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Imitating continuum in lattice models *

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Abstract. Lattice models as well as few-body models with a finite Hilbert space do not provide a continuum description of the two-body decay channel. Instead, the diagonalization of the Hamiltonian yields a discrete spectrum which hides, however, a lot of information about the relevant continuum. We show a method which extracts the effective pion-pion potential and applies it to the pion-pion scattering amplitude.

As a toy model to study the relation between continuum and discrete spectrum we are using a schematic quasispin model inspired by the Nambu – Jona-Lasinio model but restricted to a finite number of quarks occupying a finite number of states in the Dirac sea and in the valence space.

1 Introduction

The diagonalization of the Hamiltonian in few-body models with a finite Hilbert space yields a discrete spectrum. There is, however, a lot of hidden information about the continuum and we have to develop a reliable method how to extract it. For this purpose we show a possible method how to extract the effective pion-pion potential and the pion-pion scattering amplitude from the discrete spectrum. The method relies on the first order Born approximation or on its suitable generalization. The Luescher formula [1] known in the literature, for example, is a special case of the (generalized) first order Born approximation.

The simplest two-level model of chiral symmetry breaking is a schematic quasispin model similar to the Nambu – Jona-Lasinio model and it is developed in the spirit of the Lipkin model [2] known from nuclear physics as a test different approximate approaches. Our model is characterized by a finite number of quarks occupying a finite number of states in the Dirac sea and in the valence space (due to a sharp momentum cutoff and periodic boundary condition). This allows us to use the first quantization and an explicit wavefunction.

Most low-lying states in the excitation spectrum can be interpreted as multi-pion states and one can deduce the effective pion-pion interaction and scattering length. However, the intruder states can be recognized as sigma-meson excitations or their admixtures to multi-pion states.

* Talk delivered by M. Rosina

The lesson learned from the toy model can be useful in a similar problem in lattice calculations – how to extract effective potential and scattering amplitudes from the discrete excitation spectrum.

2 The two-level quasispin model

In this section we repeat some properties of the two-level quasispin model which we have presented in previous Bled Workshops [3,4]. We partially use those results and partially add some new ones (arguments using N-dependence of spectra) in order to discuss the relation between the discrete spectrum and the continuum in the two-body channel.

We are aiming at a finite-dimensional N-body Hilbert space, therefore we enclose $N = \mathcal{N}$ quarks in a periodic box $\mathcal{V}$ and use a sharp momentum cutoff Λ, leading to a finite number $\mathcal{N} = \mathcal{N}_x N_s N_c N_f$ of states in the Dirac sea and the same number of states in the valence "shell". Here $\mathcal{N}_x = \mathcal{V}\, 4\pi\Lambda^3/(3\,(2\pi)^3$ is the number of spacial states in each "shell", we have $N_s = 2$ helicities, $N_c = 3$ colours and we restrict the simple model to $N_f = 1$ flavour. Then $N = \mathcal{N} = 6\mathcal{N}_x = \mathcal{V}\Lambda^3/\pi^2$.

Furthermore, we take all quark kinetic energies equal to $\frac{3}{4}\Lambda$ and neglect the interaction terms which change the individual quark momenta:

$$H = \sum_{k=1}^{N}\Big(\gamma_5(k)h(k)\,\tfrac{3}{4}\Lambda + m_0\beta(k)\Big) - \tfrac{2G}{\mathcal{V}}\sum_{k,l=1}^{N}\Big(\beta(k)\beta(l) + i\beta(k)\gamma_5(k)\cdot i\beta(l)\gamma_5(l)\Big)$$

Here $h = \boldsymbol{\sigma}\cdot\mathbf{p}/p$ is helicity and γ_5 and β are Dirac matrices. We use the popular model parameters close to [5,6], $\Lambda = 648$ MeV, $G = 40.6$ MeV fm, $m_0 = 4.58$ MeV, which yield the phenomenological values of quark constituent mass, quark condensate and pion mass both in full Nambu – Jona-Lasinio model as well as in our quasispin model (using in both cases the Hartree-Fock + RPA approximations). It has been shown in [3] that in the large N limit the exact results of our quasispin model tend in fact to the Hartree-Fock + RPA values.

It is usually overlooked that the following operators obey (quasi)spin commutation relations $j_x = \frac{1}{2}\,\beta\ ,\quad j_y = \frac{1}{2}\,i\beta\gamma_5\ ,\quad j_z = \frac{1}{2}\,\gamma_5$. The (quasi)spin commutation relations are also obeyed by separate sums over quarks with right and left helicity as well as by the total sum ($\alpha = x,\ y,\ z$)

$$R_\alpha = \sum_{k=1}^{N}\frac{1+h(k)}{2}j_\alpha(k)\ ,\quad L_\alpha = \sum_{k=1}^{N}\frac{1-h(k)}{2}j_\alpha(k)\ ,\quad J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^{N} j_\alpha(k)\,.$$

The model Hamiltonian can then be written as

$$H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2)\quad . \tag{1}$$

It commutes with R^2 and L^2 but not with R_z and L_z. Nevertheless, it is convenient to work in the basis $|\,R, L, R_z, L_z\,\rangle$.The Hamiltonian matrix elements can be easily calculated using the angular momentum algebra. By diagonalisation we then obtain the energy spectrum of the system.

