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Edge-transitive maps of low genus*

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Abstract

Graver and Watkins classified edge-transitive maps on closed surfaces into fourteen types. In this note we study these types for maps in orientable and non-orientable surfaces of small genus, including the Euclidean and hyperbolic plane. We revisit both finite and infinite one-ended edge-transitive maps. For the finite ones we give precise description that should enable their enumeration for a given number of edges. Edge-transitive maps on surfaces with small genera are classified in the paper.

Keywords: Edge-transitive map, edge-transitive tessellation, map on surface, symmetry type graph.

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*Dedicated to Branko Grünbaum on the occasion of his 80th birthday.

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1 Introduction

In this paper we are interested in maps (see [16] or [5, Section 17.10] for definitions) whose automorphism group acts transitively on the edges. In addition to combinatorial finite maps on compact surfaces we consider also the (geometric) *tessellations* of the spherical, Euclidean or hyperbolic plane, where every automorphism can be extended to an isometry of the tessellation. We shall consider only locally finite, one-ended tessellations (see [7, 18, 3]). For multi-ended planar maps, as well as for definitions of one- and multi-ended maps, see [6]. Additionally, unless explicitly specified, we will consider *non-degenerate* maps and tessellations, i.e. maps with all valences, co-valences and Petrie valences at least 3.

An edge in a map has *edge-symbol* $(f_1 \cdot f_2;\, v_1 \cdot v_2)$ if it is incident to vertices of valence v_1 and v_2 and faces of size f_1 and f_2 (see [9]). A map is called *edge-homogeneous* if all its edges have the same edge symbol $(f_1 \cdot f_2;\, v_1 \cdot v_2)$. Note that this requires vertex valences to alternate between v_1 and v_2 around each face, and face sizes to alternate between f_1 and f_2 around each vertex. In particular, v_1, v_2, f_1, f_2 must satisfy the condition: if v_1 or v_2 is odd, then $f_1 = f_2$, and if f_1 or f_2 is odd, then $v_1 = v_2$. Call $(f_1 \cdot f_2;\, v_1 \cdot v_2)$ *admissible* if it satisfies this condition with all numbers at least 3.

An edge-transitive map is obviously edge-homogeneous. The converse is not true; for example, the map on the torus obtained by identifying opposite sides of an $m \times n$ chessboard has constant edge-symbol $(4 \cdot 4; 4 \cdot 4)$, but it is not edge-transitive when $m \neq n$, since there is no map automorphism taking 'horizontal' edges to 'vertical' edges. On the other hand we have the following geometric result of Grünbaum and Shepherd [10]:

Theorem 1.1. Given any admissible edge-symbol $(f_1 \cdot f_2;\, v_1 \cdot v_2)$, there is a unique (up to map isomorphism) *universal tessellation* $M(f_1 \cdot f_2;\, v_1 \cdot v_2)$ of the sphere, Euclidean plane, or hyperbolic plane by congruent f_1-gons and f_2-gons, such that all map automorphisms are isometries and the map is edge-transitive with the given edge-symbol. The map has reflections with respect to any line joining a vertex and the center of an incident face. If $f_1 = f_2$ it also has a reflection with respect to the lines determined by the edges and if $v_1 = v_2$ it has reflections with respect to any line connecting centers of incident faces.

The proof is mostly a matter of constructing an f_1-gon and f_2-gon with congruent sides and the appropriate internal angles. The theorem has the following consequences:

Corollary 1.2. If the finite map M is homogeneous with edge symbol $(f_1 \cdot f_2;\, v_1 \cdot v_2)$, there is a finitely generated group A of fixed-point free automorphisms of the universal tessellation $M(f_1 \cdot f_2;\, v_1 \cdot v_2)$, such that M is isomorphic to the quotient $M(f_1 \cdot f_2;\, v_1 \cdot v_2)/A$. If A is a normal subgroup of the automorphism group of $M(f_1 \cdot f_2;\, v_1 \cdot v_2)$ then the map M is edge-transitive.

Proof. The covering transformations of the universal map $M(f_1 \cdot f_2;\, v_1 \cdot v_2)$ of M are fixed-point free automorphisms. When A is normal, each automorphism of the universal map projects to an automorphism of M. □

Corollary 1.3. Any finite, edge-homogeneous map M is regularly covered by a finite edge-transitive map.

Proof. Given a map M let A be any group such that $M = M(f_1 \cdot f_2;\, v_1 \cdot v_2)/A$. The group A is determined up to conjugacy and has finite index in the automorphism group Γ of $M(f_1 \cdot f_2;\, v_1 \cdot v_2)$. Let B be the intersection of all conjugates of A in Γ. Then B is normal of finite index in Γ. It follows that $M(f_1 \cdot f_2;\, v_1 \cdot v_2)/B$ is an edge-transitive finite map regularly covering M. □

The proposition below follows directly from Theorem 1.1 and [9].

Proposition 1.4. Every edge-symbol that can be realized by a finite map on a closed surface with negative Euler characteristic is admissible for a tessellation of the hyperbolic plane.

2 Combinatorial vs. geometric edge-transitivity

The notion of edge-transitivity of maps or tessellations may be understood with two different meanings: combinatorial and geometric, depending on the choice of combinatorial automorphism or the geometric symmetry group.

Our point of departure is the combinatorial meaning where the map M is considered as an abstract collection of flags together with three fixed-point free involutions that we denote by 0,1,2 (see [5]) with the product 02 being a fixed-point free involution, too. This is equivalent to identifying the map M with its 3-edge-colored *flag graph* $G(M)$ in which the flags of M are the vertices of $G(M)$ and the edges of color i are determined by the involution i. Clearly the only axiom of maps is observed as the fact that a 3-edge-colored 3-regular graph results from a map if and only if it is connected and the 02-two-factor of $G(M)$ is 'quadrilateral', i.e. consists entirely of 4-cycles. Here automorphisms are defined as bijections of flags that preserve the colored adjacencies. However, they can also be viewed as color-preserving automorphisms of $G(M)$. This is a general description of a finite map on a closed surface. We recall the following important facts:

Theorem 2.1. Let M be a map with e edges and let Aut$M$ be its automorphism group or order $m = |\text{Aut}M|$. Let k be the number of its flag-orbits.

