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**PARALLELISM OF STABLE
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Abstract

Stable traces were investigated in [Stable traces as a model for self-assembly of polypeptide nanoscale polyhedrons, MATCH Commun. Math. Comput. Chem. 70 (2013) 317–330] as a mathematical model for an innovative biotechnological procedure. Two open problems posed there are discussed in the present paper. It is proved that graphs that admit parallel stable traces are precisely Eulerian graphs with minimum degree at least 4. It is also proved that a sufficient condition for a graph to admit an antiparallel stable trace is to contain an even spanning tree. Here a parallel (antiparallel) stable trace is a double trace with three additional conditions—having no retracing, no repetition, and traversing every edge twice in the same (opposite) direction.

Keywords: Eulerian circuit; parallel stable trace; antiparallel stable trace; even spanning tree

AMS Subject Classification (2010): 05C45, 05C05

1 Introduction

All graphs considered in this paper will be connected, finite, and simple, that is, without loops and multiple edges. If v is a vertex of a graph G , then its degree will be denoted by $d_G(v)$ or $d(v)$ for short if G will be clear from the context. The minimum and the maximum degree of G are denoted with $\delta(G)$ and $\Delta(G)$, respectively. A *directed graph* is a graph, where edges have a direction associated with them. In formal terms a directed graph is a pair $G = (V, A)$, where V is a set of vertices and A is a set of ordered pairs of vertices, called *arcs*. A maximal connected subgraph of G is called a *component* of G , while a vertex which separates two other vertices of the same component is a *cutvertex*, and an edge separating its ends is a *bridge*. A maximal connected subgraph without a cutvertex is called a *block*. Thus, every block of a graph G is either a maximal 2-connected subgraph, or a bridge (with its ends), or an isolated vertex. For other general terms and concepts from graph theory not recalled here we refer to [8].

A *double trace* in a graph G is a circuit which traverses every edge exactly twice. We say that a double trace contains a *retracing* if it has an immediate succession of an edge e by its parallel copy. Further, if v is a vertex of a graph G with a double trace T and u and w are two different neighbors of v , then we say that T contains a *repetition through v* if the vertex sequence $u \rightarrow v \rightarrow w$ appears twice in T in any direction ($u \rightarrow v \rightarrow w$ or $w \rightarrow v \rightarrow u$). We next define a *proper trace* as a double trace that has no retracing and a *stable trace* as a proper trace without repetitions through its vertices.

In order to present a mathematical model for the biotechnological procedure from [3] the graphs that admit stable traces were characterized in [4] as follows:

Theorem 1.1 [4, Theorem 3.1] *A connected graph G admits a stable trace if and only if $\delta(G) \geq 3$.*

Let now T be a double trace of a graph G . Then every edge $e = uv$ of G is traversed exactly twice. If in both cases e is traversed in the same direction (either both times from u to v or both times from v to u) we say that e is a *parallel edge* (with respect to T). If this is not the case we say that e is an *antiparallel edge*. A condition that all the edges of G are of the same type is called a *parallelism*. A double trace T is a *parallel double trace* if every edge of G is parallel and an *antiparallel double trace* if every edge of G is antiparallel.

In relation with parallelism of double traces in [4] two related open problems were posed. The first problem, [4, Problem 5.6] asks for a characterization of graphs which admit parallel stable traces. In Section 2 we solve this problem (Theorem 2.2) by proving that a connected graph admits a parallel stable trace if and only if it is Eulerian and its minimum degree is at least 4. The second open problem [4, Problem 5.7] asks for a characterization of graphs which admit antiparallel stable traces. In this direction we prove in Section 3 that a sufficient condition for a graph to admit an antiparallel

stable trace is to have an even spanning tree. We wonder whether this condition is also necessary and present some other results about even spanning trees.

2 Graphs that admit parallel stable traces

The following was observed in [4]:

Proposition 2.1 [4, Proposition 5.4] *A connected graph G admits a parallel proper trace if and only if G is Eulerian.*

It was also proved in [4] that a graph G admits a parallel double trace if and only if G is Eulerian.

Our main result reads as follows:

Theorem 2.2 *A graph G admits a parallel stable trace if and only if G is Eulerian and $\delta(G) \geq 4$.*

Proof. Suppose that a graph G admits a parallel stable trace. By definition, every stable trace is a proper trace. Thus by Proposition 2.1, G is Eulerian and hence by Theorem 1.1 we infer that $\delta(G) \geq 4$.

For the converse assume that G fulfills the conditions of the theorem. We proceed by induction on $\Delta = \Delta(G)$.

Let $\Delta = 4$. Then $\delta(G) = \Delta(G) = 4$. By Proposition 2.1, G admits a parallel proper trace T' . If T' is not already a stable trace, T' contains a repetition through v , for some vertex v of G . To make the argument more transparent, assume first that T' contains a unique vertex v with a repetition through. If a vertex v with $d_G(v) = 4$ has a repetition through in T' , then it is not difficult to see that v has two repetitions through in T' . Let v_1, v_2, v_3 , and v_4 be the neighbors of v . Without loss of generality, we can assume that $A = v_1 \rightarrow v \rightarrow v_2$ is the first and $B = v_3 \rightarrow v \rightarrow v_4$ is the second repetition through v in T' . That means that sequences A and B appear twice in T' . Because T' is a parallel proper trace, there are only two possibilities how occurrences of A and B are arranged in T' . These possibilities are $AABB$ (Fig. 1, left) and $ABAB$ (Fig. 2, left). Note that we left out all the other vertices in Figs. 1 and 2.

