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Reachability relations, transitive digraphs and groups

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Abstract

In [6] it was shown that properties of digraphs such as growth, property **Z**, and number of ends are reflected by the properties of certain reachability relations defined on the vertices of the corresponding digraphs.

In this paper we study these relations in connection with certain properties of automorphism groups of transitive digraphs. In particular, one of the main results shows that if a transitive digraph admits a nilpotent subgroup of automorphisms with finitely many orbits, then its nilpotency class and the number of orbits are closely related to particular properties of reachability relations defined on the digraphs in question.

The obtained results have interesting implications for Cayley digraphs of certain types of groups such as torsion-free groups of polynomial growth.

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1 Introduction and preliminaries

In [2], highly arc-transitive digraphs were considered from several different viewpoints, leading to – besides many nice results – a number of interesting problems. One of these problems, which remained open for a very long time and was finally settled in [4], concerned a certain reachability relation defined on the edges of digraphs. A subset of the authors of this paper also worked on this 'reachability problem' [5] and several other questions concerning highly-arc-transitive digraphs. In [6], as an offspring of our considerations, we became interested in reachability relations defined on vertices rather than edges, which we review in the sequel.

A *digraph* is an ordered pair $D = (V(D), E(D))$, where $V(D)$ is the vertex-set and $E(D) \subseteq V(D) \times V(D)$ is the edge-set. Note that a digraph can have loops (v, v) as well as pairs of 'oppositely directed' edges of the form (u, v) and (v, u). We also emphasize that with this definition our digraphs are always simple in the sense that between two vertices there can be at most one edge in each direction. Digraphs considered in this paper are connected in the sense that their underlying undirected graphs are connected.

By $\mathrm{Aut}(D)$ we denote the automorphism group of a digraph D. We say that D is *transitive* if some $H \subseteq \mathrm{Aut}(D)$ acts transitively on the vertices of D. Also, if $g \in \mathrm{Aut}(D)$, then ${}^{g}v$ denotes the image of $v \in V(D)$ under g and ${}^{H}v$ denotes the orbit of v under some subset $H \subseteq \mathrm{Aut}(D)$.

To make sure that no ambiguity arises, we explicitely define Cayley digraphs as they are understood in this paper. The *Cayley digraph* $Cay(G, S)$ of a group G with respect to a generating set S has the group G as its vertex set and the edges are given by right multiplication by the generators: $E(Cay(G, S)) = \{(g, gs) | s \in S\}$. If $Cay(G, S)$ is defined in this way, then G acts regularly on $Cay(G, S)$ by left multiplication.

A *walk* $W = (v_0, \varepsilon_1, v_1, \dots, \varepsilon_n, v_n)$ from v_0 to v_n of length $n \geq 0$ (denoted by $|W|$) is a sequence of $n + 1$ (not necessarily pairwise distinct) vertices $v_0, v_1, \dots, v_n \in V(D)$, and n indicators $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{1, -1\}$ such that for all $j \in \{1, 2, \dots, n\}$ we have

$$\varepsilon_j = 1 \quad \Rightarrow \quad (v_{j-1}, v_j) \in E(W),$$
$$\varepsilon_j = -1 \quad \Rightarrow \quad (v_j, v_{j-1}) \in E(W).$$

W is called a *closed walk* if $v_0 = v_n$. Intuitively, a walk is a traversal in the digraph from vertex to vertex along edges, where indicators 1 and -1 tell whether the traversal respects the direction of edges or not. The formal definition of a walk as above has been chosen in order to make proofs unambiguous. If the vertices of a walk W are pairwise different then W is called a *path*. A walk (or a path) is *directed* if its indicators are all equal to 1 or to -1, and is *alternating* if the values of the indicators alternate.

Let $W = (v_0, \varepsilon_1, v_1, \dots, \varepsilon_n, v_n)$ be a walk. We let the *inverse walk* of W be $W^{-1} = (v_n, -\varepsilon_n, v_{n-1}, \dots, -\varepsilon_1, v_0)$. Moreover, for $0 \leq i \leq j \leq n$, the subsequence

$${}_iW_j = (v_i, \varepsilon_{i+1}, \dots, \varepsilon_j, v_j)$$

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of W is called a *subwalk*. Furthermore, let $W' = (u_0, \delta_1, u_1, \ldots, \delta_m, u_m)$ be a walk such that $u_0 = v_n$. Then the *concatenation* of W and W' is the walk

$$W \cdot W' = (v_0, \varepsilon_1, v_1, \ldots, v_{n-1}, \varepsilon_n, u_0, \delta_1, u_1, \ldots, \delta_m, u_m)$$

of length $n + m$.

We now introduce two families of reachability relations defined on vertices of a digraph. Let $W = (v_0, \varepsilon_1, v_1, \ldots, \varepsilon_n, v_n)$ be a walk. The *weight* of the walk W is defined as

$$\zeta(W) = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n.$$

Let k be a nonnegative integer. We say that a vertex $u \in V(D)$ is R_k^+*-related* to a vertex $v \in V(D)$, in symbols

