

On the rank two geometries of the groups $\mathrm{PSL}(2, q)$: part I

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Abstract

We determine all firm and residually connected rank 2 geometries on which $\mathrm{PSL}(2, q)$ acts flag-transitively, residually weakly primitively and locally two-transitively, where one of the maximal parabolic subgroups is isomorphic to $E_q : \frac{(q-1)}{(2, q-1)}$, where E_q denotes an elementary abelian group of order q , or $D_{2n(q)}$, the dihedral group of order $2n(q)$ where $n(q) := \frac{(q\pm 1)}{\gcd(2, q-1)}$ for some prime-power q .

Keywords: Projective special linear groups, coset geometries, locally s -arc-transitive graphs.

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1 Introduction

Following Tits' geometric interpretation of the exceptional complex Lie groups [23, 25], Francis Buekenhout generalized in [2] and [3] certain aspects of this theory in order to achieve a combinatorial understanding of all finite simple groups. Since then, two main approaches have been developed in diagram geometry. One is to classify geometries over a given diagram, mainly geometries extending buildings (see for example [7], chapter 22 for a survey and [24] for the theory of buildings). Another approach is to classify coset geometries for a given group under certain conditions. Rules for such classifications have been stated by Buekenhout in [5] and [6].

Since 1993, several people, including Olivier Bauduin, Francis Buekenhout, Philippe Cara, Michel Dehon, Maya Gailly, Harald Gottschalk, Xavier Miller, Koen Vanmeerbeek and the authors, have classified geometries under the following assumptions. The geometries obtained must be firm, residually connected, flag-transitive and residually weakly primitive, and they must satisfy the intersection property of rank two. Moreover, since

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2000, Buekenhout, Cara, Dehon and Leemans decided to add one further axiom: the geometries must also be locally two-transitive. For a detailed survey of this work, we refer to [9]. Most of these classifications have been obtained using a series of CAYLEY or more recently MAGMA programs (see [12] for the programs, and [1] for MAGMA). We refer to [18] and its references for the most recent classifications related to the nine smallest sporadic simple groups. For his doctorate thesis, Leemans classified in a series of papers the residually weakly primitive geometries of the Suzuki simple groups (see [17] and its references). That classification does not rely on computer algebra. The aim of this paper is to start a similar classification for the groups $\text{PSL}(2, q)$. We present the classification of rank two coset geometries for the groups $\text{PSL}(2, q)$, with q a power of a prime, satisfying all of the above conditions but the intersection property. We choose to study these simple groups because, after the Suzuki groups, they have the simplest subgroup structure and are therefore the groups for which a theoretical classification of their geometries is more likely to succeed. Moreover, these groups are embedded in lots of other simple groups, especially in sporadic groups. We focus on the cases where one of the maximal parabolic subgroups is isomorphic to $E_q : \frac{(q-1)}{(2, q-1)}$ (where $(2, q-1)$ denotes the greatest common divisor of 2 and $q-1$) or $D_{2n(q)}$. The other cases will be analysed in a subsequent paper.

The paper is organised as follows. In Section 2, we recall the basic definitions and notations needed in this paper. In Section 3, we give a sketch proof of our main result:

Theorem 1.1. *Let $G \cong \text{PSL}(2, q)$ and $\Gamma(G; \{G_0, G_1, G_0 \cap G_1\})$ be a locally two-transitive RWPRI coset geometry of rank two. If G_0 is isomorphic to one of $E_q : \frac{(q-1)}{(2, q-1)}$ or $D_{2n(q)}$, then Γ is isomorphic to one of the geometries appearing in Table 1.*

The Buekenhout diagrams of the geometries are given in the statements of Propositions 5.3, 5.8 and 5.12. In Section 4, we recall the subgroup lattice of $\text{PSL}(2, q)$, and we give the two-transitive representations of the maximal subgroups. In Section 5, we prove Theorem 1.1, which is based on the proof of Propositions 5.3, 5.8 and 5.12. For that, we determine the rank 2 RWPRI and $(2T)_1$ geometries of $\text{PSL}(2, q)$ and their number, up to isomorphism and up to conjugacy. We also determine their Buekenhout diagrams.

The geometries obtained in Theorem 1.1 are also locally 2-arc-transitive graphs in the sense of [15]. These graphs are interesting in their own right because of the numerous connections they have with other fields of mathematics (see [15] for more details). We also refer to the classification of these graphs for almost simple groups with socle a Ree simple group $\text{Ree}(q)$ (see [14]). In terms of locally 2-arc-transitive graphs, we obtain here the classification of these graphs with one vertex-stabilizer maximal in $\text{PSL}(2, q)$ and isomorphic to $E_q : \frac{(q-1)}{(2, q-1)}$ or $D_{2n(q)}$. The last column of Table 1 gives, for each geometry Γ , the value of s such that Γ is a locally s -arc-transitive graph but not a locally $(s+1)$ -arc-transitive.

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2 Definitions and notations

In this section, we recall the basic notions on coset geometries and give the definitions needed to understand this paper.

