ANALYTIC DISCS WITH BOUNDARIES IN A GENERATING CR-MANIFOLD

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Abstract

The problem of perturbing an analytic disc with boundary in a CR-submanifold of $\mathbb{C}^n$ is considered. A theorem by Globevnik on the perturbation by analytic discs along maximal real submanifolds of $\mathbb{C}^n$ is generalized and used in various applications: (i) it is proved that every energy functional minimizing disc in $\mathbb{C}^n$ with free boundary in a Lagrangian submanifold of $\mathbb{C}^n$ and all partial indices greater or equal to -1 is holomorphic, (ii) a new proof and a generalization of a result by Pang on the Kobayashi extremal discs is given, (iii) perturbations of analytic varieties with boundaries in a totally real torus in $\mathbb{C}^2$ fibered over the unit circle ∂D are considered. Also, some results by Baouendi, Rothschild and Trepreau on the family of analytic discs attached to a CR-submanifold of $\mathbb{C}^n$ of a positive CR-dimension are globalized.

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Contents

1. Introduction

Given an analytic disc in $\mathbb{C}^n$ with boundary in a generating CR-submanifold $M \subseteq \mathbb{C}^n$, one would like to describe the family of all nearby analytic discs in $\mathbb{C}^n$ attached to M. This problem is the object of a considerable research in the recent years. The following list of authors and their papers related to the problem is not at all meant to be complete : Alexander [**Ale2**], Baouendi, Rothschild and Trepreau [**Bao-Rot-Tre**], Bedford [**Bed1, Bed2**], Bedford and Gaveau [**Bed-Gav**], Eliashberg [**Eli**], Forstnerič [**For2, For3**], Globevnik [**Glo1, Glo2**], Gromov [**Gro**], Y-G.Oh [**Oh1, Oh2**], Tumanov [**Tum**].

The technique of perturbing an analytic disc with boundary in a given manifold has found several applications in the problems of the analysis of several complex variables. Two, probably the most known problems, where this technique can be used, are the problem of describing the polynomial hull of a given set in $\mathbb{C}^n$ and the problem of extending CR-functions from a given CR-submanifold of $\mathbb{C}^n$ into some open subset of $\mathbb{C}^n$. Recently has J.Globevnik in his paper [**Glo1**], which was inspired by the work [**For2**] by F.Forstnerič, found very elegant sufficient conditions on a given analytic disc p with boundary in a maximal real submanifold M of $\mathbb{C}^n$ which imply finite dimensional parametrization of all nearby holomorphic discs attached to M.

To each, not necessary holomorphic, disc p with boundary in a maximal real submanifold $M \subseteq \mathbb{C}^n$ one associates n integers $k_1, \ldots, k_n$ called the partial indices of the disc p. Their sum $k := k_1 + \cdots + k_n$ is called the total index of the disc p. A part of Globevnik's work [**Glo1**] is the theorem in which he proves that if the pull-back bundle $p^*(TM)$ of the tangent bundle TM is trivial and if all partial indices of the disc p are greater or equal to 0, then there exists an $n + k$ dimensional parametrization of all nearby discs of the form

$$p + \text{analytic disc}$$

with boundary in M. Later we proved that Globevnik's theorem extends in the same form to the case where the pull-back bundle $p^*(TM)$ is non-trivial. The final version of the result was given by Y-G.Oh in [**Oh1**], where the Globevnik's result is generalized, but using a different approach, to the case where all partial indices are greater or equal to -1 and arbitrary pull-back bundle $p^*(TM)$. This theorem, together with the papers [**For2, For3**] by Forstnerič, represents the starting point of the present thesis and is reproved in its most general known form, using only Forstnerič's and Globevnik's technique, in section 5, Theorem 1.

The present work is organized as follows. Section 2 introduces the notation and terminology we use throughout the work. In section 3 the maximal real bundle over the unit circle $\partial D \subseteq \mathbb{C}$ and its partial indices are defined, and in section 4 some computations of the partial indices of a maximal real bundle over ∂D are given. As already mentioned, in section 5 we reprove the generalized version of Globevnik's theorem using his and Forstnerič's technique of perturbing analytic discs with maximal real boundary conditions. In the following sections we give some applications of Theorem 1.

In section 6 we apply Theorem 1 to the problem of perturbing analytic varieties with boundaries in a totally real torus in $\mathbb{C}^2$ fibered over the unit circle $\partial D \subseteq \mathbb{C}$. In section 7 we consider energy functional minimizing discs in $\mathbb{C}^n$ with Lagrangian boundary conditions and prove that the condition that all partial indices of the disc p are greater or equal to -1 implies that the disc p is in fact holomorphic. Section 8 considers stationary discs which are, following Slodkowski [**Slo4**], related to the problem of describing the polynomial hull of a fibration over the unit circle ∂D with the fibers in $\mathbb{C}^n$ and to the problem of finding Kobayashi extremal discs through a given point in an open set in $\mathbb{C}^n$. Using Theorem 1 again, we reprove and generalize a result by Pang, [**Pan1**], on the Kobayashi extremal discs. In the next section we give several examples which show that the immediate generalization of the continuity method to describe the polynomial hull of a set fibered over the unit circle with fibers in $\mathbb{C}^n$, $n \geq 2$, as used by Forstnerič, [**For3**], in the case of one dimensional fibers, is even in some relatively simple cases impossible. In the last section we first extend Globevnik's results to the case of analytic disc attached to a generating CR-submanifold of $\mathbb{C}^n$ and then also generalize some results by Baouendi, Rothschild and Trepreau, [**Bao-Rot-Tre**], to large analytic discs with boundaries in a generating CR-submanifold of $\mathbb{C}^n$.

