n(G)$) is called the positive (resp. negative) inertia index of G. In this paper, we introduce three types of congruent transformations for graphs that keep the positive inertia index and negative inertia index. By using these congruent transformations, we determine all graphs with exactly two positive eigenvalues and one zero eigenvalue.

Keywords: Congruent transformation, positive (negative) inertia index, nullity.

Math. Subj. Class.: 05C50

*The authors are grateful to the anonymous referees for their useful and constructive comments, which have considerably improved the presentation of this paper.

†Supported by the Scientific Research Projects of Universities in Xinjiang Province (No. XJEDU2019Y030).

‡Corresponding author. Supported by the National Natural Science Foundation of China (Nos. 11671344, 11531011).

§Supported by the China Postdoctoral Science Foundation (No. 2019M652556), and the Postdoctoral Research Sponsorship in Henan Province (No. 1902011).

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1 Introduction

All graphs considered here are undirected and simple. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The order of G is the number of vertices of G, denoted by $|G|$. For $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ the *neighborhood* of v, $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighborhood* of v and $d(v) = |N_G(v)|$ the degree of v. A vertex of G is said to be *pendant* if it has degree 1. By $\delta(G)$ we mean the minimum degree of vertices of G. As usual, we denote by $G + H$ the disjoint union of two graphs G and H, $K_{n_1,\dots,n_l}$ the complete multipartite graph with l parts of sizes $n_1, \dots, n_l$, and K_n, C_n, P_n the complete graph, cycle, path on n vertices, respectively.

The *adjacency matrix* of G, denoted by $A(G) = (a_{ij})$, is the square matrix with $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Clearly, $A(G)$ is a symmetric matrix with zeros on the diagonal, and thus all the eigenvalues of $A(G)$ are real, which are defined to be the *eigenvalues* of G. The multiset consisting of eigenvalues along with their multiplicities is called the *spectrum* of G denoted by $\mathrm{Spec}(G)$. To characterize graphs in terms of their eigenvalues has always been of the great interests for researchers, for instance to see [2, 4, 5, 8, 9] and references therein.

The *inertia* of a graph G is defined as the triplet $\mathrm{In}(G) = (p(G), n(G), \eta(G))$, where $p(G)$, $n(G)$ and $\eta(G)$ are the numbers of positive, negative and zero eigenvalues (including multiplicities) of G, respectively. Traditionally $p(G)$ (resp. $n(G)$) is called the *positive* (resp. *negative*) *inertia index* of G and $\eta(G)$ is called the *nullity* of G. Obviously, $p(G) + n(G) = r(G) = n - \eta(G)$ if G has n vertices, where $r(G)$ is the rank of $A(G)$. Let B and D be two real symmetric matrices of order n. Then D is called *congruent* to B if there is an real invertible matrix C such that $D = C^T BC$. Traditionally we say that D is obtained from B by congruent transformation. The famous Sylvester's law of inertia states that the inertia of two matrices is unchanged by congruent transformation.

Since the adjacency matrix $A(G)$ of G has zero diagonal, we have $p(G) \geq 1$ if G has at least one edge. One of the attractive problems is to characterize those graphs with a few positive eigenvalues. In [9] Smith characterized all graphs with exactly one positive eigenvalue. Recently, Oboudi [6] completely determined the graphs with exactly two non-negative eigenvalues, i.e., those graphs satisfying $p(G) = 1$ and $\eta(G) = 1$ or $p(G) = 2$ and $\eta(G) = 0$.

In this paper, we introduce three types congruent transformations for graphs. By using these congruent transformations and Oboudi's results in [6], we completely characterize the graphs satisfying $p(G) = 2$ and $\eta(G) = 1$.

2 Preliminaries

In this section, we will introduce some notions and lemmas for the latter use.

Theorem 2.1 (Interlacing theorem [1]). *Let G be a graph of order n and H be an induced subgraph of G with order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Lemma 2.2 ([1]). *Let H be an induced subgraph of graph G. Then $p(H) \leq p(G)$.*

Lemma 2.3 ([3]). *Let G be a graph containing a pendant vertex, and let H be the induced subgraph of G obtained by deleting the pendant vertex together with the vertex adjacent to it. Then $p(G) = p(H) + 1$, $n(G) = n(H) + 1$ and $\eta(G) = \eta(H)$.*

Lemma 2.4 (Sylvester's law of inertia). *If two real symmetric matrices A and B are congruent, then they have the same positive (resp., negative) inertia index, the same nullity.*

Theorem 2.5 ([9]). *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Let G_1 be a graph containing a vertex u and G_2 be a graph of order n that is disjoint from G_1. For $1 \leq k \leq n$, the *k-joining* graph of G_1 and G_2 with respect to u, denoted by $G_1(u) \odot^k G_2$, is a graph obtained from $G_1 \cup G_2$ by joining u to arbitrary k vertices of G_2. By using the notion of k-joining graph, Yu et al. [11] completely determined the connected graphs with at least one pendant vertex that have positive inertia index 2.

Theorem 2.6 ([11]). *Let G be a connected graph with pendant vertices. Then $p(G) = 2$ if and only if $G \cong K_{1,r}(u) \odot^k K_{n_1,\ldots,n_l}$, where u is the center of $K_{1,r}$ and $1 \leq k \leq n_1 + \cdots + n_l$.*

Theorem 2.7 ([6]). *Let G be a graph of order $n \geq 2$ with eigenvalues $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. Assume that $\lambda_3(G) < 0$, then the following hold:*

(1) If $\lambda_1(G) > 0$ and $\lambda_2(G) = 0$, then $G \cong K_1 + K_{n-1}$ or $G \cong K_n \setminus e$ for $e \in E(K_n)$;
(2) If $\lambda_1(G) > 0$ and $\lambda_2(G) < 0$, then $G \cong K_n$.

Let $\mathcal{H}$ be set of all graphs satisfying $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ (in other words, $p(G) = 2$ and $\eta(G) = 0$). Oboudi [6] determined all the graphs of $\mathcal{H}$. To give a clear description of this characterization, we introduce the class of graphs G_n defined in [6].

For every integer $n \geq 2$, let $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ be two disjoint complete graphs with vertex set $V = \{v_1, \ldots, v_{\lceil \frac{n}{2} \rceil}\}$ and $W = \{w_1, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\}$. G_n is defined to be the graph obtained from $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ by adding some edges distinguishing whether n is even or not below:

(1) If n is even, then add some new edges to $K_{\frac{n}{2}} + K_{\frac{n}{2}}$ satisfying

$$\emptyset = N_W(v_1) \subset N_W(v_2) = \{w_{\frac{n}{2}}\} \subset N_W(v_3) = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}\} \subset \cdots \subset N_W(v_{\frac{n}{2}-1}) = \{w_{\frac{n}{2}}, \ldots, w_3\} \subset N_W(v_{\frac{n}{2}}) = \{w_{\frac{n}{2}}, \ldots, w_2\}.$$

(2) If n is odd, then add some new edges to $K_{\frac{n+1}{2}} + K_{\frac{n-1}{2}}$ satisfying

$$\emptyset = N_W(v_1) \subset N_W(v_2) = \{w_{\frac{n-1}{2}}\} \subset N_W(v_3) = \{w_{\frac{n-1}{2}}, w_{\frac{n-1}{2}-1}\} \subset \cdots \subset N_W(v_{\frac{n+1}{2}-1}) = \{w_{\frac{n-1}{2}}, \ldots, w_2\} \subset N_W(v_{\frac{n+1}{2}}) = \{w_{\frac{n-1}{2}}, \ldots, w_1\}.$$

By deleting the maximum (resp. minimum) degree vertex from G_{n+1} if n is an even (resp. odd), we obtain G_n. It follows the result below.

Remark 2.8 (See [6]). G_n is an induced subgraph of G_{n+1} for every $n \geq 2$.

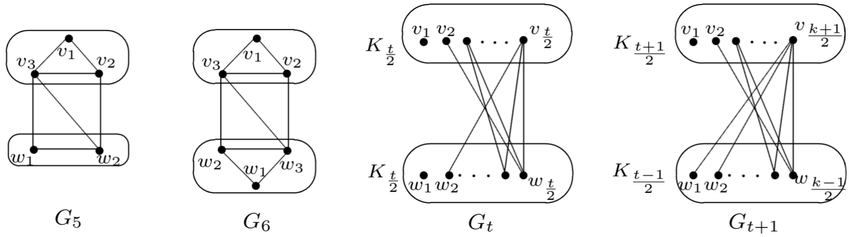


Figure 1: G_5, G_6, G_t and G_{t+1}.

For example, $G_2 \cong 2K_1$, $G_3 \cong P_3$ and $G_4 \cong P_4$. The graphs G_5 and G_6 are shown in Figure 1. In general, G_t and G_{t+1} are also shown in Figure 1 for an even number t.

Let G be a graph with vertex set $\{v_1, \dots, v_n\}$. By $G[K_{t_1}, \dots, K_{t_n}]$ we mean the *generalized lexicographic product* of G (by $K_{t_1}, K_{t_2}, \dots, K_{t_n}$), which is the graph obtained from G by replacing the vertex v_j with K_{t_j} and connecting each vertex of K_{t_i} to each vertex of K_{t_j} if v_i is adjacent to v_j in G.

Theorem 2.9 ([6]). *Let $G \in \mathcal{H}$ of order $n \geq 4$ with eigenvalues $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$.*

(1) If G is disconnected, then $G \cong K_p + K_q$ for some integers $p, q \geq 2$;

(2) If G is connected, there exist some positive integers s and $t_1, \dots, t_s$ such that $G \cong G_s[K_{t_1}, \dots, K_{t_s}]$ where $3 \leq s \leq 12$ and $t_1 + \cdots + t_s = n$.

Furthermore, Oboudi gave all the positive integers $t_1, \dots, t_s$ such that $G_s[K_{t_1}, \dots, K_{t_s}] \in \mathcal{H}$ in Theorems 3.4 – 3.14 of [6].

Let $\mathcal{G}$ be the set of all graphs with positive inertia index $p(G) = 2$ and nullity $\eta(G) = 1$. In next section, we introduce some new congruent transformations for graph that keep to the positive inertia index. By using such congruent transformations we characterize those graphs in $\mathcal{G}$ based on $\mathcal{H}$.

3 Three congruent transformations of graphs

In this section, we introduce three types of congruent transformations for graphs.

Lemma 3.1 ([10]). *Let u, v be two non-adjacent vertices of a graph G. If u and v have the same neighborhood, then $p(G) = p(G-u)$, $n(G) = n(G-u)$ and $\eta(G) = \eta(G-u)+1$.*

Remark 3.2. Two non-adjacent vertices u and v are said to be *congruent vertices of I-type* if they have the same neighbors. Lemma 3.1 implies that if one of congruent vertices of I-type is deleted from a graph then the positive and negative inertia indices left unchanged, but the nullity reduces just one. Conversely, if we add a new vertex that joins all the neighbors of some vertex in a graph (briefly we refer to add a vertex of I-type in what follows) then the positive and negative inertia indices left unchanged, but the nullity adds just one. The graph transformation of deleting or adding vertices of I-type is called the (graph) *transformation of I-type*.

Since $\operatorname{Spec}(K_s) = [(s-1)^1, (-1)^{s-1}]$. By applying the transformation of I-type, we can simply find the inertia of $K_{n_1,n_2,\dots,n_s}$.

Corollary 3.3. *Let $G = K_{n_1,n_2,\ldots,n_s}$ be a multi-complete graph where $n_1 \geq n_2 \geq \cdots \geq n_s$ and $i_0 = \min\{1 \leq i \leq s \mid n_i \geq 2\}$. Then G has the inertia index:* $\mathrm{In}(G) = (p(G), \eta(G), n(G)) = (1, n_{i_0} + n_{i_0+1} + \cdots + n_s - s + i_0 - 1, s - 1)$.

The following transformation was mentioned in [4], but the author didn't prove the result. For the completeness we give a proof below.

Lemma 3.4. *Let $\{u, v, w\}$ be an independent set of a graph G. If $N(u)$ is a disjoint union of $N(v)$ and $N(w)$, then $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$.*

Proof. Since u, v, w are not adjacent to each other, we may assume that $(0, 0, 0, \alpha^T)$, $(0, 0, 0, \beta^T)$ and $(0, 0, 0, \gamma^T)$ are the row vectors of $A(G)$ corresponding to the vertices u, v, w, respectively. Thus $A(G)$ can be written as

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & \alpha^T \\ 0 & 0 & 0 & \beta^T \\ 0 & 0 & 0 & \gamma^T \\ \alpha & \beta & \gamma & A(G - u - v - w) \end{pmatrix}.$$

Since $N(u) = N(v) \cup N(w)$ and $N(v) \cap N(w) = \emptyset$, we have $\alpha = \beta + \gamma$. By letting the u-th row (resp. u-th column) minus the sum of the v-th and w-th rows (resp. the sum of the v-th and w-th columns) of $A(G)$, we get that $A(G)$ is congruent to

$$\begin{pmatrix} 0 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & 0 & \beta^T \\ 0 & 0 & 0 & \gamma^T \\ \mathbf{0} & \beta & \gamma & A(G - u - v - w) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & A(G - u) \end{pmatrix}.$$

Thus $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$ by Lemma 2.4. □

Remark 3.5. The vertex u is said to be a *congruent vertex of II-type* if there exist two non-adjacent vertices v and w such that $N(u)$ is a disjoint union of $N(v)$ and $N(w)$. Lemma 3.4 implies that if one congruent vertex of II-type is deleted from a graph then the positive and negative indices left unchanged, but the nullity reduces just one. Conversely, if there exist two non-adjacent vertices v and w such that $N(v)$ and $N(w)$ are disjoint, we can add a new vertex u that joins all the vertices in $N(v) \cup N(w)$ (briefly we refer to add a vertex of II-type in what follows), then the positive and negative inertia indices left unchanged, but the nullity adds just one. The graph transformation of deleting or adding vertices of II-type is called the (graph) *transformation of II-type*.

An induced quadrangle $C_4 = uvxy$ of G is called *congruent* if there exists a pair of independent edges, say uv and xy in C_4, such that $N(u) \setminus \{v, y\} = N(v) \setminus \{u, x\}$ and $N(x) \setminus \{y, v\} = N(y) \setminus \{x, u\}$, where uv and xy are called a pair of *congruent edges* of C_4. We call the vertices in a congruent quadrangle the *congruent vertices of III-type*.

Lemma 3.6. *Let u be a congruent vertex of III-type in a graph G. Then $p(G) = p(G - u)$, $n(G) = n(G - u)$ and $\eta(G) = \eta(G - u) + 1$.*

Proof. Let $C_4 = uvxy$ be the congruent quadrangle of G containing the congruent vertex u. Then $(0, 1, 0, 1, \alpha^T)$, $(1, 0, 1, 0, \alpha^T)$, $(0, 1, 0, 1, \beta^T)$, $(1, 0, 1, 0, \beta^T)$ are the row vectors of $A(G)$ corresponding to the vertices u, v, x and y, respectively. Thus $A(G)$ can be presented by

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & \alpha^T \\ 1 & 0 & 1 & 0 & \alpha^T \\ 0 & 1 & 0 & 1 & \beta^T \\ 1 & 0 & 1 & 0 & \beta^T \\ \alpha & \alpha & \beta & \beta & A(G-u-v-x-y) \end{pmatrix}.$$

By letting the u-th row (resp. u-th column) minus the x-th row (resp. x-th column) of $A(G)$, and letting the v-th row (resp. v-th column) minus the y-th row (resp. y-th column) of $A(G)$, we obtain that $A(G)$ is congruent to

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha^T - \beta^T \\ 0 & 0 & 0 & 0 & \alpha^T - \beta^T \\ 0 & 0 & 0 & 1 & \beta^T \\ 0 & 0 & 1 & 0 & \beta^T \\ \alpha - \beta & \alpha - \beta & \beta & \beta & A(G-u-v-x-y) \end{pmatrix}.$$

Again, by letting the u-th row (resp. u-th column) minus the v-th row (resp. v-th column) of B, and adding the y-th row (resp. y-th column) to the v-th row (resp. v-th column) of B, we obtain that B is congruent to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & 1 & 0 & \alpha^T \\ 0 & 1 & 0 & 1 & \beta^T \\ 0 & 0 & 1 & 0 & \beta^T \\ \mathbf{0} & \alpha & \beta & \beta & A(G-u-v-x-y) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & A(G-u) \end{pmatrix}.$$

Thus $p(G) = p(G-u)$, $n(G) = n(G-u)$ and $\eta(G) = \eta(G-u) + 1$ by Lemma 2.4. □

Remark 3.7. The Lemma 3.6 confirms that if a congruent vertex of III-type is deleted from a graph then the positive and negative inertia indices left unchanged, but the nullity reduces just one. Conversely, if we add a new vertex to a graph that consists of a congruent quadrangle with some other three vertices in this graph (briefly we refer to add a vertex of III-type in what follows) then the positive and negative inertia indices left unchanged, but the nullity adds just one. The graph transformation of deleting or adding vertices of III-type is called the (graph) *transformation of III-type*.

Remark 3.2, Remark 3.5 and Remark 3.7 provide us three transformations of graphs that keep the positive and negative inertia indices and change the nullity just one. By applying these transformations we will construct the graphs in $\mathcal{G}$. Let $\mathcal{G}_1$ be the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of I-type, $\mathcal{G}_2$ be the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of II-type and $\mathcal{G}_3$ be the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of III-type. At the end of this section, we would like to give an example to illustrate the constructions of the graphs in $\mathcal{G}_i$ $(i = 1, 2, 3)$.