In the first case construct T from T' in G as follows. Let $e' = xy$ be an arbitrary (oriented) edge of T' . If $x, y \in V(G) \setminus \{v_1, v_2, v_3, v_4\}$, then we put xy into T . Put one occurrence of $v_1 \rightarrow v \rightarrow v_2$ and one occurrence of $v_3 \rightarrow v \rightarrow v_4$ in T as well. Replace the remaining occurrences with $v_1 \rightarrow v \rightarrow v_4$ and $v_3 \rightarrow v \rightarrow v_2$, respectively, such that T stays connected, see Fig. 1, right.

We construct T similarly in the second case, see Fig. 2, right.

We claim that in both cases T is a parallel stable trace of G . Note first that any edge e that appears in T has its unique corresponding edge e' in T' . Any edge $e = xy$ in T , where $x \neq v$ and $y \neq v$, is traversed twice in the same direction in T because it is

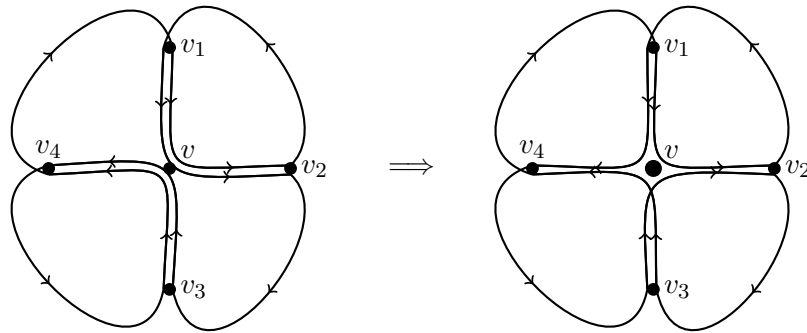


Figure 1: Construction of a stable trace from a proper trace (case $AAB B$)

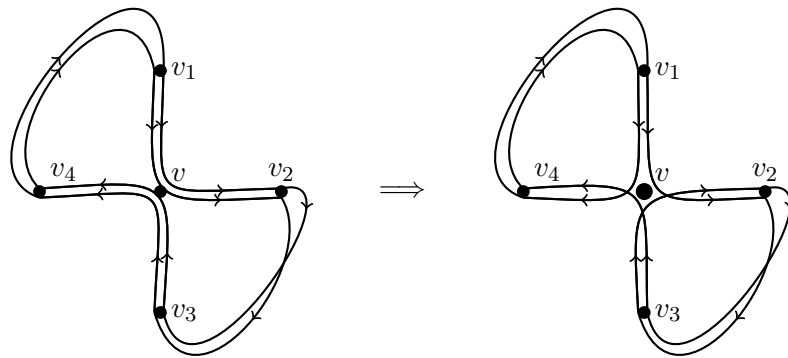


Figure 2: Construction of a stable trace from a proper trace (case $ABAB$)

traversed twice in the same direction in T' . Four remaining edges are traversed twice in the same direction by construction. Hence T is a parallel double trace. It is also clear that T is a proper trace. Vertex v is proper by construction and if any other vertex in T would not be proper, T' would not be proper. (If there would be a retracing of an edge e in T , it would lead to a retracing of its corresponding edge e' in T' .) Finally we

need to verify that T is stable. Vertex v has no repetition through by construction and if any other vertex $u \neq v$ in T would have a repetition through, already T' would have a repetition through $u \neq v$. We conclude that T is a parallel stable trace of G . We have thus proved that if a 4-regular graph G admits a parallel proper trace T' with a single vertex with repetition through, then G also admits a parallel stable trace.

We now proceed by a second induction on the number $D_{\max}(T')$ of vertices with a repetition through in T' . Let $D_{\max}(G) \geq 2$ and let v be an arbitrary vertex with a repetition through in T' . Then construct a parallel double trace T from T' by reconstructing T' in v as described above. Note that $D_{\max}(T) < D_{\max}(T')$ and hence T can be transformed into a parallel stable trace by the induction on $D_{\max}(T')$. Hence any graph G with $\delta(G) = \Delta(G) = 4$ admits a parallel stable trace.

Assume now that $\Delta \geq 6$ and that any graph H with $\Delta(H) < \Delta$ which fulfills the conditions of Theorem 2.2 admits a parallel stable trace. To make the argument more transparent, assume first that a graph G , which fulfills the conditions of Theorem 2.2, contains a unique vertex v of degree Δ . Let v_1, \dots, v_Δ be the neighbors of v and consider two cases.

Case 1: $\Delta \equiv 2 \pmod{4}$.

Construct the graph G' from G as follows. Remove from G the vertex v , add two new vertices v' and v'' , connect them by an edge, connect v' with $v_1, \dots, v_{\frac{\Delta}{2}}$, and connect v'' with the remaining neighbors of v , see Fig. 3.