$$uR_k^+ v,$$

if there exists a walk W from u to v such that $\zeta(W) = 0$, and that for every $0 \leq j \leq |W|$ we have $\zeta({}_0W_j) \in [0, k]$. For a given pair of vertices u, v, the set of all such walks is denoted by $R_k^+[u, v]$. Analogously we say that u is R_k^-*-related* to v, in symbols $uR_k^- v$, if there exists a walk W such that $\zeta(W) = 0$, and that for every $0 \leq j \leq |W|$ we have $\zeta({}_0W_j) \in [-k, 0]$. For a given pair of vertices u, v, the set of all such walks is denoted by $R_k^-[u, v]$. Note that R_k^+ and R_k^- are equivalence relations. Their equivalence classes are denoted by $R_k^+(v)$ and $R_k^-(v)$, $v \in V(D)$, respectively. If D is transitive, then the equivalence classes of R_k^+ (and similarly of R_k^-) form an imprimitivity system for $\mathrm{Aut}(D)$. Observe that the sequences $(R_k^+)_{k \in \mathbb{Z}^+}$ and $(R_k^-)_{k \in \mathbb{Z}^+}$ are ascending: for all k we have $R_k^+ \subseteq R_{k+1}^+$ and $R_k^- \subseteq R_{k+1}^-$. Their respective unions

$$R^+ = \bigcup_{k \in \mathbb{Z}^+} R_k^+ \quad \text{and} \quad R^- = \bigcup_{k \in \mathbb{Z}^+} R_k^-$$

are thus also equivalence relations, and their equivalence classes form imprimitivity systems for $\mathrm{Aut}(D)$ whenever D is transitive. As was shown in [6], $R^+ = R_k^+$ holds whenever $R_k^+ = R_{k+1}^+$. In this case, the smallest nonnegative integer k such that $R_k^+ = R^+$ holds is called the *exponent* $\exp^+(D)$ of D. If $R_k^+ \neq R^+$ for all k, then we set $\exp^+(D) = \infty$. The exponent $\exp^-(D)$ is defined analogously. We say that the relation R_k^+ (R^+, R_k^-, R^-) is *universal* if $uR_k^+ v$ $(uR^+ v, uR_k^- v, uR^- v)$ holds for any pair $u, v \in V(D)$. We mention (see [6]) that all of the above relations are universal, provided that the digraph in question is connected and has a loop at every vertex.

In [6] it was also shown that properties of the two sequences of equivalence relations $(R_k^+)_{k \in \mathbb{Z}^+}$ and $(R_k^-)_{k \in \mathbb{Z}^+}$ are strongly related to other properties of digraphs such as having property **Z**, the number of ends, growth conditions and vertex degree.

Furthermore, in [8] the relations $R_{a,b}$ were studied, where a is a non-positive integer or $a = -\infty$ and b is a non-negative integer or $b = \infty$. We say that a vertex u is $R_{a,b}$-related to a vertex v if there exists a 0-weighted walk from u to v such that every subwalk starting at u has weight in the interval $[a, b]$.

The *distance* $\mathrm{dist}_D(u, v)$ between vertices u and v in a connected digraph D is the length of a shortest path from u to v in the underlying undirected graph. The *growth function* $f_D(v, n), n \geq 0$, with respect to some $v \in V(D)$ is given by

$$f_D(v, n) = |\{u \in V(D) \mid \mathrm{dist}_D(v, u) \leq n\}|.$$

If D is transitive, then this function does not depend on a particular vertex $v \in V(D)$. In this case we denote it by $f_D(n)$.

We say that a transitive digraph D has *polynomial growth* if there are positive constants c and d such that

$$f_D(n) \leq cn^d$$

holds for all $n \geq 0$. The digraph D has *exponential growth* if there is a constant $c > 1$ such that

$$f_D(n) > c^n$$

holds for all $n \geq 0$. If the growth function of a digraph D grows faster than any polynomial but D does not have exponential growth, then we say that D has *intermediate growth*. In the case of polynomial growth it can be shown that there always exist constants c_1, c_2 and an integer $d \geq 1$ such that

$$c_1 n^d \leq f_D(n) \leq c_2 n^d$$

holds for all $n \geq 0$. We call this integer d the *growth degree* of D. We remark that the definitions concerning growth coincide with the usual definitions in the context of undirected graphs.

Let D be a digraph and let τ be a partition of the vertex set of D. The *quotient digraph* D_τ of D with respect to τ is the digraph with vertex set τ and $(u_\tau, v_\tau) \in E(D)$ if and only if there exist vertices $u \in u_\tau$ and $v \in v_\tau$ such that $(u, v) \in E(D)$. If $W = (v_0, \varepsilon_1, v_1, \varepsilon_2, \ldots, \varepsilon_n, v_n)$ is a walk in D, then the *quotient walk* W_τ of W is defined to be the walk $W = ((v_0)_\tau, \varepsilon_1, (v_1)_\tau, \varepsilon_2, \ldots, \varepsilon_n, (v_n)_\tau)$. Note that for every j, $0 \leq j \leq |W| = |W_\tau|$, we have $\zeta({}_0W_j) = \zeta({}_0(W_\tau)_j)$. We emphasize that we consider these quotient digraphs as simple digraphs in the sense that if there are several edges in the same direction between two sets in τ, then the quotient digraph contains exactly one directed edge between the respective vertices. But of course these quotient graphs might contain loops if there is an edge $(u, v) \in E(D)$ for some $u \in v_\tau$.

Let G be a group acting transitively on D and let H be a normal subgroup of G. Then the orbits of H on $V(D)$ give rise to an imprimitivity system τ of G on $V(D)$. The respective quotient digraph D_τ is usually denoted by D_H.

As mentioned above, if D is transitive, then R^+ and R^- give rise to imprimitivity systems of $\mathrm{Aut}(D)$ on D. The respective quotient digraphs are denoted by D/R^+ and D/R^- and can be described easily (see e. g. [8]). The digraph D/R^+ either is (1) a finite directed cycle or (2) a two-way infinite directed line or (3) an infinite regular directed tree with indegree 1 and outdegree > 1. Considering R^- the first two possibilities are the same, but if D/R^- is neither of these digraphs, then it is a regular tree with outdegree 1 and indegree > 1.