Example 3.8. We know the path P_4, with spectrum $\mathrm{Spec}(P_4) = \{1.6180, 0.6180, -0.6180, -1.6180\}$, is a graph belonging to $\mathcal{H}$. By adding a vertex u of I-type to P_4 we obtain $H_1 \in \mathcal{G}_1$ (see Figure 2) where $\mathrm{Spec}(H_1) = \{1.8478, 0.7654, 0, -0.7654, -1.8478\}$,

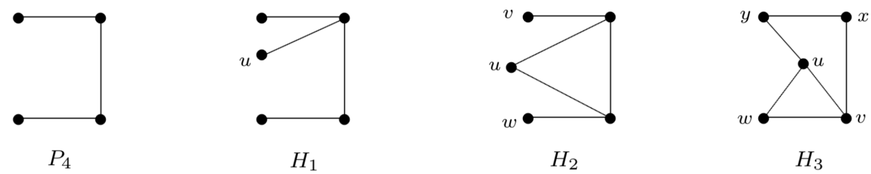


Figure 2: The graphs P_4, H_1, H_2 and H_3.

adding a vertex u of II-type to P_4 we obtain $H_2 \in \mathcal{G}_2$ where $\mathrm{Spec}(H_2) = \{2.3028, 0.6180, 0, -1.3028, -1.6180\}$. Finally, by adding a vertex u of III-type to P_4 we obtain $H_3 \in \mathcal{G}_3$, where $\mathrm{Spec}(H_3) = \{2.4812, 0.6889, 0, -1.1701, -2\}$. In fact, uv and xy is a pair of independent edges in H_3. Clearly, $N(u) \setminus \{v, y\} = N(v) \setminus \{u, x\} = \{w\}$ and $N(x) \setminus \{y, v\} = N(y) \setminus \{x, u\} = \emptyset$. Thus $C_4 = uvxy$ is a congruent quadrangle of H_3.

Clearly, $G = K_{1,2} \cup P_2$ is a non-connected graph in $\mathcal{G}$, and all such graphs we collect in $\mathcal{G}^- = \{G \in \mathcal{G} \mid G \text{ is disconnected}\}$. Additionally, H_1 and H_2 shown in Figure 2 are graphs with pendant vertex belonging to $\mathcal{G}$, and all such graphs we collect in $\mathcal{G}^+ = \{G \in \mathcal{G} \mid G \text{ is connected with a pendant vertex}\}$. In next section, we firstly determine the graphs in $\mathcal{G}^-$ and $\mathcal{G}^+$.

4 The characterization of graphs in $\mathcal{G}^-$ and $\mathcal{G}^+$

The following result completely characterizes the disconnected graphs of $\mathcal{G}$.

Theorem 4.1. *Let G be a graph of order $n \geq 5$. Then $G \in \mathcal{G}^-$ if and only if $G \cong K_s + K_t + K_1, H + K_1$ or $K_s + K_{n-s} \setminus e$ for $e \in E(K_{n-s})$, where $H \in \mathcal{H}$ is connected and $s + t = n - 1$, $s, t \geq 2$.*

Proof. All the graphs displayed in Theorem 4.1 have two positive and one zero eigenvalues by simple observation. Now we prove the necessity.

Let $G \in \mathcal{G}^-$, and $H_1, H_2, \ldots, H_k$ $(k \geq 2)$ the components of G. Since $\lambda_1(H_i) \geq 0$ for $i = 1, 2, \ldots, k$ and $\lambda_4(G) < 0$, G has two or three components and so $k \leq 3$.

First assume that $G = H_1 + H_2 + H_3$. It is easy to see that G has exactly one isolated vertex due to $\eta(G) = 1$ and $p(G) = 2$. Without loss of generality, let $H_3 \cong K_1$. Since $\lambda_3(G) = 0$ and $\lambda_1(H_i) > 0$ $(i = 1, 2)$, we have $\lambda_2(H_1) < 0$ and $\lambda_2(H_2) < 0$. By Theorem 2.7 (2), $G \cong K_s + K_t + K_1$ as desired, where $s + t = n - 1$ and $s, t \geq 2$.

Next assume that $G = H_1 + H_2$. If $H_1 \cong K_1$, then

$$\lambda_1(G) = \lambda_1(H_2) \geq \lambda_2(G) = \lambda_2(H_2) > \lambda_3(G) = \\ 0 = \lambda_1(H_1) > \lambda_4(G) = \lambda_3(H_2) < 0.$$

Thus $H_2 \cong H \in \mathcal{H}$, and so $G \cong H + K_1$ as desired. If $|H_i| \geq 2$ for $i = 1, 2$, then one of $\lambda_2(H_1)$ and $\lambda_2(H_2)$ is equal to zero and another is less than zero because $\lambda_3(G) = 0$ and $\lambda_4(G) < 0$. Without loss of generality, let $\lambda_2(H_1) < 0$ and $\lambda_2(H_2) = 0$. We have $\lambda_3(H_1) \leq \lambda_2(H_1) < 0$, in addition, $\lambda_3(H_2) < 0$ since $\eta(G) = 1$. By Theorem 2.7 (2), $H_1 \cong K_s$ for some $s \geq 2$ and by Theorem 2.7 (1), $H_2 \cong K_{n-s} \setminus e$.

We complete this proof. □

In terms of Theorem 2.6, we will determine all connected graphs with a pendant vertex satisfying $p(G) = 2$ and $\eta(G) = d$ for any positive integer d.

Theorem 4.2. *Let G be a connected graph of order n with a pendant vertex. Then $p(G) = 2$ and $\eta(G) = d \geq 1$ if and only if $G \cong K_{1,r}(u) \odot^k K_{n_1,\dots,n_l}$, where $r + n_1 + n_2 + \cdots + n_l - (l+1) = d$.*

Proof. Let $G = K_{1,r}(u) \odot^k K_{n_1,\dots,n_l}$ and vu is a pendant edge of G. By deleting v and u from G we obtain $H = G - \{u, v\} = (r-1)K_1 \cup K_{n_1,\dots,n_l}$. It is well known that $p(K_{n_1,\dots,n_l}) = 1$ and $\eta(K_{n_1,\dots,n_l}) = n_1 + \cdots + n_l - l$. From Lemma 2.3, we have

$$p(G) = p(H) + 1 = p(K_{n_1,\dots,n_l}) + 1 = 2,$$
$$\eta(G) = \eta(H) = (r-1) + (n_1 + \cdots + n_l - l) = d.$$

Conversely, let G be a graph with a pendant vertex and $p(G) = 2$. By Theorem 2.6, we have $G \cong K_{1,r}(u) \odot^k K_{n_1,\dots,n_l}$. According to the arguments above, we know that $\eta(G) = r + n_1 + n_2 + \cdots + n_l - (l+1) = d$. □

From Theorem 4.2, it immediately follows the result that completely characterizes the graphs in $\mathcal{G}^+$.

Corollary 4.3. *A connected graph $G \in \mathcal{G}^+$ if and only if $G \cong K_{1,2}(u) \odot^k K_{n-3}$ or $G \cong K_{1,1}(u) \odot^k K_{n-2} \setminus e$ for $e \in E(K_{n-2})$.*

Proof. By Theorem 4.2, we have $G \in \mathcal{G}^+$ if and only if $G \cong K_{1,r}(u) \odot^k K_{n_1,\dots,n_l}$, where $r + n_1 + n_2 + \cdots + n_l - (l+1) = 1$ and $r, l, n_1, \dots, n_l \geq 1$. It gives two solutions: one is $r = 2$, $n_1 = n_2 = \cdots = n_l = 1$ and $l = n - 3$ which leads to $G \cong K_{1,2}(u) \odot^k K_{n-3}$; another is $r = 1$, $n_1 = 2$, $n_2 = \cdots = n_l = 1$ and $l = n-2$ which leads to $G \cong K_{1,1}(u) \odot^k K_{n-2} \setminus e$ for $e \in E(K_{n-2})$. □

Let $\mathcal{G}^*$ denote the set of all connected graphs in $\mathcal{G}$ without pendant vertices. Then $\mathcal{G} = \mathcal{G}^- \cup \mathcal{G}^+ \cup \mathcal{G}^*$. Therefore, in order to characterize $\mathcal{G}$, it remains to consider those graphs in $\mathcal{G}^*$.

5 The characterization of graphs in $\mathcal{G}^*$

First we introduce some symbols which will be persisted in this section. Let $G \in \mathcal{G}^*$. The eigenvalues of G can be arranged as:

$$\lambda_1(G) \geq \lambda_2(G) > \lambda_3(G) = 0 > \lambda_4(G) \geq \cdots \geq \lambda_n(G).$$

We choose $v^* \in V(G)$ such that $d_G(v^*) = \delta(G) = t$, and denote by $X = N_G(v^*)$ and $Y = V(G) - N_G[v^*]$. Then $t = |X| \geq 2$ since G has no pendant vertices. In addition, $|Y| > 0$ since otherwise G would be a complete graph. First we characterize the induced subgraph $G[Y]$ in the following result.

Lemma 5.1. $G[Y] \cong K_{n-t-1} \setminus e, K_1 + K_{n-t-2}$ *or* K_{n-t-1}.

Proof. First we suppose that Y is an independent set. If $|Y| \geq 3$, then $\lambda_4(G) \geq \lambda_4(G[Y \cup \{v^*\}]) = 0$ by Theorem 2.1, a contradiction. Hence $|Y| \leq 2$, and so $G[Y] \cong K_1$ or $G[Y] \cong K_2 \setminus e = 2K_1$.

Next we suppose that $G[Y]$ contains some edges. We distinguish the following three situations.

If $\lambda_2(G[Y]) > 0$, we have $p(G[Y]) \geq 2$. For any $x \in X$, the induced subgraph $G[\{v^*, x\} \cup Y]$ has a pendant vertex v^* by our assumption. By Lemma 2.2 and Lemma 2.3, we have $p(G) \geq p(G[\{v^*, x\} \cup Y]) = p(G[Y]) + 1 \geq 3$, a contradiction.

If $\lambda_2(G[Y]) < 0$, by Theorem 2.7 (2) we have $G[Y] \cong K_{n-t-1}$ as desired.

At last assume that $\lambda_2(G[Y]) = 0$. If $\lambda_3(G[Y]) < 0$, by Theorem 2.7 (1), we have $G[Y] \cong K_{n-t-1} \setminus e, K_1 + K_{n-t-2}$ as desired. If $\lambda_3(G[Y]) = 0$, by Lemma 2.3 we have $p(G[\{v^*, x\} \cup Y]) = p(G[Y]) + 1 = 2$ and $\eta(G[\{v^*, x\} \cup Y]) = \eta(G[Y]) \geq 2$, which implies that $\lambda_4(G) \geq \lambda_4(G[\{v^*, x\} \cup Y]) = 0$, a contradiction.

We complete this proof. □

First assume that $Y = \{y_1\}$. If $G[X] = K_t$, then $G = K_n \setminus v^*y_1$. However $K_n \setminus v^*y_1 \notin \mathcal{G}^*$ since $p(K_n \setminus v^*y_1) = 1$. Thus there exist $x_1 \not\sim x_2$ in X. Then $N_G(x_1) = N_G(x_2)$ and $N_G(v^*) = N_G(y_1)$. It follows that $\eta(G) \geq 2$ by Lemma 3.1. Next assume that $Y = \{y_1, y\}$ is an independent set. We have $N_G(v^*) = N_G(y_1) = N_G(y)$ since $d_G(y_1), d_G(y) \geq d_G(v^*) = \delta(G)$. Thus, by Lemma 3.1 we have $\eta(G) = \eta(G - y_1) + 1 = \eta(G - y_1 - y) + 2 \geq 2$. Thus we only need to consider the case that $G[Y]$ contains at least one edge. Concretely, we distinguish three situations in accordance with the proof of Lemma 5.1:

(a) $G[Y] \cong K_{n-t-2} + K_1$ in case of $\lambda_2(G[Y]) = 0$ and $\lambda_3(G[Y]) < 0$, where $n - t - 2 \geq 2$;

(b) $G[Y] \cong K_{n-t-1} \setminus e$ in case of $\lambda_2(G[Y]) = 0$ and $\lambda_3(G[Y]) < 0$, where $|Y| = n - t - 1 \geq 3$;

(c) $G[Y] \cong K_{n-t-1}$ in case of $\lambda_2(G[Y]) < 0$, where $|Y| = n - t - 1 \geq 2$.

In the following, we deal with situation (a) in Lemma 5.2, (b) in Lemma 5.3 and (c) in Lemma 5.4, 5.7 and Lemma 5.15. We will see that the graph $G \in \mathcal{G}^*$ illustrated in (a) and (b) can be constructed from some $H \in \mathcal{H}$ by the graph transformations of I-, II- and III-type, but (c) can not.

Lemma 5.2. *If $G[Y] \cong K_{n-t-2} + K_1$, where $n - t - 2 \geq 2$, then $G \in \mathcal{G}_1$.*

Proof. Since $G[Y]$ is isomorphic to $K_{n-t-2} + K_1$ $(n - t - 2 \geq 2)$, Y exactly contains one isolated vertex of $G[Y]$, say y. We have $N_G(v^*) = N_G(y)$ and thus y is a congruent vertex of I-type. By Lemma 3.1, we have $p(G) = p(G - y)$ and $\eta(G) = \eta(G - y) + 1$. Notice that $G - y$ is connected, we have $G - y \in \mathcal{H}$, and so $G \in \mathcal{G}_1$. Such a graph G, displayed in Figure 3 (1), we call the v^**-graph of I-type*. □

In Figure 3 and Figure 5, two ellipses joining with one full line denote some edges between them. A vertex and an ellipse joining with one full line denote some edges between them, and with two full lines denote that this vertex joins all vertices in the ellipse. Two vertices join with same location of an ellipse denote that they have same neighbours in this ellipse.

It needs to mention that the v^*-graph of I-type characterized in Lemma 5.2, is a graph obtained from $H \in \mathcal{H}$ by adding a new vertex joining the neighbors of a minimum degree vertex of H.

For $S \subseteq V(G)$ and $u \in V(G)$, let $N_S(u) = N_G(u) \cap S$ and $N_S[u] = N_G[u] \cap S$.

Lemma 5.3. *Let $G[Y] \cong K_{n-t-1} \setminus e$, where $n - t - 1 \geq 3$ and $e = yy'$. Then $G \in \mathcal{G}_1$ if $N_X(y) = N_X(y')$ and $G \in \mathcal{G}_2$ otherwise.*

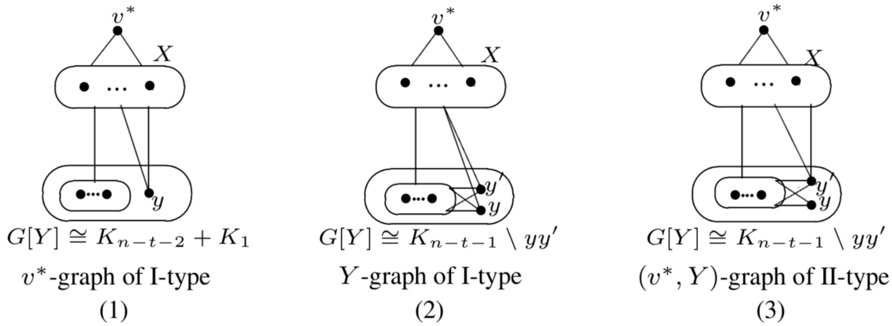


Figure 3: The structure of some graphs.

Proof. Since $n - t - 1 \geq 3$, there is $y^* \in Y$ other than y and y'. It is clear that $N_G(y) = N_X(y) \cup (Y \setminus \{y, y'\})$ and $N_G(y') = N_X(y') \cup (Y \setminus \{y, y'\})$, and thus $N_G(y) = N_G(y')$ if and only if $N_X(y) = N_X(y')$. We consider the following cases.

Case 1. $N_X(y) = N_X(y')$.

By assumption, $N_G(y) = N_G(y')$, thus y and y' are congruent vertices of I-type. By Lemma 3.1, we have $p(G) = p(G - y)$ and $\eta(G) = \eta(G - y) + 1$. Since $G - y$ is connected, we have $G - y \in \mathcal{H}$ and so $G \in \mathcal{G}_1$. Such a G, displayed in Figure 3 (2), we call the *Y-graph of I-type*.

Case 2. $N_X(y) \neq N_X(y')$.