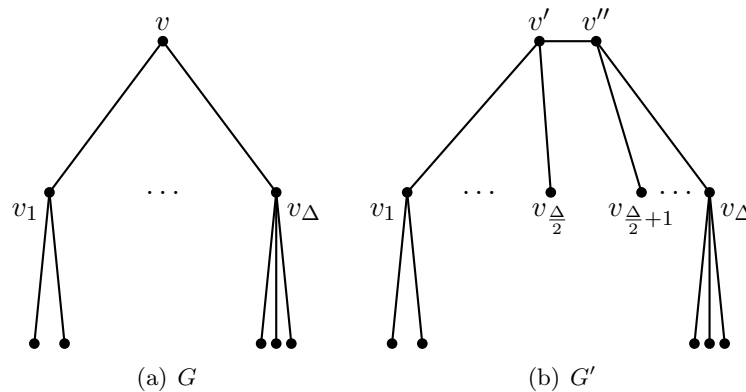


Figure 3: Construction from the proof of Theorem 2.2 for the case $\Delta \equiv 2 \pmod{4}$

Note that in G' all the vertices but v' and v'' are of the same degree as in G , while $d_{G'}(v') = \frac{\Delta}{2} + 1$ and $d_{G'}(v'') = \frac{\Delta}{2} + 1$. It follows that $\Delta(G') < \Delta$. Since $\Delta \geq 6$, we also infer that $\delta(G') \geq 4$ (to be more precise we infer that $d_{G'}(v'), d_{G'}(v'') \geq 4$, while degree of other vertices is unchanged). Because $\Delta \equiv 2 \pmod{4}$, the degrees $d_{G'}(v') = d_{G'}(v'') = \frac{\Delta}{2} + 1$ are even, hence G is Eulerian and by the induction assumption on Δ ,

G' admits a parallel stable trace T' .

Construct a parallel stable trace T in G from T' similar as in the base of induction (where $\Delta = 4$). Put every edge from T' not incident with v' and v'' into T and replace uv' , $v'u$, uv'' , and $v''u$ with uv , vu , uv , and vu , respectively. Finally, ignore the two occurrences of the edge $v'v''$ (or $v''v'$) from T' in T .

To show that T is really a parallel stable trace of G , note first that any edge e that appears in G has its unique corresponding edge e' in G' (edge $e' \neq v'v''$ does not appear in G) and is therefore traversed twice in the same direction in T . Also if there would be a retracing of an edge e in T , it would lead to a retracing of its corresponding edge e' in T' , which is not possible since T' is parallel stable trace. Hence T is a parallel proper trace. To verify that T is also stable, let $e = xy$ and $f = yz$ be two consecutive edges of T . If e and f are not incident with v , or if $x = v$ or $z = v$, then e and f does not give a repetition through y because otherwise we would have a repetition through y in T' . Hence the only unchecked option is that $y = v$. Then $x = v_i$ and $z = v_j$ are two neighbors of v . Depend on the origin of obtaining of e and f consider two subcases. In the first subcase let $i, j \leq \lceil \frac{\Delta}{2} \rceil$. Then e, f were obtained from the edges $v_i v'$, $v' v_j$ which do not have a repetition through v' hence e, f do not have a repetition through v . Analogous conclusion holds when $i, j > \lceil \frac{\Delta}{2} \rceil$ (just replace v' with v'' in the argument). In the second subcase let $i \leq \lceil \frac{\Delta}{2} \rceil < j$. Then e, f in T were constructed from $v_i v'$, $v' v''$, $v'' v_j$ in T' . Since also $v' v''$ is traversed exactly twice in T' , fact that e and f have a repetition through v would mean that we have a repetition through v' in T' , a contradiction. We therefore showed that T is a parallel stable trace of G .

Case 2: $\Delta \equiv 0 \pmod{4}$.

Construct the graph G' from G as follows. Remove from G the vertex v , and add three new vertices v' , v'' , and v''' . Connect v'' with v' and v''' by an edge, connect v' with $v_1, \dots, v_{\frac{\Delta}{2}-1}$, connect v'' with $v_{\frac{\Delta}{2}}$ and $v_{\frac{\Delta}{2}+1}$, and connect v''' with the remaining neighbors of v , see Fig. 4.

Similarly as in the first case note that in G' all the vertices except v' , v'' , and v''' are of the same degree as in G , while $d_{G'}(v') = d_{G'}(v''') = \frac{\Delta}{2}$ and $d_{G'}(v'') = 4$. It follows that $\Delta(G') < \Delta$. Since $\Delta \geq 6$, we also infer that $\delta(G') \geq 4$. Because $\Delta \equiv 0 \pmod{4}$, the degrees $d_{G'}(v') = d_{G'}(v''') = \frac{\Delta}{2}$ are even, hence G is Eulerian. By the induction assumption on Δ , G' admits a parallel stable trace T' .

We next construct a trace T in G from T' . Let $e = xy$ be an arbitrary (oriented) edge of T' . If $x, y \in V(G') \setminus \{v', v'', v'''\}$, then put xy into T . Let $u \neq v', v'', v'''$. If $e = uv'$, then replace e with uv in T . Similarly replace edges of the form $v'u$, uv'' , $v''u$, uv''' , and $v'''u$ with vu , uv , vu , uv , and vu , respectively. Finally, the two occurrences of the edges $v'v''$ (or $v''v'$) and $v''v'''$ (or $v'''v''$) from T' are ignored in T .