2 Motivation and main result

The aim of this paper is to investigate the interplay between properties of groups and properties of reachability relations in their Cayley digraphs.

For example, as a consequence of the last paragraph of the previous section, we immediately see that the quotient digraphs with respect to R^+ of Cayley digraphs of finitely generated groups with polynomial or intermediate growth are either finite directed cycles or directed lines. Further, from [6, Theorem 4.12] we know that a finitely generated group

G has exponential growth if for at least one Cayley digraph D of G, at least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite.

Additionally, by Gromov's important result [3], a finitely generated group has polynomial growth if and only if it contains a normal nilpotent subgroup of finite index. Hence the following question arises naturally: What can be said about properties of our reachability relations in Cayley digraphs of finitely generated groups with polynomial growth?

In fact we carry out our considerations by assuming that nilpotent groups act with finitely many orbits on digraphs. The results for Cayley digraphs of groups with polynomial growth are then obtained as corollaries. The main result of this paper is the following theorem.

To avoid ambiguity, we recall the definition of nilpotent groups: For a group $G = G^0$, let $G^{i+1} = [G^0, G^i]$ for $i \geq 0$. If $G = G^0 \triangleright G^1 \triangleright \ldots \triangleright G^r \triangleright G^{r+1} = 1$ then we say that G is *nilpotent of class* r.

Theorem 2.1. *Let a group G act transitively on a connected digraph D, and let $N \trianglelefteq G$ be a normal nilpotent subgroup of class r acting with m orbits on D, where $1 \leq m < \infty$. Then* $\exp^+(D) = \exp^-(D) \leq m(r+2) - 1$.

Although we are mainly interested in properties of our relations in Cayley digraphs of finitely generated groups, we emphasize that - with the exception of those results explicitely formulated for finitely generated groups - we never assume that the graphs in consideration are locally finite.

3 Auxiliary results

In this section we prove several results which will be useful for our main investigations, carried out in Section 4.

Lemma 3.1. *Let D be a digraph with minimal in- and outdegree at least* 1 *and let $k \geq 1$ be an integer. Then for any two vertices $u, v \in V(D)$ we have that uR_k^+v if and only if there exists a walk $W \in R_k^+[u, v]$ that is a concatenation of walks of the form* $(u_0, 1, u_1, 1, \ldots, 1, u_k, -1, u_{k+1}, -1, \ldots, -1, u_{2k})$. *An analogous result holds for the relation R_k^-.*

Proof. We prove the assertion for the relation R_k^+ and leave the analogous proof for R_k^- to the reader.

To this end suppose uR_k^+v and let $W' = (u_0, \varepsilon_1, u_1, \varepsilon_2, u_2, \varepsilon_3, \ldots, \varepsilon_n, u_n) \in R_k^+[u, v]$. Observe that, since the minimal in- and outdegrees of D are at least 1, there is a directed walk of any prescribed positive or negative weight starting at any vertex of D.

A walk $W \in R_k^+[u, v]$, as described in the statement of the lemma, can now be obtained from W' inductively by inserting a concatenation of such a directed walk of appropriate length with its inverse at each vertex u_i for which $\varepsilon_i \neq \varepsilon_{i+1}$ and $\zeta({}_0W'_i)$ is not 0 or k.

□

For a group G, a positive integer k, and subsets $S, T \subseteq G$ let $ST = \{st | s \in S, t \in T\}$, $S^k = \underbrace{S \cdots S}_{k}$ and $S^{-k} = \underbrace{S^{-1} \cdots S^{-1}}_{k}$.

Corollary 3.2. *Let $D = \mathrm{Cay}(G, S)$ be a Cayley digraph of a group G with respect to the generating set S. Then for any integer $k \geq 1$ and any $g \in G$ we have that $R_k^+(g) = gR_k^+(1) = g\langle S^k S^{-k} \rangle$ and $R_k^-(g) = gR_k^-(1) = g\langle S^{-k} S^k \rangle$.*

Proof. We prove the assertions for R_k^+ and leave the analogous proof for R_k^- to the reader. The fact that $R_k^+(g) = gR_k^+(1)$ is obvious since G has a natural left regular action on D while $R_k^+(1) = \langle S^k S^{-k} \rangle$ follows from Lemma 3.1. □

Lemma 3.3. *Let D be a digraph and let τ be a partition of the vertex set of D. Suppose that for each $u, v \in V(D)$ with $(u_\tau, v_\tau) \in E(D_\tau)$ there exist $u' \in u_\tau$ and $v' \in v_\tau$ such that (u, v') and (u', v) are arcs of D. Then for each $k \geq 1$ and each $u, v \in V(D)$ we have that $u_\tau R_k^+ v_\tau$ if and only if there exists some $w \in v_\tau$ such that $uR_k^+ w$. An analogous result holds for the relation R_k^-.*

Proof. We prove the result for R_k^+ and leave the analogous proof for R_k^- to the reader. Let $k \geq 1$ and let $u, v \in V(D)$.

Suppose first that for some $w \in v_\tau$ we have that $uR_k^+ w$ and let $W \in R_k^+[u, w]$. Since $v_\tau = w_\tau$, the walk W_τ is contained in $R_k^+[u_\tau, v_\tau]$. This proves one implication.

Suppose now that $u_\tau R_k^+ v_\tau$ and let $\bar{W} = (u_\tau, \varepsilon_1, \bar{x}_1, \varepsilon_2, \ldots, \bar{x}_n, \varepsilon_{n+1}, v_\tau) \in R_k^+[u_\tau, v_\tau]$. Then by assumption one can successively find representatives $x_i \in \bar{x}_i$ and $w \in v_\tau$ such that $W = (u, \varepsilon_1, x_1, \varepsilon_2, x_2, \varepsilon_3, \ldots, x_n, \varepsilon_{n+1}, w) \in R_k^+[u, w]$. □

Remark 3.4. Observe that the condition of the above lemma is satisfied if τ consists of the orbits of some group acting on D.