	$G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$			$q = p^n,$ with p prime	
$G_0 \cap G_1$	G_1	‡ Geom. up to conj.	‡ Geom. up to isom.	Extra condi- tions on q	locally(G, s)-arc- transitive graphs
$(q-1)$	$E_q : (q-1)$	1	1	$q = 2^n$	$s = 2$
$(q-1)$	$D_{2(q-1)}$	1	1	$q = 2^n$	$s = 3$
	$G_0 \cong D_{2 \frac{q-1}{(2, q-1)}}$			$q = p^n,$ with p prime	
$G_0 \cap G_1$	G_1	‡ Geom. up to conj.	‡ Geom. up to isom.	Extra condi- tions on q	locally(G, s)-arc- transitive graphs
$(q-1)$	$E_q : (q-1)$	1	1	$q = 2^n$	$s = 3$
2	S_3	1	1	$q = 4$	$s = 2$
2	2^2	1	1	$q = 4$	$s = 3$
2^2	D_{12}	2	1	$q = 13$	$s = 3$
D_8	S_4	2	1	$q = 17$	$s = 7$
D_8	S_4	2	1	$q = 25$	$s = 3$
D_{10}	A_5	1	1	$q = 16$	$s = 3$
D_{10}	A_5	2	1	$q = 31$	$s = 3$
	$G_0 \cong D_{2 \frac{q+1}{(2, q+1)}}$			$q = p^n,$ with p prime	
$G_0 \cap G_1$	G_1	‡ Geom. up to conj.	‡ Geom. up to isom.	Extra condi- tions on q	locally(G, s)-arc- transitive graphs
2	S_3	1	1	$q = 5$	$s = 2$
2	2^2	1	1	$q = 5$	$s = 3$
3	A_4	1	1	$q = 5$	$s = 3$
2^2	D_{12}	2	1	$q = 11$	$s = 3$
D_6	S_4	2	1	$q = 17$	$s = 3$
D_8	S_4	2	1	$q = 23$	$s = 3$
D_{10}	A_5	2	1	$q = 19, 29$	$s = 3$

Table 1: The RWPRI and $(2T)_1$ geometries with $G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$ or $D_{2n(q)}$.

2.1 Coset geometries, their properties and Buekenhout diagram

A general reference for diagram geometries and their properties is [7]. Let $I = \{1, \dots, n\}$ be a finite set, called the *type set*. Its elements are called *types*. Let G be a group and $(G_i)_{i \in I}$ be a collection of distinct subgroups of G . Let $X := \{G_i g : g \in G, G_i \in (G_i)_{i \in I}\}$ be the set of their cosets. We define a *pregeometry* $\Gamma = \Gamma(G; (G_i)_{i \in I}) = (X, *, t)$ provided with a *type function* $t : G_i g \mapsto i$ and an *incidence relation* $* \subset X \times X$, such that

$$G_i g * G_j h \text{ if and only if } G_i g \cap G_j h \neq \emptyset.$$

The number $n = |I|$ is called the *rank* of Γ . A *flag* \mathcal{F} of Γ is a set of pairwise incident elements, and $t(\mathcal{F}) := \{t(x) : x \in \mathcal{F}\}$ is called its *type*. A flag \mathcal{C} with $t(\mathcal{C}) = I$ is called a *chamber*. If every flag of Γ is contained in a chamber, Γ is called a (*coset*) *geometry*. A geometry is *firm* (resp. *thin*, *thick*) if any flag which is not a chamber is contained in at least two (resp. exactly two, at least three) chambers.

The *residue* of a flag \mathcal{F} of Γ is the geometry $\Gamma_{\mathcal{F}}$ consisting of the elements of $\Gamma \setminus \mathcal{F}$ incident with all elements of \mathcal{F} , together with the restricted type-function and induced incidence relation. Let \mathcal{F} be a flag of type $J \subset I$. Then $\Gamma_{\mathcal{F}}$ is a geometry over the typeset $I - J$. A coset geometry Γ is called *residually connected* if the incidence graph of every residue of rank at least two is connected. For any $J \subset I$, we set $G_J := \bigcap_{j \in J} G_j$, $B := G_I$. We call $\mathcal{L}(\Gamma) := \{G_J : J \subset I\}$ the *sublattice* (of the subgroup lattice of G) *spanned* by the collection $(G_i)_{i \in I}$. The group B is said to be the *Borel subgroup* of $\mathcal{L}(\Gamma)$. We say that $\mathcal{L}(\Gamma)$ is *strongly boolean* if, for any two elements of $\mathcal{L}(\Gamma)$, their lowest upper bound in $\mathcal{L}(\Gamma)$ is the subgroup that they generate in G . The following result is used to check whether a pregeometry Γ is a residually connected geometry.

Lemma 2.1. [24] *Let $\Gamma = \Gamma(G; (G_i)_{i \in I})$ be a pregeometry. Then Γ is a residually connected pregeometry if and only if $\mathcal{L}(\Gamma)$ is strongly boolean.*

We call G *flag-transitive* if G acts transitively on the set of flags of type J for all subsets J of I .

In this paper, we are interested only in rank two geometries. The following lemma is obvious.

Lemma 2.2. *Let $\Gamma = \Gamma(G; \{G_0, G_1\})$ be a pregeometry of rank two. Then Γ is a geometry and G acts flag-transitively on it.*

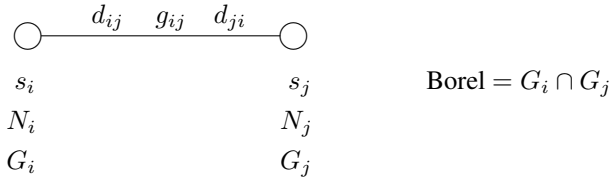
We call Γ *primitive* provided that G acts primitively on the set of i -elements of Γ , for each $i \in I$. Moreover, we call Γ *residually primitive* (RPRI) if each residue Γ_F of a flag F is primitive for the group induced on Γ_F by the stabilizer G_F of F . We say that Γ is *weakly primitive* (WPRI) if G_i is maximal in G for at least one $i \in I$. Moreover, Γ is said to be *residually weakly primitive* (RWPRI) provided that $\Gamma_{\mathcal{F}}$ is WPRI for every flag \mathcal{F} .

We say that Γ satisfies the *intersection property of rank two* $(IP)_2$ if every residue of rank two is either a partial linear space or a generalized digon. Note that this condition excludes all $2 - (v, k, \lambda)$ designs, $\lambda \geq 2$, except the generalized digons.

We call Γ *locally 2-transitive* and we write $(2T)_1$ for this, provided that the stabilizer $G_{\mathcal{F}}$ of any flag \mathcal{F} of rank $n - 1$ acts 2-transitively on the residue $\Gamma_{\mathcal{F}}$.

