ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 25 (2025) #P4.05
https://doi.org/10.26493/1855-3974.3424.d5e
(Also available at http://amc-journal.eu)

Tetravalent distance magic graphs of small order and an infinite family of examples*

Ksenija Rozman †
Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia and
University of Primorska, FAMNIT, Koper, Slovenia

Primož Šparl
Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia and
University of Ljubljana, Faculty of Education, Ljubljana, Slovenia and
University of Primorska, Andrej Marušič Institute, Koper, Slovenia

Received 4 July 2024, accepted 20 November 2024, published online 4 August 2025

Abstract

A graph of order n is *distance magic* if it admits a bijective labeling of its vertices with integers from 1 to n such that each vertex has the same sum of the labels of its neighbors.

This paper contributes to the long term project of characterizing all tetravalent distance magic graphs. With the help of a computer we find that out of almost nine million connected tetravalent graphs up to order 16 only nine are distance magic. In fact, besides the six well known wreath graphs there are only three other examples, one of each of the orders 12, 14 and 16. We introduce a generalization of wreath graphs, the so-called quasi wreath graphs, and classify all distance magic graphs among them. This way we obtain infinitely many new tetravalent distance magic graphs. Moreover, the two non-wreath graphs of orders 12 and 14 are quasi wreath graphs while the one of order 16 can be obtained from a quasi wreath graph of order 14 using a simple construction due to Kovář, Fronček and Kovářová.

Keywords: Distance magic, tetravalent, quasi wreath graph.

Math. Subj. Class. (2020): 05C78

*This work is supported in part by the Slovenian Research and Innovation Agency (Young researchers program, research program P1-0285 and research projects J1-2451, J1-3001 and J1-50000).

†Corresponding author.

E-mail addresses: ksenija.rozman@pef.uni-lj.si (Ksenija Rozman), primoz.sparl@pef.uni-lj.si (Primož Šparl)

1 Introduction

Graph labelings are assignments of labels (in most cases integers) to vertices and/or edges of graphs. They were first introduced in mid 1960s and have since attracted many mathematicians, as is witnessed by the wide variety of graph labeling variants studied and thousands of papers on the topic published (a list is compiled in the comprehensive survey by Gallian [4]). In this paper we focus on distance magic labelings which were first studied independently by Vilfred [14] and by Miller, Rodger and Simanjuntak [10] under the names sigma labelings and 1-vertex magic vertex labelings, respectively (see also [4, Section 5.6]).

According to the definition which has been commonly used ever since its introduction in 2009 in [13] (but see Section 2, where we give a somewhat different definition which appears to be more convenient for our purposes), a *distance magic labeling* of a graph of order n is a bijective labeling of vertices with integers from 1 to n, such that the *weight* of each vertex v (that is, the sum of the labels of the neighbors of v) is equal to a constant κ, called the *magic constant*. A graph is said to be *distance magic* if there exists a distance magic labeling of its vertices. It is well known and easy to see that in the case of regular graphs of valency r, the distance magic constant is $r(n+1)/2$, implying that there are no r-regular distance magic graphs with r odd [10]. While it is clear that the only connected 2-regular distance magic graph is the 4-cycle, the situation with 4-regular graphs seems to be much more intriguing. In fact, even though Rao proposed the problem of characterizing all 4-regular distance magic graphs already back in 2008 [11], we thus far only have a few partial results, for instance [1, 2, 6, 8, 12] (we do mention that there are several other results on distance magic graphs in general, see for instance [4, 5, 9]).

The first natural step in the investigation of the above mentioned problem is to determine the orders for which connected tetravalent distance magic graphs exist. That at least one example exists for each even order $n \geq 6$ is easy to see as one can simply take the wreath graph $W(n/2)$ of order n (see Section 2 for the definition). For odd orders it was first shown in [6] (using a computer) that there are no examples up to order 15 and then using a nice simple construction that at least one example exists for each odd order $n \geq 17$.

One of the next natural steps is to investigate small tetravalent distance magic graphs and try to identify nice infinite families to which these graphs belong. Using a computer we verify (see Section 3) that distance magic examples are extremely rare among all tetravalent graphs, at least for small orders. In fact, up to order 16 there are only nine connected distance magic tetravalent graphs, namely the wreath graphs $W(n)$, $3 \leq n \leq 8$, and three additional examples, one of each of the orders 12, 14 and 16. None of these three examples belongs to any of the three well-known infinite families of tetravalent distance magic graphs known thus far. Namely, in [1], the authors explored the distance magic property among certain products of graphs, particularly direct products of two cycles. By [1, Theorem 3.3] there exist distance magic direct products of two cycles of orders 12 and 16, namely $C_3 \times C_4$ and $C_4 \times C_4$, respectively. However, it is easy to verify that the first one is isomorphic to the wreath graph $W(6)$, while the second one is not connected (it is in fact isomorphic to two copies of $W(4)$). Moreover, as shown in [12, Theorem 1.1], the smallest distance magic Cartesian product of two cycles is $C_3 \square C_6$ of order 18, and similarly, as shown in [8, Theorem 1.1], the smallest non-wreath tetravalent circulant is of order 24. Observe that these three families of tetravalent distance magic graphs consist of vertex-transitive examples (a graph is *vertex-transitive* if its automorphism group acts transitively on its vertex set).

In Section 3 we introduce the family of so-called *quasi wreath graphs* (which contains the above non-wreath graphs of orders 12 and 14) and classify its distance magic members. The following is our main result (see Section 3 for definitions).

Theorem 1.1. *The quasi wreath graph $QW(S)$ is distance magic if and only if it consists of any number of segments of length congruent to* 3 *modulo* 4 *and an even number of segments of length congruent to* 1 *modulo* 4.

The family of quasi wreath graphs thus provides infinitely many tetravalent distance magic graphs. In fact, as we explain in Section 6, it provides at least one non-wreath distance magic example for each even order $n \geq 12$, except for $n = 16$. Moreover, the only non-wreath example of order 16 can be obtained from the distance magic quasi wreath graph of order 14 using the simple construction from [6].

