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Embedding of orthogonal Buekenhout-Metz unitals in the Desarguesian plane of order q^2

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Abstract

A unital, that is a 2-$(q^3+1, q+1, 1)$ block-design, is embedded in a projective plane π of order q^2 if its points are points of π and its blocks are subsets of lines of π, the point-block incidences being the same as in π. Regarding unitals $\mathcal{U}$ which are isomorphic, as a block-design, to the classical unital, T. Szőnyi and the authors recently proved that the natural embedding is the unique embedding of $\mathcal{U}$ into the Desarguesian plane of order q^2. In this paper we extend this uniqueness result to all unitals which are isomorphic, as block-designs, to orthogonal Buekenhout-Metz unitals.

Keywords: Unital, embedding, finite Desarguesian plane.

Math. Subj. Class.: 51E05, 51E20

1 Introduction

A *unital* is a set of q^3+1 points equipped with a family of subsets, each of size $q+1$, such that every pair of distinct points are contained in exactly one subset of the family. In Design Theory, such subsets are usually called *blocks* so that unitals are 2-$(q^3+1, q+1, 1)$ block-designs. A unital $\mathcal{U}$ is *embedded* in a projective plane π of order q^2, if its points are points of π, its blocks are subsets of lines of π and the point-block incidences being the same as in π.

Sufficient conditions for a unital to be embeddable in a projective plane are given in [21]. Computer aided searches suggest that there should be plenty of unitals, especially for small values of q, but those embeddable in a projective plane are quite rare, see [3, 6, 27]. Very recently, the GAP package UnitalSz was released [25]. This package contains methods for the embeddings of unitals in the finite projective plane.

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In the finite Desarguesian projective plane of order q^2, a unital arises from a unitary polarity: the points of the unital are the absolute points, and the blocks are the non-absolute lines of the polarity. This unital is called *classical unital*. The following result comes from [23].

Theorem 1.1. *Let $\mathcal{U}$ be a unital embedded in* $\mathrm{PG}(2, q^2)$ *which is isomorphic, as a block-design, to a classical unital. Then $\mathcal{U}$ is the classical unital of* $\mathrm{PG}(2, q^2)$.

Buekenhout [11] constructed unitals in any translation planes with dimension at most two over their kernel by using the Andrè/Bruck-Bose representation. Buekenhout's work was completed by Metz [24] who was able to prove by a counting argument that when the plane is Desarguesian then Buekenhout's construction provides not only the classical unital but also non-classical unitals in $\mathrm{PG}(2, q^2)$ for all $q > 2$. These unitals are called *Buekenhout-Metz unitals*, and they are the only known unitals in $\mathrm{PG}(2, q^2)$. With the terminology in [5], an *orthogonal Buekenhout-Metz unital* is a Buekenhout-Metz unital arising from an elliptic quadric in Buekehout's construction.

In this paper, we prove the following result:

Main Theorem. Let $\mathcal{U}$ be a unital embedded in $\mathrm{PG}(2, q^2)$ which is isomorphic, as block-design, to an orthogonal Buekenhout-Metz unital. Then $\mathcal{U}$ is an orthogonal Buekenhout-Metz unital.

Our approach is different from that adopted in [23]. Our idea is to exploit two different models of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(5, q)$, one of them is a variant of the so-called $\mathrm{GF}(q)$-linear representation. We start off with a representation of a non-classical Buekenhout-Metz unital given in one of these models of $\mathrm{PG}(2, q^2)$, then we exhibit a linear collineation of $\mathrm{PG}(5, q)$ that takes this representation to a representation of a classical unital in the other model of $\mathrm{PG}(2, q^2)$. At this point to finish the proof we only need some arguments from the proof of Theorem 1.1 together with the characterization of the orthogonal Buekenhout-Metz unitals due to Casse, O'Keefe, Penttila and Quinn [12, 29].

2 Preliminary results

The study of unitals in finite projective planes has been greatly aided by the use of the Andrè/Bruck-Bose representation of these planes [1, 9, 10]. Let $\mathrm{PG}(4, q)$ denote the projective 4-dimensional space over the finite field $\mathrm{GF}(q)$, and let Σ be some fixed hyperplane of $\mathrm{PG}(4, q)$. Let $\mathcal{N}$ be a line spread of Σ, that is a collection of $q^2 + 1$ mutually skew lines of Σ. We consider the following incidence structure: the *points* are the points of $\mathrm{PG}(4, q)$ not in Σ, the *lines* are the planes of $\mathrm{PG}(4, q)$ which meet Σ in a line of $\mathcal{N}$ and *incidence* is defined by inclusion. This incidence structure is an affine translation plane of order q^2 which is at most two-dimensional over its kernel. It can be completed to a projective plane $\pi(\mathcal{N})$ by the addition of an ideal line L_∞ whose points are the elements of the spread $\mathcal{N}$. Conversely, any translation plane of order q^2 with $\mathrm{GF}(q)$ in its kernel can be modeled this way [9]. Moreover, it is well known that the resulting plane is Desarguesian if and only if $\mathcal{N}$ is a Desarguesian spread [10].

Our first step is to outline the usual representation of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(5, q)$ due to Segre [30] and Bose [7]. While such representation is usually thought of in a projective setting, algebraic dimensions are more amenable to an introductory discussion of it, so we will mainly take a vector space approach along all this section.

Look at $\mathrm{GF}(q^2)$ as the two-dimensional vector space over $\mathrm{GF}(q)$ with basis $\{1, \epsilon\}$, so that every $x \in \mathrm{GF}(q^2)$ is uniquely written as $x = x_0 + x_1\epsilon$, for $x_0, x_1 \in \mathrm{GF}(q)$. Then the vectors (x, y, z) of $V(3, q^2)$ are viewed as the vectors $(x_1, x_2, y_1, y_2, z_1, z_2)$ of $V(6, q)$ where

$$\begin{aligned} x &= x_0 + x_1\epsilon, \\ y &= y_0 + \epsilon y_1 \text{ and} \\ z &= z_0 + \epsilon z_1. \end{aligned}$$

Therefore the points of $\mathrm{PG}(2, q^2)$ are two-dimensional subspaces in $V(6, q)$, and hence lines of $\mathrm{PG}(5, q)$, the five-dimensional projective space arising from $V(6, q)$. Such lines are the members of a Desarguesian line-spread $\mathcal{S}$ of $\mathrm{PG}(5, q)$ which gives rise to a point-line incidence structure $\Pi(\mathcal{S})$ where points are the elements of $\mathcal{S}$, and lines are the three-dimensional subspaces of $\mathrm{PG}(5, q)$ spanned by two elements of $\mathcal{S}$, incidence being inclusion. Obviously, $\Pi(\mathcal{S}) \simeq \mathrm{PG}(2, q^2)$, and $\Pi(\mathcal{S})$ is the $\mathrm{GF}(q)$-*linear representation of* $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(5, q)$. Since $\mathrm{PG}(5, q)$ is naturally embedded in $\mathrm{PG}(5, q^2)$, we also have an embedding of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(5, q^2)$ via $\Pi(\mathcal{S})$.