We claim that T is a parallel stable trace of G . Note first that any edge e that appears in G has its unique corresponding edge e' in G' . Clearly, $e' \neq v'v''$ and $e' \neq v''v'''$. Since e' is traversed twice in the same direction in T' , the edge e is traversed

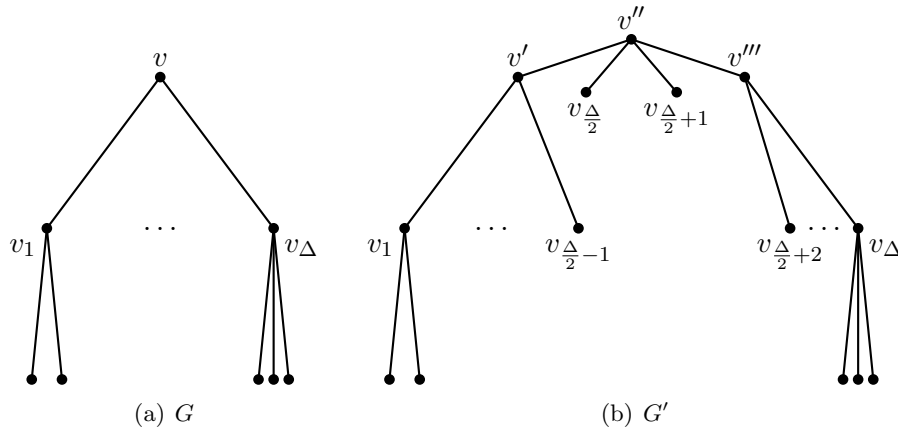


Figure 4: Construction from the proof of Theorem 2.2 for the case $\Delta \equiv 0 \pmod{4}$

twice in the same direction in T . Hence T is a parallel double trace. It is also clear that T is a proper trace because otherwise T' would not be proper. (If there would be a retracing of an edge e in T , it would lead to a retracing of its corresponding edge e' in T' .) Finally we need to verify that T is stable. Let $e = xy$ and $f = yz$ be two consecutive edges of T . If $\{x, y, z\} \cap \{v\} = \emptyset$, then e and f does not give a repetition through y because otherwise we would have a repetition through y in T' . The same conclusion holds if $x = v$ or $z = v$. Assume hence that $y = v$. Let $x = v_i$ and $z = v_j$ and consider two subcases. In the first subcase let $i, j \leq \frac{\Delta}{2} - 1$. Then e and f were obtained from the edges $v_i v'$ and $v' v_j$ which do not have a repetition through v' , hence e and f do not have a repetition through v . Analogous conclusion holds when $i, j \in \{\frac{\Delta}{2}, \frac{\Delta}{2} + 1\}$ or $i, j > \frac{\Delta}{2} + 1$ (just replace v' with v'' or v''' in the argument). In the second subcase let $i \leq \frac{\Delta}{2} - 1$ and $j \in \{\frac{\Delta}{2}, \frac{\Delta}{2} + 1\}$. Then e and f were constructed from $v_i v'$, $v' v''$, and $v'' v_j$ in T' . Recall that $v' v''$ is traversed exactly twice in T' . Hence if e and f would have a repetition through v , we would have a repetition through v' in T' , a contradiction. Analogous conclusion holds when $i > \frac{\Delta}{2} + 1$ and $j \in \{\frac{\Delta}{2}, \frac{\Delta}{2} + 1\}$ or $i \leq \frac{\Delta}{2} - 1$ and $j > \frac{\Delta}{2} + 1$ (just replace v' with v'' or v''' in the argument). We conclude that T is a parallel stable trace of G .

We have thus proved that if G has a single vertex of degree Δ , then G admits a parallel stable trace.

We now proceed by a second induction on the number $D_{\max}(G)$ of vertices of maximum degree of a graph G . Let $D_{\max}(G) \geq 2$ and let v be an arbitrary vertex of degree Δ . Then construct a graph G' from G in the same way as above. Note that $D_{\max}(G') < D_{\max}(G)$ and hence G admits a parallel stable trace by the induction on $D_{\max}(G)$. \square

3 Graphs that admit antiparallel stable traces

In this section we present a sufficient condition for graphs to admit antiparallel stable traces.

Already in 1895 Tarry [6] observed that every graph admits an antiparallel double trace. In [7] Thomassen characterized graphs that admit antiparallel proper traces (thus solving a problem posed by Ore [5]):

Theorem 3.1 [7, Theorem 3.3] *A graph G admits an antiparallel proper trace if and only if $\delta(G) \geq 2$ and G has a spanning tree T such that each component of $G - E(T)$ either has an even number of edges or contains a vertex v with $d_G(v) \geq 4$.*

Theorem 3.1 was later generalized by Fan and Zhu in [1]. We are interested in antiparallel stable traces. Suppose that a graph G admits an antiparallel stable trace. By definition every stable trace is a proper trace as well. Hence by Theorem 3.1, $\delta(G) \geq 2$ and G has a spanning tree T such that each component of $G - E(T)$ either has an even number of edges or contains a vertex v , $d_G(v) \geq 4$. By Theorem 1.1 we also have $\delta(G) \geq 3$.

To find a sufficient condition for the existence of antiparallel stable traces, we first consider cubic graphs:

Proposition 3.2 *A cubic graph G admits an antiparallel stable trace if and only if G has a spanning tree T such that each component of $G - E(T)$ has an even number of edges.*

Proof. Suppose first that a cubic graph G admits an antiparallel stable trace. By Theorem 3.1, G has a spanning tree T such that each component of $G - E(T)$ has an even number of edges.

Conversely, let G be an arbitrary cubic graph which has a spanning tree T such that each component of $G - E(T)$ has an even number of edges. Then by Theorem 3.1, G admits an antiparallel proper trace S . Since all the vertices of G are of degree 3 and S is proper, it is straightforward to see that S is also a stable trace. \square

We next present a construction of a cubic graph from an arbitrary graph G with $\delta(G) \geq 3$.

