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A note on quotients of strongly regular graphs

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Abstract

We give examples of vertex-transitive strongly regular graphs with a normal quotient which is neither complete nor strongly regular.

Keywords: Strongly regular graph.

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1 Results

A *strongly regular graph* is a graph that is not complete and for which each vertex has valency k and there exist integers λ, μ such that each pair of adjacent vertices have λ common neighbours and each pair of non-adjacent vertices have μ common neighbours. Such a graph is usually denoted by $srg(v, k, \lambda, \mu)$ where v is the number of vertices.

One common method for studying graphs is by taking quotients. Given a partition $\mathcal{B}$ of the vertex set of a graph Γ, the *quotient graph* is the graph whose vertices are the parts of the partition $\mathcal{B}$ and two parts B_1 and B_2 are joined by an edge if there exist $v \in B_1$ and $w \in B_2$ such that v is adjacent to w in the original graph Γ. When $\mathcal{B}$ is the set of orbits of a normal subgroup N of some group G of automorphisms of Γ we denote the quotient by Γ_N and refer to it as a *normal quotient*. It was shown in [5] that if Γ is a strongly regular graph with a group of automorphisms G which acts transitively on the vertex set and edge set of Γ then for a nontrivial normal subgroup N of G, the normal quotient Γ_N is either a complete graph or a strongly regular graph. The purpose of this note is to show that edge-transitivity is indeed required.

In Example 1.1, we provide a vertex-transitive, edge-intransitive strongly regular graph Γ where we take G to be the full automorphism group and obtain a normal quotient which is neither strongly regular nor complete. The graph Γ also has the following interesting properties:

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$$
\begin{pmatrix}
0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0 \\
1,0,1,1,1,1,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0 \\
1,1,0,1,0,0,1,1,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0,0,0 \\
1,1,1,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,0,0,0,0,0,1,1,0,0,0,0,0,1,1,1,1,1 \\
1,1,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,0,0,0,1,0,1,1,1,0,0,1,1,1,0,0 \\
1,1,0,0,1,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,1,1,0,0,1,1,0,0,1,1,1,0,0,1,1 \\
1,0,1,0,0,0,0,0,1,0,1,0,0,0,1,0,0,1,0,1,0,1,1,0,0,0,0,1,1,1,0,1,0,0,1,0 \\
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1,0,0,1,0,0,0,0,0,0,1,0,1,1,0,1,0,0,1,1,0,0,0,1,0,0,0,1,1,0,1,1,0,0,0,1 \\
1,0,0,0,1,0,1,1,0,1,0,0,0,0,0,0,0,1,0,0,1,0,1,1,1,0,0,0,0,1,1,0,1,0,0,1 \\
1,0,0,0,1,0,0,1,0,0,0,0,0,1,1,0,1,1,1,0,0,1,0,0,0,1,0,1,1,0,0,0,1,0,0,1 \\
1,0,0,0,0,1,0,1,1,1,0,0,0,0,0,0,1,0,0,0,1,0,1,1,0,1,0,1,1,0,0,0,0,1,1,0 \\
1,0,0,0,0,1,0,0,0,1,0,1,0,0,1,1,1,1,0,1,0,0,0,0,1,0,0,0,0,1,1,0,0,1,1,0 \\
1,0,0,0,0,0,1,0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,1,1,1,1,1,0,0,0,0,1,0,0,0,0 \\
0,1,1,0,0,0,0,1,0,1,0,0,0,1,1,0,0,0,0,1,1,0,1,0,0,0,1,1,0,0,0,0,1,0,1,1 \\
0,1,1,0,0,0,0,0,1,0,0,1,1,1,0,0,0,1,0,0,1,0,0,1,0,0,0,1,0,1,1,1,1,0,0,0 \\
0,1,0,1,0,0,1,0,0,0,1,1,0,1,0,0,1,0,0,0,0,0,1,1,0,0,1,0,1,0,0,0,0,1,1,1 \\
0,1,0,1,0,0,0,1,0,1,0,1,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,1,1,1,1,0,1,0,0 \\
0,1,0,0,1,0,1,0,1,1,0,0,0,1,0,1,0,0,0,0,0,1,0,1,0,1,0,0,1,1,0,0,1,1,0,0 \\
0,1,0,0,1,0,0,0,1,0,1,0,1,0,1,1,1,0,1,0,0,0,0,0,1,0,0,0,1,1,0,0,0,0,1,1 \\
0,1,0,0,0,1,1,1,1,0,0,1,0,0,0,0,0,0,1,1,0,0,0,1,1,0,0,1,0,0,1,0,0,0,1,1 \\
0,1,0,0,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,0,0,0,0,0,0,1,0,1,0,0,1,0,1,1,0,0 \\
0,1,0,0,0,0,0,1,0,1,1,0,1,0,1,0,1,1,0,1,0,1,0,0,1,1,1,0,0,0,0,1,0,0,0,0 \\
0,0,1,1,1,0,0,0,0,0,1,0,0,1,1,0,0,0,0,0,1,1,0,1,0,1,0,1,0,0,1,0,0,1,1,0 \\
0,0,1,1,0,1,0,0,0,0,0,1,1,0,1,0,0,0,0,1,0,0,1,1,1,0,0,0,1,1,0,0,1,0,0,1 \\
0,0,1,0,1,1,0,1,1,0,0,0,0,0,1,1,0,1,0,0,0,0,0,1,0,0,0,0,1,0,1,1,0,1,0,1 \\
0,0,1,0,1,0,1,0,0,1,0,1,1,0,0,1,1,0,0,0,0,1,1,0,1,0,0,0,0,0,0,1,0,1,0,1 \\
0,0,1,0,1,0,1,0,0,1,0,1,1,0,0,0,0,1,1,1,1,0,0,0,0,1,1,0,0,0,1,0,0,0,1,0 \\
0,0,1,0,0,1,1,1,0,0,1,0,0,1,0,0,1,0,1,1,1,0,0,0,0,1,0,0,0,0,0,1,0,1,0,1 \\
0,0,1,0,0,1,0,0,1,1,1,0,0,1,0,0,1,0,1,0,0,1,1,0,1,0,1,0,1,0,0,0,1,0,0,0 \\
0,0,0,1,1,1,1,0,0,1,0,0,0,0,1,0,1,0,1,0,0,0,0,1,0,0,1,1,0,1,0,0,1,0,1,0 \\
0,0,0,1,1,0,0,1,1,0,1,1,0,0,0,1,1,0,0,1,0,0,1,0,0,1,0,0,0,0,1,1,0,0,1,0 \\
0,0,0,1,1,0,0,1,1,0,0,0,1,1,0,0,0,1,1,1,0,0,1,0,1,0,1,1,0,1,0,0,0,0,0,0 \\
0,0,0,1,0,1,1,1,0,0,0,0,1,1,0,1,0,1,0,0,1,1,0,0,1,0,0,0,1,0,0,1,1,0,0,0 \\
0,0,0,1,0,1,0,0,1,1,1,1,0,0,0,1,0,1,0,0,1,1,0,0,0,1,1,1,0,1,0,0,0,0,0,0
\end{pmatrix}
$$

Figure 1: The matrix A.

1. Γ is a Cayley graph for three different isomorphism types of groups.
2. $\mathrm{Aut}(\Gamma)$ contains five conjugacy classes of regular subgroups, of which 4 are normal subgroups.
3. $\mathrm{Aut}(\Gamma)$ contains two isomorphic regular subgroups of shape $C_3^2 \rtimes C_2^2$ for which one is normal in $\mathrm{Aut}(\Gamma)$ while the other is not, that is, Γ is both a *normal Cayley graph* and a *nonnormal Cayley graph* for isomorphic groups.

Other examples of Cayley graphs that are both normal and nonnormal Cayley graphs for isomorphic groups are given in [1, 6].