- There are $4e$ flags.
- For any pair of flags there is at most one automorphism mapping one flag into the other.
- Let a be a flag-orbit of M, let i be one of the three involutions and let $b = i(a)$. Then b is a flag-orbit and i restricted to a is a bijection between a and b.
- The group of automorphisms Aut$M$ partitions the set of flags into k equal sized orbits. The size of each orbit is equal to m, where $km = 4e$.

However, if we work with tessellations $\mathcal{T}$ of the sphere, Euclidean plane or hyperbolic plane we may consider the additional structure of metric space and require the edges to be given as segments of geodesics. The transitivity in this case is considered with respect to the group of isometries fixing $\mathcal{T}$. Note that any tessellation $\mathcal{T}$ of the hyperbolic plane $\mathcal{H}$ can be embedded as a combinatorial map on the Euclidean plane $\mathcal{E}$, however the group of automorphisms of $\mathcal{T}$ that extend to isometries of $\mathcal{H}$ in general does not coincide with the group of automorphisms of $\mathcal{T}$ that extend to isometries of $\mathcal{E}$.

For example, it is possible to tessellate the hyperbolic plane with equilateral triangles, seven of them around each vertex. This tessellation, denoted by $\mathcal{T}_{3,7}$, has the Coxeter group

$$[3,7] = \langle \rho_0, \rho_1, \rho_2 \,|\, (\rho_0\rho_2)^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^7 = id\rangle$$

as its symmetry group; see [11]. In particular, the group $[3,7]$ acts transitively on the faces of $\mathcal{T}_{3,7}$. It is also possible to partition the Euclidean plane into triangles, seven of them around each vertex; however, in this case the tiling cannot be normal in the sense of [10, Section 3.2] (see also [18, p. 174]), and therefore, the symmetry group cannot be transitive on the faces.

Following the discussion in the previous section, every edge-transitive map on the projective plane is the quotient of a universal edge-transitive map on the sphere by the antipodal involution; each edge-transitive map on the torus is the quotient of a universal edge-transitive map on the Euclidean plane by the group generated by two linearly independent translations; and every edge-transitive map on the Klein bottle is the quotient of a universal edge-transitive map on the Euclidean plane by the group generated by two glide reflections with congruent translation component and parallel axes. For more details see for example [10], [17] and [22].

As we shall see, there are admissible edge-symbols on the sphere and the Euclidean plane which are not admissible on the projective plane and Klein bottle, respectively. Similarly, in general admissible edge-symbols on the hyperbolic plane are not admissible on each surface with negative Euler characteristic. In the following sections we describe the cases when $\mathcal{S}$ is the sphere or the Euclidean plane. When $\mathcal{S}$ is the hyperbolic plane the problem is considerably more difficult. Part of the difficulty is explained by the next proposition.

Proposition 2.2. Each edge-symbol that is admissible for a tessellation of the hyperbolic plane can be realized by infinitely many finite maps (on some closed surfaces with negative Euler characteristic).

Proof. We recall that the isometries of the hyperbolic plane can be interpreted as matrices. As a consequence of [19, Table II], the symmetry groups of the edge-transitive tessellations of the hyperbolic plane are isomorphic to finitely generated matrix groups. In particular, the symmetry group of any edge-transitive tessellation of the hyperbolic plane is residually finite (see [13], [21, Chapter 4]).

The proposition follows from the same argument as [20, Theorem 6.3], for T given by the set of elements stabilizing a given (base) face of each size and a given (base) vertex of each size. If necessary, and depending on the edge-symbol, we may require T to contain some specific finite set of extra elements to guarantee that it is non-degenerate. □

The discussion above shows in general that for an edge-symbol $(f_1 \cdot f_2; v_1 \cdot v_2)$ to be admissible it suffices to verify whether there is a tessellation with edge-symbol $(f_1 \cdot f_2; v_1 \cdot v_2)$.

Grünbaum and Shephard [9] classified admissible edge-symbol $(f_1 \cdot f_2; v_1 \cdot v_2)$ with $f_1 \geq f_2 \geq 3, v_1 \geq v_2 \geq 3$ as follows:

1. v_1, v_2, f_1, f_2, all even.
2. $v_1 = v_2$, even, $f_1 \geq f_2$, at least one odd.
3. $f_1 = f_2$, even, $v_1 \geq v_2$, at least one odd.
4. $v_1 = v_2, f_1 = f_2$, all odd.

3 The 14 types of edge-transitive maps and symmetry type graphs

We now turn our attention to the classification of edge-transitive maps according to the type of automorphisms fixing a given vertex, edge, face or Petrie circuit, or equivalently, according to the symmetry type.

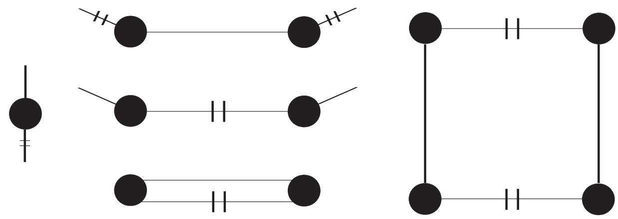

Figure 1: The edge-square and its quotients.

According to [6, 19] there are 14 types of edge-transitive maps respectively denoted by 1, 2, 2^*, 2^P, $2ex$, 2^*ex, 2^Pex, 3, 4, 4^*, 4^P, 5, 5^*, 5^P. The type containing the dual (resp. opposite) of a map with a given type C is denoted by C^* (resp. C^P). These types correspond, respectively, to 1, 2_{12}, 2_{01}, 2_1, 2_2, 2_0, 2, 4_F, 4_{Hd}, 4_H, 4_{Hp}, 4_{Gd}, 4_G, 4_{Gp} in [16].

In our approach, in order to determine the same classification of edge-transitive maps, we use a powerful combinatorial tool described next.