Let G be an arbitrary graph, $\delta(G) \geq 3$ and $\Delta(G) > 3$. To make the argument more transparent, assume first that G contains a unique vertex v with $d(v) > 3$. We construct a *cubic of G* as follows. Denote $d(v)$ with k . Let v_1, \dots, v_k be the neighbors of v in G . Put every vertex of $V(G) - \{v\}$ into a cubic of G . Replace v in a cubic of G with $k - 2$ vertices w_1, \dots, w_{k-2} . Put every edge not incident with v in a cubic of G . Connect w_i with w_{i-1} for $2 \leq i \leq k - 2$. Connect w_1 with v_1 and v_2 , and connect w_{k-2} with v_{k-1} and v_k . Finally connect w_i with v_{i+1} for $2 \leq i \leq k - 3$. It is not difficult to see that the constructed graph is cubic. If G has more than one vertex of degree

greater than 3, use the same procedure on each of them, see Fig. 5. Note that if G is cubic graph, then a cubic of G is isomorphic to G . We also point out that a graph G can have more than one cubic of G .

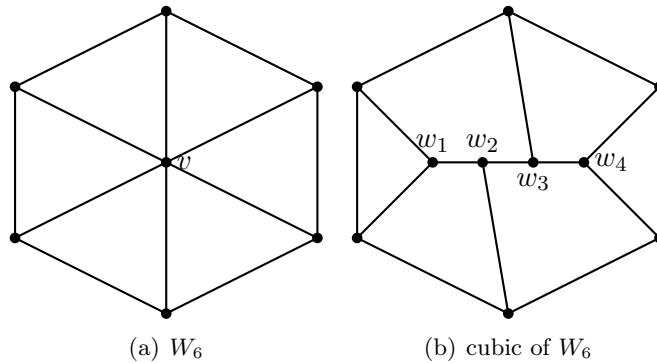


Figure 5: Construction of a cubic of W_6

Lemma 3.3 *Let G be a graph with $\Delta(G) \leq 5$ which admits an antiparallel stable trace. Then at least one cubic of G admits an antiparallel stable trace.*

Proof. Let G be an arbitrary graph with $\Delta(G) \leq 5$ which admits an antiparallel stable trace T . By Theorem 1.1, $\delta(G) \geq 3$. We proceed by induction on the number k of vertices of degree greater than 3.

Let $k = 0$. Then G is cubic and by construction, a cubic of G is isomorphic to G . Assume now that $k = 1$. Denote the unique vertex of degree greater than 3 with v and proceed by the second induction on $\Delta = \Delta(G) = d_G(v)$. We have to consider two cases.

Let first $d(v) = 4$. Denote the neighbors of v with v_1, v_2, v_3 , and v_4 . It is straightforward to see that up to isomorphism, there is only one way how T behaves in v . T contains the next four sequences: $v_1 \rightarrow v \rightarrow v_2$, $v_2 \rightarrow v \rightarrow v_3$, $v_3 \rightarrow v \rightarrow v_4$, and $v_4 \rightarrow v \rightarrow v_1$. Otherwise T would obviously contain a retracing or a repetition through v . In every cubic of G , v is replaced with two new adjacent vertices v' and v'' . Connect v' with v_1 and v_2 , and connect v'' with v_3 and v_4 .

We next construct a trace T' in G' as follows. Replace the above mentioned sequences from T with $v_1 \rightarrow v' \rightarrow v_2$, $v_2 \rightarrow v' \rightarrow v'' \rightarrow v_3$, $v_3 \rightarrow v'' \rightarrow v_4$, and $v_4 \rightarrow v'' \rightarrow v' \rightarrow v_1$ in T' , respectively. Leave all the other parts of T untouched. Then T' is an antiparallel stable trace in G' .

In the second case $d(v) = 5$. Denote the neighbors of v with v_1, v_2, v_3, v_4 , and v_5 . Similarly as in the first case we observe that T contains next sequences: $v_1 \rightarrow v \rightarrow v_2$, $v_2 \rightarrow v \rightarrow v_3$, $v_3 \rightarrow v \rightarrow v_4$, $v_4 \rightarrow v \rightarrow v_5$, and $v_5 \rightarrow v \rightarrow v_1$. In every cubic of G , v is

replaced with three new vertices v' , v'' , and v''' . Connect v'' with v' and v''' . Connect v' with v_1 and v_2 , connect v'' with v_3 , and connect v''' with v_4 and v_5 .

Similarly as in the first case construct a trace T' in G' as follows. Again replace the above five mentioned sequences from T with $v_1 \rightarrow v' \rightarrow v_2$, $v_2 \rightarrow v' \rightarrow v'' \rightarrow v_3$, $v_3 \rightarrow v'' \rightarrow v''' \rightarrow v_4$, $v_4 \rightarrow v''' \rightarrow v_5$, and $v_5 \rightarrow v''' \rightarrow v'' \rightarrow v' \rightarrow v_1$ in T' , respectively. Leave all the other parts of T untouched. It is not difficult to see that T' is an antiparallel stable trace in G' .

Assume now that $k > 1$ and that for any graph H with $\Delta(H) \leq 5$ which admits an antiparallel stable trace and has less than k vertices of degree greater than 3, at least one cubic of H admits an antiparallel stable trace as well. Let v be an arbitrary vertex of degree greater than 3. Then construct G' in the same way as above. The graph G' then admits an antiparallel stable trace and has less than k vertices of degree greater than 3. By induction assumption, at least one cubic of G' admits an antiparallel stable trace T' . Since every cubic of G' is also a cubic of G , at least one cubic of G admits an antiparallel stable trace. \square

Note that the construction of stable traces in cubics of graph from the proof of Lemma 3.3 cannot be generalized to graphs with vertices of degree greater than 5. Indeed, if a graph G which admits an antiparallel stable trace contains a vertex v of degree 6 (denote its neighbors with v_1, \dots, v_6), the next problem can occur. Sequences $v_1 \rightarrow v \rightarrow v_2$, $v_2 \rightarrow v \rightarrow v_3$, and $v_3 \rightarrow v \rightarrow v_1$ can all appear in a stable trace T . At least one repetition through one of the new vertices (through w_2 to be more accurate) would then appear, if in the proof described construction is used.