Lemma 3.5. *Let a group G act transitively on a digraph D and let H be a normal subgroup of G such that each of its subgroups is normal in G. Then* $\exp^+(D) \leq \exp^+(D_H) + 1$ *and* $\exp^-(D) \leq \exp^-(D_H) + 1$.

Proof. We prove the result for $\exp^+(D)$. The proof for $\exp^-(D)$ is analogous and is left to the reader. If $\exp^+(D_H) = \infty$, there is nothing to prove. We may thus assume that $\exp^+(D_H) = k$ for some integer $k \geq 0$.

To show that $\exp^+(D) \leq k + 1$ let $u \in V(D)$ and $v \in R^+(u)$ be arbitrary. Consider the equivalence class $B = R_{k+1}^+(u)$ and the H-orbit ${}^H u$. Note that both of these sets are blocks of imprimitivity for the action of G on $V(D)$. Let K be the setwise stabilizer in H of the set B. Note that the K-orbit of u is ${}^K u = {}^H u \cap B$ and is thus a block of imprimitivity for G. Moreover, by assumption on H the subgroup K is normal in G, and so the block system generated by the block ${}^K u$ coincides with the block system given by the orbits of K. Consequently, any two vertices within the same H-orbit are R_{k+1}^+ related if and only if they belong to the same K-orbit.

We first show that $\exp^+(D_K) \leq k$. If this is not the case, then there exists ${}^K w \in V(D_K)$ such that ${}^K w \in R_{k+1}^+({}^K u) \setminus R_k^+({}^K u)$. By Lemma 3.3 there exists $w' \in {}^K w$ such that $uR_{k+1}^+ w'$. Moreover, since $\exp^+(D_H) = k$ there exists $z \in {}^H w' = {}^H w$ such that $uR_k^+ z$. But then $zR_{k+1}^+ w'$, and so ${}^K z = {}^K w' = {}^K w$, implying that ${}^K w \in R_k^+({}^K u)$, a contradiction.

Hence $\exp^+(D_K) \leq k$. But then ${}^K v \in R_k^+({}^K u) = R_{k+1}^+({}^K u)$ in D_K, and by Lemma 3.3 there exists some $x \in {}^K v$ such that $uR_{k+1}^+ x$. Since $x \in {}^K v$ we have that $xR_{k+1}^+ v$, and so $uR_{k+1}^+ v$ holds.

Since u and v were arbitrary subject to the condition that $uR^+ v$, this shows that $\exp^+(D) \leq k + 1$. □

Lemma 3.6. *Let a group G act transitively on a digraph D with finite exponents* $\exp^+(D)$ *and* $\exp^-(D)$. *Furthermore, let τ denote the imprimitivity system of G on $V(D)$ which is* *induced by the equivalence classes with respect to R^+ or R^-. Then every $g \in G$ which leaves invariant at least one block of τ leaves invariant all blocks of τ.*

Proof. Since the exponents $\exp^+(D)$ and $\exp^-(D)$ are both finite, [6, Corollary 3.5] implies that $R^+ = R^-$, and so the discussion from the last paragraph of the first section implies that D_τ is a finite cycle or the two-way infinite directed line. Hence, the only automorphism of D_τ which fixes a vertex is the identity. On the other hand, every automorphism $g \in G$ which leaves invariant a block of τ induces an automorphism of D_τ fixing a vertex of D_τ, and the result follows. □

4 R^+ and R^- in transitive digraphs

We start with a simple observation concerning Cayley digraphs of abelian groups.

Proposition 4.1. *Let G be an abelian group acting transitively on a digraph D. Then* $\exp^+(D) = \exp^-(D) = 1$.

Proof. Since G is abelian, D is a Cayley graph of G. Then Corollary 3.2 implies that $R_1^+ = R_1^-$ and [6, Corollary 3.4] implies that $R^+ = R_1^+ = R_1^- = R^-$, as claimed. □

We now generalise this result to nilpotent groups.

Theorem 4.2. *Let G be a nilpotent group of class r acting transitively on a digraph D. Then* $\exp^+(D) = \exp^-(D) \leq r + 1$.

Proof. We first show that $\exp^+(D) \leq r + 1$. The proof is carried out by induction on r. If $r = 0$, then G is an abelian group and Proposition 4.1 applies.

Suppose now that $r \geq 1$. As $G^{(r+1)} = 1$, we have that $H = G^{(r)}$ is contained in the center of G, and so each of its subgroups is normal in G. Hence Lemma 3.5 implies that $\exp^+(D) \leq \exp(D_H) + 1$. Now, the quotient group G/H is a nilpotent group of class $r - 1$ and acts transitively on the quotient digraph D_H. By induction hypothesis we thus have that $\exp^+(D_H) \leq r$. Consequently, $\exp^+(D) \leq r + 1$, as claimed.

The fact that $\exp^-(D) \leq r + 1$ follows by analogous arguments. Then [6, Corollary 3.5], implies that $\exp^+(D) = \exp^-(D)$. □

The next example shows that the bound from the above theorem is tight, that is, for every positive integer r there exists a nilpotent group G of class r and a digraph D on which G acts transitively such that $\exp^+(D) = r + 1 = \exp^-(D)$ holds.