Actually, we will use a different embedding of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(5, q^2)$ which is more suitable for computation.

In $V(6, q^2)$, let $\widehat{V}$ be the set of all vectors (x, x^q, y, y^q, z, z^q) with $x, y, z \in \mathrm{GF}(q^2)$. With the usual sum and multiplication by scalars from $\mathrm{GF}(q)$, $\widehat{V}$ is a six-dimensional vector space over $\mathrm{GF}(q)$. On the other hand, $V(6, q)$ is naturally embedded in $V(6, q^2)$. Therefore, the question arises whether there exists an invertible endomorphism of $V(6, q^2)$ that takes $\widehat{V}$ to $V(6, q)$. The affirmative answer is given by the following proposition.

Proposition 2.1. *$\widehat{V}$ is linearly equivalent to $V(6, q)$ in $V(6, q^2)$.*

Proof. Write $V(6, q)$ as the direct sum $W^{(1)} \oplus W^{(2)} \oplus W^{(3)}$, with

$$\begin{aligned} W^{(1)} &= \{(a, b, 0, 0, 0, 0) : a, b \in \mathrm{GF}(q)\} \\ W^{(2)} &= \{(0, 0, a, b, 0, 0) : a, b \in \mathrm{GF}(q)\} \\ W^{(3)} &= \{(0, 0, 0, 0, a, b) : a, b \in \mathrm{GF}(q)\}. \end{aligned}$$

Clearly, each $W^{(i)}$ is isomorphic to $V(2, q) = \{(a, b) : a, b \in \mathrm{GF}(q)\}$. Take a basis $\{u_1, u_2\}$ of $V(2, q)$ together with a Singer cycle σ of $V(2, q)$. Since σ has two distinct eigenvalues, both in $\mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$, we find two linearly independent eigenvectors v_1, v_2 that form a basis for $V(2, q^2)$. Such a basis $\{v_1, v_2\}$ is called a *Singer basis* with respect to $V(2, q)$ [15]. In this context, $V(2, q) = \big\{xv_1 + x^q v_2 : x \in \mathrm{GF}(q^2)\big\}$ [14].

Applying this argument to $W^{(i)}$ with $i = 1, 2, 3$, gives a Singer basis $\{v_1^{(i)}, v_2^{(i)}\}$ of $W^{(i)}$ such that $W^{(i)} = \{xv_1^{(i)} + x^q v_2^{(i)} : x \in \mathrm{GF}(q^2)\}$. In this basis we have

$$V(6, q) = \{xv_1^{(1)} + x^q v_2^{(1)} + yv_1^{(2)} + y^q v_2^{(2)} + zv_1^{(3)} + z^q v_2^{(3)} : x, y, z \in \mathrm{GF}(q^2)\}. \quad (2.1)$$

Now, the result follows from the fact that the change from any basis of $V(6, q^2)$ to the basis $\{v_1^{(i)}, v_2^{(i)} : i = 1, 2, 3\}$ is carried out by an invertible endomorphism over $\mathrm{GF}(q^2)$. □

We call the vector space $\widehat{V}$ the *cyclic representation of* $V(6, q)$ *over* $\mathrm{GF}(q^2)$.

To state Proposition 2.1 in terms of projective geometry, let $\mathrm{PG}(5,q)$ denote the projective space arising from $V(6,q)$. Also, let $\mathrm{PG}(\widehat{V}) = \{\langle v\rangle_q : v \in \widehat{V}\}$ be the five-dimensional projective space whose points are the one-dimensional $\mathrm{GF}(q)$-subspaces spanned by vectors in $\widehat{V}$.

Corollary 2.2. $\mathrm{PG}(\widehat{V})$ *is projectively equivalent to* $\mathrm{PG}(5,q)$ *in* $\mathrm{PG}(5,q^2)$.

We call the the projective space $\mathrm{PG}(\widehat{V})$ the *cyclic representation of* $\mathrm{PG}(5,q)$ *over* $\mathrm{GF}(q^2)$.

Recall that a 2×2 *q-circulant* (or *Dickson*) *matrix* over $\mathrm{GF}(q^2)$ is a matrix of the form

$$D = \begin{pmatrix} d_1 & d_2 \\ d_2^q & d_1^q \end{pmatrix}$$

with $d_1, d_2 \in \mathrm{GF}(q^2)$.

Let $\mathcal{B}$ denote the basis $\{v_1^{(i)}, v_2^{(i)} : i = 1,2,3\}$ of $\widehat{V}$.

Proposition 2.3. *In the basis $\mathcal{B}$, the matrix associated to any endomorphism of $\widehat{V}$ is of the form*

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}, \tag{2.2}$$

where D_{ij} is a 2×2 q-circulant matrix over $\mathrm{GF}(q^2)$.

Proof. It is easily seen that any matrix of type (2.2) is associated to an endomorphism of $\widehat{V}$.

Conversely, take an endomorphism τ of $V(6,q^2)$ and let $T = (t_{ij})$, $t_{ij} \in \mathrm{GF}(q^2)$, be the matrix of τ in the basis $\mathcal{B}$. For a generic array $\mathbf{x} = (x, x^q, y, y^q, z, z^q) \in \widehat{V}$,

$$T\mathbf{x}^t = \begin{pmatrix} \vdots \\ t_{k,1}x + t_{k,2}x^q + t_{k,3}y + t_{k,4}y^q + t_{k,5}z + t_{k,6}z^q \\ \vdots \end{pmatrix}, \text{for } k = 1,\dots,6.$$

If $y = z = 0$, a necessary condition for $T\mathbf{x}^t \in \widehat{V}$ is

$$(t_{k,1}x + t_{k,2}x^q)^q = t_{k+1,1}x + t_{k+1,2}x^q\ ,$$

for $k = 1,3,5$, that is,

$$(t_{k,2}^q - t_{k+1,1})x + (t_{k,1}^q - t_{k+1,2})x^q = 0,$$

for $k = 1,3,5$ and for all $x \in \mathrm{GF}(q^2)$. This shows that the polynomial in x of degree q on the left hand side of the last equation has at least q^2 roots. Therefore, it must be the zero polynomial. Hence $t_{k+1,1} = t_{k,2}^q$ and $t_{k+1,2} = t_{k,1}^q$, for $k = 1,3,5$. To end the proof, it is enough to repeat the above argument for $x = z = 0$ and then for $x = y = 0$. □

Next we exhibit quadratic forms on $V(6,q^2)$ which induce quadratic forms on $\widehat{V}$.

The vector space $V(2n,q)$ has precisely two (nondegenerate) quadratic forms, and they differ by their Witt-index, that is the dimension of their maximal totally singular subspaces;

see [22, 32]. These dimensions are $n-1$ and n, and the quadratic form is *elliptic* or *hyperbolic*, respectively. In terms of the associated projective space $\mathrm{PG}(2n-1,q)$, the elliptic (resp. hyperbolic) quadratic form defines an *elliptic* (resp. *hyperbolic*) quadric of $\mathrm{PG}(2n-1,q)$.