First suppose that exactly one of $N_X(y)$ and $N_X(y')$ is empty, say $N_X(y) = \emptyset$ and $N_X(y') \neq \emptyset$. Then yy^* is a pendant edge of the induced subgraph $G[X \cup \{y, y', y^*, v^*\}]$. By Lemma 2.2 and Lemma 2.3, we have

$$2 = p(G) \geq p(G[X \cup \{y, y', y^*, v^*\}]) = p(G[X \cup \{y', v^*\}]) + 1 \geq 2.$$

Thus

$$p(G[X \cup \{y, y', y^*, v^*\}]) = 2 \quad \text{and}$$
$$p(G[X \cup \{y', v^*\}]) = 1.$$

We see that $\lambda_2(G[X \cup \{y', v^*\}]) = 0$ (since otherwise $\lambda_2(G[X \cup \{y', v^*\}]) < 0$ and then $G[X \cup \{y', v^*\}]$ is a complete graph, but $y' \not\sim v^*$). If $\lambda_3(G[X \cup \{y', v^*\}]) = 0$, we have

$$\eta(G[X \cup \{y, y', y^*, v^*\}]) = \eta(G[X \cup \{y', v^*\}]) \geq 2,$$

which implies

$$\lambda_4(G) \geq \lambda_4(G[X \cup \{y, y', y^*, v^*\}]) = 0,$$

a contradiction. If $\lambda_3(G[X \cup \{y', v^*\}]) < 0$, then $G[X \cup \{y', v^*\}] \cong K_{t+2} \setminus e$ or $K_{t+1} + K_1$ by Theorem 2.7 (1). Notice that $G[X \cup \{y', v^*\}]$ is connected, we get $G[X \cup \{y', v^*\}] \cong K_{t+2} \setminus e$ where $e = v^*y'$. Thus $N_X(y') = X$ and so $N_G(y') = X \cup (Y \setminus \{y, y'\}) = N_G(v^*) \cup N_G(y)$ is a disjoint union. Additionally, $\{y', v^*, y\}$ is an independent set in G, we see that y' is a congruent vertex of II-type. Thus $p(G) = p(G - y')$ and $\eta(G) =$

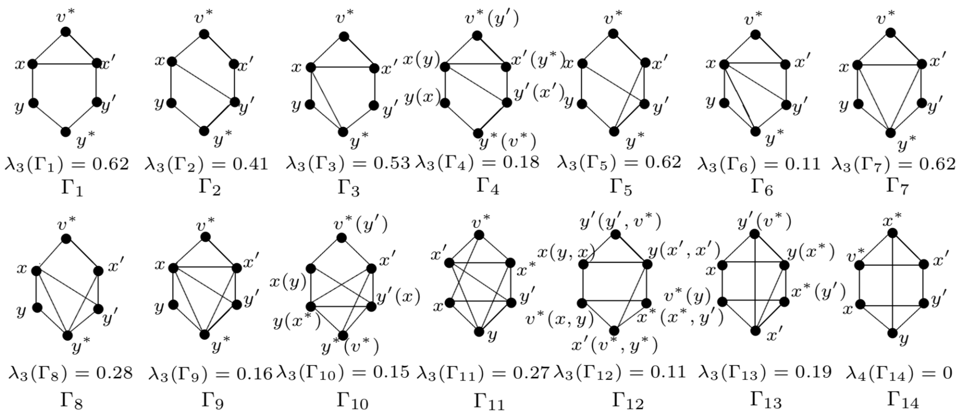


Figure 4: The graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_{14}$.

$\eta(G - y') + 1$ by Lemma 3.4. This implies that $G - y' \in \mathcal{H}$, and so $G \in \mathcal{G}_2$. Such a G, displayed in Figure 3 (3), we call the (v^*, Y)*-graph of II-type*.

Next suppose that $N_X(y), N_X(y') \neq \emptyset$, without loss of generality, assume that $N_X(y') \setminus N_X(y) \neq \emptyset$. Then there exists $x' \in N_X(y') \setminus N_X(y)$. Thus $x' \sim y'$ and $x' \not\sim y$. Now by taking some $x \in N_X(y)$, we see that $C_6 = v^*xyy^*y'x'$ is a 6-cycle in G. Note that x may joins each vertex in $\{x', y', y^*\}$ and x' may joins y^*. By distinguishing different situations in according with the number of edges we have

$$G[v^*, x, y, y^*, y', x'] \cong \begin{cases} C_6 & \text{no edge;} \\ \Gamma_1 \text{ or } \Gamma_2 & \text{one edges;} \\ \Gamma_3, \Gamma_4 \text{ or } \Gamma_5 & \text{two edges;} \\ \Gamma_6, \Gamma_7 \text{ or } \Gamma_8 & \text{three edges;} \\ \Gamma_9 & \text{four edges.} \end{cases}$$

However C_6 and $\Gamma_1, \ldots, \Gamma_8$ and Γ_9 are all forbidden subgraphs of G (see Figure 4).

We complete this proof. □

It remains to characterize the graph $G \in \mathcal{G}^*$ satisfying $G[Y] \cong K_{n-t-1}$. Such a graph G we call X*-complete* if $G[X]$ is also complete graph, and X*-imcomplete* otherwise. The following result characterizes the X-imcomplete graphs.

Lemma 5.4. *Let $G[Y] \cong K_{n-t-1}$, where $n - t - 1 \geq 2$, and G is X-imcomplete. Then $G \in \mathcal{G}_1$ if there exist two non-adjacent vertices $x_1 \not\sim x_2$ in $G[X]$ such that $N_Y(x_1) = N_Y(x_2)$ and $G \in \mathcal{G}_3$ otherwise.*

Proof. Let $X = \{x_1, x_2, \ldots, x_t\}$ and $Y = \{y_1, y_2, \ldots, y_{n-t-1}\}$. Then $V(G) = \{v^*\} \cup X \cup Y$ and Y induces K_{n-t-1}. Let x and x' be two non-adjacent vertices in X. Since $d_G(x) \geq d_G(v^*)$ and $n - t - 1 \geq 2$, we have $|N_Y(x)| \geq 1$ and $|Y| \geq 2$, respectively. First we give some claims.

Claim 5.5. *If $x \not\sim x'$ in $G[X]$ then one of $N_Y(x)$ and $N_Y(x')$ includes another. If $N_Y(x) \subset N_Y(x')$ then $|N_Y(x)| = 1$ and $N_Y(x') = Y$.*

Proof. On the contrary, let $y \in N_Y(x) \setminus N_Y(x')$ and $y' \in N_Y(x') \setminus N_Y(x)$, then $G[v^*, x, x', y, y'] \cong C_5$. Thus one of $N_Y(x)$ and $N_Y(x')$ includes another. Now assume that $N_Y(x) \subset N_Y(x')$. If $|N_Y(x)| \geq 2$, say $\{y, y'\} \subseteq N_Y(x)$, then $x' \sim y, y'$ and exists $y^* \in N_Y(x') \setminus N_Y(x)$. Thus $G[v^*, x, x', y, y', y^*] \cong \Gamma_{10}$ (see Figure 4). However $p(\Gamma_{10}) = 3$. Hence $|N_Y(x)| = 1$, and we may assume that $N_Y(x) = \{y\}$. If $N_Y(x') \neq Y$, then there exists $y' \in Y \setminus N_Y(x')$. Also, there exists $y^* \in N_Y(x') \setminus N_Y(x)$. We have $G[v^*, x, x', y, y', y^*] \cong \Gamma_4$ (see the labels in the parentheses of Figure 4), but $p(\Gamma_4) = 3$. Thus $N_Y(x') = Y$. □

Claim 5.6. *If $x \not\sim x'$ in $G[X]$ then $N_X(x) = N_X(x')$.*

Proof. On the contrary, we may assume that $x^* \in N_X(x') \setminus N_X(x)$. Then $x^* \sim x'$ and $x^* \not\sim x$, thus $|N_Y(x)| \geq 2$ since $|N_G(x)| \geq t$. By Claim 5.5, we have $N_Y(x^*), N_Y(x') \subseteq N_Y(x)$. Then either $N_Y(x^*) = N_Y(x') = N_Y(x)$ or one of $N_Y(x^*)$ and $N_Y(x')$ is a proper subset of $N_Y(x)$ (without loss of generality, assume that $N_Y(x^*) \subset N_Y(x)$, and then $|N_Y(x^*)| = 1$ and $N_Y(x) = Y$ by Claim 5.5).

Suppose that $N_Y(x) = N_Y(x^*) = N_Y(x')$. Take $y, y' \in N_Y(x)$, we see that $G[v^*, x, x^*, x', y, y'] \cong \Gamma_{11}$ (see Figure 4). However $p(\Gamma_{11}) = 3$.

Suppose that $|N_Y(x^*)| = 1$ and $N_Y(x) = Y$. Let $N_Y(x^*) = \{y\}$ and there exists another $y' \in Y$. Then $G[v^*, x, x^*, x', y, y']$ is isomorphic Γ_{13} (see Figure 4) if $x' \sim y, y'$, or isomorphic to Γ_{12} (see Figure 4) if $x' \sim y$ and $x' \not\sim y'$, or isomorphic to Γ_{14} (see Figure 4) if $x' \not\sim y$ and $x' \sim y'$. However $p(\Gamma_{12}) = p(\Gamma_{13}) = 3$ and $\lambda_4(\Gamma_{14}) = 0$. We are done. □

Now we distinguish the following cases to prove our result.

Case 1. There exist $x_1 \not\sim x_2$ such that $N_Y(x_1) = N_Y(x_2)$.

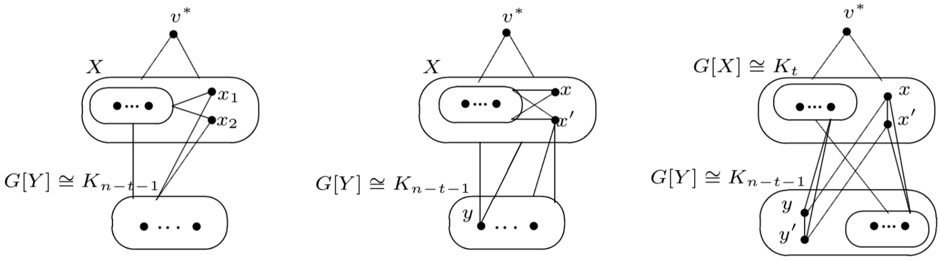
Since $x_1 \not\sim x_2$, we have $N_X(x_1) = N_X(x_2)$ by Claim 5.6, so $N_G(x_1) = N_G(x_2)$. Thus x_1 and x_2 are congruent vertices of I-type. By Lemma 3.1, $p(G) = p(G - x_1)$ and $\eta(G) = \eta(G - x_1) + 1$. Thus $G - x_1 \in \mathcal{H}$ and so $G \in \mathcal{G}_1$. Such a G, displayed in Figure 5 (1), we call the X-graph of I-type.

Case 2. For each pair of $x \not\sim x' \in X$, $N_Y(x) \neq N_Y(x')$.

By Claim 5.5, without loss of generality, assume that $N_Y(x) \subset N_Y(x')$ and then $N_Y(x) = \{y\}$ and $N_Y(x') = Y$. Thus $y \sim x, x'$ and furthermore we will show that $X \subseteq N_G(y)$. In fact, let $x^* \in X \setminus \{x, x'\}$ (if any), if $x \not\sim x^*$, we have $N_Y(x^*) \supseteq N_Y(x) = \{y\}$ by Claim 5.6. Thus $y \sim x^*$. Otherwise, $x \sim x^*$ and thus $x' \sim x^*$ since $N_X(x) = N_X(x')$ by Claim 5.6. Now take $y' \in Y \setminus \{y\}$. If $y \not\sim x^*$, then $G[v^*, x, x', x^*, y, y']$ is isomorphic to Γ_{12} (see the first labels in the parentheses of Figure 4) while $x^* \not\sim y'$, or isomorphic to Γ_{13} (see the labels in the parentheses of Figure 4) while $x^* \sim y'$, but $p(\Gamma_{12}) = p(\Gamma_{13}) = 3$. It follows that $N_G(y) = X \cup (Y \setminus \{y\})$ since Y induces a clique.

On the other hand, since $d_G(x) \geq |X| = t$, $x \not\sim x'$ and $N_Y(x) = \{y\}$, we have $N_X(x) = X \setminus \{x, x'\}$ and so $N_X(x') = X \setminus \{x, x'\}$ by Claim 5.6. Thus $N_G(x) = (X \setminus \{x, x'\}) \cup \{v^*, y\}$ and $N_G(x') = (X \setminus \{x, x'\}) \cup Y \cup \{v^*\}$. Hence the quadrangle $C_4 = xv^*x'y$ is congruent, where xv^* and $x'y$ is a pair of congruent edges of C_4. It gives that x, v^*, x', y are congruent vertices of III-type. By Lemma 3.6, we have $p(G) = p(G - x)$ and $\eta(G) = \eta(G - x) + 1$ thus $G - x \in \mathcal{H}$, and so $G \in \mathcal{G}_3$. Such a G, displayed in Figure 5 (2), we call the (v^*, X, Y)-graph of III-type.

We complete this proof. □



X-graph of I-type (1) (v^*, X, Y)-graph of III-type (2) (X, Y)-graph of III-type (3)

Figure 5: The structure of some graphs.

At last we focus on characterizing X-complete graph $G \in \mathcal{G}^*$, i.e., $G[X] \cong K_t$ and $G[Y] \cong K_{n-t-1}$. A X-complete graph $G \in \mathcal{G}^*$ is called *reduced* if one of $N_Y(x_i)$ and $N_Y(x_j)$ is a subset of another for any $x_i \neq x_j \in X$ and *non-reduced* otherwise. Thus the X-complete graphs are partitioned into a disjoint union of the reduced and non-reduced X-complete graphs. Concretely, for a reduced X-complete graph $G \in \mathcal{G}^*$, we may assume that $\emptyset = N_Y(v^*) \subseteq N_Y(x_1) \subseteq N_Y(x_2) \subseteq \cdots \subseteq N_Y(x_t)$; for a non-reduced (X, Y)-complete graph $G \in \mathcal{G}^*$, there exist some $x \neq x' \in X$ such that $N_Y(x) \setminus N_Y(x') \neq \emptyset$ and $N_Y(x') \setminus N_Y(x) \neq \emptyset$. Such vertices x and x' are called *non-reduced vertices*. It remains to characterize the reduced and non-reduced X-complete graphs in what follows.

Lemma 5.7. *Let $G \in \mathcal{G}^*$ be a non-reduced X-complete graph and x, x' be non-reduced vertices. Then $G \in \mathcal{G}_3$.*

Proof. Since x, x' are non-reduced vertices, there exist $y \in N_Y(x) \setminus N_Y(x')$ and $y' \in N_Y(x') \setminus N_Y(x)$. Then x, x', y', y induces C_4 (see Figure 5 (3)). It suffices to verify that C_4 is congruent. Clearly, $N_G(x) \supset (X \setminus \{x\}) \cup \{v^*\}$ and $N_G(x') \supset (X \setminus \{x'\}) \cup \{v^*\}$. If there exists $y^* \in N_Y(x) \setminus N_Y(x')$ other than y, then $G[v^*, x, x', y', y, y^*] \cong \Gamma_{12}$ (see the second labels in the parentheses of Figure 4), however Γ_{12} is a forbidden subgraph of G. Hence $N_Y(x) \setminus N_Y(x') = \{y\}$. Similarly, $N_Y(x') \setminus N_Y(x) = \{y'\}$. On the other aspect, $x \in N_X(y) \setminus N_X(y')$ and $x' \in N_X(y') \setminus N_X(y)$. If there exists $x^* \in N_X(y) \setminus N_X(y')$ other than x, then $G[v^*, x, x', x^*, y, y'] \cong \Gamma_{10}$ (see the labels in the parentheses of Figure 4), however Γ_{10} is a forbidden subgraph of G. Hence $N_X(y) \setminus N_X(y') = \{x\}$. Similarly, $N_X(y') \setminus N_X(y) = \{x'\}$. Hence $N_X(y) \setminus \{x\} = N_X(y') \setminus \{x'\}$. Note that $N_G(y) \supset Y \setminus \{y\}$ and $N_G(y') \supset Y \setminus \{y'\}$, we have $N_G(y) \setminus \{y', x\} = (Y \setminus \{y, y'\}) \cup (N_X(y) \setminus \{x\}) = N_G(y') \setminus \{x', y\}$. Hence the quadrangle $C_4 = xx'y'y$ is congruent, where xx' and $y'y$ is a pair of congruent edges. It follows that x, x', y', y are congruent vertices of III-type. By Lemma 3.6, we have $p(G) = p(G - x)$ and $\eta(G) = \eta(G - x) + 1$. Thus $G - x \in \mathcal{H}$, and so $G \in \mathcal{G}_3$. Such a G, displayed in Figure 5 (3), we call the (X, Y)-graph of III-type.

We complete this proof. □

To characterize the reduced X-complete graph, we need the notion of canonical graph which is introduced in [7]. For a graph G, a relation ρ on $V(G)$ we mean that $u\rho v$ iff $u \sim v$ and $N_G(u) \setminus v = N_G(v) \setminus u$. Clearly, ρ is symmetric and transitive. In accordance with ρ, the vertex set is decomposed into classes:

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_k, \tag{5.1}$$

where $v_i \in V_i$ and $V_i = \{x \in V(G) \mid x\rho v_i\}$. By definition of ρ, V_i induces a clique K_{n_i} where $n_1 + n_2 + \cdots + n_k = n = |V(G)|$, and vertices of V_i join that of V_j iff $v_i \sim v_j$ in G. We call the induced subgraph $G[\{v_1, v_2, \ldots, v_k\}]$ as the *canonical graph* of G, denoted by G_c. Thus $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$ is a *generalized lexicographic product* of G_c (by $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$).