Let M be a map with automorphism group Aut$M$. In general the quotient $M/$Aut$M$ is no longer a map in sense of our definition. Namely, it may correspond to a topological map on a surface with boundary. The quotient $G(M)/$Aut$M$ is a 3-edge-colored pre-graph $T(M)$ that we call the *symmetry type graph* of M and the canonical projection $\pi : G(M) \to G(M)/$Aut$M$ is not only a graph morphism but is also a covering projection. (Some edges of $G(M)$ may fold to semi-edges of $T(M)$). The basics of pre-graphs and covering graphs one may find in [18]. Pre-graphs are graphs with dangling edges.

The symmetry type graphs have been introduced before with a slightly different meaning (semi-edges are replaced by loops) and in more general setting under the name of *Delaney-Dress graph* [4].

The following theorem shows how one can obtain the number and structure of vertex-, edge- or face-orbits.

Theorem 3.1. Let $T_i(M)$ denote the pre-graph obtained from the symmetry type graph $T(M)$ of a map M by removing all i-edges. The number of connected components of $T_0(M)$ represents the number of vertex orbits of M, the number of connected components of $T_1(M)$ represents the number of edge orbits of M while the number of connected components of $T_2(M)$ represents the number of face-orbits of M.

Proof. Let $G_i(M)$ denote the graph obtained from $G(M)$ by removing i-edges. Connected components of $G_i(M)$ project onto connected components of $T_i(M)$ implying that for each two components projecting onto the same component of $T_i(M)$ there is an automorphism in Aut$M$ taking one component into another, while for any two components projecting onto different components of $T_i(M)$ there is no automorphism in Aut$M$. □

The following observation follows immediately form the above theorem.

Corollary 3.2. The symmetry type graph of an edge-transitive map has 1,2, or 4 vertices.

In particular there are five admissible quotient pre-graphs $T_1(M)$ that are depicted in Figure 1. In case of edge-transitive maps these five quotient pre-graphs can be augumented to obtain symmetry type pre-graphs in exactly 14 possible ways. It may be easily shown

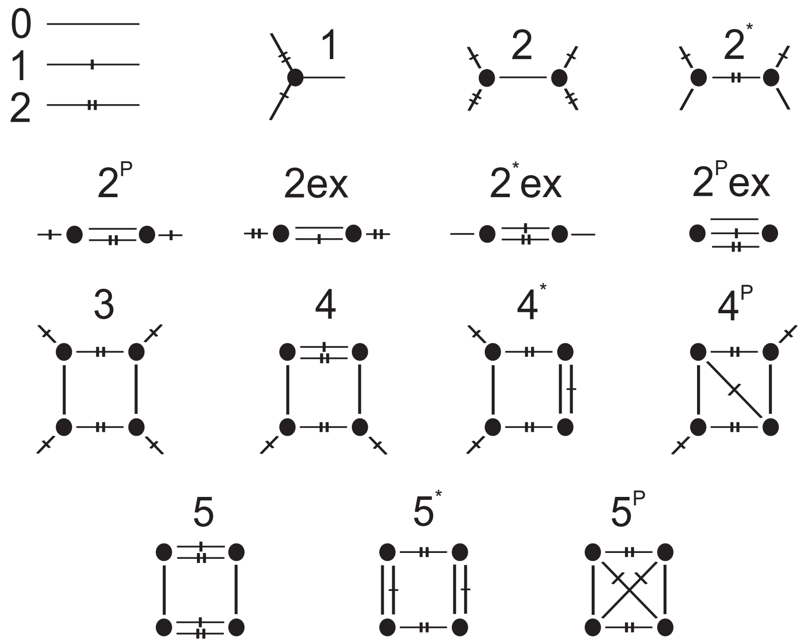

Figure 2: Symmetry type graphs for edge-transitive maps.

that these 14 symmetry pre-graphs correspond exactly to the 14 types of edge-transitive maps (see [6, 19, 16]). The pre-graphs are depicted in Figure 2.

Only type 1 consists of flag-transitive maps. However, the following types contain vertex-transitive maps: 1, 2^*, 2^P, $2ex$, 2^*ex, 2^Pex, 4^*, 4^P, 5^*, 5^P. Furthermore, types with arc-transitive maps are: 1, 2^*, 2^P, 2^*ex, 2^Pex (an arc is a pair of incident vertex and edge).

From the symmetry type graphs we can easily determine, whether a type allows self-duality of a particular map of that type. If by changing colors 0 and 2 in a type graph T we obtain a different type graph T', then the dual of a map of the initial type always changes the type and the map cannot be self-dual. On the other hand, if $T = T'$ the type allows self-dual maps. From the Figure 2 we can see that only the types 1, 2^P, 2^Pex, 3, 4^P and 5^P allow self-dual maps.

In the following sections we analyze the types and edge-symbols that appear in the tessellations and in surfaces with Euler characteristic $\chi \geq -2$. The types that appear in each surface are listed in Table 5.

4 Edge-transitive tessellations

Recall that an *edge-transitive tessellation* is a finite or locally finite (normal) map on the sphere, Euclidean plane or hyperbolic plane, such that the isometry group preserving the map acts transitively on the edges. The finite edge-transitive tessellations reside on a sphere, while the infinite ones lie either on the Euclidean or on the hyperbolic plane.

Table 1 shows the nine edge-transitive tessellations of the sphere, sorted by the number of edges. Note that they are precisely the platonic solids, their medials (Me) and the duals of the medials (Du(Me)) (see [16, Sections 2, 4.1] for the definitions of dual and medial).

f_1	f_2	v_1	v_2	**Name**	**Symbol**	**Vertices**	**Edges**	**Type**
3	3	3	3	tetrahedron	T	4	6	1
4	4	3	3	hexahedron	H	8	12	1
3	3	4	4	octahedron	$Du(H)$	6	12	1
4	3	4	4	cuboctahedron	$Me(H)$	12	24	2*
4	4	4	3	rhombic dodecahedron	$Du(Me(H))$	14	24	2
5	5	3	3	dodecahedron	D	20	30	1
3	3	5	5	icosahedron	$Du(D)$	12	30	1
5	3	4	4	icosidodecahedron	$Me(D)$	30	60	2*
4	4	5	3	rhombic triacontahedron	$Du(Me(D))$	32	60	2

Table 1: Edge-transitive maps in the sphere.

In Table 2 we present the five Euclidean cases. They coincide with the regular tessellations, the medial of one of them and the dual of this medial.