Fix a basis $\{1,\epsilon\}$ for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$, and write $x=x_0+\epsilon x_1$, for $x\in\mathrm{GF}(q^2)$ with $x_0,x_1\in\mathrm{GF}(q)$. Here, ϵ is taken such that $\epsilon^2=\xi$ with ξ a nonsquare in $\mathrm{GF}(q)$ for q odd, and that $\epsilon^2+\epsilon=s$ with $s\in C_1$ and $s\neq 1$ for q even, where C_1 stands for the set of elements in $\mathrm{GF}(q)$ with absolute trace 1. Furthermore, Tr denotes the trace map $x\in\mathrm{GF}(q^2)\to x+x^q\in\mathrm{GF}(q)$.

Proposition 2.4. *Let $\alpha,\beta\in\mathrm{GF}(q^2)$ satisfy the following conditions:*

$$\begin{cases} 4\alpha^{q+1}+(\beta^q-\beta)^2 \text{ is nonsquare in } \mathrm{GF}(q), \text{ for } q \text{ odd},\\ \alpha^{q+1}/(\beta^q+\beta)^2\in C_0 \text{ with } \beta\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q), \text{ for } q \text{ even,}\end{cases}$$

where C_0 stands for the set of elements in $\mathrm{GF}(q)$, q even, with absolute trace 0. Let $Q_{\alpha,\beta}$ be the quadratic form on $V(6,q^2)$ given by

$$\begin{aligned} &Q_{\alpha,\beta}(X_1,X_2,Y_1,Y_2,Z_1,Z_2)=\\ &\quad\delta^qX_1Z_2+\delta X_2Z_1+\alpha\delta Y_1^2+\alpha^q\delta^qY_2^2+\mathrm{Tr}(\delta\beta)Y_1Y_2,\end{aligned} \tag{2.3}$$

with $\delta=\epsilon$ or $\delta=1$ according as q is odd or even. then the restriction $\widehat{Q}_{\alpha,\beta}$ of $Q_{\alpha,\beta}$ on $\widehat{V}$ defines an elliptic quadratic form on $\widehat{V}$.

Proof. Two cases are treated separately according as q is odd or even.

If q is odd, let $b_{\alpha,\beta}$ denote the symmetric bilinear form on $V(6,q^2)$ associated to $Q_{\alpha,\beta}$. The matrix of $b_{\alpha,\beta}$ in the canonical basis is

$$B_{\alpha,\beta}=\begin{pmatrix} O_2 & O_2 & E\\ O_2 & A_{\alpha,\beta} & O_2\\ \overline{E} & O_2 & O_2\end{pmatrix},$$

with

$$E=\begin{pmatrix}0 & \epsilon^q\\ \epsilon & 0\end{pmatrix},\quad \overline{E}=\begin{pmatrix}0 & \epsilon\\ \epsilon^q & 0\end{pmatrix}\quad\text{and}\quad A_{\alpha,\beta}=\begin{pmatrix}2\alpha\epsilon & \mathrm{Tr}(\epsilon\beta)\\ \mathrm{Tr}(\epsilon\beta) & 2\alpha^q\epsilon^q\end{pmatrix}.$$

A straightforward computation shows that $B_{\alpha,\beta}$ induces a symmetric bilinear form on $\widehat{V}$. Let $\widehat{Q}_{\alpha,\beta}$ denote the resulting quadratic form on $\widehat{V}$.

Since $\det A_{\alpha,\beta}=4\alpha^{q+1}+(\beta^q-\beta)^2$ is nonsquare in $\mathrm{GF}(q)$, it follows that $Q_{\alpha,\beta}$ is nondegenerate. Hence $\widehat{Q}_{\alpha,\beta}$ is nondegenerate, as well. Let H be the four-dimensional subspace $\{(x,x^q,0,0,z,z^q)\,:\,x,z\in\ GF(q^2)\}$ of $\widehat{V}$. Then the restriction of $\widehat{Q}_{\alpha,\beta}$ on H is a hyperbolic quadratic form, as $L_1=\{(x,x^q,0,0,0,0)\,:\,x\in\mathrm{GF}(q^2)\}$ and $L_2=\{(0,0,0,0,z,z^q)\,:\,z\in\ GF(q^2)\}$ are totally isotropic subspaces with trivial intersection. The orthogonal space of H with respect to $b_{\alpha,\beta}$ is $L=\{(0,0,y,y^q,0,0)\,:\,y\in\mathrm{GF}(q^2)\}$. By [22, Proposition 2.5.11], $\widehat{Q}_{\alpha,\beta}$ is elliptic if and only if the restriction of $\widehat{Q}_{\alpha,\beta}$ on L is elliptic, that is,

$$\mathrm{Tr}(\alpha\epsilon y^2+\epsilon\beta y^{q+1})=0 \tag{2.4}$$

has no solution $y \in \mathrm{GF}(q^2)$ other than 0.

Write $y = y_0 + \epsilon y_1$, $\alpha = a_0 + \epsilon a_1$ and $\beta = b_0 + \epsilon b_1$ with $y_0, y_1, a_0, a_1, b_0, b_1 \in \mathrm{GF}(q)$. As $\epsilon^q = -\epsilon$ and $\epsilon^2 = \xi$, we have

$$\begin{aligned} y^q &= y_0 - \epsilon y_1 \\ y^{q+1} &= y_0^2 - \xi y_1^2 \\ y^2 &= y_0^2 + \xi y_1^2 + 2\epsilon y_0 y_1 \\ y^{2q} &= y_0^2 + \xi y_1^2 - 2\epsilon y_0 y_1 \\ \alpha\epsilon y^2 &= \xi(2a_0 y_0 y_1 + a_1(y_0^2 + \xi y_1^2)) + \epsilon(a_0(y_0^2 + \xi y_1^2) + 2\xi a_1 y_0 y_1) \\ \alpha^q \epsilon^q y^{2q} &= \xi(2a_0 y_0 y_1 + a_1(y_0^2 + \xi y_1^2)) - \epsilon(a_0(y_0^2 + \xi y_1^2) + 2\xi a_1 y_0 y_1), \end{aligned}$$

whence

$$\mathrm{Tr}(\alpha\epsilon y^2) = 2\xi(2a_0 y_0 y_1 + a_1(y_0^2 + \xi y_1^2)).$$

Moreover,

$$\mathrm{Tr}(\epsilon\beta y^{q+1}) = 2\xi b_1(y_0^2 - \xi y_1^2).$$

Then Equation (2.4) has a nontrivial solution $y \in \mathrm{GF}(q^2)$ if and only if $(y_0, y_1) \neq (0, 0)$ with $y_0, y_1 \in \mathrm{GF}(q)$ is a solution of

$$(a_1 + b_1)y_0^2 + 2a_0 y_0 y_1 + \xi(a_1 - b_1)y_1^2 = 0. \tag{2.5}$$

By a straightforward computation, (2.5) occurs if and only if $4\alpha^{q+1} + (\beta^q - \beta)^2 = u^2$ for some $u \in \mathrm{GF}(q)$. But the latter equation contradicts our hypothesis. Therefore, Equation (2.4) has no nontrivial solution in $\mathrm{GF}(q^2)$ and hence $\widehat{Q}_{\alpha,\beta}$ is elliptic.