Let G be a reduced X-complete graph. From (5.1) we have $G = G_c[K_{n_1}, K_{n_2}, \ldots, K_{n_k}]$, where $G_c = G[\{v_1, v_2, \ldots, v_k\}]$ and $V_i = \{x \in V(G) \mid x\rho v_i\}$ induces clique K_{n_i}. Without loss of generality, assume $v_1 = v^*$. Let $X_c = N_{G_c}(v_1)$ and $Y_c = \{v_2, v_3, \ldots, v_k\} \setminus X_c$. Clearly, $G_c[X_c]$ is a clique since X_c is a subset of X and X induces a clique in G. Furthermore, $G_c[Y_c]$ is a clique since Y_c is a subset of Y and Y induces a clique in G. Thus G_c is also a X_c-complete graph. Additionally, since G is reduced, G_c is also reduced. Let $t_c = d_{G_c}(v_1)$ and $X_c = \{x_1, x_2, \ldots, x_{t_c}\}$, $Y_c = \{y_1, y_2, \ldots, y_{k-t_c-1}\}$. We may assume $N_{Y_c}(v_1) \subset N_{Y_c}(x_1) \subset \cdots \subset N_{Y_c}(x_{t_c})$ and $N_{X_c}(y_1) \subset \cdots \subset N_{X_c}(y_{k-t_c-1})$. Therefore,

$$0 = |N_{Y_c}(v_1)| < |N_{Y_c}(x_1)| < \cdots < |N_{Y_c}(x_{t_c})| \leq |Y_c| = k - t_c - 1, \tag{5.2}$$

and

$$0 \leq |N_{X_c}(y_1)| < |N_{X_c}(y_2)| < \cdots < |N_{X_c}(y_{k-t_c-1})| \leq |X_c| = t_c. \tag{5.3}$$

From Equation (5.2), we have $t_c \leq k - t_c - 1$. Similarly, $k - t_c - 2 \leq t_c$ from Equation (5.3). Thus $k - 2 \leq 2t_c \leq k - 1$, and so $t_c = \lceil \frac{k}{2} \rceil - 1$.

If k is even, then $t_c = \frac{k}{2} - 1$. From Equation (5.2), we have $|N_{Y_c}(x_i)| = i$ for $i = 1, 2, \ldots, t_c$. Thus we may assume that

$$\begin{aligned} N_{Y_c}(v_1) &= \emptyset, \\ N_{Y_c}(x_1) &= \{y_{\frac{k}{2}}\}, \\ &\vdots \\ N_{Y_c}(x_{\frac{n}{2}-2}) &= \{y_{\frac{k}{2}}, \ldots, y_3\}, \\ N_{Y_c}(x_{\frac{n}{2}-1}) &= \{y_{\frac{k}{2}}, \ldots, y_2\}. \end{aligned}$$

This implies that $G \cong G_k$ where G_k is defined in Section 2. Similarly, $G \cong G_k$ if k is odd. Thus we obtain the following result.

Lemma 5.8. *Let G be a reduced X-complete graph. Then $G_c \cong G_k$ where $k \geq 2$ is determined in (5.1).*

Let $G \in \mathcal{G}^*$ be a reduced X-complete graph. The following lemma gives a characterization for G. First we cite a result due to Oboudi in [5].

Lemma 5.9 ([5]). *Let $G = G_3[K_{n_1}, K_{n_2}, K_{n_3}]$, where n_1, n_2, n_3 are some positive integers. Then the following hold:*

(1) If $n_1 = n_2 = n_3 = 1$, that is $G \cong P_3$, then $\lambda_3(G) = -\sqrt{2}$;

(2) If $n_1 = n_2 = 1$ and $n_3 \geq 2$, then $\lambda_3(G) = -1$;

(3) If $n_1n_2 > 1$*, then* $\lambda_3(G) = -1$.

We know that any graph G is a generalized lexicographic product of its canonical graph, i.e., $G = G_c[K_{n_1}, K_{n_2}, \dots, K_{n_k}]$. We also have $G_c = G_k$ if G is reduced X-complete by Lemma 5.8. Furthermore, the following result prove that $4 \le k \le 13$.

Lemma 5.10. *Let* $G \in \mathcal{G}^*$ *be a reduced* X*-complete graph. Then there exists* $4 \le k \le 13$ *such that* $G = G_k[K_{n_1}, K_{n_2}, \dots, K_{n_k}]$.

Proof. By Lemma 5.8, $G = G_k[K_{n_1}, K_{n_2}, \dots, K_{n_k}]$ for some k. If $k = 1$ or 2 then $G \cong K_n \notin \mathcal{G}^*$, and so $k \ge 3$. If $k = 3$, then $G = G_3[K_{n_1}, K_{n_2}, K_{n_3}]$. Thus $\lambda_3(G) < 0$ by Lemma 5.9, a contradiction. Hence $k \ge 4$. On the other hand, since $G_c = G_k$ is an induced subgraph of G, we have $\lambda_4(G_k) \le \lambda_4(G) < 0$ by Theorem 2.1. Note that G_{14} is an induced subgraph of G_k (by Remark 2.8) for $k \ge 15$, we have $\lambda_4(G_k) \ge \lambda_4(G_{14}) = 0$. It implies that $k \le 13$. □

Next we consider the converse of Lemma 5.10. In other words, we will try to find the values of $n_1, \dots, n_k$ such that $p(G_k[K_{n_1}, \dots, K_{n_k}]) = 2$ and $\eta(G_k[K_{n_1}, \dots, K_{n_k}]) = 1$, where $4 \le k \le 13$ and $n = n_1 + n_2 + \cdots + n_k$. For the simplicity, we use notation in [8] to denote

$$G_{2s}[K_{n_1}, \dots, K_{n_{2s}}] = B_{2s}(n_1, \dots, n_s; n_{s+1}, \dots, n_{2s}) \quad \text{and}$$
$$G_{2s+1}[K_{n_1}, \dots, K_{n_{2s+1}}] = B_{2s+1}(n_1, \dots, n_s; n_{s+1}, \dots, n_{2s}; n_{2s+1}).$$

By Remark 3.2 in [6], we know

$$\begin{aligned} H_0 &= B_{2s}(n_1, \dots, n_s; n_{s+1}, \dots, n_{2s}) \\ &\cong B_{2s}(n_{s+1} \dots, n_{2s}; n_1, \dots, n_s) = H_0' \quad \text{and} \\ H_1 &= B_{2s+1}(n_1, \dots, n_s; n_{s+1}, \dots, n_{2s}; n_{2s+1}) \\ &\cong B_{2s+1}(n_{s+1}, \dots, n_{2s}; n_1, \dots, n_s; n_{2s+1}) = H_1'. \end{aligned}$$

In what follows, we always take H_0 and H_1, in which $(n_1, \dots, n_s)$ is prior to $(n_{s+1}, \dots, n_{2s})$ in dictionary ordering, instead of H_0' and H_1'. For example we use $B_6(4,3,2;4,3,1)$ instead of $B_6(4,3,1;4,3,2)$ and $B_7(5,3,2;5,2,4;8)$ instead of $B_7(5,2,4;5,3,2;8)$.

For $4 \le k \le 13$, let

$$\mathcal{B}_k(n) = \{G = B_k(n_1, \dots, n_k) \mid n = n_1 + \cdots + n_k, n_i \ge 1\}.$$

Let $\mathcal{B}_k^+(n)$, $\mathcal{B}_k^{00}(n)$, $\mathcal{B}_k^0(n)$ and $\mathcal{B}_k^-(n)$ denote the set of graphs in $\mathcal{B}_k(n)$ satisfying $\lambda_3(G) > 0$ for $G \in \mathcal{B}_k^+(n)$, $\lambda_4(G) = \lambda_3(G) = 0$ for $G \in \mathcal{B}_k^{00}(n)$, $\lambda_4(G) < \lambda_3(G) = 0$ for $G \in \mathcal{B}_k^0(n)$ and $\lambda_3(G) < 0$ for $G \in \mathcal{B}_k^-(n)$, respectively. Clearly, $\mathcal{B}_k(n) = \mathcal{B}_k^+(n) \cup \mathcal{B}_k^{00}(n) \cup \mathcal{B}_k^0(n) \cup \mathcal{B}_k^-(n)$ is disjoint union and $G = G_k[K_{n_1}, K_{n_2}, \dots, K_{n_k}] \in \mathcal{B}_k^0(n)$ if $G \in \mathcal{G}^*$ is a reduced X-complete graph by Lemma 5.10. In what follows, we further show that $n \le 13$. First, one can verify the following result by using computer.

Lemma 5.11. $\mathcal{B}_k^0(14) = \emptyset$ *for* $4 \le k \le 13$ *(it means that there are no reduced* X*-complete graphs of order* 14*).*

Proof. For $4 \le k \le 13$, the k-partition of 14 gives a solution $(n_1, n_2, \dots, n_k)$ of the equation $n_1 + n_2 + \cdots + n_k = 14$ that corresponds a graph $G = B_k(n_1, n_2, \dots, n_k) \in \mathcal{B}_k(14)$. By using computer, we exhaust all the graphs of $\mathcal{B}_k(14)$ to find that there is no any graph $G \in \mathcal{B}_k(14)$ with $\lambda_4(G) < \lambda_3(G) = 0$. It implies that $\mathcal{B}_k^0(14) = \emptyset$. □

In [6], Oboudi gave all the integers $n_1, \ldots, n_k$ satisfying $\lambda_2(B_k(n_1, \ldots, n_k)) > 0$ and $\lambda_3(B_k(n_1, \ldots, n_k)) < 0$ for $4 \le k \le 9$. For simplicity, we only cite this result for $k = 5$ and the others are listed in Appendix B.

Theorem 5.12 ([6]). *Let $G = B_5(n_1, n_2; n_3, n_4; n_5)$, where n_1, n_2, n_3, n_4, n_5 are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_5(a, w; 1, 1; 1)$; *(6)* $B_5(a, 1; x, w; 1)$; *(11)* $B_5(x, w; 1, d; 1)$;
(2) $B_5(a, x; 1, d; 1)$; *(7)* $B_5(a, 1; x, y; e)$; *(12)* $B_5(x, w; 1, 1; e)$;
(3) $B_5(a, x; 1, y; z)$; *(8)* $B_5(a, 1; 1, d; e)$; *(13)* $B_5(1, b; 1, d; 1)$;
(4) $B_5(a, x; 1, 1; e)$; *(9)* $B_5(w, x; y, 1; e)$; *(14)* $B_5(1, b; 1, x; y)$;
(5) $B_5(a, 1; c, 1; e)$; *(10)* $B_5(x, b; 1, 1; 1)$; *(15)* $B_5(1, x; 1, y; e)$;

(16) 63 *specific graphs:* 13 *graphs of order* 10*,* 25 *graphs of order* 11*, and* 25 *graphs of order* 12*,*

where $a, b, c, d, e, x, y, z, w$ are some positive integers such that $x \le 2$, $y \le 2$, $z \le 2$ and $w \le 3$.

Lemma 5.13. *Let $G \in \mathcal{B}_k(n)$, where $4 \le k \le 9$ and $n \ge 14$. If $G \notin \mathcal{B}_k^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$.*

Proof. We prove this lemma by induction on n. If $n = 14$, since $G \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$, our result is obviously true by taking $\Gamma = G$. Let $n \ge 15$ and $G' \in \mathcal{B}_k(n-1)$ be an induced subgraph of G. If $G' \notin \mathcal{B}_k^-(n-1)$, then G' has an induced subgraph $\Gamma \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$ by induction hypothesis, and so does G. Hence it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_k(n-1) \setminus \mathcal{B}_k^-(n-1)$ for $n \ge 15$ in the following. We will prove that there exists $G' \in \mathcal{B}_5(n-1) \setminus \mathcal{B}_5^-(n-1)$ for $n \ge 15$, and it can be similarly proved for the other k which we keep in the Appendix B.

Let $G = B_5(n_1, n_2; n_3, n_4; n_5) \in \mathcal{B}_5(n)$. Then one of

$$H_1 = B_5(n_1 - 1, n_2; n_3, n_4; n_5), \quad H_2 = B_5(n_1, n_2 - 1; n_3, n_4; n_5),$$
$$H_3 = B_5(n_1, n_2; n_3 - 1, n_4; n_5), \quad H_4 = B_5(n_1, n_2; n_3, n_4 - 1; n_5) \text{ and}$$
$$H_5 = B_5(n_1, n_2; n_3, n_4; n_5 - 1)$$

must belong to $\mathcal{B}_5(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_5^-(n-1)$ for $i = 1, 2, \ldots, 5$. Then H_i is a graph belonging to (1)–(15) in Theorem 5.12 since $|H_i| = n - 1 \ge 14$.

First we consider H_1. If H_1 is a graph belonging to (1) of Theorem 5.12, then $H_1 = B_5(a, w; 1, 1; 1)$ where $n_1 - 1 = a$, $n_2 = w$, $n_3 = n_4 = n_5 = 1$, and hence $G = B_5(a+1, w; 1, 1; 1) \in \mathcal{B}_5^-(n)$, a contradiction. Similarly, H_1 cannot belong to (2)–(8) of Theorem 5.12. If H_1 is a graph belonging to (9) of Theorem 5.12, then $H_1 = B_5(w, x; y, 1; e)$ where $n_1 - 1 = w$, $n_2 = x$, $n_3 = y$, $n_4 = 1$, $n_5 = e$. Since $w \le 3$, we have $n_1 \le 4$. If $n_1 < 4$ then $w + 1 \le 3$ and $G = B_5(w+1, x; y, 1; e) \in \mathcal{B}_5^-(n)$, a contradiction. Now assume that $n_1 = 4$. Then $H_1 = B_5(3, x; y, 1; e)$. Since $x, y \in \{1, 2\}$, we have $G \in \{B_5(4, 1; 1, 1; e), B_5(4, 2; 1, 1; e), B_5(4, 1; 2, 1; e), B_5(4, 2; 2, 1; e)\}$. However $B_5(4, 1; 1, 1; e)$, $B_5(4, 2; 1, 1; e)$, $B_5(4, 1; 2, 1; e)$ belong to (4), (5) of Theorem 5.12 which contradicts our assumption. Thus $G = B_5(4, 2; 2, 1; e)$. By Theorem 5.12, $G =$

$B_5(4,2;2,1;e) \notin \mathcal{B}_5^-(n)$, and also its induced subgraph $B_5(4,2;2,1;e-1) \notin \mathcal{B}_5^-(n-1)$, a contradiction. Hence H_1 belongs to (10) – (15) of Theorem 5.12, from which we see that $n_1 - 1$ is either x or 1. Thus $n_1 \leq 3$ due to $x \leq 2$.

By the same method, we can verify that $n_2 \leq 3$ if $H_2 \in \mathcal{B}_5^-(n-1)$; $n_3 \leq 3$ if $H_3 \in \mathcal{B}_5^-(n-1)$; $n_4 \leq 3$ if $H_4 \in \mathcal{B}_5^-(n-1)$ and $n_5 \leq 2$ if $H_5 \in \mathcal{B}_5^-(n-1)$. Hence $n = n_1 + \cdots + n_5 \leq 14$, a contradiction. We are done. □

Lemma 5.14 ([6]). *If* $n \geq 14$*, then* $\mathcal{B}_k^-(n) = \emptyset$ *for* $10 \leq k \leq 13$.

Lemma 5.15. *Given* $4 \leq k \leq 13$*,* $\mathcal{B}_k^0(n) = \emptyset$ *for* $n \geq 14$ *(it means that there are no reduced* X*-complete graphs of order* $n \geq 14$*).*

Proof. Let $G \in \mathcal{B}_k^0(n)$ and $n \geq 14$. Then $\lambda_4(G) < \lambda_3(G) = 0$. First we assume that $4 \leq k \leq 9$. Since $G \notin \mathcal{B}_k^-(n)$, G has an induced subgraphs $\Gamma \in \mathcal{B}_k(14) \setminus \mathcal{B}_k^-(14)$ by Lemma 5.13. Thus $\lambda_3(\Gamma) \geq 0$. Furthermore, we have $\lambda_3(\Gamma) = 0$ since otherwise $0 < \lambda_3(\Gamma) \leq \lambda_3(G)$. Additionally, $\lambda_4(\Gamma) \leq \lambda_4(G) < 0$, we have $\Gamma \in \mathcal{B}_k^0(14)$, contrary to Lemma 5.11. Next we assume that $10 \leq k \leq 13$. By deleting $n - 14$ vertices from G, we may obtain an induced subgraph $\Gamma \in \mathcal{B}_k(14)$. By Lemma 5.14, we have $\lambda_3(\Gamma) \geq 0$, and then $\lambda_3(\Gamma) = 0$ by the arguments above. Additionally, $\lambda_4(\Gamma) \leq \lambda_4(G) < 0$, we have $\Gamma \in \mathcal{B}_k^0(14)$ which also contradicts Lemma 5.11. □

By Lemma 5.15, we know that, for any reduced X-complete graph $G \in \mathcal{G}^*$, there exists $4 \leq k \leq 13$ and $n \leq 13$ such that $G \in \mathcal{B}_k^0(n)$. Let

$$\mathcal{B}^* = \{G = B_k(n_1, n_2, \ldots, n_k) \in \mathcal{B}_k^0(n) \mid 4 \leq k \leq 13 \text{ and } n \leq 13\}.$$

Thus $G \in \mathcal{G}^*$ is a reduced X-complete graph if and only if $G \in \mathcal{B}^*$.

Remark 5.16. Clearly, $\mathcal{B} = \cup_{4 \leq k \leq 13, n \leq 13} \mathcal{B}_k(n)$ contains finite graphs. By using computer we can exhaust all the graphs of $\mathcal{B}$ to find out the graphs in $\mathcal{B}^*$. We list them in Table 1.

Recall that $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are the set of connected graphs each of them is obtained from some $H \in \mathcal{H}$ by adding one vertex of I, II, III-type, respectively. Summarizing Lemmas 5.2, 5.3, 5.4, 5.15 and Theorem 4.2, finally we give the characterization of the connected graphs in $\mathcal{G}$.