To see that the condition described before Proposition 3.2 is not sufficient in the general case, consider the wheel graph W_5 . This graph fulfils that condition. On the other hand, computations made by computer program, based on backtracking, showed us that W_5 does not admit an antiparallel stable trace. We can also prove this theoretically. An arbitrary cubic of W_5 , denote it with W'_5 , has eight vertices and 12 edges. Every spanning tree T' in W'_5 has seven edges, hence $W'_5 - E(T')$ has five edges. By Theorem 3.1, W'_5 does not admit an antiparallel proper trace, hence it does not admit an antiparallel stable trace as well. From Lemma 3.3 it then follows, that W_5 does not admit an antiparallel stable trace as well. We have thus proved that the condition before Proposition 3.2 is not sufficient in general.

Let we call spanning trees that fulfills the condition from Proposition 3.2 as *even spanning trees*. We next prove three lemmas about even spanning trees.

Lemma 3.4 *Let G be a graph with $\delta(G) = 3$ and $\Delta(G) = 4$, which has an even spanning tree. If v is a unique vertex of degree 4, then at least one cubic of G has an even spanning tree.*

Proof. Let T be an even spanning tree of G . Denote the neighbors of v with v_1, v_2, v_3 , and v_4 , and the edges connecting them to v with e_1, e_2, e_3 , and e_4 , respectively.

We claim that at least one cubic of G has an even spanning tree. In a cubic of G , v is replaced with two adjacent vertices v' and v'' . Denote the edge connecting them with e . Without loss of generality, v' and v'' can be adjacent to neighbors of v in any order (as long as they are both adjacent to exactly two of them). Depending on an even spanning tree T , we will determine how v' , v'' , and neighbors of v are connected with each other in a cubic of G .

Construct T' in a cubic of G from T as follows. First put every edge from T in T' . Then also put the new edge e in T' .

We claim that there exists an arrangement of neighbors of v between v' and v'' in a cubic of G , such that T' is spanning tree in a cubic of G . T' is connected because T is connected. Every vertex from G , except v , lies in T' because every vertex from G lies in T . Because e is in T' , v' and v'' lie in T' . It is not difficult to see that if adding an edge e in T' would make a cycle C in T' , C would already be in T . We conclude that T' is spanning tree in a cubic of G (we still did not arrange neighbors of v to v' and v''). We claim that for at least one cubic of G , denote it with G' , T' is an even spanning tree. Next we have to determine, how v' , v'' and neighbors of v are adjacent in G' , so that T' would be an even spanning tree.

We first notice that $G - E(T)$ and $G' - E(T')$ distinguish only in v . Next we observe that every vertex of G' lies in exactly one component of $G' - E(T')$. From the construction of T' it is also obvious, that if v lies in component of $G - E(T)$ without any edge, then v' and v'' lie in two components of $G' - E(T')$ without any edge, hence no matter how we connect the neighbors of v to v' and v'' , the tree T' is an even spanning tree.

Assume now that v lies in a component of $G - E(T)$ with even number of edges (> 0). Denote this component with B and consider two cases. In the first case let B and v have exactly one common edge f ; without loss of generality let $f = e_1$ (that means that e_2 , e_3 and e_4 are edges of T). Connect f and e_2 with v' in G' , and connect e_3 and e_4 with v'' in G' . Then v' would lie in an even component of $G' - E(T')$ and v'' would lie in a component of $G' - E(T')$ without any edge, hence T' is an even spanning tree.

In the second case B and v have more than one common edge. That means that we can without loss of generality assume that e_1 and e_2 are not edges of T . Those common edges lie on some edge-disjoint paths in $G' - E(T') - v$. We have to consider two subcases. In the first subcase at least two of those paths, say P' and P'' , have another common vertex u . We may assume that e_1 lies on P' and e_2 lies on P'' . Connect v' with e_1 and e_3 , and connect v'' with e_2 and e_4 in G' . Then the component B stays connected in $G' - E(T')$, and v' and v'' lie in a common component of $G' - E(T')$ with even number of edges, hence T' is an even spanning tree.

In the second subcase, v is the only vertex where the paths on which common edges of B and v intersect in $G' - E(T')$. If some of these paths are of odd length, there should be an even number of them, because there is an even number of edges in B . If there is no such paths we could arrange v_1, v_2, v_3 , and v_4 between v' and v'' in any order.

Otherwise, it follows that there are two paths P' and P'' of odd length (if there were four, v would not lie in T , because no edge incident with v would be in T). Without loss of generality e_1 lies on P' and e_2 lies on P'' . Connect v' with e_1 and e_2 , and connect v'' with e_3 and e_4 in G' . Then v' and v'' lie on two even components of $G' - E(T')$, hence T' is an even spanning tree. We conclude that G' has an even spanning tree T' . \square

Lemma 3.5 *Let G be a graph with $\delta(G) = 3$ and $\Delta(G) \geq 4$, which has an even spanning tree. If v is a unique vertex of degree greater than 3, then at least one cubic of G has an even spanning tree as well.*

Proof. Let T be an even spanning tree of G . We proceed by induction on $\Delta = \Delta(G) = d(v)$.

The base of the induction follows by Lemma 3.4.