For q even, the above approach still works up to some differences due to the fact that the well known formula solving equations of degree 2 fails in even characteristic. For completeness, we give all details.

If q is even, the restriction of $Q_{\alpha,\beta}$ on $\widehat{V}$ is a quadratic form $\widehat{Q}_{\alpha,\beta}$ on $\widehat{V}$, and the matrix of the associated bilinear form b_β is

$$B_\beta = \begin{pmatrix} O_2 & O_2 & E \\ O_2 & A_\beta & O_2 \\ E & O_2 & O_2 \end{pmatrix},$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_\beta = \begin{pmatrix} 0 & \mathrm{Tr}(\beta) \\ \mathrm{Tr}(\beta) & 0 \end{pmatrix}.$$

Since $\beta \notin \mathrm{GF}(q)$, a straightforward computation shows that the radical of b_β is trivial, which gives $\widehat{Q}_{\alpha,\beta}$ is nonsingular. As for the odd q case, the orthogonal space of H with respect to b_β is L. Therefore, $\widehat{Q}_{\alpha,\beta}$ is elliptic if and only if

$$\mathrm{Tr}(\alpha y^2 + \beta y^{q+1}) = 0 \tag{2.6}$$

has no nontrivial solution $y \in \mathrm{GF}(q^2)$.

As before, let $y = y_0 + \epsilon y_1, \alpha = a_0 + \epsilon a_1$ and $\beta = b_0 + \epsilon b_1$ with $y_0, y_1, a_0, a_1, b_0, b_1 \in \mathrm{GF}(q)$. As $\epsilon^q = \epsilon + 1$ and $\epsilon^2 = \epsilon + s$, with $s \in C_1$, we have

$$\begin{aligned} y^q &= y_0 + y_1 + \epsilon y_1 \\ y^{q+1} &= y_0^2 + y_0 y_1 + s y_1^2 \\ y^2 &= y_0^2 + s y_1^2 + \epsilon y_1^2 \\ y^{2q} &= y_0^2 + (s+1) y_1^2 + \epsilon y_1^2 \\ \alpha y^2 &= a_0 y_0^2 + s(a_0 + a_1) y_1^2 + \epsilon(a_0 y_1^2 + a_1 y_0^2 + (s+1) a_1 y_1^2) \\ \alpha^q y^{2q} &= a_0 y_0^2 + s(a_0 + a_1) y_1^2 + (a_0 y_1^2 + a_1 y_0^2 + (s+1) a_1 y_1^2) \\ &\quad + \epsilon(a_0 y_1^2 + a_1 y_0^2 + (s+1) a_1 y_1^2), \end{aligned}$$

whence

$$\mathrm{Tr}(\alpha y^2) = a_0 y_1^2 + a_1 y_0^2 + (s+1) a_1 y_1^2,$$

and

$$\mathrm{Tr}(\beta y^{q+1}) = b_1 (y_0^2 + y_0 y_1 + s y_1^2).$$

Therefore, Equation (2.6) has a nontrivial solution in $\mathrm{GF}(q^2)$ if and only if

$$(a_1 + b_1) y_0^2 + b_1 y_0 y_1 + (a_0 + a_1 + s a_1 + s b_1) y_1^2 = 0.$$

Assume $y = y_0 \in \mathrm{GF}(q)$ is a nontrivial solution of (2.6). Then $a_1 = b_1$. This gives

$$\frac{\alpha^{q+1}}{(\beta^q + \beta)^2} = \frac{a_0^2}{a_1^2} + \frac{a_0}{a_1} + s \in C_1,$$

a contradiction since

$$\frac{a_0^2}{a_1^2} + \frac{a_0}{a_1} \in C_0.$$

Assume that $y = y_0 + \epsilon y_1 \in \mathrm{GF}(q^2)$, with $y_1 \neq 0$, is a solution of (2.6). Then $y_0 y_1^{-1}$ is a solution of

$$(a_1 + b_1) X^2 + b_1 X + a_0 + a_1 + s(a_1 + b_1) = 0, \tag{2.7}$$

where $b_1 \neq 0$.

Let $Y = (a_1 + b_1) b_1^{-1} X$. Replacing X by Y in (2.7) gives $Y^2 + Y + d = 0$ where

$$d = \frac{a_0^2 + a_1 a_0 + s a_1^2}{b_0^2} + \frac{a_0^2 + a_1^2}{b_0^2} + \frac{a_0 + a_1}{b_0} + s.$$

Here, $d \in C_1$ by

$$\frac{a_0^2 + a_1 a_0 + s a_1^2}{b_0^2} = \frac{\alpha^{q+1}}{(\beta^q + \beta)^2} \in C_0.$$

This shows that Equation (2.7) has no nontrivial solution in $\mathrm{GF}(q)$. Hence Equation (2.6) has no nontrivial solution in $\mathrm{GF}(q^2)$, as well. Therefore $\widehat{Q}_{\alpha,\beta}$ is elliptic. □

Let $\widehat{\mathcal{Q}}_{\alpha,\beta}$ stand for the elliptic quadric in $\mathrm{PG}(\widehat{V})$ defined by the quadratic form $\widehat{Q}_{\alpha,\beta}$ on $\widehat{V}$. Then the coordinates of the points of $\mathrm{PG}(\widehat{V})$ that lie on $\widehat{\mathcal{Q}}_{\alpha,\beta}$ satisfy the equation

$$\delta^q X Z^q + \delta X^q Z + \alpha \delta Y^2 + \alpha^q \delta^q Y^{2q} + \mathrm{Tr}(\delta\beta) Y^{q+1} = 0, \tag{2.8}$$

with $\delta = \epsilon$ or $\delta = 1$ according as q is odd or even.

3 The $\mathrm{GF}(q)$-linear representation of Buekenhout-Metz unitals

In the light of Proposition 2.1, we introduce another incidence structure $\Pi(\widehat{\mathcal{S}})$.

Let $\widehat{\phi}$ be the bijective map defined by

$$\widehat{\phi}\colon \quad \begin{array}{ccc} V(3,q^2) & \longrightarrow & \widehat{V} \\ (x,y,z) & \longmapsto & (x,x^q,y,y^q,z,z^q) \end{array} .$$

By Proposition 2.1, $\widehat{\phi}$ is the field reduction of $V(3,q^2)$ over $\mathrm{GF}(q)$ in the basis $\{v_1^{(i)}, v_2^{(i)}, i = 1,2,3\}$ of $V(6,q^2)$.