Theorem 5.17. *Let* G *be a connected graph of order* $n \geq 5$*. Then* $G \in \mathcal{G}$ *if and only if* G *is isomorphic to one of the following graphs listed in (1), (2) and (3):*

(1) $K_{1,2}(u) \odot^k K_{n-3}$ *or* $K_{1,1}(u) \odot^k K_{n-2} \setminus e$ *for* $e \in E(K_{n-2})$*;*

(2) the graphs belonging to $\mathcal{G}_1, \mathcal{G}_2$ *or* $\mathcal{G}_3$*;*

(3) the 802 *specific graphs belonging to* $\mathcal{B}^*$ *some of which we list in Table 1.*

If G^* is obtained from $G \in \mathcal{G}$ by adding one vertex of I, II or III-type, then the positive and negative indices of G^* left unchanged, but the nullity adds just one. Repeating this process, we can get a class of graphs which has two positive eigenvalues and s zero eigenvalues, where $s \geq 2$ is any integer. However, by using the I, II and III-type (graph) transformations, we can not get all such graphs. For example, $H = B_{10}(1,1,2,3,2;1,1,1,1,1)$ is a graph satisfying $p(H) = 2$ and $\eta(H) = 2$ that can not be constructed by above (graph) transformation. Hence the characterization of graphs with $p(H) = 2$ and $\eta(H) = s$ (especially $\eta(H) = 2$) is also an attractive problem.

Table 1: All graphs of $\mathcal{B}^*$.

k	$\mathcal{B}^*$	Number
4	$B_4(3,2;3,2)$; $B_4(4,3;2,2)$, $B_4(4,3;3,1)$; $B_4(5,4;2,1)$, $B_4(5,2;2,3)$, $B_4(3,4;2,3)$, $B_4(4,1;3,4)$, $B_4(5,2;4,1)$; $B_4(7,3;2,1)$, $B_4(4,6;2,1)$, $B_4(7,2;2,2)$, $B_4(3,6;2,2)$, $B_4(4,2;2,5)$, $B_4(3,3;2,5)$, $B_4(7,2;3,1)$, $B_4(3,6;3,1)$, $B_4(6,1;3,3)$, $B_4(6,1;4,2)$.	18
5	$B_5(2,2;2,2;1)$; $B_5(2,3;1,2;2)$, $B_5(3,3;2,1;1)$; $B_5(3,4;1,1;2)$, $B_5(3,4;1,2;1)$, $B_5(1,3;1,3;3)$, $B_5(2,2;1,3;3)$, $B_5(2,4;2,1;2)$, $B_5(4,2;3,1;1)$; $B_5(4,5;1,1;1)$, $B_5(2,5;1,1;3)$, $B_5(4,3;1,1;3)$, $B_5(1,4;1,2;4)$, $B_5(3,2;1,2;4)$, $B_5(2,5;1,3;1)$, $B_5(4,3;1,3;1)$, $B_5(1,4;1,4;2)$, $B_5(3,2;1,4;2)$, $B_5(5,2;2,1;2)$, $B_5(3,1;2,3;3)$, $B_5(3,1;2,5;1)$, $B_5(4,1;3,2;2)$; $B_5(3,7;1,1;1)$, $B_5(6,4;1,1;1)$, $B_5(2,7;1,1;2)$, $B_5(6,3;1,1;2)$, $B_5(2,4;1,1;5)$, $B_5(3,3;1,1;5)$, $B_5(2,7;1,2;1)$, $B_5(6,3;1,2;1)$, $B_5(1,6;1,2;3)$, $B_5(5,2;1,2;3)$, $B_5(1,3;1,2;6)$, $B_5(2,2;1,2;6)$, $B_5(1,6;1,3;2)$, $B_5(5,2;1,3;2)$, $B_5(2,4;1,5;1)$, $B_5(3,3;1,5;1)$, $B_5(2,2;1,6;2)$, $B_5(2,7;2,1;1)$, $B_5(7,2;2,1;1)$, $B_5(4,2;2,1;4)$, $B_5(2,3;2,1;5)$, $B_5(5,1;2,3;2)$, $B_5(5,1;2,4;1)$, $B_5(3,2;3,1;4)$, $B_5(6,1;3,2;1)$.	47
6	See Table 2 of Appendix A	138
7	See Table 3 of Appendix A	161
8	See Table 4 of Appendix A	205
9	See Table 5 of Appendix A	124
10	See Table 6 of Appendix A	78
11	$B_{11}(1,1,1,2,1;1,1,1,1,1;1)$, $B_{11}(2,1,1,1,1;1,1,1,1,1;1)$; $B_{11}(1,1,1,1,3;1,1,1,1,1;1)$, $B_{11}(1,1,1,2,2;1,1,1,1,1;1)$, $B_{11}(1,1,2,1,2;1,1,1,1,1;1)$, $B_{11}(1,1,2,2,1;1,1,1,1,1;1)$, $B_{11}(1,1,3,1,1;1,1,1,1,1;1)$, $B_{11}(1,2,1,1,2;1,1,1,1,1;1)$, $B_{11}(1,2,2,1,1;1,1,1,1,1;1)$, $B_{11}(1,3,1,1,1;1,1,1,1,1;1)$, $B_{11}(2,1,1,1,2;1,1,1,1,1;1)$, $B_{11}(2,2,1,1,1;1,1,1,1,1;1)$, $B_{11}(1,1,1,1,2;1,1,1,1,1;2)$, $B_{11}(1,1,2,1,1;1,1,1,1,1;2)$, $B_{11}(1,2,1,1,1;1,1,1,1,1;2)$, $B_{11}(1,1,1,1,1;1,1,1,1,1;3)$, $B_{11}(1,1,1,1,2;1,1,1,1,2;1)$, $B_{11}(1,1,1,2,1;1,1,1,1,2;1)$, $B_{11}(1,1,2,1,1;1,1,1,1,2;1)$, $B_{11}(1,2,1,1,1;1,1,1,1,2;1)$, $B_{11}(1,1,2,1,1;1,1,2,1,1;1)$, $B_{11}(1,2,1,1,1;1,1,2,1,1;1)$, $B_{11}(2,1,1,1,1;1,1,2,1,1;1)$, $B_{11}(1,2,1,1,1;1,2,1,1,1;1)$.	24
12	$B_{12}(1,1,1,1,1,2;1,1,1,1,1,1)$, $B_{12}(1,1,1,1,2,1;1,1,1,1,1,1)$, $B_{12}(1,1,1,2,1,1;1,1,1,1,1,1)$, $B_{12}(1,1,2,1,1,1;1,1,1,1,1,1)$, $B_{12}(1,2,1,1,1,1;1,1,1,1,1,1)$, $B_{12}(2,1,1,1,1,1;1,1,1,1,1,1)$.	6
13	$B_{13}(1,1,1,1,1,1;1,1,1,1,1,1;1)$.	1

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Appendix A Five tables

Appendix A contains 5 tables, in which there are 706 specific graphs: 4 graphs of order 10, 32 graphs of order 11, 150 graphs of order 12, and 520 graphs of order 13.

Table 2: $k = 6$.

n	$\mathcal{B}^*$
10	$B_6(1,2,2;1,2,2)$, $B_6(2,2,1;1,2,2)$;
11	$B_6(1,3,3;1,1,2)$, $B_6(2,3,2;1,1,2)$, $B_6(3,3,1;1,1,2)$, $B_6(1,3,3;1,2,1)$, $B_6(2,3,2;1,2,1)$, $B_6(3,3,1;1,2,1)$, $B_6(2,1,1;1,3,3)$, $B_6(3,2,1;2,1,2)$, $B_6(2,2,2;2,2,1)$, $B_6(3,1,2;3,1,1)$;
12	$B_6(1,4,4;1,1,1)$, $B_6(2,4,3;1,1,1)$, $B_6(3,4,2;1,1,1)$, $B_6(4,4,1;1,1,1)$, $B_6(1,2,4;1,1,3)$, $B_6(1,4,2;1,1,3)$, $B_6(2,2,3;1,1,3)$, $B_6(2,4,1;1,1,3)$, $B_6(3,2,2;1,1,3)$, $B_6(4,2,1;1,1,3)$, $B_6(1,3,1;1,2,4)$, $B_6(2,1,2;1,2,4)$, $B_6(3,1,1;1,2,4)$, $B_6(1,4,2;1,3,1)$, $B_6(2,2,3;1,3,1)$, $B_6(2,4,1;1,3,1)$, $B_6(3,2,2;1,3,1)$, $B_6(4,2,1;1,3,1)$, $B_6(2,1,2;1,4,2)$, $B_6(3,1,1;1,4,2)$, $B_6(2,3,3;2,1,1)$, $B_6(4,1,3;2,1,1)$, $B_6(4,3,1;2,1,1)$, $B_6(2,3,2;2,1,2)$, $B_6(3,2,2;2,1,2)$, $B_6(4,1,2;2,1,2)$, $B_6(2,3,1;2,1,3)$, $B_6(4,1,1;2,1,3)$, $B_6(3,1,3;2,2,1)$, $B_6(3,2,2;2,2,1)$, $B_6(3,3,1;2,2,1)$, $B_6(3,1,1;2,2,3)$, $B_6(2,3,1;2,3,1)$, $B_6(3,2,2;3,1,1)$, $B_6(4,2,1;3,1,1)$, $B_6(4,1,1;4,1,1)$;
13	$B_6(1,3,6;1,1,1)$, $B_6(1,6,3;1,1,1)$, $B_6(2,3,5;1,1,1)$, $B_6(2,6,2;1,1,1)$, $B_6(3,3,4;1,1,1)$, $B_6(3,6,1;1,1,1)$, $B_6(4,3,3;1,1,1)$, $B_6(5,3,2;1,1,1)$, $B_6(6,3,1;1,1,1)$, $B_6(1,2,6;1,1,2)$, $B_6(1,6,2;1,1,2)$, $B_6(2,2,5;1,1,2)$, $B_6(2,6,1;1,1,2)$, $B_6(3,2,4;1,1,2)$, $B_6(4,2,3;1,1,2)$, $B_6(5,2,2;1,1,2)$, $B_6(6,2,1;1,1,2)$, $B_6(1,2,3;1,1,5)$, $B_6(1,3,2;1,1,5)$, $B_6(2,2,2;1,1,5)$, $B_6(2,2,5;1,2,1)$, $B_6(2,6,1;1,2,1)$, $B_6(3,2,4;1,2,1)$, $B_6(4,2,3;1,2,1)$, $B_6(5,2,2;1,2,1)$, $B_6(2,3,1;1,1,5)$, $B_6(3,2,1;1,1,5)$, $B_6(1,2,6;1,2,1)$, $B_6(1,6,2;1,2,1)$, $B_6(6,2,1;1,2,1)$, $B_6(1,5,1;1,2,3)$, $B_6(2,1,4;1,2,3)$, $B_6(3,1,3;1,2,3)$, $B_6(4,1,2;1,2,3)$, $B_6(5,1,1;1,2,3)$, $B_6(2,1,1;1,2,6)$, $B_6(1,5,1;1,3,2)$, $B_6(2,1,4;1,3,2)$, $B_6(3,1,3;1,3,2)$, $B_6(4,1,2;1,3,2)$, $B_6(5,1,1;1,3,2)$, $B_6(2,2,2;1,5,1)$, $B_6(2,3,1;1,5,1)$, $B_6(3,2,1;1,5,1)$, $B_6(2,1,1;1,6,2)$, $B_6(2,2,5;2,1,1)$, $B_6(2,5,2;2,1,1)$, $B_6(3,1,5;2,1,1)$, $B_6(3,2,4;2,1,1)$, $B_6(3,3,3;2,1,1)$, $B_6(3,4,2;2,1,1)$, $B_6(3,5,1;2,1,1)$, $B_6(4,2,3;2,1,1)$, $B_6(4,3,2;2,1,1)$, $B_6(5,2,2;2,1,1)$, $B_6(6,1,2;2,1,1)$, $B_6(6,2,1;2,1,1)$, $B_6(2,2,4;2,1,2)$, $B_6(2,5,1;2,1,2)$, $B_6(3,1,4;2,1,2)$, $B_6(3,2,3;2,1,2)$, $B_6(6,1,1;2,1,2)$, $B_6(2,2,3;2,1,3)$, $B_6(3,1,3;2,1,3)$, $B_6(2,2,2;2,1,4)$, $B_6(3,1,2;2,1,4)$, $B_6(2,2,1;2,1,5)$, $B_6(3,1,1;2,1,5)$, $B_6(2,5,1;2,2,1)$, $B_6(4,2,2;2,2,1)$, $B_6(5,1,2;2,2,1)$, $B_6(5,2,1;2,2,1)$, $B_6(3,1,3;2,2,2)$, $B_6(4,1,2;2,2,2)$, $B_6(5,1,1;2,2,2)$, $B_6(3,1,2;2,2,3)$, $B_6(4,1,2;2,3,1)$, $B_6(4,2,1;2,3,1)$, $B_6(3,1,2;2,3,2)$, $B_6(4,1,1;2,3,2)$, $B_6(3,1,2;2,4,1)$, $B_6(3,2,1;2,4,1)$, $B_6(3,1,1;2,4,2)$, $B_6(3,3,2;3,1,1)$, $B_6(3,4,1;3,1,1)$, $B_6(6,1,1;3,1,1)$, $B_6(3,3,1;3,2,1)$, $B_6(4,2,1;3,2,1)$, $B_6(5,1,1;3,2,1)$, $B_6(4,1,1;3,3,1)$.

Table 3: $k = 7$.