2 Preliminaries

In this paper we consider only finite, connected, simple and undirected graphs. First we give some basic notation that we use. For a graph Γ we denote its vertex set by $V(\Gamma)$, its edge set by $E(\Gamma)$, and the neighborhood of a vertex $u \in V(\Gamma)$, which is the set of vertices adjacent to u, by $N(u)$. The ring of residue classes modulo n, where n is a positive integer, is denoted by $\mathbb{Z}_n$.

As mentioned in the Introduction we now give a slightly different definition of a distance magic labeling of a tetravalent graph, which however leads to the same definition of being distance magic. Let $\Gamma = (V, E)$ be a tetravalent graph of order n and let $N = \{1-n, 3-n, \ldots, n-1\}$. Then a *distance magic labeling* of Γ is a bijection $\ell\colon V \longrightarrow N$ such that the weight of each vertex is equal to 0.

Clearly, if ℓ is a distance magic labeling according to this definition, then setting $\tilde{\ell}(u) = (\ell(u)+n+1)/2$ for each vertex u one gets a distance magic labeling according to [13] and conversely, if $\tilde{\ell}$ is a distance magic labeling according to [13], then setting $\ell(u) = 2\tilde{\ell}(u) - 1 - n$ for each vertex u one gets a distance magic labeling according to our definition. Throughout the rest of this paper we will thus always be working with our definition of a distance magic labeling. We point out that in this setting the weight of each vertex for such a labeling is 0.

For $n \geq 3$, the *wreath graph* $W(n)$ is a tetravalent graph of order $2n$ with vertex set $V = \{u_i \colon i \in \mathbb{Z}_n\} \cup \{v_i \colon i \in \mathbb{Z}_n\}$ where for each $i \in \mathbb{Z}_n$ it holds that $N(u_i) = N(v_i) = \{u_{i\pm1}, v_{i\pm1}\}$, all indices computed modulo n. Clearly, the graphs $W(n)$, $n \geq 3$, are all distance magic, since one can simply label the pair u_i, v_i by $2n-2i-1$ and $-(2n-2i-1)$ (but see also [8, Theorem 1.1]).

3 The quasi wreath graphs

In 1999 Meringer [7] developed an algorithm enabling construction of regular graphs of given valency up to a specified small enough order. The database of all connected tetravalent graphs of small orders is available at [7] (see also [3]). It turns out that there are 906 331 connected tetravalent graphs up to order 15. To analyze which of them are distance magic we used the following idea pointed out in [8, Lemma 2.1] and its immediate corollary.

Lemma 3.1 ([8]). *Let $\Gamma = (V, E)$ be a regular graph of even valency and order n. Then Γ is distance magic if and only if* 0 *is an eigenvalue of Γ and there exists an eigenvector* *for the eigenvalue* 0 *with the property that a certain permutation of its entries results in the arithmetic sequence* $1-n, 3-n, 5-n, \ldots, n-3, n-1$.

Corollary 3.2. *Let* $\Gamma = (V, E)$ *be a regular graph of even valency and order* n. *Suppose that* 0 *is an eigenvalue of* Γ *and let* B *be the basis of the corresponding eigenspace. If there exist integers* i, j *with* $1 \leq i < j \leq n$ *such that* $b(i) = b(j)$ *for all* $b \in B$ *then* Γ *is not distance magic.*

Using a computer one finds that out of the $906\,331$ connected tetravalent graphs up to order 15 all but 9 graphs are ruled out as candidates for distance magic graphs by Corollary 3.2. In particular, there are $1, 1, 1, 2, 2, 2$ candidates of order $6, 8, 10, 12, 14, 15$, respectively. We already know that the 5 wreath graphs $W(n)$, $3 \leq n \leq 7$, are distance magic which leaves us with two candidates of order 15 and one candidate of each of the orders 12 and 14. For the two graphs of order 15 a computer check reveals that they do not satisfy the conditions of Lemma 3.1 and are thus not distance magic (which is consistent with [6]). On the other hand the remaining two graphs of orders 12 and 14 are indeed distance magic. They are presented in Figure 1 where a distance magic labeling for each of them is given. In view of the fact that these are the only two connected tetravalent distance magic graphs up to order 15 which are not wreath graphs and do not belong to any other well-known infinite family of tetravalent graphs it makes sense to introduce the following family of graphs and investigate which of its members are distance magic.

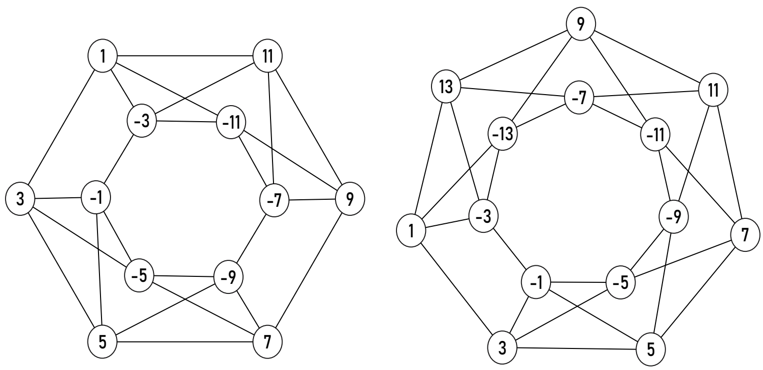

Figure 1: The quasi wreath graphs $QW([0, 1, 1, 0, 1, 1])$ and $QW([0, 1, 1, 1, 1, 1, 1])$.