Following [2], the *Buekenhout diagram* of a firm, residually connected, flag-transitive geometry Γ is a graph together with additional structure, whose vertices are the elements of I , which is further described as follows. To each vertex $i \in I$, we attach the *order* s_i that

is $|\Gamma_F| - 1$, where F is any flag of type $I \setminus \{i\}$, the number n_i of varieties of type i , that is the index of G_i in G , and the subgroup G_i . Elements i, j of I are not joined by an edge of the diagram provided a residue Γ_F of type $\{i, j\}$ is a generalized digon. Otherwise, i and j are joined by an edge endowed with three positive integers d_{ij}, g_{ij}, d_{ji} , where g_{ij} (the gonality) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i, j\}$ and d_{ij} (resp. d_{ji}), the i -diameter (resp. j -diameter) is the greatest distance from some fixed i -element (resp. j -element) to any other element in the incidence graph of Γ_F . On a picture of the diagram, this structure will often be depicted as follows.



If $g_{ij} = d_{ij} = d_{ji} = n$, then Γ_F is called a *generalized n -gon* and we do not write d_{ij} and d_{ji} . If $d_{ij} = g_{ij} = 3, d_{ji} = 4$ and $s_i = 1$, we write c instead of $g_{ij} d_{ij} d_{ji}$.

We classify the rank two geometries $\Gamma = \Gamma(G; \{G_0, G_1\})$ for $G = \text{PSL}(2, q)$, with q a power of a prime, under the following conditions. The geometry Γ must be firm, residually connected, and the group G must act flag-transitively and residually weakly primitively on Γ . Moreover, Γ must be locally two-transitive. Our search for such geometries is based on the following obvious lemma.

Lemma 2.3. *If $\Gamma = \Gamma(G; \{G_0, G_1\})$ is residually weakly primitive, then Γ is firm and residually connected.*

Hence Lemmas 2.2 and 2.3 imply that we only have to test the (RWPRI) and $(2T)_1$ conditions.

Let G be a group and $\text{Aut}(G)$ be its automorphism group. The coset geometries $\Gamma(G; \{G_0, G_1\})$ and $\Gamma(G; \{G'_0, G'_1\})$ are *conjugate* (resp. *isomorphic*) provided there exists an element $g \in G$ (resp. $g \in \text{Aut}(G)$) such that $\{G^g_0, G^g_1\} = \{G'_0, G'_1\}$ (resp. $\{g(G_0), g(G_1)\} = \{G'_0, G'_1\}$). We classify geometries up to conjugacy and up to isomorphism. That is, for each triple $\{G_0, G_1, G_0 \cap G_1\}$, we give the number of classes of geometries with respect to conjugacy and isomorphism.

From now on, we denote $\Gamma = \Gamma(G; \{G_0, G_1\})$ by $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ and we say this a coset geometry of rank two.

Throughout this paper, we use the notation of the ATLAS [11] for groups.

2.2 Locally s -arc-transitive graphs

The following definitions are taken from [15]. Let \mathcal{G} be a finite simple undirected connected graph. Denote by V (resp. E) its vertex-set (resp. edge-set). The edge-set may be identified with a subset of unordered pairs of elements of V . An s -arc is an ordered $(n+1)$ -tuple $(\alpha_0, \dots, \alpha_n)$ of vertices such that $\{\alpha_{i-1}, \alpha_i\}$ is an edge of \mathcal{G} for all $i = 1, \dots, n$ and $\alpha_{j-1} \neq \alpha_{j+1}$ for all $j = 1, \dots, n - 1$. Let G be a subgroup of the automorphism group $\text{Aut}(\mathcal{G})$ of \mathcal{G} . The graph \mathcal{G} is said to be (G, s) -arc-transitive if G is transitive on the set of s -arcs of \mathcal{G} ; also, \mathcal{G} is said to be s -arc-transitive if it is $(\text{Aut}(\mathcal{G}), s)$ -arc-transitive. Similarly \mathcal{G} is said to be $(G, 1)$ -arc-transitive if G is transitive on the 1-arcs of \mathcal{G} , that is on

the ordered pairs (α_0, α_1) where $\{\alpha_0, \alpha_1\}$ is an edge of \mathcal{G} . Given $G \leq \text{Aut}(\mathcal{G})$, we call \mathcal{G} *locally (G, s) -arc-transitive* if \mathcal{G} contains an s -arc and given any two s -arc α and β starting at the same vertex v , there exists an element $g \in G_v$ mapping α to β . We say \mathcal{G} is *locally s -arc-transitive* if it is locally (G, s) -arc-transitive for some $G \leq \text{Aut}(\mathcal{G})$.

Obviously, 1-arc transitivity is equivalent to flag-transitivity if the graph \mathcal{G} is seen as a rank two geometry whose elements of type 0 (resp. 1) are the vertices (resp. edges) of \mathcal{G} . Moreover, local 2-arc transitivity is equivalent to property $(2T)_1$ defined in the previous section.

3 Sketch of the proof of Theorem 1.1

Let G be a group. Let G_0 and G_1 be subgroups of G and let $G_{01} := G_0 \cap G_1$. The RWPRI condition in rank two requires that either G_0 or G_1 is a maximal subgroup of G and that G_{01} is a maximal subgroup of G_0 and G_1 . The $(2T)_1$ condition requires that both G_0 and G_1 act two-transitively on the respective cosets of G_{01} . We use the following method to determine all the RWPRI and $(2T)_1$ geometries of the projective special linear group $\text{PSL}(2, q)$. We choose a subgroup G_0 and we insist that it must be a maximal subgroup of $\text{PSL}(2, q)$. Then we determine all the possibilities for G_{01} . They are the maximal subgroups of G_0 such that the action of G_0 on the cosets of G_{01} in G_0 is two-transitive. Finally, we look at all the subgroups G_1 which contain G_{01} as a maximal subgroup and act two-transitively on the cosets of G_{01} in G_1 . Therefore, to achieve our goal, we first must determine what are the possibilities for G_0 . These are the maximal subgroups of G . They are given in the next section, as well as their maximal subgroups, the possibilities for G_{01} . In Section 5, we then determine what are the possibilities for G_1 .