In Example 1.2 we provide an infinite family of strongly regular graphs where we take G to be a vertex-transitive proper subgroup of the full automorphism group and obtain normal quotients which are neither strongly regular nor complete.

Example 1.1. Let Γ be the strongly regular graph with adjacency matrix A given in Figure 1 which has parameters $srg(36, 14, 4, 6)$. The adjacency matrix was retrieved from [7].

$$
\begin{pmatrix}
0,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0 \\
1,0,1,1,1,1,1,0,1,0,1,1,1,1,0,0,0,0 \\
1,1,0,1,1,1,1,1,0,0,0,0,1,0,1,1,1,0 \\
1,1,1,0,1,1,0,1,1,0,0,0,1,0,0,1,1,1 \\
1,1,1,1,0,1,0,0,0,1,1,0,0,1,1,1,0,1 \\
1,1,1,1,1,0,0,0,0,1,0,1,0,1,1,0,1,1 \\
1,1,1,0,0,0,0,0,1,1,1,1,0,1,1,1,1,0 \\
1,0,1,1,0,0,0,0,0,1,1,1,1,0,1,1,1,1 \\
1,1,0,1,0,0,1,0,0,1,1,1,0,1,0,1,1,1 \\
1,0,0,0,1,1,1,1,1,0,0,0,0,1,1,1,1,1 \\
1,1,0,0,1,0,1,1,1,0,0,1,1,0,1,1,0,1 \\
1,1,0,0,0,1,1,1,1,0,1,0,1,0,1,0,1,1 \\
0,1,1,1,0,0,0,1,0,0,1,1,0,1,1,1,1,1 \\
0,1,0,0,1,1,1,0,1,1,0,0,1,0,1,1,1,1 \\
0,0,1,0,1,1,1,1,0,1,1,1,1,1,0,0,0,1 \\
0,0,1,1,1,0,1,1,1,1,1,0,1,1,0,0,1,0 \\
0,0,1,1,0,1,1,1,1,1,0,1,1,1,0,1,0,0 \\
0,0,0,1,1,1,0,1,1,1,1,1,1,1,1,0,0,0
\end{pmatrix}
$$

Figure 2: The matrix B.

According to a GAP [3] calculation, $\mathrm{Aut}(\Gamma)$ equals

$$
\left\langle
\begin{array}{l}
(2,15)(3,7)(4,9)(5,12)(6,14)(8,11)(10,13)(16,23)(17,32)(18,27)(19,21)(20,26)(22,25) \\
\quad (31,35)(34,36), \\
(3,5)(4,6)(7,12)(8,11)(9,14)(10,13)(16,21)(17,20)(18,22)(19,23)(25,27)(26,32)(28,29) \\
\quad (30,33)(31,34)(35,36), \\
(1,25,32,2,26,27)(3,15,5,4,24,6)(7,21,33,18,13,31)(8,14,28,19,20,36)(9,11,35,17,23,29) \\
\quad (10,22,30,16,12,34)
\end{array}
\right\rangle
$$

which has shape $C_6^2 \rtimes C_2^2$. (By a group G of shape $H \rtimes K$ we mean that G has a normal subgroup H and a subgroup K such that $H \cap K = 1$. Since this does not specify how K acts on H there may be more than one isomorphism class of groups of a given shape.) In fact, $G \cong \mathbb{Z}_6^2 \rtimes \langle \sigma, \tau \rangle$ acting on $\mathbb{Z}_6^2$, with $\mathbb{Z}_6^2$ acting regularly on itself and $(a,b)^\tau = (-a,-b)$ and $(a,b)^\sigma = (b,a)$. Thus $\mathrm{Aut}(\Gamma)$ is vertex-transitive and is a Cayley graph for $H_1 = \mathbb{Z}_6^2$. The joining set is

$$
\{(0,5),(0,1),(0,3),(5,0),(1,0),(3,0),(1,3),(5,3),(3,1),(3,5),\\
(1,5),(5,1),(2,4),(4,2)\}.
$$

Since $\mathrm{Aut}(\Gamma)_{(0,0)} = \langle \tau, \sigma \rangle$ has five orbits on this set, $\mathrm{Aut}(\Gamma)$ has five orbits on edges.