Construction 3.3. *Let* $m \geq 3$ *be a positive integer and let* $S = [s_0, s_1, \ldots, s_{m-1}]$ *be a sequence such that* $s_i \in \{0, 1\}$ *for all* $i \in \mathbb{Z}_m$, $s_0 = 0$, $s_{m-1} = 1$ *and for all* $i \in \mathbb{Z}_m \setminus \{m-1\}$, *at least one of* s_i, s_{i+1} *is equal to* 1. *Then the* quasi wreath graph $QW(S)$ *is the tetravalent graph of order* $2m$ *with vertex set* $\{x_i : i \in \mathbb{Z}_m\} \cup \{y_i : i \in \mathbb{Z}_m\}$ *in which the edges are defined as follows:*

- *for each* $i \in \mathbb{Z}_m$ *we have the edges* $\{x_i, x_{i+1}\}$ *and* $\{y_i, y_{i+1}\}$,
- *for each* $i \in \mathbb{Z}_m$ *with* $s_i = 0$ *we have the edges* $\{x_i, y_i\}$ *and* $\{x_{i+1}, y_{i+1}\}$ *and*
- *for each* $i \in \mathbb{Z}_m$ *with* $s_i = 1$ *we have the edges* $\{x_i, y_{i+1}\}$ *and* $\{x_{i+1}, y_i\}$,

where all the subscripts are computed modulo m.

Note that the wreath graph $W(3)$ is isomorphic to the quasi wreath graph $QW([0,1,1])$ and that the graphs from Figure 1 are isomorphic to the quasi wreath graphs $QW([0,1,1,0,1,1])$ and $QW([0,1,1,1,1,1,1])$. In the rest of this paper we abbreviate the term quasi wreath graph to *QW-graph*. We also simplify the notation and write $QW(a_1, a_2, \ldots, a_r)$ instead of $QW([\underbrace{0,1,\ldots,1}_{a_1}, \underbrace{0,1,\ldots,1}_{a_2}, \ldots, \underbrace{0,1,\ldots,1}_{a_r}])$. Note that with this notation, $m = a_1 + a_2 + \ldots + a_r$ and the above three graphs are $QW(3)$, $QW(3,3)$ and $QW(7)$, respectively.

As already mentioned, we determine all distance magic QW-graphs in this paper. Before starting this analysis we introduce some additional notation. Let $\Gamma = QW(S)$ be a QW-graph of order $2m$. For each $i \in \mathbb{Z}_m$ we let $B_i = \{x_i, y_i\}$. The sets B_i, $i \in \mathbb{Z}_m$, are called *blocks* of Γ. Suppose $i \in \mathbb{Z}_m$ is such that $s_i = 0$ and let $j \geq 1$ be the smallest integer such that $s_{i+j} = 0$ (the index computed modulo m). Then the subgraph of Γ induced on $B_{i+1} \cup B_{i+2} \cup \cdots \cup B_{i+j}$ is called a *segment* of Γ and is said to be of *length* j. Note that the number of segments of Γ equals the number of $i \in \mathbb{Z}_m$ such that $s_i = 0$ and that a QW-graph $QW(a_1, a_2, \ldots, a_r)$ has r segments of lengths $a_1, a_2, \ldots, a_r$.

Finally, for a QW-graph $\Gamma = QW(S)$ of order $2m$ and for a labeling ℓ of its vertices (not necessarily distance magic) we let $\ell_i = \ell(x_i) + \ell(y_i)$ for each $i \in \mathbb{Z}_m$ and call it the *block label* of B_i. We also make an agreement that the indices of s_i, x_i, y_i, ℓ_i and B_i are always computed modulo m.

4 A necessary condition

In this section we give a necessary condition for a QW-graph to be distance magic. First we point out two properties of distance magic labelings of QW-graphs that will be useful later on.

Lemma 4.1. *Let $QW(S)$ be a distance magic QW-graph and let ℓ be a corresponding distance magic labeling. Then for each $i \in \mathbb{Z}_m$ such that $s_i = 0$ we have that $\ell_{i+1} \neq -\ell_{i+2}$.*

Proof. By way of contradiction, suppose that for some $i \in \mathbb{Z}_m$ we have that $s_i = 0$ and $\ell_{i+1} = -\ell_{i+2}$, that is, $\ell(x_{i+1}) + \ell(y_{i+1}) + \ell_{i+2} = 0$. Considering the neighbors of x_{i+1} we have that $\ell(x_i) + \ell(y_{i+1}) + \ell_{i+2} = 0$. This implies that $\ell(x_i) = \ell(x_{i+1})$, a contradiction. □

Lemma 4.2. *Let $QW(S)$ be a distance magic QW-graph and let ℓ be a corresponding distance magic labeling. Then for each $i \in \mathbb{Z}_m$ the following holds:*

$$\ell_{i+2} = \begin{cases} -\ell_i\colon & s_i = s_{i+1} = 1 \\ -\frac{1}{2}(\ell_i + \ell_{i+1})\colon & s_i = 0, \quad s_{i+1} = 1 \\ -2\ell_i - \ell_{i+1}\colon & s_i = 1, \quad s_{i+1} = 0. \end{cases} \tag{4.1}$$

Proof. The case of $s_i = s_{i+1} = 1$ is clear (simply considering the neighbors of x_{i+1}). Suppose now that $s_i = 0$ and $s_{i+1} = 1$. Considering the neighbors of x_{i+1} and y_{i+1} we have that $\ell(x_i) + \ell(y_{i+1}) + \ell_{i+2} = 0 = \ell(y_i) + \ell(x_{i+1}) + \ell_{i+2}$. This yields $\ell_i + \ell_{i+1} + 2\ell_{i+2} = 0$, and therefore $\ell_{i+2} = -\frac{1}{2}(\ell_i + \ell_{i+1})$. The case of $s_i = 1$, $s_{i+1} = 0$ is analogous and is left to the reader. □

Proposition 4.3. *Let $\Gamma = QW(S)$ be a distance magic QW-graph and let ℓ be a corresponding distance magic labeling. Then all of the segments of Γ are of odd length. Moreover, the number of segments whose length is congruent to 1 modulo 4 is even.*