4 Structure of subgroups of $\text{PSL}(2, q)$

To follow the approach described above, we first recall the list of subgroups of the projective special linear groups $\text{PSL}(2, q)$. We then give the list of maximal subgroups of $\text{PSL}(2, q)$ and the list of the maximal subgroups of each maximal subgroup of $\text{PSL}(2, q)$. Finally, in order to be able to check the $(2T)_1$ property, we determine the two-transitive representations of the maximal subgroups of $\text{PSL}(2, q)$.

4.1 The subgroups of $\text{PSL}(2, q)$

The subgroup structure of $\text{PSL}(2, q)$ may be found in Dickson [13] or Huppert [16]. It was first obtained in papers by Moore [20] and Wiman [27].

Lemma 4.1. [Dickson] *The group $\text{PSL}(2, q)$ of order $\frac{q(q^2-1)}{(2, q-1)}$, where $q = p^n$ (p prime), contains only the following subgroups:*

1. $q + 1$ elementary conjugate abelian subgroups of order q , denoted by E_q .
2. $\frac{q(q \pm 1)}{2}$ cyclic conjugate subgroups of order $\frac{(q \pm 1)}{(2, q-1)}$, denoted by $\frac{(q \pm 1)}{(2, q-1)}$.
3. $\frac{q(q \pm 1)}{2}$ cyclic conjugate subgroups of order d for every divisor d of $\frac{(q \pm 1)}{(2, q-1)}$, denoted by d .
4. $\frac{q(q^2-1)}{2d(2, q-1)}$ dihedral groups of order $2d$, denoted by D_{2d} for every $d > 2$ dividing $n(q) := \frac{(q \pm 1)}{(2, q-1)}$. The number of conjugacy classes of these subgroups is one or two,

depending on whether $\frac{q \pm 1}{d(2, q-1)}$ is odd or even.

5. For q odd, $\frac{q(q^2-1)}{12(2, q-1)}$ dihedral groups of order 4 denoted by 2^2 . The number of conjugacy classes of these groups is one if $q \equiv \pm 3(8)$ and two if $q \equiv \pm 1(8)$. If q is even, the groups 2^2 are in the family 6.
6. There are sets of $\frac{q^2-1}{(2,1,1)(p^k-1)}$ conjugate elementary abelian subgroups of order p^m , denoted by E_{p^m} for all natural number m , such that $1 \leq m \leq n-1$, where k is a common divisor of n and m and $(2, 1, 1)$ is equal to 2 (resp. 1, 1) if $p > 2$ and $\frac{n}{k}$ is even (resp. $p > 2$ and $\frac{n}{k}$ is odd, $p = 2$).
7. There are sets of $\frac{(q^2-1)p^{n-m}}{(2,1,1)(p^k-1)}$ conjugate subgroups $E_{p^m} : d$ which are semidirect products of an elementary abelian group E_{p^m} and a cyclic group of order d , for every natural number m such that $1 \leq m \leq n$ and every natural number d dividing $\frac{p^k-1}{(1,2,1)}$, where k is a common divisor of n and m and $(1, 2, 1)$ is equal to 1 (resp. 2, 1) if $p > 2$ and $\frac{n}{k}$ is even (resp. $p > 2$ and $\frac{n}{k}$ is odd, $p = 2$).
8. For q odd or $q = 4^m$, $\frac{q(q^2-1)}{12(2, q-1)}$ subgroups isomorphic to A_4 . The number of conjugacy classes of these groups is one if $q \equiv \pm 3(8)$ or q even and two if $q \equiv \pm 1(8)$.
9. For $q \equiv \pm 1(8)$, two classes of $\frac{q(q^2-1)}{24(2, q-1)}$ symmetric groups S_4 , of order 24.
10. For $q \equiv \pm 1(5)$, two classes of $\frac{q(q^2-1)}{60(2, q-1)}$ alternating groups A_5 (of order 60) and for $q = 4^m$ one class of $\frac{q(q^2-1)}{60(2, q-1)}$ alternating groups A_5 .
11. $\frac{q(q^2-1)}{p^w(p^{2w}-1)}$ groups $\text{PSL}(2, p^w)$ for every divisor w of r . The number of conjugacy classes of these groups is two (resp. one) if $p > 2$ and $\frac{r}{w}$ is even (resp. $p > 2$ and $\frac{r}{w}$ is odd, $p = 2$).
12. For q odd, two classes of $\frac{q(q^2-1)}{2p^w(p^{2w}-1)}$ groups $\text{PGL}(2, p^w)$ for every even divisor $2w$ of r .

Remark 4.2. A_5 subgroups are given either by case 10 (when $q \equiv \pm 1(5)$ and $q = 4^m$) or by case 11 (when $q \equiv 0(5)$) of Lemma 4.1. Also, case 12 is a subcase of case 11 provided q is even.

Remark 4.3. In the cases 6 and 7 of Lemma 4.1, the numbers of conjugacy classes are not given. The number of conjugacy classes of the elementary abelian subgroups E_{p^m} given by Dickson (see [13], §260) is wrong. Take for instance $\text{PSL}(2, 64)$ and the subgroups isomorphic to E_{16} . Dickson's result gives 651 classes of such subgroups. Using MAGMA [1] we find there are 11 conjugacy classes of subgroups E_{16} . Ten classes are of length 4095 and one is of length 1365, the lengths given by Dickson.

Notice that Dickson does not give the number of conjugacy classes of the subgroups $E_{p^m} : d$, except in the particular case where $m = n$ and $d = \frac{p^n-1}{(2, q-1)}$. There are $q+1$ subgroups $E_q : \frac{q-1}{(2, q-1)}$, all conjugate.

4.2 Maximal subgroups of $\text{PSL}(2, q)$

Table 2 and 3 give lists of the maximal subgroups of $\text{PSL}(2, q)$ in the case where q is even or odd respectively. This result is known; see for example [22].