Now $\mathrm{Aut}(\Gamma)$ has a normal subgroup N of order two generated by the element $(3,3) \in \mathbb{Z}_6^2$ and which is the centre of $\mathrm{Aut}(\Gamma)$. The group N has 18 orbits of length two on the 36 vertices of Γ and the set of neighbours of $(0,0)$ contains the three N-orbits $\{(0,3),(3,0)\}$, $\{(1,5),(5,1)\}$ and $\{(2,4),(4,2)\}$. Hence, Γ_N is a valency 11 graph on 18 vertices of diameter 2 but is not strongly regular. Indeed there are no feasible parameters for strongly regular graphs on 18 vertices which are not complete multipartite [4, p227]. The matrix B given in Figure 2 is the adjacency matrix for Γ_N.

Not only is Γ a Cayley graph for H_1, which is normal in $\mathrm{Aut}(\Gamma)$, we also have that $H_2 = \langle (2,0),(0,2),(3,0)\tau,(0,3)\tau \rangle$, $H_3 = \langle (0,2),(2,0),(3,0)\sigma \rangle$ and $H_4 = \langle (2,0),(0,2),(2,5)\sigma\tau \rangle$ are normal subgroups of $\mathrm{Aut}(\Gamma)$ that act regularly on $V\Gamma$. The subgroup H_2 has shape $C_3^2 \rtimes C_2^2$, while $H_3 \cong H_4$ have shape $C_3^2 \rtimes C_4$. Finally, $H_5 = \langle (2,0),(0,2),(1,0)\tau,(0,1) \rangle \cong H_2$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$ which is not normal. Thus Γ is a Cayley graph for three different isomorphism types of groups. A Magma [2] calculation shows that H_1, H_2, H_3, H_4 and the subgroups conjugate to H_5 are the only regular subgroups of $\mathrm{Aut}(\Gamma)$.

The automorphism group of Γ_N is isomorphic to $S_2 \times S_4 \times S_3$, which is vertex-transitive and has three orbits on edges. Note that $\mathrm{Aut}(\Gamma)/N < \mathrm{Aut}(\Gamma_N)$. The automorphism group contains 4 conjugacy classes of regular subgroups, none of which are normal in $\mathrm{Aut}(\Gamma_N)$. One class is isomorphic to $C_3^2 \times C_2$, and there are three classes of subgroups with shape $C_3^2 \rtimes C_2$, with two of the classes being isomorphic to each other. Representatives of these four conjugacy classes are H_i/N for $i = 1, 2, 3, 4$. Note that $H_2/N = H_5N/N$.

Example 1.2. Let $\Gamma = H(2, m)$, the Hamming graph with m^2 vertices and suppose that m is not a prime. Then Γ is a strongly regular graph with parameters $(m^2, 2(m-1), m-2, 2)$. Let $G = M_1 \times M_2$, with $M_1 \cong M_2 \cong C_m$, act regularly on the set of vertices of Γ. Let $N_1 \leqslant M_1$ and $N_2 \leqslant M_2$ and $N = N_1 \times N_2 \lhd G$. Consider the graph Γ_N. Then Γ_N is the cartesian product of K_r and K_k where $|M_1 : N_1| = r$ and $|M_2 : N_2| = k$. The adjacent vertices $(a, b_1), (a, b_2)$ in Γ_N have $k-2$ common neighbours, namely the vertices of the form (a, b) with $b \neq b_1, b_2$. However, the adjacent vertices $(a_1, b), (a_2, b)$ have $r-2$ common neighbours, these being the vertices of the form (a, b) with $a \neq a_1, a_2$. Hence for $r \neq k$, the graph Γ_N is not strongly regular.

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