Assume now that $\Delta > 4$. Denote the neighbors of v with v_1, \dots, v_Δ and construct G' as follows. Remove the vertex v from G , add two new vertices v' and v'' , and connect them by an edge e . Connect v' with two neighbors of v , and connect v'' with the remaining ones. Similarly as in the proof of Lemma 3.4, v' and v'' can be adjacent to neighbors of v in any order (as long as v' is adjacent to exactly two of them and v'' is adjacent to all of the remaining ones). Depending on an even spanning tree T , we will again determine how the neighbors of v are connected to v' and v'' in G' .

We claim that G' is a graph with $\delta(G') = 3$, $\Delta(G') < \Delta$ and a unique vertex v'' of degree greater than 3, which has an even spanning tree T . It is not difficult to see that G' fulfills the first three conditions. Similarly as in the proof of Lemma 3.4, we next prove that G' has an even spanning tree. The only difference between the arguments is in the last subcase, because here can be more than two paths of odd length in B adjacent to v . However, their number is still even, so we connect two of them to v' and all the others (also paths of even length) to v'' .

We conclude that G' has an even spanning tree T' . By induction assumption, at least one cubic of G' has an even spanning tree T'' . Since every cubic of G' is also a cubic of G , at least one cubic of G has an even spanning tree. \square

Lemma 3.6 *If a graph G with $\delta(G) \geq 3$ has an even spanning tree, then at least one cubic of G has an even spanning tree.*

Proof. Let T be an even spanning tree of G . We proceed by induction on the number k of vertices of degree greater than 3.

Let $k = 0$. Then G is cubic and a cubic of G is isomorphic to G . Let next $k = 1$. By Lemma 3.5 at least one cubic of G has an even spanning tree.

Assume now that $k > 1$. Analogously as in case $k = 1$, we replace one of high degree vertices and construct a graph G' with $k - 1$ vertices of degree greater than 3

and an even spanning tree T' . By induction assumption, also cubic of G' has such a spanning tree. Since every cubic of G' is also a cubic of G , at least one cubic of G has an even spanning tree. \square

Using Lemmas 3.4, 3.5, and 3.6, we get:

Theorem 3.7 *If a graph G has an even spanning tree T and $\delta(G) \geq 3$, then G admits an antiparallel stable trace.*

Proof. If G is a cubic graph, then Proposition 3.2 claims that G admits an antiparallel stable trace. Assume now that $\Delta(G) \geq 4$. By Lemma 3.6 at least one cubic of G , denote it with G' , has an even spanning tree T' . Moreover, because G' is cubic, by Proposition 3.2, G' admits an antiparallel stable trace S' . We next construct a stable trace S in G as in the first case of the proof of Theorem 2.2 (by ignoring occurrences of edges newly created in a cubic of G). It is straightforward to see that S is an antiparallel stable trace. \square

To conclude the section we pose:

Problem 3.8 *Is it true that a graph G admits an antiparallel stable trace if and only if $\delta(G) \geq 3$ and G has an even spanning tree T ?*

4 Concluding remarks

In this section we present two concepts for constructing parallel stable traces. Unfortunately, when proving Theorem 2.2, we found examples of graphs, where either the first or the second concept cannot be applied. So both concepts presented here cannot be used in general.

The first idea how to construct parallel stable traces goes as follows. Let G be an Eulerian graph with n vertices (denoted with v_1, \dots, v_n) fulfilling conditions of Theorem 2.2 and let T be an Eulerian circuit of G . T induces a set of functions $\Pi = \{\pi_1, \dots, \pi_n\}$, where $\pi_i : V(G) \setminus \{v_i\} \rightarrow V(G) \setminus \{v_i\}$, $\pi_i(v) = u$ if and only if $v \rightarrow v_i \rightarrow u$ is a sequence in T , for $1 \leq i \leq n$. Note that $u \neq v$, because G is simple and T traverses every edge exactly once. Construct another Eulerian circuit T' in G such that it will induce a set of functions $\Pi' = \{\pi'_1, \dots, \pi'_n\}$ with above described characteristics. In addition demand, that edges are traversed in the same direction as in T , and that if $\pi_i(v) = u$ then $\pi'_i(v) \neq u$ and $\pi'_i(u) \neq v$. Let $f = xy$ be the last traversed edge in T . Concatenate Eulerian circuits T and T' in y to get a trace S . By construction it is obvious that in S every edge is traversed twice in the same direction and that S is without any retracing and repetition. Hence, if a graph G admits two Eulerian circuits with above described characteristic, G admits parallel stable trace as well.

It turns out that, we cannot always construct a parallel stable trace of G by concatenating T and T' . For instance, the graph G from Fig. 6 has a parallel stable trace: $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1 \rightarrow v_2 \rightarrow v_4 \rightarrow v_1 \rightarrow v_5 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_6 \rightarrow v_5 \rightarrow v_2 \rightarrow v_4 \rightarrow v_6 \rightarrow v_7 \rightarrow v_9 \rightarrow v_8 \rightarrow v_6 \rightarrow v_7 \rightarrow v_{10} \rightarrow v_8 \rightarrow v_{11} \rightarrow v_7 \rightarrow v_9 \rightarrow v_{10} \rightarrow v_{11} \rightarrow v_7 \rightarrow v_{10} \rightarrow v_{11} \rightarrow v_9 \rightarrow v_8 \rightarrow v_{11} \rightarrow v_9 \rightarrow v_{10} \rightarrow v_8 \rightarrow v_6 \rightarrow v_5 \rightarrow v_3 \rightarrow v_1 \rightarrow v_5 \rightarrow v_3 \rightarrow v_4 \rightarrow v_1$, but because of the cut vertex v_6 , from any Eulerian circuit T of G we cannot construct another Eulerian circuit using the described construction.