Structure	Order	Index
$E_q : (q - 1)$	$q(q - 1)$	$q + 1$
$D_{2(q+1)}$ $q \neq 2$	$2(q + 1)$	$\frac{q(q-1)}{2}$
$D_{2(q-1)}$	$2(q - 1)$	$\frac{q(q+1)}{2}$
A_5 $q = 4^r$ and r is a prime	60	$\frac{q(q^2-1)}{60}$
$\text{PSL}(2, q') \cong \text{PGL}(2, q')$ $q' > 2, q = q'^m, m$ is prime or $q' = 2, q = q'^2$	$q'(q'^2 - 1)$	$\frac{q(q^2-1)}{q'(q'^2-1)}$

Table 2: The maximal subgroups of $\text{PSL}(2, q)$, for q even

Structure	Order	Index
$E_q : \frac{q-1}{2}$	$\frac{q(q-1)}{2}$	$q + 1$
$D_{(q+1)}$ $q \neq 7, 9$	$q + 1$	$q(q - 1)$
$D_{(q-1)}$ $q \neq 3, 5, 7, 9, 11$	$q - 1$	$\frac{q(q+1)}{2}$
A_4 if $q = p > 3$ and $q \equiv 3, 13, 27, 37(40)$ or $q = 5$	12	$\frac{q(q^2-1)}{12 \times 2}$
S_4 if $q = p > 2$ and $q \equiv \pm 1(8)$	24	$\frac{q(q^2-1)}{24 \times 2}$
A_5 if $\begin{cases} q = 5^r & r \text{ odd prime} & \text{or} \\ p = q \equiv \pm 1(5) & p \text{ prime} & \text{or} \\ q = p^2 \equiv -1(5) & p \text{ prime} \end{cases}$	60	$\frac{q(q^2-1)}{60 \times 2}$
$\text{PSL}(2, q')$ $q' > 2, q = q'^m, m$ odd prime	$\frac{q'(q'^2-1)}{2}$	$\frac{q(q^2-1)}{q'(q'^2-1)}$
$\text{PGL}(2, q')$ $q = q'^2$	$q'(q'^2 - 1)$	$\frac{q(q^2-1)}{q'(q'^2-1)}$

Table 3: The maximal subgroups of $\text{PSL}(2, q)$, for q odd

G_0	G_{01}
$E_q : \frac{q-1}{(2, q-1)}$	$E_q : \frac{q-1}{4}$ for $q \equiv 1(4)$ cyclic group of order $(q - 1)$ for q even
D_{2n}	D_{2m} if $n = 2m$ or $n = 3m$; if n is prime $D_2 = C_2$ cyclic group of order n
A_4	C_3
S_4	D_6 D_8 A_4
A_5	D_{10} A_4
$\text{PSL}(2, q)$	$E_q : \frac{q-1}{(2, q-1)}$ C_3 if $q = 2$ S_4 if $q = 7$ A_5 if $q = 9, 11$
$\text{PGL}(2, q)$	$E_q : (q - 1)$ $\text{PSL}(2, q)$ for q odd

Table 4: The two-transitive actions of the maximal subgroups of $\text{PSL}(2, q)$

4.3 Two-transitive representations of the maximal subgroups of $\text{PSL}(2, q)$

The first lemma is obvious but will be used often in the next section as a necessary condition to have a two-transitive action.

Lemma 4.4. *Let G be a group and let H be a subgroup of G . If G acts 2-transitively on the cosets of H in G , then $|G|$ must be divisible by $[G:H]([G:H] - 1)$.*

A group G is said to act *regularly* on a set Ω if G is transitive on Ω and the stabilizer in G of a point $p \in \Omega$ is the identity.

Lemma 4.5. [26] *Let (G, Ω) be a permutation group which is transitive over Ω and let G be abelian. Then G is regular. Moreover, if G is 2-transitive then $|\Omega| = 2$.*

We now provide the classification (existence and uniqueness) all two-transitive representations of every maximal subgroup of $\text{PSL}(2, q)$, result borrowed from [10].

Lemma 4.6. *Let G_0 be a maximal subgroup of $\text{PSL}(2, q)$. Then G_0 acts two-transitively on the cosets of the subgroups G_{01} given in Table 4.*

5 Proof of Theorem 1.1

In this section, we determine the rank 2 RWPRI and $(2T)_1$ geometries of $\text{PSL}(2, q)$. We break down the task by classifying those geometries with a fixed subgroup G_0 . Since we may assume without loss of generality that G_0 is maximal in G , we follow Tables 2 and 3 that give all maximal subgroups of $\text{PSL}(2, q)$. The number of RWPRI and $(2T)_1$ geometries of rank 2 depends on the value of $q = p^n$. More precisely, it usually depends

		$G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$			$q = p^n,$ with p prime
	G_{01}	G_1	‡ Geom. up to conj.	‡ Geom. up to isom.	Extra conditions on q
Γ_1	$(q-1)$	$E_q : (q-1)$	1	1	$q = 2^n$
Γ_2	$(q-1)$	$D_{2(q-1)}$	1	1	$q = 2^n$

Table 5: The RWPRI and $(2T)_1$ geometries with $G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$.

on whether q is even or odd. Knowing that $q = p^n$ with p prime, the two cases are $q = 2^n$ or q odd.

5.1 The case where $G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$

By Lemma 4.6 the possibilities for G_{01} are $E_q : \frac{q-1}{4}$ provided $q \equiv 1(4)$ and the cyclic group of order $(q-1)$ provided q is even.