The points of $\mathrm{PG}(2,q^2)$ are mapped by $\widehat{\phi}$ to the two-dimensional $\mathrm{GF}(q)$-subspaces of $\widehat{V}$ of the form

$$\{(\lambda x, \lambda^q x^q, \lambda y, \lambda^q y^q, \lambda z, \lambda^q z^q) : \lambda \in \mathrm{GF}(q^2)\}, \text{ for } x,y,z \in \mathrm{GF}(q^2),$$

and hence lines of $\mathrm{PG}(\widehat{V})$. Such lines form a line-spread $\widehat{\mathcal{S}}$ of $\mathrm{PG}(\widehat{V})$. By Proposition 2.1 and Corollary 2.2, $\widehat{\mathcal{S}}$ is projectively equivalent to $\mathcal{S}$ in $\mathrm{PG}(5,q^2)$. Hence, $\widehat{\mathcal{S}}$ is also a Desarguesian line-spread of $\mathrm{PG}(\widehat{V})$. Therefore, in $\mathrm{PG}(5,q^2)$ $\Pi(\widehat{\mathcal{S}})$ is projectively equivalent to the $\mathrm{GF}(q)$-linear representation $\Pi(\mathcal{S})$ of $\mathrm{PG}(2,q^2)$.

The following lemma goes back to Singer, see [31].

Lemma 3.1. *Let ω be a primitive element of $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$ with minimal polynomial $f(T) = T^2 - p_1 T - p_0$. then the multiplication by ω in $\mathrm{GF}(q^2)$ defines a Singer cycle of $V(2,q) = \{(a,b) : a,b \in \mathrm{GF}(q)\}$ whose matrix is the companion matrix of $f(T)$.*

Proposition 3.2. *Any endomorphism of $V(3,q^2)$ with matrix $A = (a_{ij})$ defines the endomorphism of $\widehat{V}$ with matrix*

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix},$$

where $D_{ij} = \mathrm{diag}(a_{ij}, a_{ij}^q)$.

The Frobenius transformation $\psi\colon (x,y,z) \mapsto (x^q,y^q,z^q)$ of $V(3,q^2)$ defines the endomorphism of $\widehat{V}$ with matrix

$$\begin{pmatrix} \widehat{F} & 0 & 0 \\ 0 & \widehat{F} & 0 \\ 0 & 0 & \widehat{F} \end{pmatrix},$$

where

$$\widehat{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. The Singer cycle defined by a primitive element ω of $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$ acts on the $\mathrm{GF}(q)$-vector space $\{(x,x^q) : x \in \mathrm{GF}(q^2)\}$ by the matrix $D = \mathrm{diag}(\omega, \omega^q)$. For every entry a_{ij} of A, write $a_{ij} = \omega^{e(i,j)}$, $0 \le e(i,j) \le q^2 - 2$. From Lemma 3.1, the multiplication by a_{ij} in $\mathrm{GF}(q^2)$ defines the endomorphism with matrix $D^{e(i,j)} = \mathrm{diag}(a_{ij}, a_{ij}^q)$. From this the first part of the proposition follows. The second part comes from Cooperstein's paper [14]. □

Remark 3.3. From a result due to Dye [16], the stabilizer of the Desarguesian partition $\mathcal{K}$ in $\mathrm{GL}(6,q)$ is the semidirect product of the field extension subgroup $\mathrm{GL}(3,q^2)$ by the cyclic subgroup $\langle\psi\rangle$ generated by the Frobenius transformation. In terms of projective geometry, the stabilizer of the Desarguesian spread $\mathcal{S}$ in $\mathrm{PGL}(6,q)$ is $(\mathrm{GL}(3,q^2)\rtimes\langle\psi\rangle)/\,\mathrm{GF}(q)^*$ [16]. It should be noted that the center of $\mathrm{GL}(\widehat{V})$ is the subgroup $\{cI : c\in\mathrm{GF}(q)^*\}$. Proposition 3.2 provides the representation in $\mathrm{GL}(\widehat{V})$ and $\mathrm{PGL}(\widehat{V})$ of these stabilizers.

In [2] and [17] the orthogonal Buekenhout-Metz unitals are coordinatized in $\mathrm{PG}(2,q^2)$. Let L_∞ be the line of $\mathrm{PG}(2,q^2)$ with equation $Z=0$ and $P_\infty=\langle(1,0,0)\rangle_{q^2}$.

Theorem 3.4. *Let $\alpha,\beta\in\mathrm{GF}(q^2)$ such that*

$$\begin{cases}4\alpha^{q+1}+(\beta^q-\beta)^2 \text{ is nonsquare in } \mathrm{GF}(q), \text{ for } q \text{ odd},\\ \alpha^{q+1}/(\beta^q+\beta)^2\in C_0 \text{ with } \beta\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q), \text{ for } q \text{ even.}\end{cases}$$

Then

$$U_{\alpha,\beta}=\{\langle(\alpha y^2+\beta y^{q+1}+r,y,1)\rangle_{q^2} : y\in\mathrm{GF}(q^2), r\in\mathrm{GF}(q)\}\cup\{P_\infty\}$$

is an orthogonal Buekenhout-Metz unital. $U_{\alpha,\beta}$ is classical if and only if $\alpha=0$.

Conversely, every orthogonal Buekenhout-Metz unital can be expressed as $U_{\alpha,\beta}$ for some $\alpha,\beta\in\mathrm{GF}(q^2)$ which satisfy the above conditions.

We go back to the projective equivalence of $\Pi(\mathcal{S})$ and $\Pi(\widehat{\mathcal{S}})$ arising from the bijective map $\widehat{\phi}$. The line set $\widehat{\phi}(U_{\alpha,\beta})=\{\widehat{\phi}(P) : P\in U_{\alpha,\beta}\}$ can be regarded as the restriction on $U_{\alpha,\beta}$ of the $\mathrm{GF}(q)$-linear representation of $\mathrm{PG}(2,q^2)$ in $\mathrm{PG}(\widehat{V})$.

Remark 3.5. Thas [33] showed that the $\mathrm{GF}(q)$-linear representation of the classical unital is a partition of an elliptic quadric in $\mathrm{PG}(5,q)$. Thas's result is obtained here when the representation $\widehat{\phi}(U_{0,\beta})$ is used. Let $\delta=\epsilon$ for odd q, and $\delta=1$ for even q. For any $\beta\in\mathrm{GF}(q^2)$ satisfying the conditions of Theorem 3.4, $U_{0,\beta}$ is the set of absolute points of the unitary polarity associated to the Hermitian form h_β of $V(3,q^2)$ with matrix

$$H_\beta=\begin{pmatrix}0 & 0 & \delta^q\\ 0 & \mathrm{Tr}(\delta\beta) & 0\\ \delta & 0 & 0\end{pmatrix}.$$

Hence $U_{0,\beta}$ has equation

$$\delta X^qZ+\delta^qXZ^q+\mathrm{Tr}(\delta\beta)Y^{q+1}=0.$$

Let Tr denote the trace map of $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$. For any $v,v'\in V(3,q^2)$,

$$\mathrm{Tr}(h_\beta(v,v'))=\begin{cases}b_{0,\beta}(\widehat{\phi}(v),\widehat{\phi}(v')), & \text{for } q \text{ odd}\\ b_\beta(\widehat{\phi}(v),\widehat{\phi}(v')), & \text{for } q \text{ even.}\end{cases}$$

This shows that the points in $\widehat{\phi}(U_{0,\beta})$ belong to $\widehat{\mathcal{Q}}_{0,\beta}$. In particular, the line set $\widehat{\phi}(U_{\alpha,\beta})$ is a partition of $\widehat{\mathcal{Q}}_{0,\beta}$.