Example 4.3. Already for the smallest nonabelian finitely generated nilpotent group, the dihedral group D_8 of order 8 (of nilpotency class 1), this is the case. Let us write $D_8 = \langle f, a_1, a_2 | f^2 = a_1^2 = a_2^2 = 1, fa_1f^{-1} = a_1a_2, fa_2 = a_2f, a_1a_2 = a_2a_1\rangle$. Then for the Cayley digraph $D = Cay(D_8, \{f, fa_1\})$ we clearly have that $\exp^+(D) = \exp^-(D) = 2$.

In fact, this example happens to be the smallest member of the following infinite family. Let $n \geq 1$ be an integer and let G_n be the semidirect product of the elementary abelian group $\mathbb{Z}_2^n$ by the cyclic group $\mathbb{Z}_{2^{n-1}}$ generated by $G_n = \langle f, a_1, a_2, \ldots, a_n\rangle$, where f is of order 2^{n-1}, the a_i are involutions commuting with each other and $fa_if^{-1} = a_ia_{i+1}$ holds for all i, $1 \leq i < n$, while f and a_n commute. One can verify that for $S = \{f, fa_1a_2 \cdots a_n\}$ we have that $\langle S^iS^{-i}\rangle = \langle a_1, a_2, \ldots, a_i\rangle$ holds for all i, $1 \leq i \leq n$, and so Corollary 3.2 implies that $\exp^+(Cay(G_n, S)) = n$. Moreover, it can be verified that G_n is nilpotent of class $n - 1$. Indeed, we have that $G^{(i)} = \langle a_{i+1}, a_{i+2}, \ldots, a_n\rangle$ holds for each i, $1 \leq i \leq n-1$, and of course then $G^{(n)} = 1$. The Cayley graph $Cay(G_n, S)$ thus attains the bound from the above theorem.

We shall now see, that the above theorem cannot be generalized to solvable groups.

Example 4.4. The lamplighter group L is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. The standard representation for L is

$$\langle a, t | a^2, [t^m a t^{-m}, t^n a t^{-n}], m, n \in \mathbb{Z} \rangle.$$

If we consider the Cayley digraph of L with respect to the generating set $S = \{t, at\}$, then this Cayley digraph is the horocyclic product of two directed trees with indegree 1 and outdegree 2. In this digraph $R_k^+ \neq R^+$ clearly holds for all $k \in \mathbb{Z}^+$. This shows that for solvable groups we cannot expect a result like Theorem 4.2.

As was shown in [6], a connected, locally finite, transitive digraph D has exponential growth if at least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite. Hence these exponents must be finite if a connected, locally finite, transitive digraph D does not have exponential growth. So the question arises if we can find a bound on $\exp^+(D)$ and $\exp^-(D)$ which depends on the growth rate of D or on certain properties of groups acting transitively on D. In the sequel we show that this is indeed possible.

We first consider the case where a digraph D allows a transitive action of a group G containing a normal abelian subgroup, acting with finitely many orbits on D, thereby obtaining a tight bound for $\exp^+(D)$ and $\exp^-(D)$. We then explore a more general situation where a transitive group G contains a normal nilpotent subgroup acting with finitely many orbits on D. We start by proving two auxiliary results.

Lemma 4.5. *Let D be a connected digraph, and let G be a transitive subgroup of $\mathrm{Aut}(D)$ having a normal subgroup $H \lhd G$ with m, $1 \leq m < \infty$, orbits on D. If for some (and hence every) $u \in V(D)$ the set $R_1^+(u)$ is contained in ${}^H u$, then the following hold:*

(i) For every $v \in V(D)$ the set $R^+(v)$ is contained in ${}^H v$.

(ii) The quotient digraph D_H is a directed cycle.

Proof. Observe that if $m = 1$ there is nothing to prove, so we may assume $m \geq 2$.

To prove (i) we show that $R_k^+(v) \subseteq {}^H v$ for all $v \in V(D)$ and all k. We do that by induction on k. The base of induction ($k = 1$) holds by assumption. Let now $k \geq 1$ and suppose that $R_j^+(v) \subseteq {}^H v$ holds for all $j \leq k$. Pick an arbitrary vertex $v \in V(D)$ and let $w \in R_{k+1}^+(v)$. Let $v = v_0$, $w = v_n$ and choose a walk $W = (v_0, 1, v_1, \ldots, v_{n-1}, -1, v_n) \in R_{k+1}^+[v, w]$. Suppose first that for all i, $0 < i < n$, we have that $\zeta({}_0W_i) > 0$. In this case $v_1 R_k^+ v_{n-1}$, and so induction hypothesis implies that $v_{n-1} \in {}^H v_1$, that is, $v_{n-1} = {}^h v_1$ for some $h \in H$. Then $({}^h v_0, v_{n-1}) \in E(D)$, and so ${}^h v_0 R_1^+ v_n$. Then, by assumption, we have that ${}^h v_0 \in {}^H v_n$, and so $v \in {}^H w$ (recall that $v = v_0$ and $w = v_n$). Suppose now that $0 < i_1 < i_2 < \cdots < i_t = n$ are such that $\zeta({}_0W_{i_j}) = 0$. By the above argument $v_{i_1} \in {}^H v$, $v_{i_2} \in {}^H v_{i_1}, \ldots, w \in {}^H v_{i_{t-1}}$. Hence $v \in {}^H w$, which proves (i).