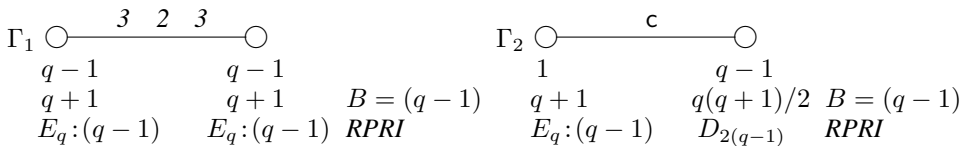
Lemma 5.1. *Let $G \cong \text{PSL}(2, q)$. If $q \equiv 1(4)$ and H is a subgroup of G such that H contains a subgroup isomorphic to $E_q : \frac{q-1}{4}$ maximally, and acts two-transitively on its cosets. Then $H \cong E_q : \frac{(q-1)}{2}$.*

Proof. Left to the reader. □

Lemma 5.2. *Let $G \cong \text{PSL}(2, q)$. If q is even and H is a subgroup of G such that H contains a subgroup isomorphic to a cyclic group of order $(q-1)$ maximally, and acts two-transitively on its cosets. Then $H \cong E_q : (q-1)$ or $H \cong D_{2(q-1)}$.*

Proof. Left to the reader. □

Proposition 5.3. *Let $G \cong \text{PSL}(2, q)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$ is isomorphic to one of the geometries appearing in Table 5. Their Buekenhout diagrams are as follows:*



Moreover, Γ_1 is a Doubling of Γ_2 and it does not satisfy the $(IP)_2$ condition.

Proof. Let $G_0 \cong E_q : \frac{(q-1)}{(2, q-1)}$. Lemma 4.6 gives two possibilities for G_{01} .

Subcase 1: $G_{01} := G_0 \cap G_1 \cong E_q : \frac{q-1}{4}$ provided $q \equiv 1(4)$.

By Lemma 5.1 the only possibility for G_1 is $E_q : \frac{q-1}{2}$. We must still check whether this geometry exists, that is whether there are two subgroups isomorphic to $E_q : \frac{q-1}{2}$ in

$\text{PSL}(2, q)$ that have a subgroup $E_q : \frac{q-1}{4}$ in common. Since $\text{PSL}(2, q)$ is simple and $E_q : \frac{q-1}{2}$ maximal, $E_q : \frac{q-1}{2}$ is self-normalized. Moreover, the group $E_q : \frac{q-1}{2}$ contains exactly one maximal subgroup of order $E_q : \frac{q-1}{4}$. The normalizer of $E_q : \frac{q-1}{4}$ in $\text{PSL}(2, q)$ is $E_q : \frac{q-1}{2}$. Therefore the number of subgroups $E_q : \frac{q-1}{2}$ containing a given subgroup $E_q : \frac{q-1}{4}$ in $\text{PSL}(2, q)$ is equal to

$$\frac{|\text{PSL}(2, q)|}{|E_q : \frac{q-1}{2}|} \cdot 1 \cdot \frac{|E_q : \frac{q-1}{4}|}{|\text{PSL}(2, q)|} = 1.$$

Thus the geometry $\Gamma(\text{PSL}(2, q); E_q : \frac{q-1}{2}, E_q : \frac{q-1}{2}, E_q : \frac{q-1}{4})$ does not exist.

Subcase 2: $G_{01} := G_0 \cap G_1 \cong (q-1)$ provided q is even.

By Lemma 5.2 the possibilities for G_1 are $E_q : (q-1)$ and $D_{2(q-1)}$.

Consider first the case where $G_1 \cong E_q : (q-1)$. We must check whether this geometry exists, that is, if there are, in $\text{PSL}(2, q)$, two subgroups isomorphic to $E_q : (q-1)$ that have a cyclic group of order $(q-1)$ in common. Since $\text{PSL}(2, q)$ is simple and $E_q : (q-1)$ maximal, $E_q : (q-1)$ is self-normalized. There is only one conjugacy class of $E_q : (q-1)$ in $\text{PSL}(2, q)$. Moreover, the group $E_q : (q-1)$ contains q maximal subgroups of order $(q-1)$ which are all conjugate in $E_q : (q-1)$. Therefore the number of subgroups $E_q : (q-1)$ containing a given cyclic subgroup of order $(q-1)$ in $\text{PSL}(2, q)$ is equal to

$$\frac{|\text{PSL}(2, q)|}{|E_q : (q-1)|} \cdot q \cdot \frac{|D_{2(q-1)}|}{|\text{PSL}(2, q)|} = 2,$$

and thus the geometry exists. Up to conjugacy there is exactly one RWPRI and $(2T)_1$ geometry $\Gamma_1 := \Gamma(\text{PSL}(2, q); E_q : (q-1), E_q : (q-1), q-1)$, and thus also exactly one up to isomorphism.

Next, let us consider the case where $G_1 \cong D_{2(q-1)}$. There is only one conjugacy class of $E_q : (q-1)$ and also one of $D_{2(q-1)}$ in $\text{PSL}(2, q)$. Since $\text{PSL}(2, q)$ is simple and $D_{2(q-1)}$ maximal, $D_{2(q-1)}$ is self-normalized. Moreover, the group $D_{2(q-1)}$ contains one maximal subgroup of order $(q-1)$. The normalizer of a cyclic group of order $(q-1)$ in $\text{PSL}(2, q)$ is a dihedral group $D_{2(q-1)}$. Therefore the number of subgroups $D_{2(q-1)}$ containing a given cyclic subgroup of order $(q-1)$ in $\text{PSL}(2, q)$ is equal to

$$\frac{|\text{PSL}(2, q)|}{|D_{2(q-1)}|} \cdot 1 \cdot \frac{|D_{2(q-1)}|}{|\text{PSL}(2, q)|} = 1$$

and thus the geometry exists. Up to conjugacy there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_2 := \Gamma(\text{PSL}(2, q); E_q : (q-1), D_{2(q-1)}, q-1)$ and thus also exactly one up to isomorphism.

Elements of type 0 in Γ_2 are the points of the projective line $PG(1, q)$. Elements of type 1 are the pairs of points of $PG(1, q)$. Incidence is symmetrized inclusion. Hence, this geometry is clearly a complete graph and its Buekenhout diagram follows. Applying Corollary 4.1 of [19] to Γ_2 , we get Γ_1 and the corresponding Buekenhout diagram (see [19], Table 1 or [21]). The $(IP)_2$ condition is clearly satisfied in Γ_2 and not in Γ_1 . \square

Observe that the geometries given in the previous theorem also satisfy the RPRI condition since $D_{2(q-1)}$ is a maximal subgroup of $\text{PSL}(2, q)$.