n	$\mathcal{B}^*$
10	$B_7(2,2,1;1,1,2;1)$, $B_7(2,1,2;2,1,1;1)$;
11	$B_7(3,3,1;1,1,1;1)$, $B_7(2,1,3;1,1,1;2)$, $B_7(2,2,2;1,1,2;1)$, $B_7(2,1,2;1,1,2;2)$,
	$B_7(1,2,1;1,1,3;2)$, $B_7(2,1,1;1,1,3;2)$, $B_7(1,2,3;1,2,1;1)$, $B_7(1,2,2;1,2,2;1)$,
	$B_7(2,1,1;1,2,3;1)$, $B_7(2,2,2;2,1,1;1)$, $B_7(3,2,1;2,1,1;1)$, $B_7(3,1,1;3,1,1;1)$;
12	$B_7(1,3,4;1,1,1;1)$, $B_7(3,1,4;1,1,1;1)$, $B_7(3,3,2;1,1,1;1)$, $B_7(1,2,4;1,1,1;2)$,
	$B_7(2,2,3;1,1,1;2)$, $B_7(2,4,1;1,1,1;2)$, $B_7(3,2,2;1,1,1;2)$, $B_7(4,2,1;1,1,1;2)$,
	$B_7(1,1,4;1,1,1;3)$, $B_7(3,1,2;1,1,1;3)$, $B_7(1,3,3;1,1,2;1)$, $B_7(2,2,3;1,1,2;1)$,
	$B_7(3,1,3;1,1,2;1)$, $B_7(1,1,3;1,1,2;3)$, $B_7(1,3,1;1,1,2;3)$, $B_7(3,1,1;1,1,2;3)$,
	$B_7(1,3,2;1,1,3;1)$, $B_7(3,1,2;1,1,3;1)$, $B_7(1,2,2;1,1,3;2)$, $B_7(1,3,1;1,1,4;1)$,
	$B_7(3,1,1;1,1,4;1)$, $B_7(2,1,4;1,2,1;1)$, $B_7(2,2,3;1,2,1;1)$, $B_7(2,3,2;1,2,1;1)$,
	$B_7(4,2,1;1,2,1;1)$, $B_7(2,1,2;1,2,1;3)$, $B_7(1,2,2;1,2,2;2)$, $B_7(1,3,1;1,2,2;2)$,
	$B_7(2,4,1;1,2,1;1)$, $B_7(2,1,2;1,2,2;2)$, $B_7(3,1,1;1,2,2;2)$, $B_7(2,1,2;1,2,3;1)$,
	$B_7(1,3,2;1,3,1;1)$, $B_7(3,1,1;1,3,2;1)$, $B_7(2,3,2;2,1,1;1)$, $B_7(2,3,1;2,1,1;2)$,
	$B_7(4,1,1;2,1,1;2)$, $B_7(3,2,1;2,2,1;1)$, $B_7(3,1,1;2,2,1;2)$;
13	$B_7(1,2,6;1,1,1;1)$, $B_7(1,5,3;1,1,1;1)$, $B_7(2,1,6;1,1,1;1)$, $B_7(2,2,5;1,1,1;1)$,
	$B_7(2,3,4;1,1,1;1)$, $B_7(2,4,3;1,1,1;1)$, $B_7(2,5,2;1,1,1;1)$, $B_7(2,6,1;1,1,1;1)$,
	$B_7(3,2,4;1,1,1;1)$, $B_7(3,3,3;1,1,1;1)$, $B_7(4,2,3;1,1,1;1)$, $B_7(5,1,3;1,1,1;1)$,
	$B_7(5,2,2;1,1,1;1)$, $B_7(6,2,1;1,1,1;1)$, $B_7(1,1,6;1,1,1;2)$, $B_7(1,4,3;1,1,1;2)$,
	$B_7(2,3,3;1,1,1;2)$, $B_7(2,4,2;1,1,1;2)$, $B_7(5,1,2;1,1,1;2)$, $B_7(1,3,3;1,1,1;3)$,
	$B_7(2,3,2;1,1,1;3)$, $B_7(1,2,3;1,1,1;4)$, $B_7(2,2,2;1,1,1;4)$, $B_7(2,3,1;1,1,1;4)$,
	$B_7(3,2,1;1,1,1;4)$, $B_7(1,1,3;1,1,1;5)$, $B_7(2,1,2;1,1,1;5)$, $B_7(1,2,5;1,1,2;1)$,
	$B_7(1,5,2;1,1,2;1)$, $B_7(2,1,5;1,1,2;1)$, $B_7(2,2,4;1,1,2;1)$, $B_7(5,1,2;1,1,2;1)$,
	$B_7(1,1,5;1,1,2;2)$, $B_7(1,2,4;1,1,2;2)$, $B_7(1,3,3;1,1,2;2)$, $B_7(1,4,2;1,1,2;2)$,
	$B_7(1,5,1;1,1,2;2)$, $B_7(5,1,1;1,1,2;2)$, $B_7(1,2,3;1,1,2;3)$, $B_7(1,3,2;1,1,2;3)$,
	$B_7(1,2,2;1,1,2;4)$, $B_7(1,1,2;1,1,2;5)$, $B_7(1,2,1;1,1,2;5)$, $B_7(2,1,1;1,1,2;5)$,
	$B_7(1,2,4;1,1,3;1)$, $B_7(1,5,1;1,1,3;1)$, $B_7(2,1,4;1,1,3;1)$, $B_7(5,1,1;1,1,3;1)$,
	$B_7(1,1,4;1,1,3;2)$, $B_7(1,2,3;1,1,3;2)$, $B_7(1,2,3;1,1,4;1)$, $B_7(2,1,3;1,1,4;1)$,
	$B_7(1,2,2;1,1,5;1)$, $B_7(2,1,2;1,1,5;1)$, $B_7(1,2,1;1,1,6;1)$, $B_7(2,1,1;1,1,6;1)$,
	$B_7(1,5,2;1,2,1;1)$, $B_7(3,2,3;1,2,1;1)$, $B_7(4,1,3;1,2,1;1)$, $B_7(4,2,2;1,2,1;1)$,
	$B_7(1,4,2;1,2,1;2)$, $B_7(2,3,2;1,2,1;2)$, $B_7(3,2,2;1,2,1;2)$, $B_7(4,1,2;1,2,1;2)$,
	$B_7(1,3,2;1,2,1;3)$, $B_7(2,2,2;1,2,1;3)$, $B_7(2,3,1;1,2,1;3)$, $B_7(3,2,1;1,2,1;3)$,
	$B_7(1,2,2;1,2,1;4)$, $B_7(1,5,1;1,2,2;1)$, $B_7(2,1,4;1,2,2;1)$, $B_7(3,1,3;1,2,2;1)$,
	$B_7(4,1,2;1,2,2;1)$, $B_7(5,1,1;1,2,2;1)$, $B_7(1,2,2;1,2,2;3)$, $B_7(2,1,1;1,2,2;4)$,
	$B_7(2,1,3;1,2,3;1)$, $B_7(3,1,3;1,3,1;1)$, $B_7(3,2,2;1,3,1;1)$, $B_7(2,2,2;1,3,1;2)$,
	$B_7(2,3,1;1,3,1;2)$, $B_7(3,1,2;1,3,1;2)$, $B_7(3,2,1;1,3,1;2)$, $B_7(2,1,3;1,3,2;1)$,
	$B_7(3,1,2;1,3,2;1)$, $B_7(2,1,2;1,3,2;2)$, $B_7(2,1,1;1,3,2;3)$, $B_7(2,1,3;1,4,1;1)$,
	$B_7(2,2,2;1,4,1;1)$, $B_7(2,3,1;1,4,1;1)$, $B_7(3,2,1;1,4,1;1)$, $B_7(2,1,2;1,4,1;2)$,
	$B_7(2,1,2;1,4,2;1)$, $B_7(2,1,1;1,4,2;2)$, $B_7(2,1,1;1,5,2;1)$, $B_7(2,4,2;2,1,1;1)$,
	$B_7(2,5,1;2,1,1;1)$, $B_7(6,1,1;2,1,1;1)$, $B_7(2,2,1;2,1,1;4)$, $B_7(3,1,1;2,1,1;4)$,
	$B_7(2,4,1;2,2,1;1)$, $B_7(5,1,1;2,2,1;1)$, $B_7(2,3,1;2,2,1;2)$, $B_7(2,2,1;2,2,1;3)$,
	$B_7(2,3,1;2,3,1;1)$, $B_7(3,2,1;2,3,1;1)$, $B_7(4,1,1;2,3,1;1)$, $B_7(3,1,1;2,4,1;1)$.

Table 4: $k = 8$.

n	$\mathcal{B}^*$
11	$B_8(1,2,1,2;1,1,1,2)$, $B_8(2,2,1,1;1,1,1,2)$, $B_8(1,2,2,1;1,1,2,1)$, $B_8(1,2,1,1;1,1,2,2)$, $B_8(2,1,2,1;1,2,1,1)$, $B_8(2,1,1,1;1,2,1,2)$;
12	$B_8(1,1,3,3;1,1,1,1)$, $B_8(1,3,1,3;1,1,1,1)$, $B_8(1,3,3,1;1,1,1,1)$, $B_8(2,1,3,2;1,1,1,1)$, $B_8(2,3,1,2;1,1,1,1)$, $B_8(3,1,3,1;1,1,1,1)$, $B_8(3,3,1,1;1,1,1,1)$, $B_8(1,1,2,3;1,1,1,2)$, $B_8(1,2,2,2;1,1,1,2)$, $B_8(1,3,2,1;1,1,1,2)$, $B_8(2,1,2,2;1,1,1,2)$, $B_8(2,2,2,1;1,1,1,2)$, $B_8(3,1,2,1;1,1,1,2)$, $B_8(1,1,1,3;1,1,1,3)$, $B_8(1,3,1,1;1,1,1,3)$, $B_8(2,1,1,2;1,1,1,3)$, $B_8(3,1,1,1;1,1,1,3)$, $B_8(1,1,3,2;1,1,2,1)$, $B_8(1,2,2,2;1,1,2,1)$, $B_8(1,3,1,2;1,1,2,1)$, $B_8(2,1,3,1;1,1,2,1)$, $B_8(2,2,2,1;1,1,2,1)$, $B_8(2,3,1,1;1,1,2,1)$, $B_8(2,1,1,1;1,1,2,3)$, $B_8(1,1,3,1;1,1,3,1)$, $B_8(1,3,1,1;1,1,3,1)$, $B_8(1,2,1,3;1,2,1,1)$, $B_8(1,2,2,2;1,2,1,1)$, $B_8(1,2,3,1;1,2,1,1)$, $B_8(2,2,1,2;1,2,1,1)$, $B_8(2,2,2,1;1,2,1,1)$, $B_8(3,2,1,1;1,2,1,1)$, $B_8(2,1,1,1;1,2,2,2)$, $B_8(2,1,1,2;1,3,1,1)$, $B_8(3,1,1,1;1,3,1,1)$, $B_8(2,1,1,1;1,3,2,1)$, $B_8(2,2,1,2;2,1,1,1)$, $B_8(3,1,2,1;2,1,1,1)$, $B_8(2,2,1,1;2,1,1,2)$, $B_8(3,1,1,1;2,1,1,2)$, $B_8(2,1,2,1;2,1,2,1)$, $B_8(2,2,1,1;2,1,2,1)$, $B_8(3,1,1,1;3,1,1,1)$;
13	$B_8(1,1,2,5;1,1,1,1)$, $B_8(1,1,5,2;1,1,1,1)$, $B_8(1,2,1,5;1,1,1,1)$, $B_8(1,2,2,4;1,1,1,1)$, $B_8(1,2,3,3;1,1,1,1)$, $B_8(1,2,4,2;1,1,1,1)$, $B_8(1,2,5,1;1,1,1,1)$, $B_8(1,3,2,3;1,1,1,1)$, $B_8(1,3,3,2;1,1,1,1)$, $B_8(1,4,2,2;1,1,1,1)$, $B_8(1,5,1,2;1,1,1,1)$, $B_8(1,5,2,1;1,1,1,1)$, $B_8(2,1,2,4;1,1,1,1)$, $B_8(2,1,5,1;1,1,1,1)$, $B_8(2,2,1,4;1,1,1,1)$, $B_8(2,2,2,3;1,1,1,1)$, $B_8(2,2,3,2;1,1,1,1)$, $B_8(2,2,4,1;1,1,1,1)$, $B_8(2,3,2,2;1,1,1,1)$, $B_8(2,3,3,1;1,1,1,1)$, $B_8(2,4,2,1;1,1,1,1)$, $B_8(2,5,1,1;1,1,1,1)$, $B_8(3,1,2,3;1,1,1,1)$, $B_8(3,2,1,3;1,1,1,1)$, $B_8(3,2,2,2;1,1,1,1)$, $B_8(3,2,3,1;1,1,1,1)$, $B_8(3,3,2,1;1,1,1,1)$, $B_8(4,1,2,2;1,1,1,1)$, $B_8(4,2,1,2;1,1,1,1)$, $B_8(4,2,2,1;1,1,1,1)$, $B_8(5,1,2,1;1,1,1,1)$, $B_8(5,2,1,1;1,1,1,1)$, $B_8(1,1,1,5;1,1,1,2)$, $B_8(1,1,4,2;1,1,1,2)$, $B_8(1,2,3,2;1,1,1,2)$, $B_8(1,2,4,1;1,1,1,2)$, $B_8(1,5,1,1;1,1,1,2)$, $B_8(2,1,1,4;1,1,1,2)$, $B_8(2,1,4,1;1,1,1,2)$, $B_8(2,2,3,1;1,1,1,2)$, $B_8(3,1,1,3;1,1,1,2)$, $B_8(4,1,1,2;1,1,1,2)$, $B_8(5,1,1,1;1,1,1,2)$, $B_8(1,1,3,2;1,1,1,3)$, $B_8(1,2,3,1;1,1,1,3)$, $B_8(2,1,3,1;1,1,1,3)$, $B_8(1,1,2,2;1,1,1,4)$, $B_8(1,2,2,1;1,1,1,4)$, $B_8(2,1,2,1;1,1,1,4)$, $B_8(1,2,1,1;1,1,1,5)$, $B_8(2,1,1,1;1,1,1,5)$, $B_8(1,1,2,4;1,1,2,1)$, $B_8(1,1,5,1;1,1,2,1)$, $B_8(1,2,1,4;1,1,2,1)$, $B_8(1,2,2,3;1,1,2,1)$, $B_8(1,5,1,1;1,1,2,1)$, $B_8(2,1,2,3;1,1,2,1)$, $B_8(2,2,1,3;1,1,2,1)$, $B_8(2,2,2,2;1,1,2,1)$, $B_8(3,1,2,2;1,1,2,1)$, $B_8(3,2,1,2;1,1,2,1)$, $B_8(3,2,2,1;1,1,2,1)$, $B_8(4,1,2,1;1,1,2,1)$, $B_8(4,2,1,1;1,1,2,1)$, $B_8(1,1,2,3;1,1,2,2)$, $B_8(1,1,3,2;1,1,2,2)$,

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n	$\mathcal{B}^*$
13	$B_8(1,1,4,1;1,1,2,2)$, $B_8(2,1,1,3;1,1,2,2)$, $B_8(2,1,2,2;1,1,2,2)$, $B_8(2,1,3,1;1,1,2,2)$, $B_8(3,1,1,2;1,1,2,2)$, $B_8(3,1,2,1;1,1,2,2)$, $B_8(4,1,1,1;1,1,2,2)$, $B_8(1,1,3,1;1,1,2,3)$, $B_8(2,1,2,1;1,1,2,3)$, $B_8(1,2,1,3;1,1,3,1)$, $B_8(2,1,2,2;1,1,3,1)$, $B_8(2,2,1,2;1,1,3,1)$, $B_8(3,1,2,1;1,1,3,1)$, $B_8(3,2,1,1;1,1,3,1)$, $B_8(2,1,1,2;1,1,3,2)$, $B_8(2,1,2,1;1,1,3,2)$, $B_8(3,1,1,1;1,1,3,2)$, $B_8(1,2,1,2;1,1,4,1)$, $B_8(2,1,2,1;1,1,4,1)$, $B_8(2,2,1,1;1,1,4,1)$, $B_8(2,1,1,1;1,1,4,2)$, $B_8(1,2,1,1;1,1,5,1)$, $B_8(1,3,2,2;1,2,1,1)$, $B_8(1,4,1,2;1,2,1,1)$, $B_8(1,4,2,1;1,2,1,1)$, $B_8(2,1,1,4;1,2,1,1)$, $B_8(2,3,2,1;1,2,1,1)$, $B_8(2,4,1,1;1,2,1,1)$, $B_8(3,1,1,3;1,2,1,1)$, $B_8(4,1,1,2;1,2,1,1)$, $B_8(5,1,1,1;1,2,1,1)$, $B_8(1,2,3,1;1,2,1,2)$, $B_8(1,3,2,1;1,2,1,2)$, $B_8(1,4,1,1;1,2,1,2)$, $B_8(1,2,2,1;1,2,1,3)$, $B_8(1,3,1,2;1,2,2,1)$, $B_8(1,4,1,1;1,2,2,1)$, $B_8(2,1,1,3;1,2,2,1)$, $B_8(2,2,1,2;1,2,2,1)$, $B_8(2,3,1,1;1,2,2,1)$, $B_8(3,1,1,2;1,2,2,1)$, $B_8(3,2,1,1;1,2,2,1)$, $B_8(4,1,1,1;1,2,2,1)$, $B_8(2,1,1,2;1,2,3,1)$, $B_8(2,2,1,1;1,2,3,1)$, $B_8(3,1,1,1;1,2,3,1)$, $B_8(2,1,1,1;1,2,3,2)$, $B_8(2,1,1,1;1,2,4,1)$, $B_8(1,3,1,2;1,3,1,1)$, $B_8(1,3,2,1;1,3,1,1)$, $B_8(2,3,1,1;1,3,1,1)$, $B_8(2,2,1,1;1,3,2,1)$, $B_8(2,2,1,1;1,4,1,1)$, $B_8(2,1,1,1;1,5,1,1)$, $B_8(2,1,1,4;2,1,1,1)$, $B_8(2,1,2,3;2,1,1,1)$, $B_8(2,1,3,2;2,1,1,1)$, $B_8(2,1,4,1;2,1,1,1)$, $B_8(2,2,2,2;2,1,1,1)$, $B_8(2,2,3,1;2,1,1,1)$, $B_8(2,3,2,1;2,1,1,1)$, $B_8(2,4,1,1;2,1,1,1)$, $B_8(3,1,1,3;2,1,1,1)$, $B_8(3,1,2,2;2,1,1,1)$, $B_8(3,2,1,2;2,1,1,1)$, $B_8(3,2,2,1;2,1,1,1)$, $B_8(3,3,1,1;2,1,1,1)$, $B_8(4,1,1,2;2,1,1,1)$, $B_8(4,2,1,1;2,1,1,1)$, $B_8(5,1,1,1;2,1,1,1)$, $B_8(2,1,1,3;2,1,1,2)$, $B_8(2,1,2,2;2,1,1,2)$, $B_8(2,1,3,1;2,1,1,2)$, $B_8(2,2,2,1;2,1,1,2)$, $B_8(3,1,1,2;2,1,1,2)$, $B_8(2,1,2,1;2,1,1,3)$, $B_8(2,1,2,2;2,1,2,1)$, $B_8(3,1,1,2;2,1,2,1)$, $B_8(3,2,1,1;2,1,2,1)$, $B_8(4,1,1,1;2,1,2,1)$, $B_8(3,1,1,1;2,1,2,2)$, $B_8(3,1,1,1;2,1,3,1)$, $B_8(2,2,2,1;2,2,1,1)$, $B_8(2,3,1,1;2,2,1,1)$, $B_8(3,1,1,2;2,2,1,1)$, $B_8(3,2,1,1;2,2,1,1)$, $B_8(4,1,1,1;2,2,1,1)$, $B_8(3,1,1,1;2,2,2,1)$, $B_8(3,1,1,1;2,3,1,1)$, $B_8(3,2,1,1;3,1,1,1)$.