Finally we show the types of tessellations in the hyperbolic plane. They almost coincide with the four types of Grünbaum Shephard [9, Theorem 1]. There are some subtle differences. Our classification works with symmetries (Delaney-Dress symbol) while theirs work with parity (of the parameters). The classification of Grünbaum and Shephard does not take on account the transitivity of the action of the symmetry group on the vertices, edges or faces of the map. Therefore we refine this classification as follows.

1.1 $f_1 > f_2, v_1 > v_2$, all even: type 3.

1.2 $f_1 > f_2, v_1 = v_2$, all even: type 2*, vertex-transitive (and arc-transitive).

1.3 $f_1 = f_2, v_1 > v_2$, all even: type 2, face-transitive

1.4 $f_1 = f_2, v_1 = v_2$, all even: type 1, regular (reflexible).

2.1 $f_1 > f_2$, one odd, $v_1 = v_2$, even: type 2*, vertex-transitive (and arc-transitive).

2.2 $f_1 \geq f_2$, both odd, $v_1 = v_2$, even: type 2*, vertex-transitive (and arc-transitive).

f_1	f_2	v_1	v_2	**Faces**	**Notation**	**Type**	**Transitivity**
4	4	4	4	squares	Sq_∞ or $\langle 4; 4\rangle$	1	flag
6	6	3	3	hexagons	H_∞ or $\langle 3; 6\rangle$	1	flag
3	3	6	6	triangles	$Du(H_\infty)$ or $\langle 6; 3\rangle$	1	flag
6	3	4	4	triangles/hexagons	$Me(H_\infty)$ or $\langle 4; 3, 6\rangle$	2*	vertex, arc
4	4	6	3	4-gons	$Du(Me(H_\infty))$ or $\langle 3, 6; 4\rangle$	2	face

Table 2: Edge-transitive tessellations of the Euclidean plane.

f_1	f_2	v_1	v_2	Name	Notation	Type	Vertices	Edges	Graph
4	4	3	3	hemi-hexahedron	$h(H)$	1	4	6	K_4
3	3	4	4	hemi-octahedron	$h(Me(T))$	1	3	6	$K_3^{(2)}$
4	3	4	4	hemi-cuboctahedron	$h(Me(H))$	2^*	6	12	$K_{2,2,2}$
4	4	4	3	hemi rhombic dodecahedron	$Du(h(Me(H)))$	2	7	12	$K_{3,4}$
5	5	3	3	hemi-dodecahedron	$h(D)$	1	10	15	$G(5,2)$
3	3	5	5	hemi-icosahedron	$h(Du(D))$	1	6	15	K_6
5	3	4	4	hemi-icosidodecahedron	$h(Me(D))$	2^*	15	30	$L(G(5,2))$
4	4	5	3	hemi rhombic triacontahedron	$Du(h(Me(D)))$	2	16	30	

Table 3: Edge-transitive maps on the projective plane.

3.1 $f_1 = f_2$, even, $v_1 > v_2$, one odd: type 2, face-transitive.

3.2 $f_1 = f_2$, even, $v_1 \geq v_2$, both odd: type 2, face-transitive.

4.1 $f_1 = f_2, v_1 = v_2$, all odd: type 1, regular (reflexible).

In the enumeration $n.x$ above the number n is taken from the list in Section 2.

We note that type 3 does not appear in the sphere or Euclidean plane, although it does in the hyperbolic plane.

5 Edge-transitive maps on the projective plane, torus, and Klein bottle

We now investigate the compact closed surfaces with Euler characteristic $\chi \geq 0$. When possible we also provide the graph-theoretical name of the graph given by the vertices and edges of the map.

5.1 The projective plane (types: 1, 2, 2^*)

Each edge-transitive map in the projective plane is obtained from the corresponding edge-transitive map on the sphere via the antipodal covering projection. If M is a map in the sphere, we denote by $h(M)$ the antipodal quotient in the projective plane and call it "hemi-M". Out of 9 edge-transitive maps on the sphere 8 of them admit a central involution that can be used for the covering. Note that the tetrahedron is the only one not containing such an involution. Table 3 shows the eight edge-transitive maps on the projective plane.

5.2 The torus (types: 1, 2, 2^*, 2^P, 2^Pex, 5, 5^*)

In [19] all possible types of edge-transitive maps on the torus are given. The maps are divided in 14 families. Each toroidal map arises as a quotient of one of the five edge-transitive maps of the Euclidean plane: (see Section 4). By splitting 3 of 14 families in two we obtain 17 families denoted by T_1 to T_{17}. The pairs of families obtained by splitting include (T_4, T_5), (T_{12}, T_{13}) and (T_{15}, T_{16}).

Following the notation by Širáň, Tucker, Watkins [19], the families are organized as follows.

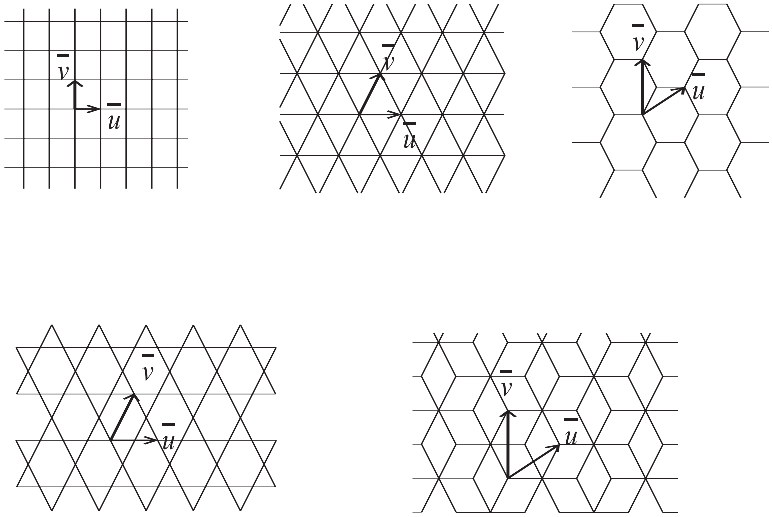

Figure 3: Generating translations.