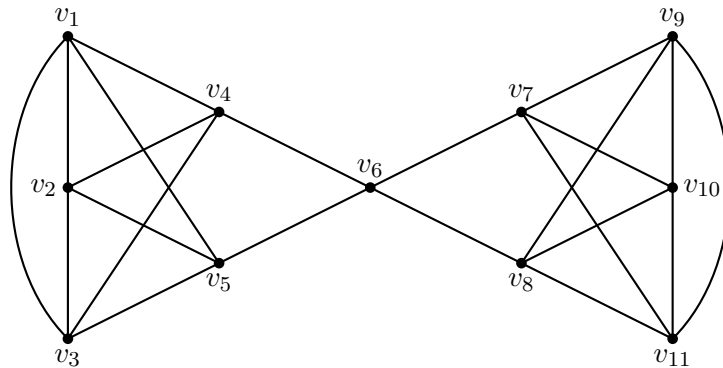


Figure 6: Graph whose parallel stable trace cannot be constructed by concatenating two Eulerian circuits

Realizing that cut vertices cause problems, we could try to use another approach. Let G be an Eulerian graph fulfilling the conditions of Theorem 2.2. Denote blocks of G with B_1, \dots, B_k and cutvertices with v_1, \dots, v_{k-1} , where for cutvertex v which separates B_i and B_j the following is true: $v \in B_i \cap B_j$. Let T_1, \dots, T_k be parallel stable traces in B_1, \dots, B_k respectively. Note again that cutvertex v which separates B_i and B_j appears in both T_i and T_j . Let first $k = 1$. Then T_1 is also a parallel stable trace of G . Assume now that $k = 2$ and let v be the unique cutvertex of G . Construct a double trace T in G as follows. Start in an arbitrary vertex of B_1 and continue on T_1 until coming to v . Traverse then every edge of T_2 until finishing in v . Traverse now the rest of the edges in T_1 . Since every edge e of G is traversed twice in the same direction in T_1 or in T_2 , the edge e is traversed twice in the same direction in T . Hence T is a parallel stable trace. Let next $e = xy$ and $f = yz$ be two consecutive edges of T . If $\{x, y, z\} \cap \{v\} = \emptyset$, then e and f does not give a repetition through y (retracing) because otherwise we would have a repetition through y (retracing) in T_1 or T_2 . The same conclusion holds if $x = v$ or $z = v$. Assume hence that $y = v$. Then e and f does not give a repetition through v (retracing) by construction. We have thus find an algorithm for construction of parallel stable traces in graphs with at most 2 blocks, with assumption that we can found a parallel stable trace in every block of graph.

We proceed by induction on the number k of blocks of graph G . Let $k > 2$ and

assume that for any graph H with strictly less than k blocks we can construct a parallel stable trace with above described construction (if we can find a parallel stable trace in every block of graph). Let v be an arbitrary cutvertex. Without loss of generality we can assume that $v \in B_1 \cap B_2$. Then construct a parallel stable trace T' in $B_1 \cup B_2$ from T_1 and T_2 the same way as above. Because $|\{T', T_3, \dots, T_k\}| < k$, by induction above described algorithm will find a parallel stable trace in G .

Again we cannot always construct a parallel stable trace of G using this construction. The problem lies in an assumption that we can find a parallel stable trace in every block of graph G . As we have seen before, graph G from Fig. 6 admits a parallel stable trace. Vertex v_6 is its unique cutvertex. Because in both blocks of G vertex v_6 is of degree 2, by Theorem 2.2 blocks of G do not admit parallel stable traces. Similar problem occurs if one or more blocks of G are bridges.

If we instead of parallel stable traces in blocks demand parallel proper traces where repetitions occur only at cutvertices (retracings if block is bridge), we can still get parallel stable trace of G when concatenating those smaller parallel proper traces together. But even this modification is not enough to produce an algorithm which would work in general. By Theorem 2.2 the graph H from Fig. 7 has a parallel stable trace. However, two of its blocks have vertices of degree 3 (v_1, v_2, v_3 , and v_4) and therefore by Proposition 2.1 do not admit neither parallel proper trace nor parallel stable trace.

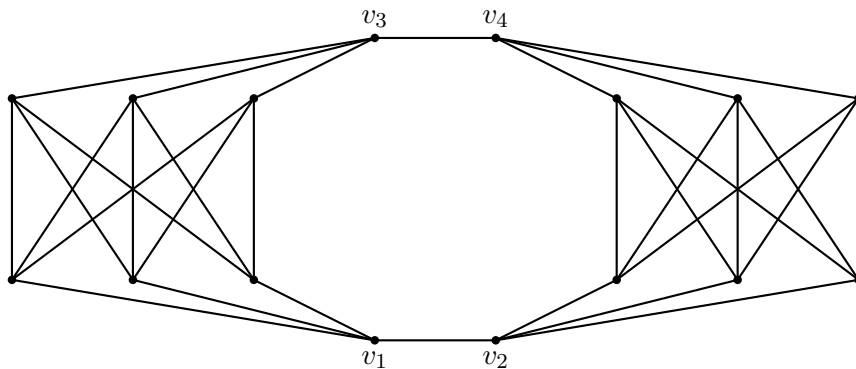


Figure 7: Graph whose parallel stable trace cannot be constructed by concatenating parallel stable traces in blocks of a graph

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