Proof. Let $k \in \mathbb{Z}_m$ be such that $|\ell_k| \geq |\ell_t|$ for all $t \in \mathbb{Z}_m$. Note that Lemma 4.1 implies that $\ell_k \neq 0$. Let i, j be the smallest integers such that $i \geq 1$, $j \geq 0$ and $s_{k-i} = s_{k+j} = 0$. Therefore, the subgraph induced on $B_{k-i+1} \cup B_{k-i+2} \cup \cdots \cup B_{k+j}$ is a segment (of length $i+j$) containing the block B_k having the maximum absolute value of a block label among all blocks of Γ. Let $a = \ell_{k-i+1}$ and $b = \ell_{k-i+2}$. Since $s_{k-i+r} = 1$ for all r with $1 \leq r < i+j$, Lemma 4.2 implies that $\ell_{k-i} = -a-2b$ and that

$$\ell_{k-i+r} = \begin{cases} -b\colon & r \equiv 0 \pmod 4 \\ a\colon & r \equiv 1 \pmod 4 \\ b\colon & r \equiv 2 \pmod 4 \\ -a\colon & r \equiv 3 \pmod 4 \end{cases} \tag{4.2}$$

for all r with $1 \leq r \leq i+j$ (see Figure 2). Moreover,

$$\ell_{k+j+1} = \begin{cases} 2a+b\colon & i+j \equiv 0 \pmod 4 \\ 2b-a\colon & i+j \equiv 1 \pmod 4 \\ -2a-b\colon & i+j \equiv 2 \pmod 4 \\ -2b+a\colon & i+j \equiv 3 \pmod 4. \end{cases} \tag{4.3}$$

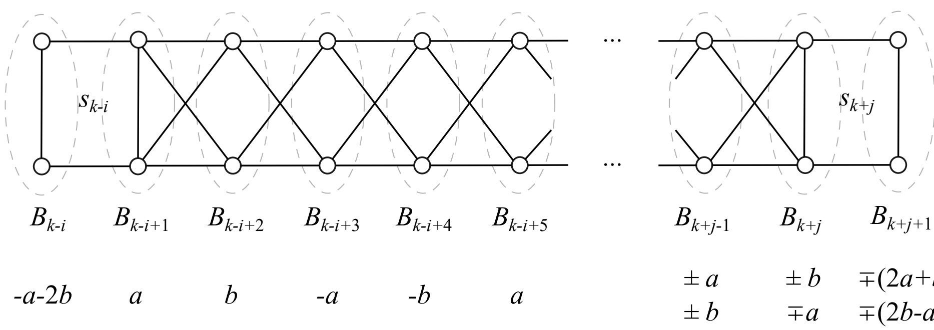

Figure 2: Block labels of $QW(S)$.

Suppose $i+j$ is even. Then (4.3) implies that $|\ell_{k+j+1}| = |2a+b|$, and so

$$\max\{|a|, |b|\} = |\ell_k| \geq \max\{|\ell_{k-i}|, |\ell_{k+j+1}|\} = \max\{|a+2b|, |2a+b|\}$$

implies that $b = -a$, contradicting Lemma 4.1. Therefore, $i+j$ is odd, and so (4.3) implies that $|\ell_{k+j+1}| = |2b-a|$. It follows that

$$\max\{|a|, |b|\} = |\ell_k| \geq \max\{|\ell_{k-i}|, |\ell_{k+j+1}|\} = \max\{|2b+a|, |2b-a|\}.$$

Therefore $|a| > |b|$, and consequently $b = 0$. Moreover, (4.2) and (4.3) imply that if $i+j \equiv 1 \pmod 4$ then $\ell_{k+j} = a$ and $\ell_{k+j+1} = -a$, while if $i+j \equiv 3 \pmod 4$ then

$\ell_{k+j} = -a$ and $\ell_{k+j+1} = a$. Note that this implies that the sum of labels of all vertices of this segment equals 0 if $i + j \equiv 3 \pmod 4$ and a otherwise. Moreover, (4.3) implies that the next segment also contains a block having the maximum absolute value of a block label among all blocks of Γ, and so an inductive approach shows that all segments are of odd length, as claimed. Since Γ is distance magic, the sum of labels of all vertices of the graph is 0, showing that the number of segments whose length is congruent to 1 modulo 4 is even. □

5 A sufficient condition

In this section we show that the condition from Proposition 4.3 is in fact sufficient for a QW-graph to be distance magic. We first introduce the following two terms. Let $\Gamma = QW(S)$ be a QW-graph. A segment of Γ is said to be of *type A* if its length is congruent to 3 modulo 4, and is said to be of *type B* if its length is congruent to 1 modulo 4. The next result shows that if Γ satisfies the condition of Proposition 5.1, then it is distance magic. This is done by providing a particular distance magic labeling for such graphs. To simplify the notation when working with pairs of integers we make the convention of writing the pair $\langle -x, -y\rangle$ as $(-1)\langle x, y\rangle$ and the pair $\langle x + x', y + y'\rangle$ as $\langle x, y\rangle + \langle x', y'\rangle$.

Proposition 5.1. *Let $\Gamma = QW(S)$ be a QW-graph. If Γ only has segments of types A and B and has an even number of segments of type B, then Γ is distance magic.*

Proof. Suppose all segments of Γ are of types A and B and there is an even number of segments of type B. Let $m = |S|$ and let t be the number of segments of Γ. By definition there exist integers $k_1, k_2, \dots, k_t$, where $0 = k_1 < k_2 < k_3 < \cdots < k_t < m - 1$, such that for each j with $0 \le j < m$, we have that $s_j = 0$ if and only if $j \in \{k_1, k_2, \dots, k_t\}$.

For each i with $1 \le i \le t$ we call the segment containing all the blocks B_j with $k_i + 1 \le j \le k_{i+1}$ the *i-th segment* of Γ (with the understanding that $k_{t+1} = m$) and we let b_i be the number of segments of type B preceding the i-th segment where by definition $b_1 = 0$. Furthermore, we let Γ_i be the subgraph of Γ induced on $B_{k_i} \cup B_{k_i+1} \cup \cdots \cup B_{k_{i+1}-1}$ (see Figure 3).