We now prove (ii). Let ${}^H v$ be an H-orbit. Since D is connected and H has at least two orbits which are blocks of imprimitivity for G, there exists an H-orbit ${}^H w \neq {}^H v$ such that $({}^H w, {}^H v) \in E(D_H)$. It follows that there exists a vertex $w' \in {}^H w$ with $(w', v) \in E(D)$. Consequently, the quotient digraph D_H must have indegree one (for otherwise we obtain a vertex $x \notin {}^H w$ which is R_1^+-related to w'). Since D_H is finite, it is a simple directed cycle. □

Lemma 4.6. *Let D be a digraph, and let G be a transitive subgroup of $\mathrm{Aut}(D)$ having an abelian normal subgroup $H \lhd G$ with m, $1 \leq m < \infty$, orbits on D. If for some (and hence any) $u \in V(D)$ the set $R_1^+(u)$ is contained in ${}^H u$, then $exp^+(D) = exp^-(D) \leq m$.*

Proof. We prove that $exp^+(D) \leq m$ and leave the analogous proof that $exp^-(D) \leq m$ to the reader.

If $m = 1$, then D is a Cayley digraph of an abelian group, so Proposition 4.1 applies. We can thus assume that $m \geq 2$. Let $\Delta = {}^H u$ for some $u \in V(D)$.

We first construct an auxiliary digraph D^* with vertex set Δ and an edge (w, v) whenever there exists a directed path of length m in D from w to v. The restriction of H on Δ acts regularly on Δ. The digraph D^* thus is a Cayley digraph of an abelian group (possibly disconnected). Therefore $\exp^+(D^*) \leq 1$ by Proposition 4.1.

Now, let vR^+w for some $v, w \in V(D)$ and let us show that in this case vR_m^+w holds. By definition of R^+ we have that vR_k^+w holds for some integer k. Then Lemma 3.1 implies that there exists a walk in $R_k^+[v, w]$ which is a concatenation of walks of the form $W = (v_0, 1, v_1, 1, \ldots, 1, v_k, -1, v_{k-1}, -1, \ldots, -1, v_{2k})$. By transitivity it suffices to prove that $v_0 R_m^+ v_{2k}$. Let t, r with $0 \leq r < m$ be the integers such that $k = tm + r$. By Lemma 4.5 the vertices $v_0, v_m, v_{2m}, \ldots, v_{tm}$ and $v_{2k}, v_{2k-m}, v_{2k-2m}, \ldots, v_{2k-tm}$ all belong to the H-orbit ${}^H v_0$. Hence $v_{2k} = {}^h v_0$, $v_{tm} = {}^{h_1} v_0$ and $v_{2k-tm} = {}^{h_2} v_0$ for some $h, h_1, h_2 \in H$. Now, ${}_0W_{tm} \cdot ({}^{h_1 h_2^{-1}}({}_{(2k-tm)}W_{2k}))$ is a walk from v_0 to $x = {}^{h_1 h_2^{-1}} v_{2k} = {}^{h_1 h_2^{-1} h} v_0$. As H is abelian, $x = {}^{h h_2^{-1} h_1} v_0$. Therefore, ${}^{h h_2^{-1}}({}_{tm}W_{2k-tm}) \in R_r^+[x, v_{2k}]$, and so $r < m$ implies that $v_0 R_m^+ v_{2k}$ if and only if $v_0 R_m^+ x$, that is, we can assume $r = 0$. Since $v_0 \in {}^H v$, we have $v_0 = {}^{h_0} v$ for some $h_0 \in H$. It follows that the walk W corresponds to a walk $W^* \in R_t^+[h_0, h_1 h_2^{-1} h h_0]$ in D^*. Since $\exp^+(D^*) \leq 1$, the walk W^* can be replaced by a walk in $R_1^+[h_0, h_1 h_2^{-1} h h_0]$, implying that W can be replaced by a walk in $R_m^+[v_0, v_{2k}]$.

Therefore, $R^+ \subseteq R_m^+$, implying that $R^+ = R_m^+$. Analogously, it can be shown that $R^- = R_m^-$. Then [6, Corollary 3.5] completes the proof. □

To prove the next theorem we need the following result from [6].

Proposition 4.7. *([6], Proposition 3.11) Let D be a digraph, let τ be the set of equivalence classes of R_1^+, and let $u \in V(D)$. Then, for any $v \in V(D)$ and any $k \geq 2$ we have that uR_k^+v if and only if $u_\tau R_{k-1}^+ v_\tau$. An analogous assertion holds for R_k^- when taking the quotient with respect to R_1^-.*

Theorem 4.8. *Let D be a digraph and let $G \leq \mathrm{Aut}(D)$ be a transitive subgroup having an abelian normal subgroup H acting with m, $1 \leq m < \infty$, orbits on $V(D)$. Then $\exp^+(D) = \exp^-(D) \leq m$.*

Proof. We prove that $\exp^+(D) \leq m$ and leave the analogous proof for $\exp^-(D)$ to the reader. We proceed by induction on m. If $m = 1$, then the result follows from Proposition 4.1. Suppose the assertion holds for all $n < m$, $m \geq 2$, and suppose that H has m orbits on D. If for some $u \in V(D)$ the set $R_1^+(u)$ is contained in ${}^H u$, then Lemma 4.6 applies.

Assume now that the equivalence classes with respect to R_1^+ are not contained in the orbits of H and consider the quotient digraph D/R_1^+. Let K be the kernel of the action of G on D/R_1^+ and let $N = HK/K \cong H/(H \cap K)$ be the induced faithful action of H on D/R_1^+. Observe that, since the R_1-equivalence classes are not fully contained in the H-orbits, N acts with at most $\frac{m}{2}$ orbits on D/R_1^+. By induction hypothesis (note that N is an abelian normal subgroup of G/K) we have that $\exp^+(D/R_1^+) \leq \frac{m}{2}$. By Proposition 4.7 it follows that $\exp^+(D) = \exp^+(D/R_1^+) + 1 \leq \frac{m+2}{2} \leq m$.