- $\langle 4;4\rangle$ – edge-symbol $(4 \cdot 4, 4 \cdot 4)$, 5 families, 2 type 1, 2 type 2^P, 1 type $2^P ex$,
- $\langle 6;3\rangle$ – edge-symbol $(3 \cdot 3, 6 \cdot 6)$, 3 families, 2 type 1, 1 type $2^P ex$,
- $\langle 3;6\rangle$ – edge-symbol $(6 \cdot 6, 3 \cdot 3)$, 3 families, 2 type 1, 1 type $2^P ex$,
- $\langle 4;3,6\rangle$ – edge-symbol $(3 \cdot 6, 4 \cdot 4)$, 3 families, 2 type 2, 1 type 5,
- $\langle 3,6;4\rangle$ – edge-symbol $(4 \cdot 4, 3 \cdot 6)$, 3 families, 2 type 2^*, 1 type 5^*.

For each family we provide in Table 4 a description in terms of parameters. Additionally we provide the numbers of vertices and edges of its maps and drawings of a few small representatives.

For each of the five tessellations of the Euclidean plane we consider the vectors $\bar{u}, \bar{v}$ as in Figure 3. For each of the families $\tau \in \{\langle 4;4\rangle, \langle 3;6\rangle, \langle 6;3\rangle, \langle 4;3,6\rangle, \langle 3,6;4\rangle\}$ we shall make use of the following notation (see [17]). The toroidal map obtained as a quotient of τ by the translation group T generated by the translations by vectors $a\bar{u} + b\bar{v}$ and $c\bar{u} + d\bar{v}$ will be denoted by $\tau_{\{(a,b),(c,d)\}}$. If $\tau = \langle 4;4\rangle$ and $(c,d) = (-b,a)$, and when $\tau \neq \langle 4;4\rangle$ and $(c,d) = (-b, a+b)$, the toroidal map $\tau_{\{(a,b),(c,d)\}}$ is simply denoted by $\tau_{(a,b)}$.

With the families described in this way (including the restrictions of the parameters) any given map has a unique expression as a member of its family.

5.3 The Klein bottle (types: 3)

From the discussion at Section 2, every edge-transitive map M on the Klein bottle must share its edge-symbol with that of an edge-transitive tessellation $\mathcal{T}$ on the Euclidean plane. We recall that M is the quotient of $\mathcal{T}$ by a group A generated by two glide reflections with parallel axes and equal translation component. It follows that any automorphism of M can be obtained from an automorphism of $\mathcal{T}$ that normalizes A, however, any such

Name	**Map**	**Parameters**	**Class**	**Example**	**Fig.**	**Vertices**	**Edges**
$T_1(r)$	$\langle 4;4\rangle_{(r,0)}$	$r \geq 2$	1	$\langle 4;4\rangle_{(3,0)}$	4	r^2	$2r^2$
$T_2(r)$	$\langle 4;4\rangle_{(r,r)}$	$r \geq 1$	1	$\langle 4;4\rangle_{(2,2)}$	4	$2r^2$	$4r^2$
$T_3(r,s)$	$\langle 4;4\rangle_{(r,s)}$	$r > s > 0$	$2^P ex$	$\langle 4;4\rangle_{(2,1)}$	4	r^2+s^2	$2(r^2+s^2)$
$T_4(r_1,r_2)$	$\langle 4;4\rangle_{\{(r_1,r_2),(r_2,r_1)\}}$	$r_1 > r_2 > 0, r_1 \geq 3$	2^P	$\langle 4;4\rangle_{\{(4,1),(1,4)\}}$	4	$r_1^2 - r_2^2$	$2(r_1^2 - r_2^2)$
$T_5(r_1,r_2)$	$\langle 4;4\rangle_{\{(r,r),(s,-s)\}}$	$r > s > 1$	2^P	$\langle 4;4\rangle_{\{(3,3),(2,-2)\}}$	4	$2rs$	$4rs$
$T_6(r)$	$\langle 6;3\rangle_{(r,0)}$	$r \geq 2$	1	$\langle 6;3\rangle_{(3,0)}$	5	r^2	$3r^2$
$T_7(r)$	$\langle 6;3\rangle_{(r,r)}$	$r \geq 1$	1	$\langle 6;3\rangle_{(2,2)}$	5	$3r^2$	$9r^2$
$T_8(r,s)$	$\langle 6;3\rangle_{(r,s)}$	$r > s > 0$	$2^P ex$	$\langle 6;3\rangle_{(2,1)}$	5	r^2+rs+s^2	$3(r^2+rs+s^2)$
$T_9(r)$	$\langle 3;6\rangle_{(r,0)}$	$r \geq 2$	1	$\langle 3;6\rangle_{(3,0)}$	5	$2r^2$	$3r^2$
$T_{10}(r)$	$\langle 3;6\rangle_{(r,r)}$	$r \geq 1$	1	$\langle 3;6\rangle_{(1,1)}$	5	$6r^2$	$9r^2$
$T_{11}(r,s)$	$\langle 3;6\rangle_{(r,s)}$	$r > s > 0$	$2^P ex$	$\langle 3;6\rangle_{(2,1)}$	5	$2(r^2+rs+s^2)$	$3(r^2+rs+s^2)$
$T_{12}(r)$	$\langle 4;3,6\rangle_{(r,0)}$	$r \geq 1$	2	$\langle 4;3,6\rangle_{(3,0)}$	6	$3r^2$	$6r^2$
$T_{13}(r)$	$\langle 4;3,6\rangle_{(r,r)}$	$r \geq 1$	2	$\langle 4;3,6\rangle_{(1,1)}$	6	$9r^2$	$18r^2$
$T_{14}(r,s)$	$\langle 4;3,6\rangle_{(r,s)}$	$r > s > 0$	5	$\langle 4;3,6\rangle_{(2,1)}$	6	$3(r^2+rs+s^2)$	$6(r^2+rs+s^2)$
$T_{15}(r)$	$\langle 3,6;4\rangle_{(r,0)}$	$r \geq 1$	2^*	$\langle 3,6;4\rangle_{(3,0)}$	6	$3r^2$	$6r^2$
$T_{16}(r)$	$\langle 3,6;4\rangle_{(r,r)}$	$r \geq 1$	2^*	$\langle 3,6;4\rangle_{(1,1)}$	6	$9r^2$	$18r^2$
$T_{17}(r,s)$	$\langle 3,6;4\rangle_{(r,s)}$	$r > s > 0$	5^*	$\langle 3,6;4\rangle_{(2,1)}$	6	$3(r^2+rs+s^2)$	$6(r^2+rs+s^2)$

Table 4: Edge-transitive maps on the torus.