We now assign labels to the vertices of Γ as follows (see Figure 4 for an example of such a labeling). For each i with $1 \le i \le t$ we first let

$$\langle \ell(x_{k_i}), \ell(y_{k_i})\rangle = (-1)^{b_i}(\langle 2k_i, -2k_i\rangle + \langle 1, -3\rangle), \tag{5.1}$$

$$\langle \ell(x_{k_i+1}), \ell(y_{k_i+1})\rangle = (-1)^{b_i}(\langle 2k_i, -2k_i\rangle + \langle 3, -1\rangle), \tag{5.2}$$

and

$$\langle \ell(x_{k_i+j}), \ell(y_{k_i+j})\rangle = (-1)^{b_i}(\langle 2k_i, -2k_i\rangle + \langle \alpha, -\beta\rangle), \tag{5.3}$$

for each j with $2 \le j \le k_{i+1} - k_i - 3$, where

$$\langle \alpha, \beta\rangle = \begin{cases} \langle 2j+3,\ 2j+3\rangle : & j \equiv 0 \pmod 4 \\ \langle 2j-1,\ 2j-3\rangle : & j \equiv 1 \pmod 4 \\ \langle 2j+1,\ 2j+1\rangle : & j \equiv 2 \pmod 4 \\ \langle 2j+1,\ 2j+3\rangle : & j \equiv 3 \pmod 4. \end{cases} \tag{5.3*}$$

In addition, for the i with $1 \le i \le t$ for which the i-th segment is of type B and b_i is even, we let $i' > i$ be the smallest positive integer such that the i'-th segment is also of type B, and we set

$$\langle \ell(x_{k_{i'+1}-2}), \ell(y_{k_{i'+1}-2}) \rangle = \langle 2k_{i+1}, -2k_{i+1} \rangle + \langle -1, 3 \rangle \,, \tag{5.4}$$

$$\langle \ell(x_{k_{i+1}-2}), \ell(y_{k_{i+1}-2}) \rangle = \langle 2k_{i+1}, -2k_{i+1} \rangle + \langle -3, 1 \rangle \,, \tag{5.5}$$

and

$$\langle \ell(x_{k_{i+1}-1}), \ell(y_{k_{i+1}-1}) \rangle = \langle 2k_{i'+1}, -2k_{i'+1} \rangle + \langle -3, 3 \rangle \,. \tag{5.6}$$

Note that with (5.4), (5.5) and (5.6) we labeled four vertices of the i-th segment and two vertices of the i'-th segment.
Next, for the i with $1 \le i \le t$ for which the i-th segment is of type A and of length greater than three we set

$$\langle \ell(x_{k_{i+1}-2}), \ell(y_{k_{i+1}-2}) \rangle = (-1)^{b_i}(\langle 2k_{i+1}, -2k_{i+1} \rangle + \langle -5, 7 \rangle). \tag{5.7}$$

Finally, for the i with $1 \le i \le t$ for which the i-th segment is of type B with b_i odd or of type A we set

$$\langle \ell(x_{k_{i+1}-1}), \ell(y_{k_{i+1}-1}) \rangle = \langle 2k_{i+1}, -2k_{i+1} \rangle + \langle -1, 1 \rangle \,. \tag{5.8}$$

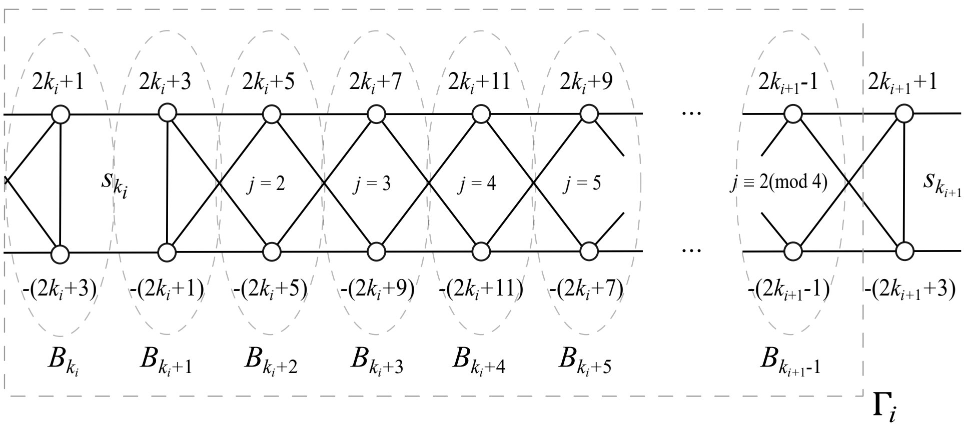

Figure 3: Labeling of segments of type A (of length greater than 3) preceded by an even number of segments of type B.

In order to prove that ℓ is a distance magic labeling of Γ we proceed by proving two claims.

Claim 1. The weight of each vertex of Γ is equal to 0.

Let $i \in \{1, \dots, t\}$. Considering the labels of vertices of any subgraph Γ_i, for each j with $2 \le j \le k_{i+1} - k_i - 1$ we have that $N(x_{k_i+j}) = N(y_{k_i+j})$. To see that these two vertices have their weight equal to 0 it is thus sufficient to consider the block labels ℓ_{k_i+j-1} and ℓ_{k_i+j+1}. Note that the block label of each block in Γ_i is 0, -2, or 2. Namely, (5.1) and (5.2) imply that

$$\ell_{k_i} = -2(-1)^{b_i} \quad \text{and} \quad \ell_{k_{i+1}} = 2(-1)^{b_i}, \tag{5.9}$$

and for each j with $2 \le j \le k_{i+1} - k_i - 3$, we have by (5.3) and (5.3*) that

$$\ell_{k_i+j} = \begin{cases} 0: & j \equiv 0 \pmod 2 \\ 2(-1)^{b_i}: & j \equiv 1 \pmod 4 \\ -2(-1)^{b_i}: & j \equiv 3 \pmod 4. \end{cases} \tag{5.10}$$

Additionally, (5.6) and (5.8) imply that

$$\ell_{k_{i+1}-1} = 0 \tag{5.11}$$

and finally for segments of length greater than three (5.4), (5.5) and (5.7) imply that

$$\ell_{k_{i+1}-2} = \begin{cases} 2(-1)^{b_i}: & \text{the } i\text{-th segment is of type A} \\ -2(-1)^{b_i}: & \text{the } i\text{-th segment is of type B.} \end{cases} \tag{5.12}$$