5.2 The case where $G_0 \cong D_{2 \frac{q-1}{(2,q-1)}}$

Recall that following Table 3 if q is odd, then $q > 11$. By Lemma 4.6, the possible subgroups for G_{01} are D_{2d} with $\frac{q-1}{(2,q-1)} = 2d$ or $3d$ and the cyclic group of order $\frac{q-1}{(2,q-1)}$. For each of these G_{01} we look for the various possible groups G_1 in one of the following four lemmas.

Lemma 5.4. *Let $G \cong \text{PSL}(2, q)$. If q is odd, $q > 11$ and H is a subgroup of G such that H contains a subgroup isomorphic to a cyclic group of order $\frac{q-1}{2}$ maximally, and acts two-transitively on its cosets. Then $H \cong D_{q-1}$.*

Proof. Left to the reader. □

When q is even, we distinguish the case $q = 4$ for clarity.

Lemma 5.5. *Let $G \cong \text{PSL}(2, 4)$. If H is a subgroup of G such that H contains a subgroup isomorphic to a C_2 maximally, and acts two-transitively on its cosets. Then $H \cong S_3$ or $H \cong 2^2$.*

Proof. Straightforward. □

Lemma 5.6. *Let $G \cong \text{PSL}(2, q)$. If $q = 2^n > 2$, $q = 1(3)$ and H is a subgroup of G such that H contains a subgroup isomorphic to $D_{2 \frac{(q-1)}{3}}$ maximally, and acts two-transitively on its cosets. Then $H \cong D_{2(q-1)}$ or $H \cong A_5$ provided $q = 16$.*

Proof. Left to the reader. □

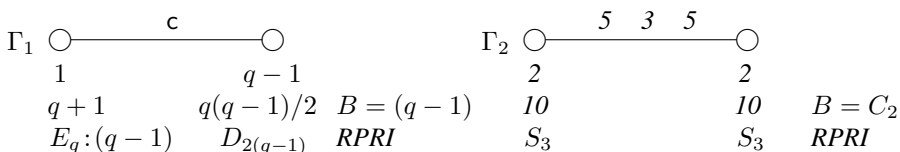
Lemma 5.7. *Assume q is odd and $q > 11$ and let $G \cong \text{PSL}(2, q)$.*

1. *If $q = 1(4)$ and H is a subgroup of G such that H contains a subgroup isomorphic to $D_{2 \frac{(q-1)}{4}}$ maximally, and acts two-transitively on its cosets. Then $H \cong D_{q-1}$, or $H \cong S_4$ provided $q = 17$.*
2. *If $q = 1(6)$ and H is a subgroup of G such that H contains a subgroup isomorphic to $D_{2 \frac{(q-1)}{6}}$ maximally, and acts two-transitively on its cosets. Then $H \cong D_{q-1}$, or $H \cong S_4$ provided $q = 25$ or $H \cong A_5$ provided $q = 31$.*

Proof. Left to the reader. □

The proof of the following Proposition is very similar to the one for Proposition 5.3. Therefore we do not give the details.

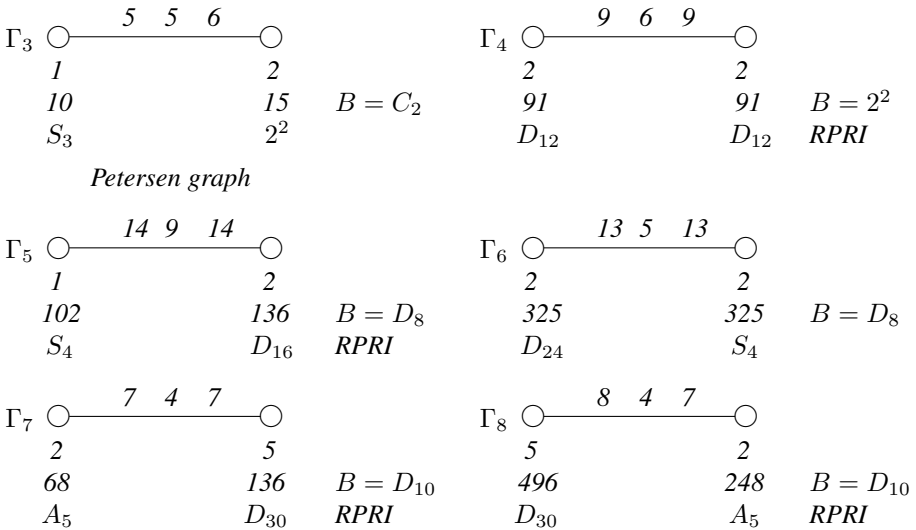
Proposition 5.8. *Let $G \cong \text{PSL}(2, q)$ with $q \neq 3, 5, 7, 9$ or 11. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong D_{2 \frac{(q-1)}{(2,q-1)}}$ is isomorphic to one of the geometries appearing in Table 6. Their Buekenhout diagrams are as follows:*



*Desargues configuration
(doubling of the Petersen graph)*

		$G_0 \cong D_{2 \frac{q-1}{(2,q-1)}}$			$q = p^n$, with p prime
	G_{01}	G_1	‡ Geom. up to conj.	‡ Geom. up to isom.	Extra conditions on q
Γ_1	$(q - 1)$	$E_q : (q - 1)$	1	1	$q = 2^n$
Γ_2	2	S_3	1	1	$q = 4$
Γ_3	2	2^2	1	1	$q = 4$
Γ_4	2^2	D_{12}	2	1	$q = 13$
Γ_5, Γ_6	D_8	S_4	2	1	$q = 17, 25$
Γ_7	D_{10}	A_5	1	1	$q = 16$
Γ_8	D_{10}	A_5	2	1	$q = 31$

Table 6: The RWPRI and $(2T)_1$ geometries with $G_0 \cong D_{2 \frac{q-1}{(2,q-1)}}$.