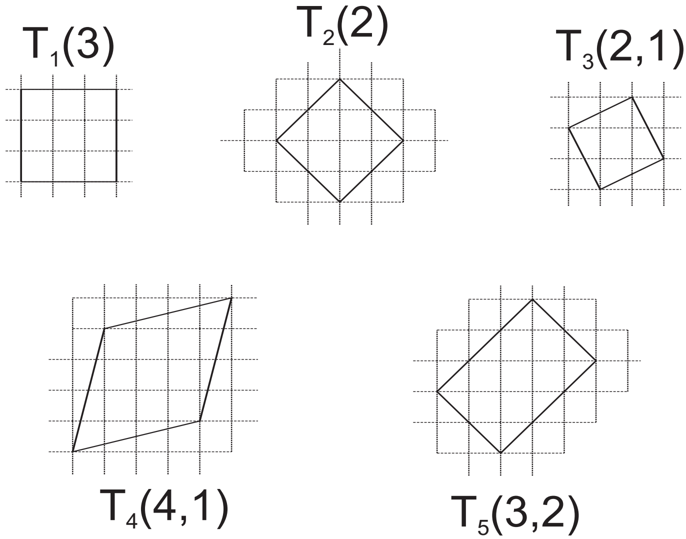

Figure 4: Examples of toroidal edge transitive maps from family $\langle 4; 4 \rangle$.

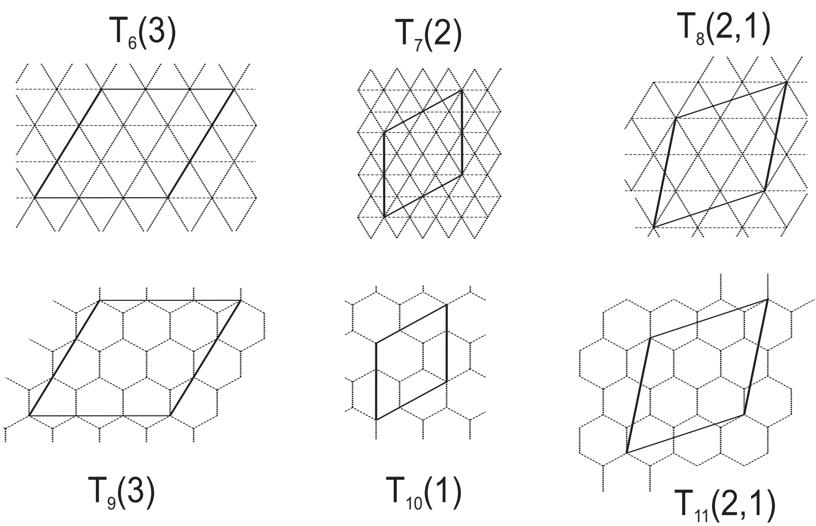

Figure 5: Examples of toroidal edge transitive maps from families $\langle 6; 3 \rangle$ and $\langle 3; 6 \rangle$.

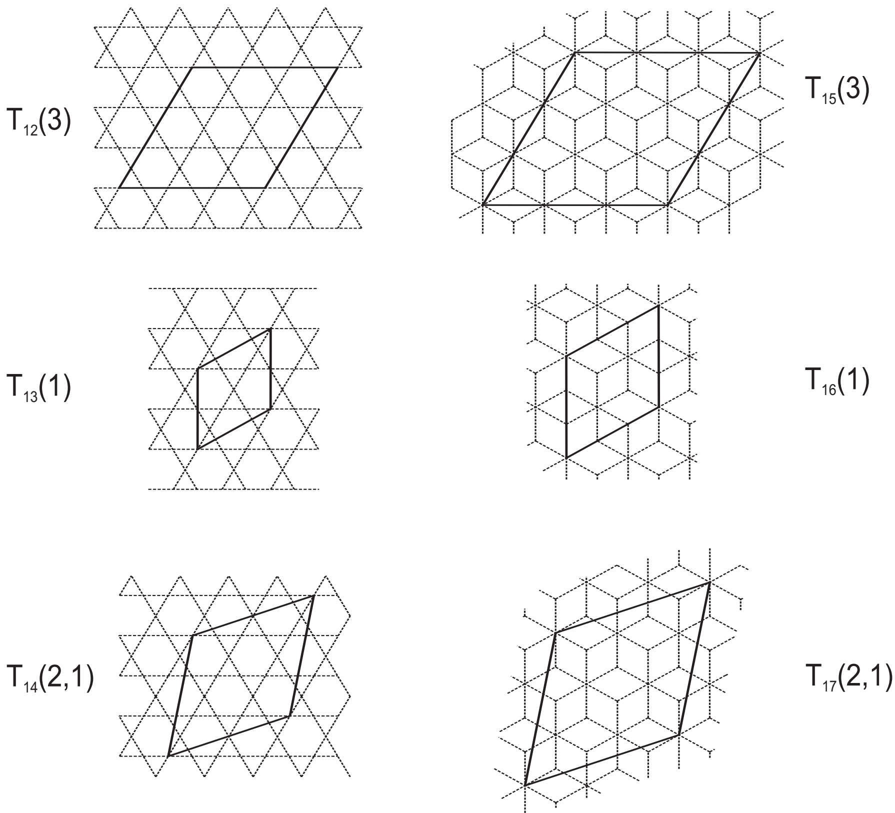

Figure 6: Examples of toroidal edge transitive maps from families $\langle 4; 3, 6\rangle$ and $\langle 3, 6; 4\rangle$.