We now verify that for each j with $0 \le j \le k_{i+1} - k_i - 1$ we indeed have that $w(x_{k_i+j}) = w(y_{k_i+j}) = 0$. For j with $3 \le j \le k_{i+1} - k_i - 4$ this follows from (5.10).
Suppose that $j \in \{0, 1\}$. By (5.1) and (5.2) we get that $\ell(x_{k_i}) + \ell(y_{k_i+1}) = 0 = \ell(y_{k_i}) + \ell(x_{k_i+1})$. The first two vertices belong to $N(x_{k_i+1})$ and $N(y_{k_i})$, while the second two belong to $N(x_{k_i})$ and $N(y_{k_i+1})$. Moreover, by (5.10) and (5.11) we have that $\ell_{k_i-1} = \ell_{k_i+2} = 0$ which means that the vertices $x_{k_i}, y_{k_i}, x_{k_i+1}$ and y_{k_i+1} all have weight 0.
Suppose that the i-th segment is of length three. Then $b_{i+1} = b_i$ (unless $i = t$ in which case b_i is even), and so (5.9) implies that $w(x_{k_i+2}) = w(y_{k_i+2}) = 0$ which in turn implies that all vertices of Γ_i have weight 0. From now on we thus assume that the i-th segment is of length at least 5, that is $k_{i+1} - k_i \ge 5$ (where we set $k_{t+1} = m$ if $i = t$).
For $j = k_{i+1} - k_i - 3$ we consider two cases. If the i-th segment is of type A, then $j \equiv 0 \pmod 4$ and $j \ge 4$. Hence, (5.10) and (5.12) imply that $\ell_{k_i+j-1} = -2(-1)^{b_i}$ and $\ell_{k_i+j+1} = 2(-1)^{b_i}$. Otherwise, $j \equiv 2 \pmod 4$ (recall that the segments are of odd length) and by (5.12), $\ell_{k_i+j+1} = -2(-1)^{b_i}$. Moreover, by (5.9) or (5.10) (depending on whether the i-th segment is of length 5 or more, respectively) we have that $\ell_{k_i+j-1} = 2(-1)^{b_i}$. In both cases, the corresponding sum is 0.
To complete the case $j = 2$ it suffices to consider the segments of length greater than 5. In that case (5.9) and (5.10) imply that $w(x_{k_i+2}) = w(y_{k_i+2}) = 0$.
Similarly, for $j = k_{i+1} - k_i - 2$, (5.10) and (5.11) imply that $\ell_{k_i+j-1} = \ell_{k_i+j+1} = 0$.
Finally, let $j = k_{i+1} - k_i - 1$. If the i-th segment is of type A, then $b_{i+1} = b_i$ (unless $i = t$ in which case b_i is even), and so (5.9) implies that $\ell_{k_{i+1}} = -2(-1)^{b_i}$, while (5.12) implies that $\ell_{k_{i+1}-2} = 2(-1)^{b_i}$. On the other hand, if the i-th segment is of type B, then $b_{i+1} = b_i + 1$ (unless $i = t$ in which case b_i is odd), and so (5.9) implies that $\ell_{k_{i+1}} = 2(-1)^{b_i}$, while (5.12) implies that $\ell_{k_{i+1}-2} = -2(-1)^{b_i}$. In both cases, the corresponding sum is 0. This finally proves Claim 1.

We now prove that the described labeling ℓ is a bijective mapping from $V(\Gamma)$ to $\{1 - n, 3 - n, \ldots, n - 1\}$, where $n = 2m$. It is clear that we lose nothing by exchanging a few labels from one segment with a few labels from another segment. To simplify this part of the proof, we do just that by letting $\tilde{\ell}$ be the labeling obtained from ℓ as follows. For each i with $1 \le i \le t$ such that the i-th segment is of type B with b_i even, let $i' > i$ be the smallest positive integer such that the i'-th segment is also of type B. Then $\tilde{\ell}$ on these two

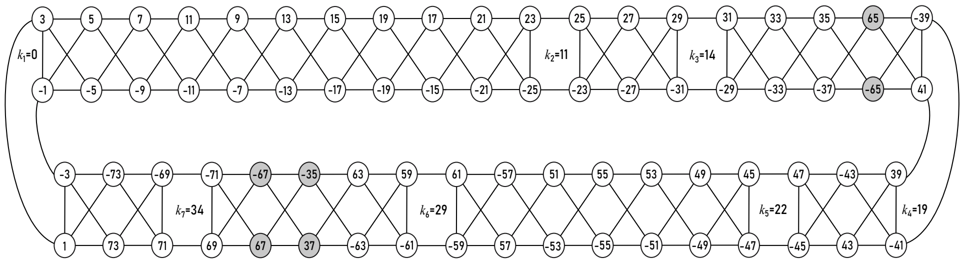

Figure 4: A distance magic labeling of $QW(11,3,5,3,7,5,3)$ $(m=37)$ consisting of 5 segments of type A and 2 segments of type B, as defined in (5.1) – (5.8).

segments is defined precisely as ℓ, except on the blocks $B_{k_{i+1}-1}$ and $B_{k_{i'+1}-2}$, for which we exchange the labels. In particular, instead of (5.4) and (5.6) we let

$$\langle\tilde{\ell}(x_{k_{i'+1}-2}),\tilde{\ell}(y_{k_{i'+1}-2})\rangle = \langle 2k_{i'+1},-2k_{i'+1}\rangle + \langle -3,3\rangle\,, \tag{5.4*}$$

$$\langle\tilde{\ell}(x_{k_{i+1}-1}),\tilde{\ell}(y_{k_{i+1}-1})\rangle = \langle 2k_{i+1},-2k_{i+1}\rangle + \langle -1,3\rangle\,. \tag{5.6*}$$