We now put in evidence the relation between the elliptic quadric $\widehat{\mathcal{Q}}_{\alpha,\beta}$ and the Buekenhout representation of $U_{\alpha,\beta}$ in the Andrè/Bruck-Bose model of $\mathrm{PG}(2,q^2)$.

The subspace $\Lambda = \{\langle(x,x^q,y,y^q,c,c)\rangle_q : c \in \mathrm{GF}(q), x,y \in \mathrm{GF}(q^2)\}$ is an hyperplane of $\mathrm{PG}(\widehat{V})$ containing the 3-dimensional subspace $\Sigma = \{\langle(x,x^q,y,y^q,0,0)\rangle_q : x,y \in \mathrm{GF}(q^2)\}$. The line set $\mathcal{N} = \{\widehat{\phi}(P) : P \in L_\infty\}$ is a Desarguesian line spread of Σ. Hence, $\mathcal{N}$ defines the Andrè/Bruck-Bose model of $\mathrm{PG}(2,q^2)$ in Λ: the points are the lines of $\mathcal{N}$ and the points of Λ not in Σ, the *lines* are the planes of Λ not in Σ which meet Σ in a line of $\mathcal{N}$ and $\mathcal{N}$ itself, *incidence* is defined by inclusion. We denote by $\pi(\mathcal{N})$ this model of $\mathrm{PG}(2,q^2)$. The set $\overline{U}_{\alpha,\beta} = \bigcup_{P\in U_{\alpha,\beta}} (\widehat{\phi}(P)\cap\Lambda)$ is the Buekenhout representation of $U_{\alpha,\beta}$ in $\pi(\mathcal{N})$.

The hyperplane Λ is the orthogonal space of the point $R = \langle(1,1,0,0,0,0)\rangle_q$ with respect the polarity associated with the quadric $\widehat{\mathcal{Q}}_{\alpha,\beta}$. Since $R \in \widehat{\mathcal{Q}}_{\alpha,\beta}$, the intersection between Λ and $\widehat{\mathcal{Q}}_{\alpha,\beta}$ is a cone $\Gamma_{\alpha,\beta}$ projecting an elliptic quadric from R and containing the spread element $\widehat{\phi}(P_\infty) = \{\langle(x,x^q,0,0,0,0)\rangle_q : x \in \mathrm{GF}(q^2)\}$ as a generator.

Proposition 3.6. *The cone $\Gamma_{\alpha,\beta}$ coincides with the Buekenhout representation $\overline{U}_{\alpha,\beta}$ of $U_{\alpha,\beta}$ in $\pi(\mathcal{N})$, that is,*

$$\bigcup_{P\in U_{\alpha,\beta}} (\widehat{\phi}(P)\cap\Lambda) = \Gamma_{\alpha,\beta}.$$

Proof. We have $\widehat{\phi}(P_\infty) = \widehat{\mathcal{Q}}_{\alpha,\beta}\cap\Sigma$. For any $P = \langle(ay^2+\beta y^{q+1},y,1)\rangle_{q^2} \in U_{\alpha,\beta}$,

$$\widehat{\phi}(P) = \{\langle(\lambda(ay^2+\beta y^{q+1}),\lambda^q(a^q y^{2^q}+\beta^q y^{q+1}),\lambda y,\lambda^q y^q,\lambda,\lambda^q)\rangle_q : \lambda \in \mathrm{GF}(q^2)\}.$$

Then $\widehat{\phi}(P)\cap\Lambda = \langle(\alpha y^2+\beta y^{q+1}+r,\alpha^q y^{2q}+\beta^q y^{q+1}+r,y,y^q,1,1)\rangle_q$. From a straightforward calculation involving Equation (2.8) of $\widehat{\mathcal{Q}}_{\alpha,\beta}$ it follows that $\widehat{\phi}(P)\cap\Lambda \in \Gamma_{\alpha,\beta}$. Since the size of $\bigcup_{P\in U_{\alpha,\beta}\setminus\{P_\infty\}} (\widehat{\phi}(P)\cap\Lambda)$ equals the size of $\Gamma_{\alpha,\beta}\setminus\widehat{\phi}(P_\infty)$ the result follows. □

Remark 3.7. The affine points of $\Gamma_{\alpha,\beta}$ satisfy the equation

$$\delta^q X + \delta X^q + \alpha\delta Y^2 + \alpha^q\delta^q Y^{2q} + \mathrm{Tr}(\delta\beta)Y^{q+1} = 0, \tag{3.1}$$

with $\delta = \epsilon$ or $\delta = 1$ according as q is odd or even. It may be observed that Equation (3.1) is the equation of the affine points of $U_{\alpha,\beta}$ [13, 20]. Equation (3.1) in homogeneous form is

$$\delta^q XZ^{2q-1} + \delta X^q Z^q + \alpha\delta Y^2 Z^{2q-2} + \alpha^q\delta^q Y^{2q} + \mathrm{Tr}(\delta\beta)Y^{q+1}Z^{q-1} = 0,$$

which is satisfied by the points of the $\mathrm{GF}(q)$-linear representation $\widehat{\phi}(U_{\alpha,\beta})$ of $U_{\alpha,\beta}$.