K(10)

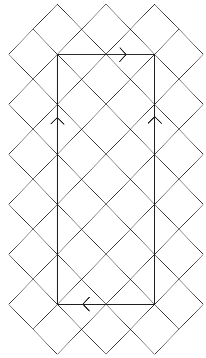

Figure 7: Edge transitive maps on the Klein bottle.

isometry of the Euclidean plane preserves the direction of the axes of the glide reflections and hence they must be either translations, half-turns or glide reflection whose (possibly trivial) translation vector is either parallel or orthogonal to the axes of the generators of A. A direct inspection shows that, no map arising from the families $\langle 3;6\rangle$, $\langle 6;3\rangle$, $\langle 4;3,6\rangle$ and $\langle 3,6;4\rangle$ can be edge-transitive when considering only such possible automorphisms. Therefore, the only possible family is $\langle 4;4\rangle$.

The only family of edge-transitive maps on the Klein bottle in the notation of [22] is $K(r) := \{4,4\}_{\backslash r,2\backslash}$ with $r \geq 2$ even, in class 3 (see $\{4,4\}_{\backslash 10,2\backslash}$ as an example in Figure 7). The number of vertices and edges of $K(r)$ are $2r$ and $4r$ respectively, and its underlying graph is the lexicographic product $C_r \circ \bar{K}_2$ of an r-cycle with an empty graph in two vertices.

We observe that in [22] it is mistakenly mentioned that the map $\{4,4\}_{\backslash r,2\backslash}$ is edge transitive if and only if r is odd. Note however that in that case there are two orbits of edges, the one containing edges intersecting the axes of the defining glide reflections, and the remaining ones. Clearly these orbits are distinct. This conclusion can also be attained by noting that the half-turn with respect to the center of an edge that intersects the axis of one of the defnining glide reflections is an automorphism of the map. Since there are $4r$ edges and $4r$ automorphisms, there cannot be only one flag orbit. On the other hand, if r is even then all edge-stabilizers are trivial implying that the map is edge transitive.

6 Conclusion

The summary of our results is presented in Table 5 which is also based on calculations using the computer package MAGMA [1] (see Tables 6,7,8). The calculations and results are in further detail explained in [15]. The main idea is that for each edge-transitive map on a surface with negative Euler characteristic the size of the automorphism group is bounded by the Hurwitz bound $168(g-1)$, where g is the orientable (or non-orientable) genus of the surface. Note that depending on the type and orientability of edge-transitive maps the upper bound can be further refined (see [15], page 59). On the other hand the number of flags cannot exceed four times the size of the group. In addition, the candidate groups must have one of the presentations stated in [19], page 14. Our approach is to use LOWINDEXNORMAL-

SUBGROUPS algorithm from MAGMA [1] starting with the presentations mentioned above, calculating quotients corresponding to normal subgroups of index up the relevant upper bound and filtering out degenerate maps and maps with higher symmetry then expected for the considered type. An alternative approach would be simply to check for generators of relevant groups in the SmallGroupLibrary (included in GAP and MAGMA) up to order less or equal the relevant upper bound. For $\chi \geq -2$ all possible groups can be found in the SmallGroupLibrary. The third approach would be to calculate edge-transitive covers of the fourteen 3-valent type graphs using the standard voltage graph technique built in the monograph by Gross and Tucker [8] and extended to graphs with semi-edges in a paper by Malnič et. al. [14].

Surface	χ	**Orientability**	**Types**									
			1	2	2^*	2^P	$2^P ex$	3	4	4^*	5	5^*
sphere	2	yes	5	2	2							
Euclidean plane	1	yes	3	1	1							
hyperbolic plane	1	yes	∞	∞	∞			∞				
projective plane	1	no	4	2	2							
torus	0	yes	∞	∞	∞	∞	∞				∞	∞
Klein bottle	0	no						∞				
	-1	no						4				
	-2	no	4	3	3	2		11	2	2		
double torus	-2	yes	6	8	8	-		5	1	1		

Table 5: Number of non-degenerate edge-transitive maps on surfaces with Euler characteristic $\chi \geq -2$.

Notation:

- $G(k)$ - simple graph G with each edge replaced by k parallel edges.
- $G[k]$ - lexicographic product of G with the empty graph on k vertices.
- B_n - bouquet of n circles.
- G_1 - vertex-face incidence graph of a cube.
- G_2 - vertex-face incidence graph with each vertex-vertex doubled.
- G_3 - skeleton of cuboctahedron
- G_4 - line graph of the Möbuis-Kantor graph
- $GP(m, n)$ - generalized Petersen graph.
- $\{(f1, f2), (v1, v2)\}_{(p1,p2)}$ extended Schlafli type. $p1, p2$ Petrie polygons.
- In the comments column, some maps are labeled by capital letters possibly with indices. Using these labels certain relations between maps are denoted. For instance, the first map in Table 6 is labeled by M_1 and the third map is its dual. In Table 7, the notation $E = \mathrm{Med}(Y)$, $\mathrm{Pe}(Y) \neq Y$ means that the corresponding map is labeled by E and is the medial of some map Y, which is not self-petrie.

Euler char.	orientable	type	graph	No. edges	extended Schläfli symb.	Comment
-1	no	3	$C_4(2)$	8	$\{(4,8),(4,4)\}_{(4,8)}$	M_1
-1	no	3	$K_{2,3}(2)$	12	$\{(4,4),(4,6)\}_{(4,12)}$	M_2
-1	no	3	$C_6(2)$	12	$\{(4,6),(4,4)\}_{(4,12)}$	Du(M_2)
-1	no	3	$K_{1,2}(4)$	8	$\{(4,4),(4,8)\}_{(4,8)}$	Du(M_1)

Table 6: Non-degenerate edge-transitive maps on a surface with Euler characteristic -1.