Claim 2. For each $i\in\{1,\dots,t\}$ the labeling $\tilde{\ell}$ maps $V(\Gamma_i)$ bijectively to

$$\{1-2k_{i+1},3-2k_{i+1},\dots,-2k_i-1\}\cup\{2k_i+1,2k_i+3,\dots,2k_{i+1}-1\}.$$

Note that by definition of $\tilde{\ell}$ it is clear that for each i the smallest absolute value of labels of vertices of Γ_i is $2k_i+1$ and belongs to the vertices x_{k_i} and y_{k_i+1}.
Suppose first that the i-th segment is of length 3. Then (5.1), (5.2) and (5.8) imply that Claim 2 clearly holds for Γ_i. Suppose now that the i-th segment is of length at least 5. Computation of $\langle\alpha,\beta\rangle$ in (5.3*) modulo 8 yields

$$\langle\alpha,\beta\rangle \equiv \begin{cases} \langle 3 \pmod 8,\ 3 \pmod 8\rangle: & j\equiv 0 \pmod 4\\ \langle 1 \pmod 8,\ 7 \pmod 8\rangle: & j\equiv 1 \pmod 4\\ \langle 5 \pmod 8,\ 5 \pmod 8\rangle: & j\equiv 2 \pmod 4\\ \langle 7 \pmod 8,\ 1 \pmod 8\rangle: & j\equiv 3 \pmod 4, \end{cases}$$

where $2\le j\le k_{i+1}-k_i-3$. Thus, the labels of the vertices from each of the sets $\{x_{k_i+j}\colon 2\le j\le k_{i+1}-k_i-3\}$ and $\{y_{k_i+j}\colon 2\le j\le k_{i+1}-k_i-3\}$ are pairwise distinct. Moreover, since the labels of the vertices from one set are positive while those of the other are negative, all of these labels are pairwise distinct. Furthermore, since we require in (5.3) that $j\ge 2$, we have by (5.3*) that $\alpha,\beta\ge 5$, and so for each i the absolute value of each label from (5.3) is greater than the absolute value of any label from (5.1) and (5.2). Hence, since $k_{i+1}-k_i$ is odd, (5.3*) implies that for all j with $0\le j\le k_{i+1}-k_i-4$ we have that $|\tilde{\ell}(x_{k_{i+1}-3})|,|\tilde{\ell}(y_{k_{i+1}-3})| > |\tilde{\ell}(x_{k_i+j})|,|\tilde{\ell}(y_{k_i+j})|$.
We now compare the labels of the vertices of $B_{k_{i+1}-2}$ and $B_{k_{i+1}-1}$ with the labels of the other vertices of Γ_i. Note that $k_{i+1}-k_i-3$ is congruent to 0 or 2 modulo 4, depending on whether the i-th segment is of type A or B, respectively. Thus (5.3) implies that

$$\langle\tilde{\ell}(x_{k_{i+1}-3}),\tilde{\ell}(y_{k_{i+1}-3})\rangle = (-1)^{b_i}(\langle 2k_{i+1}-3,-2k_{i+1}+3\rangle),\ \text{or} \tag{5.13}$$

$$\langle \tilde{\ell}(x_{k_{i+1}-3}), \tilde{\ell}(y_{k_{i+1}-3})\rangle = (-1)^{b_i}(\langle 2k_{i+1} - 5, -2k_{i+1} + 5\rangle), \tag{5.14}$$

depending on whether the i-th segment is of type A or B, respectively. Suppose first, that the i-th segment is of type A. Then (5.3) implies that

$$\langle \tilde{\ell}(x_{k_{i+1}-4}), \tilde{\ell}(y_{k_{i+1}-4})\rangle = (-1)^{b_i}(\langle 2k_{i+1} - 7, -2k_{i+1} + 5\rangle), \tag{5.15}$$

and that $|\tilde{\ell}(x_{k_{i+1}-4})|, |\tilde{\ell}(y_{k_{i+1}-4})| > |\tilde{\ell}(x_{k_i+j})|, |\tilde{\ell}(y_{k_i+j})|$ for all j with $0 \leq j \leq k_{i+1} - k_i - 5$. It follows from (5.7), (5.8), (5.13) and (5.15) that all the labels (according to $\tilde{\ell}$) of the vertices of Γ_i are pairwise distinct and that the largest absolute value of a label in Γ_i is $2k_{i+1} - 1$. We are left with the possibility that the i-th segment is of type B. Depending on whether b_i is even or odd, (5.5), (5.6*) and (5.14), or (5.4*), (5.8) and (5.14), respectively, imply that all the labels of the vertices of Γ_i are pairwise distinct and that the largest absolute value of a label in Γ_i is $2k_{i+1} - 1$. Therefore, Claim 2 indeed holds for all i with $1 \leq i \leq t$.

Note that Claim 2 implies that $\tilde{\ell}$ is a one to one correspondence from $V(\Gamma)$ to $\{1 - 2m, 3 - 2m, \ldots, 2m - 1\}$. Combining this with Claim 1 we finally have that ℓ is a distance magic labeling of Γ. □

6 Concluding remarks

Recall that $QW(3)$ and $W(3)$ are isomorphic. However, for each $n \geq 4$ the wreath graph $W(n)$ has no 3-cycles, while each (distance magic) QW-graph of order $2n$ has 3-cycles (for instance, (x_1, x_2, y_1)). This shows that Theorem 1.1 provides an infinite family of tetravalent distance magic graphs which are not wreath graphs. In fact, it is easy to see that for each even order $n \geq 18$, there exists at least one distance magic QW-graph of order n. We mention that the above argument can also be used to verify that with the exception of the examples of the form $QW(2, 2, \ldots, 2)$, all QW-graphs of order at least 8 (not just the distance magic ones) are not vertex-transitive. The family of QW-graphs thus provides infinitely many tetravalent distance magic graphs which are not vertex-transitive.