Table 5: $k=9$.

n	$\mathcal{B}^*$
11	$B_9(2,1,2,1;1,1,1,1;1)$, $B_9(2,1,1,1;1,1,1,2;1)$, $B_9(1,1,2,1;1,1,2,1;1)$, $B_9(2,1,1,1;2,1,1,1;1)$;
12	$B_9(1,2,1,3;1,1,1,1;1)$, $B_9(1,2,3,1;1,1,1,1;1)$, $B_9(2,1,2,2;1,1,1,1;1)$, $B_9(2,2,2,1;1,1,1,1;1)$, $B_9(3,2,1,1;1,1,1,1;1)$, $B_9(1,1,1,3;1,1,1,1;2)$,

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n	$\mathcal{B}^*$
12	$B_9(1,1,3,1;1,1,1,1;2)$, $B_9(2,1,1,2;1,1,1,1;2)$, $B_9(3,1,1,1;1,1,1,1;2)$, $B_9(1,2,1,2;1,1,1,2;1)$, $B_9(2,1,1,2;1,1,1,2;1)$, $B_9(1,1,2,1;1,1,1,2;2)$, $B_9(1,2,1,1;1,1,1,3;1)$, $B_9(1,1,2,2;1,1,2,1;1)$, $B_9(1,2,1,2;1,1,2,1;1)$, $B_9(2,1,1,1;1,1,2,1;2)$, $B_9(1,2,1,1;1,1,2,2;1)$, $B_9(2,1,1,1;1,1,2,2;1)$, $B_9(1,2,1,1;1,1,3,1;1)$, $B_9(3,1,1,1;1,2,1,1;1)$, $B_9(2,1,1,1;1,2,2,1;1)$, $B_9(2,2,1,1;2,1,1,1;1)$;
13	$B_9(1,1,1,5;1,1,1,1;1)$, $B_9(1,1,2,4;1,1,1,1;1)$, $B_9(1,1,3,3;1,1,1,1;1)$, $B_9(1,1,4,2;1,1,1,1;1)$, $B_9(1,1,5,1;1,1,1,1;1)$, $B_9(1,2,2,3;1,1,1,1;1)$, $B_9(1,2,3,2;1,1,1,1;1)$, $B_9(1,3,2,2;1,1,1,1;1)$, $B_9(1,4,1,2;1,1,1,1;1)$, $B_9(1,4,2,1;1,1,1,1;1)$, $B_9(2,1,1,4;1,1,1,1;1)$, $B_9(2,1,2,3;1,1,1,1;1)$, $B_9(2,2,1,3;1,1,1,1;1)$, $B_9(2,2,2,2;1,1,1,1;1)$, $B_9(2,3,1,2;1,1,1,1;1)$, $B_9(2,3,2,1;1,1,1,1;1)$, $B_9(2,4,1,1;1,1,1,1;1)$, $B_9(3,1,1,3;1,1,1,1;1)$, $B_9(3,2,1,2;1,1,1,1;1)$, $B_9(4,1,1,2;1,1,1,1;1)$, $B_9(5,1,1,1;1,1,1,1;1)$, $B_9(1,1,2,3;1,1,1,1;2)$, $B_9(1,1,3,2;1,1,1,1;2)$, $B_9(1,2,2,2;1,1,1,1;2)$, $B_9(1,3,1,2;1,1,1,1;2)$, $B_9(1,3,2,1;1,1,1,1;2)$, $B_9(2,2,1,2;1,1,1,1;2)$, $B_9(2,3,1,1;1,1,1,1;2)$, $B_9(1,1,2,2;1,1,1,1;3)$, $B_9(1,2,1,2;1,1,1,1;3)$, $B_9(1,2,2,1;1,1,1,1;3)$, $B_9(2,2,1,1;1,1,1,1;3)$, $B_9(1,1,1,2;1,1,1,1;4)$, $B_9(1,1,2,1;1,1,1,1;4)$, $B_9(2,1,1,1;1,1,1,1;4)$, $B_9(1,1,1,4;1,1,1,2;1)$, $B_9(1,1,2,3;1,1,1,2;1)$, $B_9(1,1,3,2;1,1,1,2;1)$, $B_9(1,1,4,1;1,1,1,2;1)$, $B_9(1,2,2,2;1,1,1,2;1)$, $B_9(1,2,3,1;1,1,1,2;1)$, $B_9(1,3,2,1;1,1,1,2;1)$, $B_9(1,4,1,1;1,1,1,2;1)$, $B_9(2,1,1,3;1,1,1,2;1)$, $B_9(1,1,1,3;1,1,1,2;2)$, $B_9(1,1,2,2;1,1,1,2;2)$, $B_9(1,2,1,2;1,1,1,2;2)$, $B_9(1,2,2,1;1,1,1,2;2)$, $B_9(1,3,1,1;1,1,1,2;2)$, $B_9(1,1,1,2;1,1,1,2;3)$, $B_9(1,2,1,1;1,1,1,2;3)$, $B_9(1,1,1,3;1,1,1,3;1)$, $B_9(1,1,2,2;1,1,1,3;1)$, $B_9(1,1,3,1;1,1,1,3;1)$, $B_9(1,2,2,1;1,1,1,3;1)$, $B_9(1,1,2,1;1,1,1,4;1)$, $B_9(1,1,2,3;1,1,2,1;1)$, $B_9(1,4,1,1;1,1,2,1;1)$, $B_9(2,1,1,3;1,1,2,1;1)$, $B_9(2,2,1,2;1,1,2,1;1)$, $B_9(2,3,1,1;1,1,2,1;1)$, $B_9(3,1,1,2;1,1,2,1;1)$, $B_9(3,2,1,1;1,1,2,1;1)$, $B_9(4,1,1,1;1,1,2,1;1)$, $B_9(1,3,1,1;1,1,2,1;2)$, $B_9(2,2,1,1;1,1,2,1;2)$, $B_9(1,2,1,1;1,1,2,1;3)$, $B_9(1,1,2,2;1,1,2,2;1)$, $B_9(2,1,1,2;1,1,2,2;1)$, $B_9(1,2,1,1;1,1,2,2;2)$, $B_9(2,1,1,2;1,1,3,1;1)$, $B_9(2,2,1,1;1,1,3,1;1)$, $B_9(3,1,1,1;1,1,3,1;1)$, $B_9(2,1,1,1;1,1,3,2;1)$, $B_9(2,1,1,1;1,1,4,1;1)$, $B_9(1,2,2,2;1,2,1,1;1)$, $B_9(1,3,1,2;1,2,1,1;1)$, $B_9(1,3,2,1;1,2,1,1;1)$, $B_9(2,1,1,3;1,2,1,1;1)$, $B_9(2,2,1,2;1,2,1,1;1)$, $B_9(2,3,1,1;1,2,1,1;1)$, $B_9(3,1,1,2;1,2,1,1;1)$, $B_9(1,2,1,2;1,2,1,1;2)$, $B_9(1,2,2,1;1,2,1,1;2)$, $B_9(2,1,1,2;1,2,1,1;2)$, $B_9(2,2,1,1;1,2,1,1;2)$, $B_9(2,1,1,1;1,2,1,1;3)$, $B_9(1,2,2,1;1,2,1,2;1)$, $B_9(1,3,1,1;1,2,1,2;1)$, $B_9(1,3,1,1;1,2,2,1;1)$, $B_9(2,1,1,2;1,2,2,1;1)$, $B_9(2,2,1,1;1,2,2,1;1)$, $B_9(2,1,1,2;1,3,1,1;1)$, $B_9(2,2,1,1;1,3,1,1;1)$, $B_9(2,1,1,1;1,3,1,1;2)$, $B_9(2,1,1,1;1,4,1,1;1)$, $B_9(2,3,1,1;2,1,1,1;1)$, $B_9(2,2,1,1;2,2,1,1;1)$.

Table 6: $k = 10$.

n	$\mathcal{B}^*$
12	$B_{10}(1,1,2,1,2;1,1,1,1,1)$, $B_{10}(1,2,1,2,1;1,1,1,1,1)$, $B_{10}(2,1,2,1,1;1,1,1,1,1)$, $B_{10}(1,1,1,1,2;1,1,1,1,2)$, $B_{10}(1,2,1,1,1;1,1,1,1,2)$, $B_{10}(2,1,1,1,1;1,1,1,1,2)$, $B_{10}(1,1,2,1,1;1,1,1,2,1)$, $B_{10}(1,2,1,1,1;1,1,1,2,1)$, $B_{10}(1,1,2,1,1;1,1,2,1,1)$, $B_{10}(2,1,1,1,1;1,2,1,1,1)$;
13	$B_{10}(1,1,1,1,4;1,1,1,1,1)$, $B_{10}(1,1,1,2,3;1,1,1,1,1)$, $B_{10}(1,1,1,3,2;1,1,1,1,1)$, $B_{10}(1,1,1,4,1;1,1,1,1,1)$, $B_{10}(1,1,2,2,2;1,1,1,1,1)$, $B_{10}(1,1,2,3,1;1,1,1,1,1)$, $B_{10}(1,1,3,2,1;1,1,1,1,1)$, $B_{10}(1,1,4,1,1;1,1,1,1,1)$, $B_{10}(1,2,1,1,3;1,1,1,1,1)$, $B_{10}(1,2,1,2,2;1,1,1,1,1)$, $B_{10}(1,2,2,1,2;1,1,1,1,1)$, $B_{10}(1,2,2,2,1;1,1,1,1,1)$, $B_{10}(1,2,3,1,1;1,1,1,1,1)$, $B_{10}(1,3,1,1,2;1,1,1,1,1)$, $B_{10}(1,3,2,1,1;1,1,1,1,1)$, $B_{10}(1,4,1,1,1;1,1,1,1,1)$, $B_{10}(2,1,1,1,3;1,1,1,1,1)$, $B_{10}(2,1,1,2,2;1,1,1,1,1)$, $B_{10}(2,1,1,3,1;1,1,1,1,1)$, $B_{10}(2,1,2,2,1;1,1,1,1,1)$, $B_{10}(2,2,1,1,2;1,1,1,1,1)$, $B_{10}(2,2,1,2,1;1,1,1,1,1)$, $B_{10}(2,2,2,1,1;1,1,1,1,1)$, $B_{10}(2,3,1,1,1;1,1,1,1,1)$, $B_{10}(3,1,1,1,2;1,1,1,1,1)$, $B_{10}(3,1,1,2,1;1,1,1,1,1)$, $B_{10}(3,2,1,1,1;1,1,1,1,1)$, $B_{10}(4,1,1,1,1;1,1,1,1,1)$, $B_{10}(1,1,1,2,2;1,1,1,1,2)$, $B_{10}(1,1,1,3,1;1,1,1,1,2)$, $B_{10}(1,1,2,2,1;1,1,1,1,2)$, $B_{10}(1,1,3,1,1;1,1,1,1,2)$, $B_{10}(1,2,2,1,1;1,1,1,1,2)$, $B_{10}(2,1,1,2,1;1,1,1,1,2)$, $B_{10}(1,1,1,2,1;1,1,1,1,3)$, $B_{10}(1,1,2,1,1;1,1,1,1,3)$, $B_{10}(1,1,1,2,2;1,1,1,2,1)$, $B_{10}(1,1,1,3,1;1,1,1,2,1)$, $B_{10}(1,1,2,2,1;1,1,1,2,1)$, $B_{10}(1,2,1,1,2;1,1,1,2,1)$, $B_{10}(2,1,1,1,2;1,1,1,2,1)$, $B_{10}(2,1,1,2,1;1,1,1,2,1)$, $B_{10}(2,2,1,1,1;1,1,1,2,1)$, $B_{10}(3,1,1,1,1;1,1,1,2,1)$, $B_{10}(1,1,2,1,1;1,1,1,2,2)$, $B_{10}(2,1,1,1,1;1,1,1,2,2)$, $B_{10}(2,1,1,1,1;1,1,1,3,1)$, $B_{10}(1,2,1,1,2;1,1,2,1,1)$, $B_{10}(1,2,2,1,1;1,1,2,1,1)$, $B_{10}(1,3,1,1,1;1,1,2,1,1)$, $B_{10}(2,1,1,1,2;1,1,2,1,1)$, $B_{10}(2,1,1,2,1;1,1,2,1,1)$, $B_{10}(2,2,1,1,1;1,1,2,1,1)$, $B_{10}(3,1,1,1,1;1,1,2,1,1)$, $B_{10}(1,2,1,1,1;1,1,2,2,1)$, $B_{10}(2,1,1,1,1;1,1,2,2,1)$, $B_{10}(1,2,1,1,1;1,1,3,1,1)$, $B_{10}(2,1,1,1,1;1,1,3,1,1)$, $B_{10}(1,2,1,1,2;1,2,1,1,1)$, $B_{10}(1,2,2,1,1;1,2,1,1,1)$, $B_{10}(1,3,1,1,1;1,2,1,1,1)$, $B_{10}(2,2,1,1,1;1,2,1,1,1)$, $B_{10}(2,1,1,1,1;1,2,2,1,1)$, $B_{10}(2,1,1,1,2;2,1,1,1,1)$, $B_{10}(2,1,1,2,1;2,1,1,1,1)$, $B_{10}(2,1,2,1,1;2,1,1,1,1)$, $B_{10}(2,2,1,1,1;2,1,1,1,1)$, $B_{10}(3,1,1,1,1;2,1,1,1,1)$.

Appendix B Some theorems and lemmas

Theorem B.1 ([6]). *Let $G = B_4(a_1, a_2; a_3, a_4)$, where a_1, a_2, a_3, a_4 are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_4(a, b; 1, d)$; *(3)* $B_4(a, 1; c, 1)$; *(5)* $B_4(a, 1; x, d)$; *(7)* $B_4(w, x; y, d)$;

(2) $B_4(a, x; y, 1)$; *(4)* $B_4(a, 1; w, x)$; *(6)* $B_4(w, b; x, 1)$; *(8)* $B_4(x, b; y, d)$;

(9) 25 *specific graphs:* 5 *graphs of order* 10, 10 *graphs of order* 11, *and* 10 *graphs of order* 12,

where a, b, c, d, x, y, w are some positive integers such that $x \leq 2, y \leq 2$ and $w \leq 3$.

Lemma B.2. *Let $G \in \mathcal{B}_4(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_4^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_4(14) \setminus \mathcal{B}_4^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_4(n-1) \setminus \mathcal{B}_4^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_4(n_1, n_2; n_3, n_4) \in \mathcal{B}_4(n)$. Then one of

$$H_1 = B_4(n_1 - 1, n_2; n_3, n_4), \qquad H_2 = B_4(n_1, n_2 - 1; n_3, n_4),$$
$$H_3 = B_4(n_1, n_2; n_3 - 1, n_4) \quad \text{and} \qquad H_4 = B_4(n_1, n_2; n_3, n_4 - 1)$$

must belong to $\mathcal{B}_4(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_4^-(n-1)$ $(i = 1, 2, 3, 4)$. Then H_i is a graph belonging to (1) – (8) in Theorem B.1 since $n \geq 15$.

First we consider H_1. If H_1 is a graph belonging to (1) of Theorem B.1, then $H_1 = B_4(a, b; 1, d)$ where $n_1 - 1 = a$, $n_2 = b$, $n_3 = 1$ and $n_4 = d$, hence $G = B_4(a + 1, b; 1, d) \in \mathcal{B}_4^-(n)$, a contradiction. Similarly, H_1 cannot belong to (2) – (5) of Theorem B.1. Hence H_1 is belong to (6) – (8) of Theorem B.1 from which we see that $n_1 - 1$ is either w or x. Thus $n_1 \leq 4$ due to $w \leq 3$ and $x \leq 2$.

By the same method, we can verify that $n_2 \leq 3$ if $H_2 \in \mathcal{B}_4^-(n-1)$; $n_3 \leq 4$ if $H_3 \in \mathcal{B}_4^-(n-1)$ and $n_4 \leq 3$ if $H_4 \in \mathcal{B}_4^-(n-1)$. Hence $n = n_1 + \cdots + n_4 \leq 14$, a contradiction. We are done. □

Theorem B.3 ([6]). *Let $G = B_6(a_1, a_2, a_3; a_4, a_5, a_6)$, where $a_1, \ldots, a_6$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_6(a, x, c; 1, 1, 1)$; *(6)* $B_6(x, b, 1; y, 1, 1)$; *(11)* $B_6(1, b, 1; 1, e, 1)$;

(2) $B_6(a, 1, c; 1, e, 1)$; *(7)* $B_6(x, y, 1; 1, e, 1)$; *(12)* $B_6(1, b, 1; 1, x, y)$;

(3) $B_6(a, 1, c; 1, x, y)$; *(8)* $B_6(x, y, 1; 1, 1, f)$; *(13)* $B_6(1, x, y; 1, 1, f)$;

(4) $B_6(a, 1, c; 1, 1, f)$; *(9)* $B_6(x, 1, c; y, 1, f)$;

(5) $B_6(a, 1, 1; x, e, 1)$; *(10)* $B_6(1, b, x; 1, 1, 1)$;

(14) 145 *specific graphs:* 22 *graphs of order* 10, 54 *graphs of order* 11, *and* 69 *graphs of order* 12,

where a, b, c, d, e, f, x, y are some positive integers such that $x \leq 2$ and $y \leq 2$.

Lemma B.4. *Let $G \in \mathcal{B}_6(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_6^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_6(14) \setminus \mathcal{B}_6^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_6(n-1) \setminus \mathcal{B}_6^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_6(n_1, n_2, n_3; n_4, n_5, n_6) \in \mathcal{B}_6(n)$. Then one of

$$H_1 = B_6(n_1 - 1, n_2, n_3; n_4, n_5, n_6), \qquad H_2 = B_6(n_1, n_2 - 1, n_3; n_4, n_5, n_6),$$
$$H_3 = B_6(n_1, n_2, n_3 - 1; n_4, n_5, n_6), \qquad H_4 = B_6(n_1, n_2, n_3; n_4 - 1, n_5, n_6),$$
$$H_5 = B_6(n_1, n_2, n_3; n_4, n_5 - 1, n_6) \quad \text{and} \quad H_6 = B_6(n_1, n_2, n_3; n_4, n_5, n_6 - 1)$$

must belong to $\mathcal{B}_6(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_6^-(n-1)$ $(i = 1, 2, \ldots, 6)$. Then H_i is a graph belonging to (1) – (13) in Theorem B.3 since $n \geq 15$.