Euler char.	orientable	type	graph	No. of edges	extended Schläfli symbol	Comment
-2	no	1	$K_4(2)$	12	$\{4,6\}_6$	R_1
-2	no	1	$K_4(2)$	12	$\{4,6\}_3$	R_2
-2	no	1	$C_6(2)$	12	$\{6,4\}_6$	Du(R_1)
-2	no	1	$K_{2,2,2}$	12	$\{6,4\}_3$	Du(R_2)
-2	no	2	$K_{3,4}$	12	$\{8,(3,4)\}_4$	
-2	no	2	$K_{4,6}$	24	$\{4,(4,6)\}_8$	
-2	no	2	$K_{4,6}$	24	$\{4,(4,6)\}_8$	
-2	no	2^*	$K_3(4)$	12	$\{(3,4),8\}_4$	
-2	no	2^*	G_3	24	$\{(4,6),4\}_8$	
-2	no	2^*	$K_{6,6}-3C_4$	24	$\{(4,6),4\}_8$	
-2	no	2^P	$K_3(2)$	6	$\{4,12\}_{(3,6)}$	
-2	no	2^P	B_6	6	$\{4,12\}_{(3,6)}$	
-2	no	3	$K_{2,4}(2)$	16	$\{(4,4),(4,8)\}_{(4,8)}$	A
-2	no	3	$K_{1,2}(4)$	8	$\{(4,8),(4,8)\}_{(4,4)}$	selfdual
-2	no	3	$K_{1,3}(4)$	12	$\{(4,4),(4,12)\}_{(4,6)}$	B
-2	no	3	$C_8(2)$	16	$\{(4,8),(4,4)\}_{(4,8)}$	C
-2	no	3	$C_6(2)$	12	$\{(4,12),(4,4)\}_{(4,6)}$	Du(B)
-2	no	3	$K_{2,4}(2)$	16	$\{(4,4),(4,8)\}_{(4,4)}$	D = Med(X), Pe(X) = X
-2	no	3	$K_{4,4}$	16	$\{(4,8),(4,4)\}_{(4,4)}$	Du(D)
-2	no	3	$K_{4,4}$	16	$\{(4,8),(4,4)\}_{(4,8)}$	Du(A)
-2	no	3	$K_{2,4}(2)$	16	$\{(4,4),(4,8)\}_{(4,8)}$	Du(E)
-2	no	3	$K_{4,4}$	16	$\{(4,8),(4,4)\}_{(4,8)}$	E = Med(Y), Pe(Y)$\neq$ Y
-2	no	3	$K_{2,4}(2)$	16	$\{(4,4),(4,8)\}_{(4,8)}$	Du(C)
-2	no	4	$C_4(2)$	8	$\{8,(4,4)\}_8$	Order of monod. group 128
-2	no	4	$C_4(2)$	8	$\{8,(4,4)\}_8$	Order of monod. group 512
-2	no	4^*	$K_2(8)$	8	$\{(4,4),8\}_8$	
-2	no	4^*	$K_2(8)$	8	$\{(4,4),8\}_8$	

Table 7: Non-degenerate edge-transitive maps on a non-orientable surface with Euler characteristic -2. Note that the maps of the type 2^* (4^*) are exactly the duals of the maps of the type 2 (4) in respective order.

Euler char.	orientable	type	graph	No. of edges	extended Schläfli symbol	Comment
-2	yes	1	$K_{2,2,2}(2)$	24	$\{3,8\}_{12}$	K_1
-2	yes	1	$C_4(3)$	12	$\{4,6\}_{12}$	K_2
-2	yes	1	$K_2(8)$	8	$\{4,8\}_8$	K_3
-2	yes	1	$GP(8,3)$	24	$\{8,3\}_{12}$	Du(K_1)
-2	yes	1	$C_6(2)$	12	$\{6,4\}_{12}$	Du(K_2)
-2	yes	1	$C_4(2)$	8	$\{8,4\}_8$	Du(K_3)
-2	yes	2	$K_2(5)$	5	$\{10,(5,5)\}_{10}$	
-2	yes	2	$K_{1,2}(3)$	6	$\{12,(3,6)\}_4$	
-2	yes	2	$K_{1,2}(5)$	10	$\{4,(5,10)\}_{20}$	
-2	yes	2	$K_{3,4}$	12	$\{8,(3,4)\}_8$	
-2	yes	2	$K_{2,4}(2)$	16	$\{4,(4,8)\}_4$	
-2	yes	2	G_1	24	$\{6,(3,4)\}_{12}$	
-2	yes	2	$K_{4,6}$	24	$\{4,(4,6)\}_4$	
-2	yes	2	G_2	48	$\{4,(3,8)\}_{16}$	
-2	yes	2^*	B_5	5	$\{(5,5),10\}_{10}$	
-2	yes	2^*	B_6	6	$\{(3,6),12\}_4$	
-2	yes	2^*	$C_5(2)$	10	$\{(5,10),4\}_{20}$	
-2	yes	2^*	$K_3(4)$	12	$\{(3,4),8\}_8$	
-2	yes	2^*	$K_{4,4}$	16	$\{(4,8),4\}_4$	
-2	yes	2^*	$K_8 - 4K_2$	24	$\{(3,4),6\}_{12}$	
-2	yes	2^*	$C_6[2]$	24	$\{(4,6),4\}_4$	
-2	yes	2^*	G_4	48	$\{(3,8),4\}_{16}$	
-2	yes	3	$K_{2,3}(2)$	12	$\{(4,6),(4,6)\}_{(4,6)}$	selfdual, selfpetrie
-2	yes	3	$K_{2,4}(2)$	16	$\{(4,4),(4,8)\}_{(4,8)}$	U
-2	yes	3	$K_{4,4}$	16	$\{(4,8),(4,4)\}_{(4,8)}$	Du(U)
-2	yes	3	$K_{4,6}$	24	$\{(4,4),(4,6)\}_{(4,12)}$	V
-2	yes	3	$C_6[2]$	24	$\{(4,6),(4,4)\}_{(4,12)}$	Du(V)
-2	yes	4	$K_{3,4}$	12	$\{8,(3,4)\}_8$	
-2	yes	4^*	$K_3(4)$	12	$\{(3,4),8\}_8$	

Table 8: Non-degenerate edge-transitive maps on an orientable surface with Euler characteristic -2.

There are several interesting related problems to the topic of this paper.

Problem 1. Determine the surfaces with negative Euler characteristic that contain edge-transitive maps of all possible 14 types.

Problem 2. In [2] infinitely many positive integers g are found with the property that chiral maps cannot be drawn on orientable surfaces with genus g. What can be said of numbers g for which there are no edge-transitive maps of a given class on orientable or non-orientable surfaces of genus g?

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