Table 1. The spectrum of the quasispin model with $N = 144$ and $N = 192$, and the ground state quantum numbers $R + L = N/4$

n	Parity	$(E - E_0)$[MeV] N=144	$(E - E_0)$[MeV] N=192	$\bar{V}$[MeV] N=144	$\bar{V}$[MeV] N=192
10	+	932	(942)	-9.5	(-5.4)
9	−	803	(805)	-11.7	(-7.2)
8	+	771	861	-11.3	-8.3
7	−	767	802	-8.8	-7.3
6	+	**646**	709	-11.4	-7.3
6	+	634	**655**	-12.2	-10.9
5	−	580	611	-10.0	-7.2
4	+	482	503	-10.5	-7.1
3	−	378	388	-10.1	-7.1
2	+	261	266	-10.3	-7.1
1	−	136	137		
0	+	0	0		

3 Extraction of pion-pion interaction

The average effective pion-pion potential $\bar{V}$ given in Table 1 has been extracted from the energy levels of n-pion states

$$E_{n\pi} = n\, m_\pi + \frac{n(n-1)}{2}\bar{V}.$$

An important test to distinguish one-pion and two-pion properties is the volume-dependence (N-dependence). In a larger volume, pions are more dilute pions leading to a proportionally smaller $\bar{V}$. In fact, the ratio of $\bar{V}$ in Table 1 for $N = 144$ and $N = 192$ is $10.3/7.1 = 1.45$, close to $192/144 = 1.33$. (The small discrepancy does indicate that we are not yet quite in the large-N limit and further corrections might be needed).

We calculate the s-state scattering length in the first-order Born approximation ("Lüscher formula" [1])

$$a_0 = \frac{m_\pi/2}{2\pi}\int V(\mathbf{r})\, d^3r = \frac{m_\pi}{4\pi}\bar{V}\mathcal{V}. \tag{2}$$

In our example for $N = 192$ we have $\bar{V} = -7.1$ MeV and $\mathcal{V} = \pi^2 N/\Lambda^3 = 53\,\mathrm{fm}^3$. This gives

$$a_0\, m_\pi = \frac{m_\pi^2}{4\pi}\bar{V}\mathcal{V} = -0.077. \tag{3}$$

Since there are no experiments with one-flavour pions it is tempting to compare with the two-flavour value ($I = 2$). The chiral perturbation theory (soft pions) suggests in leading order $a_0^{I=2}\, m_\pi = -m_\pi^2/16\pi f_\pi^2 = -0.0445$. Our almost twice larger value might be due to the artifact that we made up for the second flavour by replacing G with 2G. Further investigation is in progress.

4 The intruder state - the sigma meson

In the spectrum in Table 1 one can clearly distinguish the presence of the sigma meson by noticing the doubling of the positive parity states at 634 and 646 MeV for $N = 144$ (655 and 709 MeV for $N = 192$). Moreover, the states at 646 MeV (655 MeV)indicated in boldface have strong one-body transition matrix elements from the ground state. Note that going from N=144 to 192 the ordering of the two positive parity states ("σ" and "6π") has reversed because for larger N the six pions are more dilute and the energy is less depressed by attractive effective interactions between pions.

5 Relation to lattice calculations

The discrete single-particle space in our model is analogous to a lattice. The model assumption $0 \le |\mathbf{p}_i| \le \Lambda$ corresponds to the cell size (resolution)

$$a = \sqrt[3]{\frac{\mathcal{V}}{\mathcal{N}_x}} = \frac{\sqrt[3]{6\pi^2}}{\Lambda} = 1.2\,\text{fm}\,.$$

Here $\mathcal{V}/\mathcal{N}_x = \mathcal{V}/(N/6)$ is the "land" available per particle in case of 2 helicities, 3 colours and one flavour.

The periodic boundary condition in $\mathcal{V}$ corresponds to the block size

$$L = \sqrt[3]{\mathcal{V}} = \frac{\sqrt[3]{6\pi^2\mathcal{N}_x}}{\Lambda} = 3.7\,\text{fm} \approx 3a.$$

It is surprising that such a poor resolution and block size yields excellent results. One reason is that the model interaction is not very sensitive to the number of dimensions, there are no spacial correlations. In one dimension, the ratio between the block size and the cell size $\mathcal{N}_x = 32$ is much larger than $\sqrt[3]{\mathcal{N}_x} \approx 3$ but the structure of results is the same. This is a general feature of Nambu – Jona-Lasinio models.

Furthermore, we were dealing with soft pion excitations and we get an impression that in this case a high resolution is not crucial.

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