Let us consider H_3. If H_3 is a graph belonging to (1) of Theorem B.3, then $H_3 = B_6(a, x, c; 1, 1, 1)$ where $n_1 = a$, $n_2 = x$, $n_3 - 1 = c$, $n_4 = n_5 = n_6 = 1$, hence $G = B_6(a, x, c + 1; 1, 1, 1) \in \mathcal{B}_6^-(n)$, a contradiction. Similarly, H_3 cannot belong to (2) – (4) and (9) of Theorem B.3. If H_3 is a graph belonging to (10) of Theorem B.3, then $H_3 = B_6(1, b, x; 1, 1, 1)$, where $n_1 = 1$, $n_2 = b$, $n_3 - 1 = x$, $n_4 = n_5 = n_6 = 1$. Since $x \leq 2$, we have $n_3 \leq 3$. If $n_3 < 3$ then $x + 1 \leq 2$ and $G = B_6(1, b, x + 1; 1, 1, 1) \in \mathcal{B}_6^-(n)$, a contradiction. Now assume that $n_3 = 3$. Then $H_3 = B_6(1, b, 2; 1, 1, 1)$, and so $G = B_6(1, b, 3; 1, 1, 1)$. By Theorem B.3, $G \notin \mathcal{B}_6^-(n)$, and also its induced subgraph $B_6(1, b-1, 3; 1, 1, 1) \notin \mathcal{B}_6^-(n-1)$, a contradiction. Similarly, H_3 cannot belong to (13) of Theorem B.3. Hence H_3 is belong to (5) – (8) and (11) – (12) of Theorem B.3 from which we see that $n_3 - 1 \leq 1$. Thus $n_3 \leq 2$.

By the same method, we can verify that $n_1 \leq 3$ if $H_1 \in \mathcal{B}_6^-(n-1)$; $n_2 \leq 3$ if $H_2 \in \mathcal{B}_6^-(n-1)$; $n_4 \leq 2$ if $H_4 \in \mathcal{B}_6^-(n-1)$; $n_5 \leq 2$ if $H_5 \in \mathcal{B}_6^-(n-1)$ and $n_6 \leq 2$ if $H_6 \in \mathcal{B}_6^-(n-1)$. Hence $n = n_1 + \cdots + n_6 \leq 14$, a contradiction. We are done. □

Theorem B.5 ([6])**.** *Let $G = B_7(a_1, a_2, a_3; a_4, a_5, a_6; a_7)$, where $a_1, \ldots, a_7$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_7(a, 1, x; 1, e, 1; 1)$; *(4)* $B_7(x, y, 1; 1, e, 1; g)$; *(7)* $B_7(1, b, 1; 1, e, 1; g)$;

(2) $B_7(a, 1, 1; 1, e, 1; g)$; *(5)* $B_7(x, 1, 1; y, 1, 1; g)$; *(8)* $B_7(1, 1, c; 1, 1, f; 1)$;

(3) $B_7(a, 1, 1; 1, 1, x; 1)$; *(6)* $B_7(1, b, x; 1, 1, 1; g)$;

(9) 143 *specific graphs:* 18 *graphs of order* 10, 52 *graphs of order* 11, *and* 73 *graphs of order* 12,

where $a, b, c, d, e, f, g, x, y$ are some positive integers such that $x \leq 2$ and $y \leq 2$.

Lemma B.6. *Let $G \in \mathcal{B}_7(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_7^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_7(14) \setminus \mathcal{B}_7^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_7(n-1) \setminus \mathcal{B}_7^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_7(n_1, n_2, n_3; n_4, n_5, n_6; n_7) \in \mathcal{B}_7(n)$. Then one of

$$H_1 = B_7(n_1 - 1, n_2, n_3; n_4, n_5, n_6; n_7),$$
$$H_2 = B_7(n_1, n_2 - 1, n_3; n_4, n_5, n_6; n_7),$$
$$H_3 = B_7(n_1, n_2, n_3 - 1; n_4, n_5, n_6; n_7),$$

$$H_4 = B_7(n_1, n_2, n_3; n_4 - 1, n_5, n_6; n_7),$$
$$H_5 = B_7(n_1, n_2, n_3; n_4, n_5 - 1, n_6; n_7);$$
$$H_6 = B_7(n_1, n_2, n_3; n_4, n_5, n_6 - 1; n_7) \text{ and}$$
$$H_7 = B_7(n_1, n_2, n_3; n_4, n_5, n_6; n_7 - 1)$$

must belong to $\mathcal{B}_7(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_7^-(n-1)$ $(i = 1, 2, \ldots, 7)$. Then H_i is a graph belonging to (1)–(8) in Theorem B.5 since $n \geq 15$.

Let us consider H_1. If H_1 is a graph belonging to (1) of Theorem B.5, then $H_1 = B_7(a, 1, x; 1, e, 1; 1)$ where $n_1 - 1 = a$, $n_2 = 1$, $n_3 = x$, $n_4 = 1$, $n_5 = e$, $n_6 = n_7 = 1$, hence $G = B_7(a+1, 1, x; 1, e, 1; 1) \in \mathcal{B}_7^-(n)$, a contradiction. Similarly, H_1 cannot belong to (2)–(3) of Theorem B.5. If H_1 is a graph belonging to (4) of Theorem B.5, then $H_1 = B_7(x, y, 1; 1, e, 1; g)$, where $n_1 - 1 = x$, $n_2 = y$, $n_3 = n_4 = 1$, $n_5 = e$, $n_6 = 1$ and $n_7 = g$. Since $x \leq 2$, we have $n_1 \leq 3$. If $n_1 < 3$ then $x + 1 \leq 2$ and $G = B_7(x+1, y, 1; 1, e, 1; g) \in \mathcal{B}_7^-(n)$, a contradiction. Now assume that $n_1 = 3$. Then $H_1 = B_7(2, y, 1; 1, e, 1; g)$, and so $G = B_7(3, y, 1; 1, e, 1; g)$. Since $y \in \{1, 2\}$, we have $G \in \{B_7(3, 1, 1; 1, e, 1; g), B_7(3, 2, 1; 1, e, 1; g)\}$. However $B_7(3, 1, 1; 1, e, 1; g)$ belongs to (2) of Theorem B.5 which contradicts our assumption. Thus $G = B_7(3, 2, 1; 1, e, 1; g)$. By Theorem B.5, $G \notin \mathcal{B}_7^-(n)$, and also its induced subgraph $B_7(3, 2, 1; 1, e-1, 1; g)$ or $B_7(3, 2, 1; 1, e, 1; g-1)$ is not in $\mathcal{B}_7^-(n-1)$, a contradiction. Similarly, H_1 cannot belong to (5) of Theorem B.5. Hence H_1 belongs to (6)–(8) of Theorem B.5 from which we see that $n_1 - 1 \leq 1$. Thus $n_1 \leq 2$.

By the same method, we can verify that $n_2 \leq 2$ if $H_2 \in \mathcal{B}_7^-(n-1)$; $n_3 \leq 2$ if $H_3 \in \mathcal{B}_7^-(n-1)$; $n_4 \leq 2$ if $H_4 \in \mathcal{B}_7^-(n-1)$; $n_5 \leq 2$ if $H_5 \in \mathcal{B}_7^-(n-1)$, $n_6 \leq 2$ if $H_6 \in \mathcal{B}_7^-(n-1)$ and $n_7 \leq 2$ if $H_7 \in \mathcal{B}_7^-(n-1)$. Hence $n = n_1 + \cdots + n_7 \leq 14$, a contradiction. We are done. □

Theorem B.7 ([6]). *Let $G = B_8(a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8)$, where $a_1, \ldots, a_8$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_8(a, 1, 1, d; 1, 1, g, 1)$; *(2)* $B_8(1, b, 1, 1; 1, f, 1, 1)$;

(3) 134 *specific graphs:* 12 *graphs of order* 10, 42 *graphs of order* 11, *and* 80 *graphs of order* 12,

where a, b, d, f, g are some positive integers.

Lemma B.8. *Let $G \in \mathcal{B}_8(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_8^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_8(14) \setminus \mathcal{B}_8^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_8(n-1) \setminus \mathcal{B}_8^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_8(n_1, n_2, n_3, n_4; n_5, n_6, n_7, n_8) \in \mathcal{B}_8(n)$ and

$$H_1 = B_8(n_1 - 1, n_2, n_3, n_4; n_5, n_6, n_7, n_8),$$
$$H_2 = B_8(n_1, n_2 - 1, n_3, n_4; n_5, n_6, n_7, n_8),$$
$$H_3 = B_8(n_1, n_2, n_3 - 1, n_4; n_5, n_6, n_7, n_8),$$
$$H_4 = B_8(n_1, n_2, n_3, n_4 - 1; n_5, n_6, n_7, n_8),$$
$$H_5 = B_8(n_1, n_2, n_3, n_4; n_5 - 1, n_6, n_7, n_8),$$

$$H_6 = B_8(n_1, n_2, n_3, n_4; n_5, n_6 - 1, n_7, n_8),$$
$$H_7 = B_8(n_1, n_2, n_3, n_4; n_5, n_6, n_7 - 1, n_8) \quad \text{and}$$
$$H_8 = B_8(n_1, n_2, n_3, n_4; n_5, n_6, n_7, n_8 - 1).$$

If $n_3 \geq 3$, then $H_3 \in \mathcal{B}_8(n-1) \setminus \mathcal{B}_8^-(n-1)$ by Theorem B.7 as desired. If $n_3 = 2$, then at least one of n_1, n_2, n_4, n_5, n_6, n_7, n_8 is greater than 1 since $n \geq 15$, say n_2. Thus $H_2 \in \mathcal{B}_8(n-1) \setminus \mathcal{B}_8^-(n-1)$ by Theorem B.7 as desired. Hence let $n_3 = 1$. Similarly, let $n_5 = n_8 = 1$. Thus one of H_1, H_2, H_4, H_6, H_7 must belong to $\mathcal{B}_8(n-1)$. On the contrary, assume that $H_i \in \mathcal{B}_8^-(n-1)$ $(i = 1, 2, 4, 6, 7)$. Then H_i is a graph belonging to (1) – (2) in Theorem B.7 since $n \geq 15$.

Let us consider H_1. If H_1 is a graph belonging to (1) of Theorem B.7, then $H_1 = B_8(a, 1, 1, d; 1, 1, g, 1)$; where $n_1 - 1 = a$, $n_2 = n_3 = 1$, $n_4 = d$, $n_5 = n_6 = 1$, $n_7 = g$ and $n_8 = 1$, hence $G = B_8(a+1, 1, 1, d; 1, 1, g, 1) \in \mathcal{B}_8^-(n)$, a contradiction. Hence H_1 belongs to (2) of Theorem B.7 from which we see that $n_1 = 2$ due to $n_1 - 1 = 1$.

By the same method, we can verify that $n_i = 2$ if $H_i \in \mathcal{B}_8^-(n-1)$ for $i = 2, 4, 6, 7$. Hence $n = n_1 + \cdots + n_8 \leq 13$, a contradiction. We are done. □

Theorem B.9 ([6]). *Let $G = B_9(a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8; a_9)$, where $a_1, \ldots, a_9$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

(1) $B_9(1, b, 1, 1; 1, f, 1, 1; k)$*;*

(2) 59 *specific graphs:* 3 *graphs of order* 10*,* 17 *graphs of order* 11*, and* 39 *graphs of order* 12*,*

where b, f, k are some positive integers.

Lemma B.10. *Let $G \in \mathcal{B}_9(n)$, where $n \geq 14$. If $G \notin \mathcal{B}_9^-(n)$, then G has an induced subgraph $\Gamma \in \mathcal{B}_9(14) \setminus \mathcal{B}_9^-(14)$.*

Proof. By the proof of Lemma 5.13, it suffices to prove that G contains an induced subgraph $G' \in \mathcal{B}_9(n-1) \setminus \mathcal{B}_9^-(n-1)$ for $n \geq 15$ in the following.

Let $G = B_9(n_1, n_2, n_3, n_4; n_5, n_6, n_7, n_8; n_9) \in \mathcal{B}_9(n)$. On the contrary, suppose that every induced subgraphs $G' \in \mathcal{B}_9(n-1)$ of G belongs to $\mathcal{B}_9^-(n-1)$. If $n_1 \geq 3$, then $H_1 = B_9(n_1 - 1, n_2, n_3, n_4; n_5, n_6, n_7, n_8; n_9) \notin \mathcal{B}_9^-(n-1)$ by Theorem B.9, a contradiction. If $n_1 = 2$, then at least one of $n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9$ is greater than 1 since $n \geq 15$, say n_2. Thus $H_2 = B_9(n_1, n_2 - 1, n_3, n_4; n_5, n_6, n_7, n_8; n_9) \notin \mathcal{B}_9^-(n-1)$ by Theorem B.9, a contradiction. Hence $n_1 = 1$. Similarly, $n_3 = n_4 = n_5 = n_7 = n_8 = 1$. But now $G = B_9(1, n_2, 1, 1; 1, n_6, 1, 1; n_9) \in \mathcal{B}_9^-(n)$ by Theorem B.9, a contradiction. We are done. □

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Symmetries of Discrete Objects (SODO 2020)

Rotorua, New Zealand, 10–14 February 2020

https://www.math.auckland.ac.nz/~conder/SODO-2020/

A third conference on *Symmetries of Discrete Objects* will be held in New Zealand in February 2020. The conference theme is broad, and includes symmetries of graphs, maps, polytopes, Riemann/Klein surfaces, and other discrete structures such as block designs and finite geometries, with theory and applications of groups as a common thread. The first two of these conferences were held in Queenstown (NZ) in 2012 and 2016.

Venue: The venue will be Rotorua, which is a scenic and interesting city about three hours drive south of Auckland (or by a 45-minute flight from Auckland). It's also close to *Hobbiton*, the film site used for the *Hobbit* and *Lord of the Rings* movies.

Important note: February is summer in New Zealand (with daytime temperature in Rotorua in the mid-20s (centigrade) at that time of year).

The confirmed invited keynote speakers so far include:

- Anneleen De Schepper (Ghent University, Belgium)
- Dimitri Leemans (Université Libre de Bruxelles, Belgium)
- Joy Morris (University of Lethbridge, Canada)
- Primož Potočnik (University of Ljubljana, Slovenia)
- Jozef Širáň (Open University, UK, and Slovak University of Technology, Slovakia)

If you are interested in attending, please register (via the conference website) by early November 2019. (Registration fees do not have to be paid until 4th January 2020.)

Organisers: Marston Conder, Gabriel Verret

Further information: https://www.math.auckland.ac.nz/~conder/SODO-2020/

Sponsors:

- The University of Auckland
- The Marsden Fund (administered by the Royal Society of New Zealand)

Combinatorics around the q-Onsager algebra

Kranjska Gora, Slovenia, 13–18 July 2020

https://conferences.famnit.upr.si/event/15/

We are happy to announce a conference next year entitled *Combinatorics around the q-Onsager algebra*, at which we will be celebrating the 65th birthday of Paul Terwilliger. This conference will take place in beautiful Kranjska Gora, Slovenia, from July 13–18, 2020. The general theme will be the mathematical topics that Paul has worked on over the years (which all have relationships to the q-Onsager algebra). These topics include the following:

- Topics in algebraic graph theory, such as distance-regular graphs, association schemes, the subconstituent algebra, and the Q-polynomial property;
- Topics in linear algebra, such as Leonard pairs, tridiagonal pairs, billiard arrays, lowering-raising triples, and a linear algebraic approach to the orthogonal polynomials of the Askey scheme;
- Topics in Lie theory, such as the tetrahedron algebra and the Onsager algebra;
- Topics in algebras and their representations, such as the equitable presentation of $U_q(sl_2)$, the q-tetrahedron algebra, the q-Onsager algebra in mathematical physics, and the universal Askey-Wilson algebra.

The confirmed invited speakers so far include:

- Eiichi Bannai (Shanghai Jiao Tong University, China)
- Pascal Baseilhac (Université de Tours, France)
- Samuel Belliard (Université Paris Saclay, France)
- Sarah Bockting-Conrad (DePaul University, Chicago, USA)
- Ada Chan (York University, Toronto, Canada)
- Sebastian Cioabă (University of Delaware, Newark, USA)
- Darren Funk-Neubauer (Colorado State University-Pueblo, USA)
- Hau-Wen Huang (National Central University, Zhongli, Taiwan)
- Tatsuro Ito (Anhui University, Hefei, China)
- Vaughan Jones (Vanderbilt University, Nashville, USA)
- Aleksandar Jurišić (University of Ljubljana, Slovenia)
- Jack Koolen (University of Science and Technology of China, Hefei, China)
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- Mikhail Muzychuk (Ben-Gurion University of the Negev, Beer-Sheva, Israel)
- Hiroshi Nozaki (Aichi University of Education, Kariya, Japan)
- Safet Penjić (University of Primorska, Koper, Slovenia)
- Sarah Post (University of Hawaii at Mānoa, USA)
- Hjalmar Rosengren (Chalmers University of Technology, Gothenburg, Sweden)
- Supalak Sumalroj (Silpakorn University, Bangkok, Thailand)
- Hajime Tanaka (Tohoku University, Sendai, Japan)
- Luc Vinet (Université de Montréal, Canada)
- Yuta Watanabe (Tohoku University, Sendai, Japan)
- Alexei Zhedanov (Renmin University of China, Beijing, China)

In addition to invited talks, a limited number of contributed talks will also be available.

Venue: Kranjska Gora is a popular and attractive mountain and tourist sports centre nestled in the Julian Alps at the triple border point of Slovenia, Italy and Austria. In winter Alpine skiers compete and top ski jumpers break new records at nearby Planica. Summer offers cyclists the challenge of conquering the highest Slovenian mountain pass, while hikers can enjoy more than 100 km of trails that incorporate many points of interest. See https://www.kranjska-gora.si/en.

Organisers: Štefko Miklavič, Mark MacLean

Further information: https://conferences.famnit.upr.si/event/15/

This will be a satellite conference of the 8th European Congress of Mathematics (8ECM), which will be held the prior week in Portorož, Slovenia (https://www.8ecm.si).

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