Analogously it can be shown that $\exp^-(D) \leq m$. Then again [6, Corollary 3.5] completes the proof. □

Proof of Theorem 2.1

Let a group G act transitively on a connected digraph D, and let $N \trianglelefteq G$ be a normal nilpotent subgroup of class r acting with m orbits on D, where $1 \leq m < \infty$.

We first prove that $\exp^+(D) \leq m(r+2) - 1$. The proof is done by induction on m.

If $m = 1$, then the result holds by Theorem 4.2. If $m \geq 2$ we distinguish two cases, depending on the structure of D_N.

Case 1. D_N is not isomorphic to a directed cycle on $m \geq 2$ vertices.

In this case Lemma 4.5 implies that, for any $v \in V(D)$, the set $R_1^+(v)$ is not completely contained in one orbit of N. Let τ denote the imprimitivity system of G on D consisting of the equivalence classes with respect to R_1^+. Then the permutation group G_τ, induced by the action of G on τ, acts transitively on D_τ. Furthermore, N_τ acts with at most $\frac{m}{2}$ orbits. In addition N_τ is nilpotent of class at most r. Then, by induction hypothesis, $\exp^+(D_\tau) \leq \frac{m}{2}(r+2) - 1$ holds and the result follows by Proposition 4.7.

Case 2. D_N is isomorphic to a directed cycle $C = (c_1, \ldots, c_m)$ on $m \geq 2$ vertices.

Let $O_1, \ldots, O_m$ denote the orbits of N on $V(D)$ which correspond to the vertices $c_1, \ldots, c_m \in D_N$. Then of course there is no edge in D which connects two vertices which are both contained in the same orbit. Furthermore, all edges of D are directed from O_i to O_{i+1}, $1 \leq i \leq m$, where indices are taken modulo m. Then for every $v \in O_i$, $1 \leq i \leq m$, $R^+(v) \subseteq O_i$ holds. Of course $\exp^+(D) \leq m - 1$ holds if $R_{m-1}^+(v) = O_i$ for some $v \in O_i$ and some i, $1 \leq i \leq m$.

Hence we only have to consider the case when $R_{m-1}^+(v)$ is properly contained in O_i for every i, $1 \leq i \leq m$, and every vertex $v \in O_i$. By B_ι, $\iota \in \mathcal{I}$, we denote the equivalence classes of R_{m-1}^+ on O_1. For $v \in O_1$, let $\mathcal{P}(v)$ denote the set of all directed paths starting at v and containing exactly one vertex from each orbit O_i, $1 \leq i \leq m$. Since D_N is isomorphic to a directed cycle with m vertices and N acts transitively on each of its orbits, $\mathcal{P}(v) \neq \emptyset$ for all $v \in O_1$. Furthermore, for $\iota \in \mathcal{I}$ let S_ι be the subdigraph of D induced by the vertices of the union $\bigcup_{v \in B_\iota} \mathcal{P}(v)$. Note that since the sets B_ι are different equivalence classes with respect to R_{m-1}^+, the digraphs S_ι, $\iota \in \mathcal{I}$, are pairwise disjoint.

We first define $\mathcal{P}^m$ as the set of all directed paths $P = (v_1, \ldots, v_{m+1})$ in D where $v_j \in O_j$ for $1 \leq j \leq m$ and $v_{m+1} \in O_1$. Analogously we define $\mathcal{P}^{-m}$ as the set of all inverses of the paths in $\mathcal{P}^m$. Furthermore, let $\mathcal{R}_{m-1}^+$ denote the set of all walks which are contained in $R_{m-1}^+[u, v]$ for some vertices $u, v \in O_1$.

Let $v_1, v_2 \in O_1$ now satisfy $v_1 R^+ v_2$. If v_1 and v_2 are both contained in one and the same set B_ι, $\iota \in \mathcal{I}$, then of course $v_1 R_{m-1}^+ v_2$ holds.

Now let $v_1 \in B_{\iota_1}$ and $v_2 \in B_{\iota_2}$, $\iota_1 \neq \iota_2$. Then there is a walk $W \in R^+[v_1, v_2]$ which is the concatenation of finitely many paths and walks which are contained in $\mathcal{P}^m$, $\mathcal{P}^{-m}$ or $\mathcal{R}_{m-1}^+$. Let D' now be the digraph with vertex set $\mathcal{I}$ with $(\iota_1, \iota_2) \in E(D')$ whenever there exists a path $P \in \mathcal{P}^m$ with origin in B_{ι_1} and terminal vertex in B_{ι_2}. Observe that, in general, the digraph D' might not be locally finite. Nevertheless, the restriction of N to O_1 induces a transitive group acting on D' which is nilpotent of class at most r. Thus Theorem 4.2 implies that $\exp^+(D') \leq r + 1$.

Observe that, by Lemma 3.1, we can assume that the walk W is the concatenation of t paths from $\mathcal{P}^m$, followed by a walk in $\mathcal{R}^+_{m-1}$ and then t paths from $\mathcal{P}^{-m}$, for some non-negative integer t. Let $u_0, u_1, \ldots, u_{2t+1}$ be the vertices of W, contained in O_1, given in the order they are met when traversing W. Thus $u_0, u_1, \ldots, u_{t-1}$ are the origins of the paths from $\mathcal{P}^m$ while the vertices $u_{t+1}, u_{t+2}, \ldots, u_{2t}$ are the origins of the paths from $\mathcal{P}^{-m}$. The walk W thus naturally gives rise to the walk $W' = (\iota_0, \iota_1, \ldots, \iota_{t-1}, \iota_{t+1}, \iota_{t+2}, \ldots, \iota_{2t+1})$ in D', where for each i we have that $u_i \in B_{\iota_i}$ (observe that $u_t \in B_{\iota_{t+1}} = B_{\iota_t}$). Of course $W' \in R^+[\iota_0, \iota_{2t+1}]$, and so $\exp^+(D') \leq r+1$ implies that there is a walk $\bar{W}' \in R^+_{r+1}[\iota_0, \iota_{2t+1}]$. Since the sets B_{ι_i} are equivalence classes of the relation R^+_{m-1} on D it is now clear that this walk gives rise to some walk in $R^+_{m(r+2)-1}[v_1, v_2]$.