In [28], Polverino proved that the $\mathrm{GF}(q)$-linear representation of an orthogonal Buekenhout-Metz unital cover the $\mathrm{GF}(q)$-points of an algebraic hypersurface of degree four minus the complements of a line in a three-dimensional subspace. She also showed that the hypersurface is reducible if and only if the unital is classical. Polverino's result is obtained here when the representation $\widehat{\phi}(U_{0,\beta})$ is used. Let $\mathcal{F}$ be the hypersurface of $\mathrm{PG}(5,q^2)$ with equation

$$\mathcal{F}\colon \delta^q X_1 Z_1 Z_2^2 + \delta X_2 Z_1^2 Z_2 + \alpha\delta Y_1^2 Z_2^2 + \alpha^q\delta^q Y_2^2 Z_1^2 + \mathrm{Tr}(\delta\beta^q)Y_1Y_2Z_1Z_2 = 0.$$

The intersection $\widehat{\mathcal{F}}$ of $\mathcal{F}$ with $\mathrm{PG}(\widehat{V})$ consists of all points of $\mathrm{PG}(\widehat{V})$ satisfying the equation

$$\delta^q X Z^{2q+1} + \delta X^q Z^{q+2} + \alpha\delta Y^2 Z^{2q} + \alpha^q \delta^q Y^{2q} Z^2 + \mathrm{Tr}(\delta\beta^q) Y^{q+1} Z^{q+1} = 0. \quad (3.2)$$

Clearly, $\widehat{\mathcal{F}}$ contains the three-dimensional subspace Σ. By the above arguments, the $\mathrm{GF}(q)$-linear representation $\widehat{\phi}(U_{\alpha,\beta})$ covers the points in $\widehat{\mathcal{F}}$ minus the complements of $\widehat{\phi}(L_\infty)$ in Σ. Furthermore, Equation (3.2) defines an algebraic hypersurface of degree four of $\mathrm{PG}(5,q)$. A straightforward, though tedious, calculation shows that Equation (3.2) is precisely the algebraic hypersurface provided by Polverino in [28].

As elliptic quadrics in $\mathrm{PG}(\widehat{V})$ are projectively equivalent, some linear collineation τ_α of $\mathrm{PG}(\widehat{V})$ takes $\widehat{\mathcal{Q}}_{0,\beta}$ to $\widehat{\mathcal{Q}}_{\alpha,\beta}$. Actually we need such a linear collineation τ_α with some extra-property.

Proposition 3.8. *In* $\mathrm{PG}(\widehat{V})$ *there exists a linear collineation* τ_α *which takes* $\widehat{\mathcal{Q}}_{0,\beta}$ *to* $\widehat{\mathcal{Q}}_{\alpha,\beta}$*, preserves the subspaces* Λ*,* Σ*, and fixes* $\widehat{\phi}(P_\infty)$ *pointwise. Therefore it maps the cone* $\Gamma_{0,\beta}$ *into* $\Gamma_{\alpha,\beta}$.

Proof. The restriction $\widehat{Q}_{\alpha,\beta}|_L$ on the subspace $L = \{(0,0,y,y^q,0,0) : y \in \mathrm{GF}(q^2)\}$ of $\widehat{Q}_{\alpha,\beta}$ given by (2.3) is the quadratic form defined by

$$\widehat{Q}_{\alpha,\beta}|_L(y,y^q) = \alpha\delta y^2 + \alpha^q\delta^q y^{2q} + \mathrm{Tr}(\delta\beta) y^{q+1} \in \mathrm{GF}(q)$$

which is of elliptic type by the proof of Proposition 2.4. As two such forms are equivalent, some endomorphism of L maps $\widehat{Q}_{0,\beta}|_L$ to $\widehat{Q}_{\alpha,\beta}|_L$. In a natural way, as in the proof of Proposition 2.3, we may identify any endomorphism of L with a 2×2 q-circulant matrix. Doing so, the endomorphism with matrix

$$D = \begin{pmatrix} d_1 & d_2 \\ d_2^q & d_1^q \end{pmatrix},$$

where

$$\begin{aligned} d_1^{q+1} + d_2^{q+1} &= 1 \\ d_1 d_2^q &= \alpha\delta\,\mathrm{Tr}(\delta\beta)^{-1}, \end{aligned}$$

maps $\widehat{Q}_{0,\beta}|_L$ to $\widehat{Q}_{\alpha,\beta}|_L$. Let τ_α be the linear collineation of $\mathrm{PG}(\widehat{V})$ defined by the matrix

$$D_\alpha = \begin{pmatrix} I_2 & O_2 & O_2 \\ O_2 & D & O_2 \\ O_2 & O_2 & I_2 \end{pmatrix}.$$

It is easily seen that τ_α preserves the subspaces Λ, Σ, and fixes $\widehat{\phi}(P_\infty)$ pointwise, and that it maps the cone $\Gamma_{0,\beta}$ into $\Gamma_{\alpha,\beta}$. □

Remark 3.9. Bearing in mind Remark 3.3, one can ask whether τ_α is an incidence preserving map of $\Pi(\widehat{\mathcal{S}})$. The answer is negative by $d_1 d_2 \neq 0$ and Proposition 3.2. This implies that $\Gamma_{0,\beta}$ and $\Gamma_{\alpha,\beta}$ are Buekenhout representations of unitals of $\mathrm{PG}(2,q^2)$ and that they are not projectively equivalent. In particular, this provides a new proof for the existence of non-classical unitals embedded in $\mathrm{PG}(2,q^2)$.

It is clear that the image $\widehat{\mathcal{S}}^{\tau_\alpha}$ of the Desarguesian line-spread $\widehat{\mathcal{S}}$ under the linear collineation τ_α is a Desarguesian line-spread and it defines the $\mathrm{GF}(q)$-linear representation $\Pi(\widehat{\mathcal{S}}^{\tau_\alpha})$ of $\mathrm{PG}(2,q^2)$.

4 The proof of the Main Theorem

In our proof the models of $\mathrm{PG}(2,q^2)$ treated in Section 3 play a role. Two of them arose from Desarguesian line-spreads of $\mathrm{PG}(\widehat{V})$ denoted by $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}^{\tau_\alpha}$ respectively, the third was the Andrè/Bruck-Bose model $\pi(\mathcal{N})$ in the 4-dimensional subspace Λ.

In $\mathrm{PG}(2,q^2)$ consider a unital $\mathcal{U}$ isomorphic, as a block-design, to an orthogonal Buekenhout-Metz unital $U_{\alpha,\beta}$ with $\alpha \neq 0$. It is known [2, 17] that $U_{\alpha,\beta}$ has a special point which is the unique fixed point of the automorphism group of $U_{\alpha,\beta}$. Hence the automorphism group of $\mathcal{U}$ fixes a unique point of $\mathcal{U}$. Up to a change of the homogeneous coordinate system in $\mathrm{PG}(2,q^2)$, the special point of $U_{\alpha,\beta}$ is $P_\infty = \langle(1,0,0)\rangle_{q^2}$ and the tangent line of $U_{\alpha,\beta}$ at P_∞ is $L_\infty\colon Z=0$. Up to a linear collineation, $P_\infty \in \mathcal{U}$ is the fixed point of the automorphism group of $\mathcal{U}$ and L_∞ is the tangent to $\mathcal{U}$ at P_∞. Therefore, $\mathcal{U}$ and $U_{\alpha,\beta}$ share P_∞ and L_∞.