A computer search reveals that out of 8 037 418 connected tetravalent graphs of order 16 only two are not ruled out by Corollary 3.2 as candidates for being distance magic. One of them is of course $W(8)$ and is thus distance magic. The other one turns out to also be distance magic but does not belong to the family of quasi wreath graphs (by Proposition 4.3). Intriguingly, this "mysterious" graph can be derived from $QW(7)$ using a construction from [6] as follows. The proof of [6, Lemma 2.1] shows that if Γ is a tetravalent distance magic graph of order n admitting a distance magic labeling (according to our definition) such that for some 4-cycle C of Γ the sum of the labels of each pair of antipodal vertices is 0, then one can construct a tetravalent distance magic graph Γ' of order $n + 2$ from Γ by deleting the edges of C, adding two new vertices and joining them to each vertex of C. Applying this construction to the distance magic labeling of $QW(7)$ from Figure 1 and the 4-cycle (x_3, x_4, y_5, y_4) we thus obtain a tetravalent distance magic graph of order 16 which is clearly not a wreath graph (see Figure 5). It thus must be the above mentioned "mysterious" distance magic graph of order 16.

Of course, the construction from [6] can be used on any suitable 4-cycle also in larger distance magic quasi wreath graphs. However, the resulting graph of course depends on which suitable 4-cycle we choose and can sometimes be a quasi wreath graph or even a wreath graph. For instance, applying this construction to the 4-cycle (x_0, x_1, y_1, y_0) and the labeling of $QW(7)$ from Figure 1 we obtain the wreath graph $W(8)$.

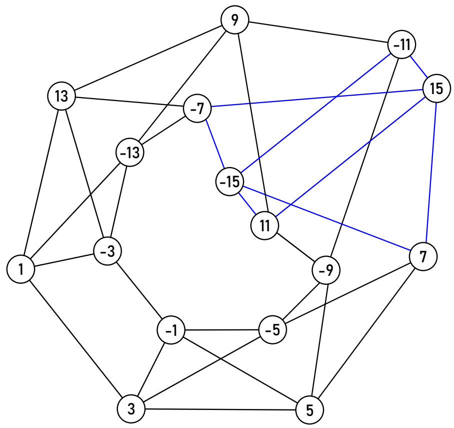

Figure 5: Applying the construction from [6] to $QW(7)$ we obtain a new distance magic graph of order 16.

Finally, an exhaustive computer search reveals that there are 4 and 21 connected tetravalent distance magic graphs of orders 17 and 18, respectively, which are neither wreath graphs nor quasi wreath graphs. It would therefore be interesting to further explore what kind of tetravalent distance magic graphs one can obtain by applying the construction from [6] (and similar other constructions) to wreath and quasi wreath graphs.

ORCID iDs

Primož Šparl https://orcid.org/0000-0002-1760-9475

References

[1] M. Anholcer, S. Cichacz, I. Peterin and A. Tepeh, Distance magic labeling and two products of graphs, *Graphs Comb.* **31** (2015), 1125–1136, doi:10.1007/s00373-014-1455-8, https://doi.org/10.1007/s00373-014-1455-8.

[2] S. Cichacz and D. Froncek, Distance magic circulant graphs, *Discrete Math.* **339** (2016), 84–94, doi:10.1016/j.disc.2015.07.002, https://doi.org/10.1016/j.disc.2015.07.002.

[3] K. Coolsaet, S. D'hondt and J. Goedgebeur, House of graphs 2.0: A database of interesting graphs and more, *Discrete Appl. Math.* **325** (2023), 97–107, doi:10.1016/j.dam.2022.10.013, https://doi.org/10.1016/j.dam.2022.10.013.

[4] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Comb.* (2022), DS6, doi:10.37236/27, https://doi.org/10.37236/27.

[5] A. Godinho and T. Singh, Some distance magic graphs, *AKCE Int. J. Graphs Comb.* **15** (2018), 1–6, doi:10.1016/j.akcej.2018.02.004, https://doi.org/10.1016/j.akcej.2018.02.004.

[6] P. Kovář, D. Fronček and T. Kovářová, A note on 4-regular distance magic graphs, *Australas. J. Comb.* **54** (2012), 127–132, https://ajc.maths.uq.edu.au/pdf/54/ajc_v54_p127.pdf.

[7] M. Meringer, Fast generation of regular graphs and construction of cages, *J. Graph Theory* **30** (1999), 137–146, doi:10.1002/(SICI)1097-0118(199902)30:2⟨137::AID-JGT7⟩3.0.CO;2-G, https://doi.org/10.1002/(SICI)1097-0118(199902)30:2<137::AID-JGT7>3.0.CO;2-G.

[8] Š. Miklavič and P. Šparl, Classification of tetravalent distance magic circulant graphs, *Discrete Math.* **344** (2021), 112557, doi:10.1016/j.disc.2021.112557, https://doi.org/10.1016/j.disc.2021.112557.

[9] Š. Miklavič and P. Šparl, On distance magic circulants of valency 6, *Discrete Appl. Math.* **329** (2023), 35–48, doi:10.1016/j.dam.2022.12.024, https://doi.org/10.1016/j.dam.2022.12.024.

[10] M. Miller, C. Rodger and R. Simanjuntak, Distance magic labelings of graphs, *Australas. J. Combin.* **28** (2003), 305–315, https://ajc.maths.uq.edu.au/pdf/28/ajc_v28_p305.pdf.

[11] S. B. Rao, Sigma graphs - a survey, in: B. D. Acharya, S. Arumugam and A. Rosa (eds.), *Labelings of Discrete Structures and Applications*, Narosa Publishing House, 2008 pp. 135–140.

[12] K. Rozman and P. Šparl, Distance magic labelings of Cartesian products of cycles, *Discrete Math.* **347** (2024), 114125, doi:10.1016/j.disc.2024.114125, https://doi.org/10.1016/j.disc.2024.114125.

[13] K. A. Sugeng, D. Froncek, M. Miller, J. Ryan and J. Walker, On distance magic labeling of graphs, *J. Comb. Math. Comb. Comput.* **71** (2009), 39–48.

[14] V. Vilfred, *$\sum$-labelled graph and Circulant Graphs*, Ph.D. thesis, University of Kerala, Trivandrum, India, 1994.