Observe that Γ_1 is one of the two geometries found in section 5.2. Geometries $\Gamma_2, \Gamma_4, \Gamma_5, \Gamma_6$ with $q = 17$ and Γ_7 are in [8]. For Γ_3 , see [4]. To the best of our knowledge Γ_6 with $q = 25$ and Γ_8 are new.

5.3 The case where $G_0 \cong D_{2 \frac{q+1}{(2,q+1)}}$

Recall that following Table 2 and Table 3 the subgroup $D_{2 \frac{q+1}{(2,q+1)}}$ is maximal in G if $q \neq 2, 7, 9$.

By Lemma 4.6, the possible subgroups for G_{01} are D_{2d} with $\frac{q+1}{(2,q+1)} = 2d$ or $3d$, and the cyclic group of order $\frac{q+1}{(2,q+1)}$. For each of these G_{01} we look for the various possible groups G_1 in one of the following three lemmas.

Lemma 5.9. *Let $G \cong \text{PSL}(2, q)$ and assume $q \neq 2, 7, 9$ as required. If H is a subgroup of G such that H contains a subgroup isomorphic to a cyclic group of order $\frac{q+1}{(2,q+1)}$ max-*

imally, and acts two-transitively on its cosets. Then $H \cong D_{2 \frac{q+1}{(2,q-1)}}$ or $H \cong A_4$ provided $q = 5$.

Proof. Left to the reader. □

Lemma 5.10. *Let $G \cong \text{PSL}(2, q)$. If $q \neq 2$, q is even, $q = -1(3)$ and H is a subgroup of G such that H contains a subgroup isomorphic to $D_{2 \frac{q+1}{(2,q-1)}}$ maximally, and acts two-transitively on its cosets. Then $H \cong D_{2(q+1)}$.*

Proof. Left to the reader. □

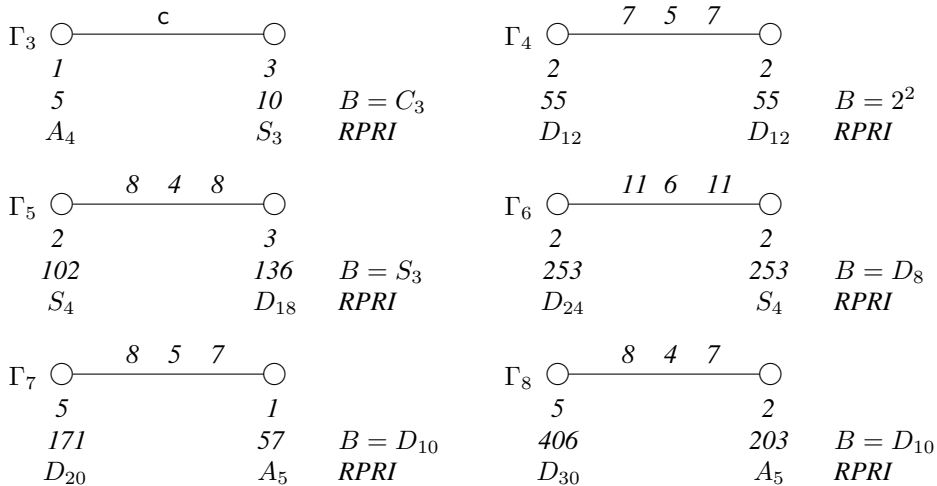
Lemma 5.11. *Let $G \cong \text{PSL}(2, q)$ with q odd and $q \neq 7, 9$.*

1. *If $q = -1(4)$ and H is a subgroup of G such that H contains a given subgroup isomorphic to $D_{2 \frac{q+1}{4}}$ maximally and acts two-transitively on its cosets. Then $H \cong D_{q+1}$ or $H \cong A_5$ provided $q = 19$.*
2. *If $q = -1(6)$ and H is a subgroup of G such that H contains a given subgroup isomorphic to $D_{2 \frac{q+1}{6}}$ maximally, and acts two-transitively on its cosets. Then $H \cong D_{q+1}$, or $H \cong S_4$ provided $q = 17, 23$ or $H \cong A_5$ provided $q = 29$ or $H \cong 2^2$ provided $q = 5$.*

Proof. Left to the reader. □

The proof of the following Proposition is very similar to the one of Proposition 5.3. Therefore we do not give the details.

Proposition 5.12. *Let $G \cong \text{PSL}(2, q)$ with $q \neq 2, 7, 9$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong D_{2 \frac{q+1}{(2,q-1)}}$ is isomorphic to one of the geometries appearing in Table 7. Their Buekenhout diagrams are as follows:*



		$G_0 \cong D_{2 \frac{q+1}{(2,q+1)}}$			$q = p^n$, with p prime
	G_{01}	G_1	# Geom. up to conj.	# Geom. up to isom.	Extra conditions on q
Γ_1	C_2	S_3	1	1	$q = 5$
Γ_2	C_2	2^2	1	1	$q = 5$
Γ_3	C_3	A_4	1	1	$q = 5$
Γ_4	2^2	D_{12}	2	1	$q = 11$
Γ_5	D_6	S_4	2	1	$q = 17$
Γ_6	D_8	S_4	2	1	$q = 23$
Γ_7, Γ_8	D_{10}	A_5	2	1	$q = 19, 29$

Table 7: The RWPRI and $(2T)_1$ geometries with $G_0 \cong D_{2 \frac{q+1}{(2,q+1)}}$.

Observe that Γ_3 is a special case of the second geometry of Proposition 5.3 provided $q = 4$. Moreover, Γ_1 and Γ_2 are also the same as geometries obtained in section 5.2 since $\text{PSL}(2, 4) \cong \text{PSL}(2, 5)$. For $\Gamma_1, \Gamma_3, \Gamma_4, \Gamma_5$ and Γ_8 with $q = 19$, see [8], and for Γ_2 and Γ_7 see [4]. Observe that Γ_7 is a truncation of Coxeter’s 57-cells. It is its vertex-edge graph also called the Perkel graph. To the best of our knowledge Γ_6 and Γ_8 provided $q = 29$ are new.

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