We interpret the isomorphism between $\mathcal{U}$ and $U_{\alpha,\beta}$ in each of the above three models of $\mathrm{PG}(2,q^2)$. The representation $\widehat{\mathcal{U}} = \{\widehat{\phi}(P) : P \in \mathcal{U}\}$ of $\mathcal{U}$ in $\Pi(\widehat{\mathcal{S}})$ is isomorphic, as a block-design, to $\widehat{U}_{\alpha,\beta} = \{\widehat{\phi}(P) : P \in U_{\alpha,\beta}\}$. The Buekenhout representation $\overline{\mathcal{U}} = \bigcup_{P\in\mathcal{U}}(\widehat{\phi}(P)\cap\Lambda)$ of $\mathcal{U}$ in $\pi(\mathcal{N})$ is isomorphic, as a block-design, to $\overline{U}_{\alpha,\beta} = \bigcup_{P\in U_{\alpha,\beta}}(\widehat{\phi}(P)\cap\Lambda)$. Here, by Proposition 3.6, $\overline{U}_{\alpha,\beta}$ is the cone $\Gamma_{\alpha,\beta}$. This gives that the representation $\widetilde{\mathcal{U}} = \{L \in \widehat{\mathcal{S}}^{\tau_\alpha} : L\cap\Lambda \subset \overline{U}\}$ of $\mathcal{U}$ in $\Pi(\widehat{\mathcal{S}}^{\tau_\alpha})$ is isomorphic, as a block-design, to $\widetilde{U}_{\alpha,\beta} = \{L \in \widehat{\mathcal{S}}^{\tau_\alpha} : L\cap\Lambda \subset \Gamma_{\alpha,\beta}\}$.

From Proposition 3.8, the lines which are the points of $\widetilde{U}_{\alpha,\beta}$ partition the elliptic quadric $\widehat{\mathcal{Q}}_{\alpha,\beta} = \widehat{\mathcal{Q}}_{0,\beta}^{\tau_\alpha}$. On the other hand, from Remark 3.5, $\widehat{\mathcal{Q}}_{0,\beta}$ is partitioned by lines which are the points of the classical unital $\widehat{U}_{0,\beta}$ in $\Pi(\widehat{\mathcal{S}})$. This yields that $\widetilde{U}_{\alpha,\beta}$ coincides with $\widehat{U}_{0,\beta}^{\tau_\alpha}$. It turns out that $\widetilde{U}_{\alpha,\beta}$ is a classical unital in $\Pi(\widehat{\mathcal{S}}^{\tau_\alpha})$, and hence $\widetilde{\mathcal{U}}$ is isomorphic, as a block-design, to the classical unital.

Now we quote the following result from [23] which was the keystone in the proof of Theorem 1.1.

Lemma 4.1. *Let $\mathcal{U}$ be a unital embedded in a Desarguesian finite projective plane π and isomorphic, as a block-design, to the classical unital. For any block B of $\mathcal{U}$, let ℓ be the line of π containing B. Then B is an orbit of a cyclic subgroup of order $q+1$ contained in the projectivity group of ℓ. This implies that B is a Baer subline of ℓ.*

We emphasize that the proof of Lemma 4.1 only uses arguments involving point-block incidences of $\mathcal{U}$ viewed as a block-design embedded in π.

Therefore, Lemma 4.1 applies to $\widetilde{\mathcal{U}}$. Thus, every block of $\widetilde{\mathcal{U}}$ is a Baer subline of $\Pi(\widehat{\mathcal{S}}^{\tau_\alpha})$, that is, a regulus of $\mathrm{PG}(\widehat{V})$. From this, each block of $\overline{\mathcal{U}}$ is the intersection of these reguli with Λ. In particular, each block of $\overline{\mathcal{U}}$ through $\widehat{\phi}(P_\infty)$ is the union of $\widehat{\phi}(P_\infty)$ with q collinear affine points, and this implies that each block of $\widehat{\mathcal{U}}$ through $\widehat{\phi}(P_\infty)$ is a regulus of $\mathrm{PG}(\widehat{V})$ whose lines are in $\widehat{\mathcal{S}}$. Under $\widehat{\phi}$, these reguli correspond to Baer sublines of $\mathrm{PG}(2,q^2)$ through P_∞. This yields that the points of $\mathcal{U}$ on each of the q^2 secant lines to $\mathcal{U}$ form a Baer subline through P_∞. By the characterization of such unitals of $\mathrm{PG}(2,q^2)$

given in [12, 29], we may conclude that $\mathcal{U}$ is a Buekenhout-Metz unital. By definition, the Buekenhout representation $\overline{\mathcal{U}}$ of $\mathcal{U}$ is a cone that project an ovoid $\mathcal{O}$ from a point of $\widehat{\phi}(P_\infty)$ not in $\mathcal{O}$. Here an *ovoid* is a set of q^2+1 points in a 3-dimensional subspace of Λ no three of which are collinear.

To conclude the proof we only need to prove that $\mathcal{O}$ is an elliptic quadric. Since the ovoids in $\mathrm{PG}(3,q)$ with odd q are elliptic quadrics, see [4, 26], we assume $q = 2^h$. In $\mathrm{PG}(3,2^h)$, there are known two ovoids, up to projectivities, namely the elliptic quadric which exist for $h \geq 1$, and the Tits ovoid which exists for odd $h \geq 3$; see [18, Chapter 10]. Let Ω be the 3-dimensional subspace of Λ containing $\mathcal{O}$. Note that $\mathcal{O} = \Omega \cap \overline{\mathcal{U}}$. Set α_∞ to be the plane $\Omega \cap \Sigma$. Then α_∞ meets $\mathcal{O}$ exactly in the point $\mathcal{O} \cap \widehat{\phi}(P_\infty)$, and it is a simple matter to show that α_∞ contains only one line $\widehat{\phi}(P)$ of $\mathcal{N}$. Also, $\widehat{\phi}(P)$ is distinct from $\widehat{\phi}(P_\infty)$. Let $\alpha_1, \dots, \alpha_q$ denote the further planes of Ω through $\widehat{\phi}(P)$. As these planes are lines of $\pi(\mathcal{N})$ through the point $\widehat{\phi}(P)$, each of them meets $\overline{\mathcal{U}}$ in 1 or $q+1$ points. This holds true for $\mathcal{O}$.

It is well known [19, Section 12.3] that in a finite Desarguesian projective plane through any point off a unital there are exactly $q+1$ tangent lines, that is, lines of the plane that intersects the unital in exactly one point. In terms of the unital $\overline{\mathcal{U}}$ this property states that there is only one plane among $\alpha_1, \dots, \alpha_q$ that meets $\mathcal{O}$ in exactly one point. Let α_1 denote this plane. Then the block $\alpha_i \cap \mathcal{O}$ of $\overline{\mathcal{U}}$, for $i = 2, \dots, q$, is the intersection of α_i with a regulus in $\mathrm{PG}(\widehat{V})$. Since that regulus does not contain $\widehat{\phi}(P)$, the block $\alpha_i \cap \mathcal{O}$ is a conic C_i of α_i, for $i = 2, \dots, q$. Thus the blocks $\alpha_i \cap \mathcal{O}$, for $i = 2, \dots, q$, are $q-1$ conics that partition all but two points of $\mathcal{O}$. By [8, Theorem 5] $\mathcal{O}$ is an elliptic quadric.

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