Since $\exp^-(D) \leq m(r+2)-1$ holds by similar arguments, [6, Corollary 3.5] implies that $\exp^+(D) = \exp^-(D)$. □

Corollary 4.9. *Let G be a finitely generated group, let N be a normal nilpotent subgroup of finite index m in G and let D denote a Cayley digraph of G with respect to some finite generating set S. Then $\exp^+(D) = \exp^-(D) \leq m(r+2)-1$ holds, where r is the nilpotency class of N.*

It is natural to ask if this bound is tight. All examples we know in fact satisfy the inequality $\exp^+(D) \leq m(r+1)$. We thus pose the following problem.

Problem 4.10. Is it true that $\exp^+(D) = \exp^-(D) \leq m(r+1)$ holds for the Cayley digraphs of groups described in Corollary 4.9?

For Cayley digraphs D of finitely generated torsion-free groups G with polynomial growth we even obtain bounds for $\exp^+(D)$ and $\exp^-(D)$ which only depend on the growth degree. To formulate the result we first have to consider $\mathrm{GL}(n, \mathbb{Z})$.

Theorem 4.11. *(see e.g. [7]) The orders of the finite subgroups of* $\mathrm{GL}(n, \mathbb{Z})$ *are bounded by some function g(n) of n alone.*

Theorem 4.12. *(see e.g. [7]) Let G be a finitely generated torsion-free group with polynomial growth of degree d. Then G contains a normal nilpotent subgroup of class less than $\sqrt{2d}$ and index at most $g(d)$, where $g(d)$ is the function of Theorem 4.11.*

Corollary 4.13. *Let G be a finitely generated torsion-free group with polynomial growth of degree d. Then for any Cayley digraph D of G,* $\exp^+(D)$ *and* $\exp^-(D)$ *are bounded by $g(d)(\sqrt{2d}+2)-1$, where $g(d)$ is the function of Theorem 4.11*

We conclude the paper with the following observations. Let $G \leq \mathrm{Aut}(D)$ act transitively on a digraph D with finite exponents $\exp^+(D)$ and $\exp^-(D)$. Then Lemma 3.6 implies that the equivalence classes of the relation $R^+ = R^-$ are orbits of a normal subgroup of G. Thus, if this relation is not universal and if the digraph has indegree or outdegree at least 2, then this normal subgroup of G is proper and not trivial. As a consequence, if G is simple, the relation $R^+ = R^-$ is universal on D. As was already mentioned above it was shown in [6, Theorem 4.12] that a connected infinite locally finite transitive digraph D has exponential growth if at least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite. At this point we recall the following problem from combinatorial group theory (see e.g. [1]), which was originally posed by R. I. Grigorchuk.

Problem 4.14. Does every finitely generated infinite simple group have exponential growth?

The following proposition then allows to formulate a conjecture which closely relates this problem to reachability relations.

Proposition 4.15. *If a finitely generated infinite simple group G does not have exponential growth, then for every finite generating set S of G there is a finite integer $k_S \geq 1$, such that $R^+_{k_S} = R^-_{k_S}$ is universal in $C(G, S)$.*

Proof. Follows immediately from [6, Theorem 4.12] and Lemma 3.6. □

Conjecture 4.16. *Let G be a finitely generated infinite group. Then there is a finite generating set S of G such that for the Cayley digraph D of G with respect to S one of the following holds:*

- *At least one of the exponents $\exp^+(D)$ or $\exp^-(D)$ is infinite and hence D has exponential growth.*
- *Both, $\exp^+(D)$ and $\exp^-(D)$ are finite and the reachability relations R^+ and R^- are not universal on D.*

Observe that by Proposition 4.15 the validity of this conjecture would provide a positive answer to Grigorchuk's problem.

References

[1] G. Baumslag, A. G. Myasnikov and V. Shpilrain, Open problems in combinatorial group theory. Second edition, Combinatorial and geometric group theory, S. Cleary, R. Gilman, A. G. Myasnikov, V. Shpilrain, eds., *Contemporary Mathematics* **296**, AMS 2002.

[2] P. J. Cameron, C. E. Praeger and N. C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs, *Combinatorica* **13** (1993), 377–396.

[3] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Etudes Sci. Publ. Math.* **53** (1981), 53–78.

[4] M. DeVos, B. Mohar and R. Samal, Reachability in highly arc-transitive digraphs, arXiv:1110.2945.

[5] A. Malnič, D. Marušič, R. Möller, N. Seifter, V. Trofimov and B. Zgrablič, Highly arc transitive digraphs: reachability, topological groups, *Europ. J. Combinatorics* **26** (2005), 19–28.

[6] A. Malnič, D. Marušič, N. Seifter, P. Šparl and B. Zgrablič, Reachability relations in digraphs, *Europ. J. Combin.* **29** (2008), 1566–1581.

[7] A. Mann, *How groups grow*, LMS Lecture Notes Ser. 395, Cambridge Uni. Press, 2012.

[8] N. Seifter and V. I. Trofimov, Reachability relations and the structure of transitive digraphs, *Electron. J. Combin.* **16** (2009), R26.