2. Notation and terminology

Let $D = \{z \in \mathbb{C}; |z| < 1\}$ and let ∂D denote the unit circle in $\mathbb{C}$, the boundary of D. If K is either $\overline{D}$ or ∂D, and $0 < \alpha < 1$, we denote by $C^{0,\alpha}(K)$ the Banach algebra of Hölder continuous complex-valued functions on K with finite Lipschitz norm of exponent α

$$\|f\|_\alpha = \sup_{x \in K} |f| + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \ .$$

For every $m \in \mathbb{N} \cup \{0\}$ we also define the algebra

$$C^{m,\alpha}(K) = \{f \in C^m; \|f\|_{m,\alpha} = \sum_{|j| \leq m} \|D^j f\|_\alpha < \infty\} \ .$$

The subalgebra of the real-valued functions from $C^{m,\alpha}(K)$ will be denoted by $C^{m,\alpha}_{\mathbb{R}}(K)$.

Let $A(D)$ denote the disk algebra and let $A(\partial D) = \{f|_{\partial D}; f \in A(D)\}$. We define

$$A^{m,\alpha}(D) = C^{m,\alpha}(\overline{D}) \cap A(D)$$

and

$$A^{m,\alpha}(\partial D) = C^{m,\alpha}(\partial D) \cap A(\partial D) \ .$$

Note that if $f \in A(D)$, then $f \in A^{m,\alpha}(D)$ if and only if $f|_{\partial D} \in A^{m,\alpha}(\partial D)$, [**Gol**].

We will also need some other not so standard function spaces. Let $r(\xi)$, $\xi \in \partial D \setminus \{1\}$, denote the principal branch of the square root, i.e., the complex plane is cut along the positive real line and $r(-1) = i$. Let $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ be the space consisting of the real continuous functions on $\partial D \setminus \{1\}$ with the property that a continuous function g on $\partial D \setminus \{1\}$ is in $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ if and only if there exists an odd function g_o in $C_{\mathbb{R}}^{m,\alpha}(\partial D)$, i.e., $g_o(-\xi) = -g_o(\xi),\ \xi \in \partial D$, such that

$$g(\xi) = g_o(r(\xi)) \quad (\xi \in \partial D \setminus \{1\}) \ .$$

In other words, this is the space of continuous functions g on $\partial D \setminus \{1\}$ such that

a) there exist the limits

$$\lim_{\theta \to 0^+} g(e^{i\theta}) \quad \text{and} \quad \lim_{\theta \to 2\pi^-} g(e^{i\theta}) \tag{1}$$

which we denote by $g(1^+)$ and $g(1^-)$, respectively, and are related by the equation

$$g(1^+) + g(1^-) = 0 \ ,$$

b) the function

$$(Hg)(\xi) := \begin{cases} g(\xi^2) & ; \ \ \mathrm{Im}\xi \geq 0 \\ -g(\xi^2) & ; \ \ \mathrm{Im}\xi < 0 \end{cases} \tag{2}$$

is in $C_{\mathbb{R}}^{m,\alpha}(\partial D)$.

Obviously $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ is an $\mathbb{R}$-linear space and for the norm on it we take

$$\|g\|_{m,\alpha} := \|Hg\|_{m,\alpha} \quad (g \in \mathcal{E}_{\mathbb{R}}^{m,\alpha}) \ .$$

So $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ is a Banach space that is via H isometrically isomorphic to the closed subspace of odd functions in $C_{\mathbb{R}}^{m,\alpha}(\partial D)$.

Remark. Another equivalent description of the space $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ can be given in terms of Fourier series. Namely, each element $g \in \mathcal{E}_{\mathbb{R}}^{m,\alpha}$ has a unique expension of the form

$$\sum_{k=-\infty}^{\infty} c_k e^{i(2k+1)\theta/2}$$

where the sum

$$\sum_{k=-\infty}^{\infty} c_k e^{i(2k+1)\theta}$$

represents the Fourier series of some odd function $g_o \in C_{\mathbb{R}}^{m,\alpha}(\partial D)$. We will refer to $c_k,\ k \in \mathbb{Z}$, as the Fourier coefficients of the function g.

One can define a Hilbert transform T on $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$. Let T_o be the standard harmonic conjugate function operator on $C_{\mathbb{R}}^{m,\alpha}(\partial D)$. Then

$$T : \mathcal{E}_{\mathbb{R}}^{m,\alpha} \longrightarrow \mathcal{E}_{\mathbb{R}}^{m,\alpha}$$

is defined by

$$Tg = H^{-1} T_o H g \qquad (g \in \mathcal{E}_{\mathbb{R}}^{m,\alpha}) \ .$$

Note that T_o takes the subspace of odd functions in $C_{\mathbb{R}}^{m,\alpha}(\partial D)$ into itself. Thus for every $g \in \mathcal{E}_{\mathbb{R}}^{m,\alpha}$ the function

$$H(g + iTg) := Hg + iHTg = Hg + iT_o(Hg)$$

is an odd function on ∂D from the space $A^{m,\alpha}(\partial D)$. We denote the space of functions of the form

$$g + iTg \qquad (g \in \mathcal{E}_{\mathbb{R}}^{m,\alpha})$$

by $\mathcal{A}^{m,\alpha}$. Observe that all functions from the space $\mathcal{A}^{m,\alpha}$ are of the form $r(\xi)f(\xi)$ for some $f \in A^{m,\alpha}(\partial D)$. Observe also that since for any two functions g and h from $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ the following identity holds

$$(Hg)(\xi)(Hh)(\xi) = g(\xi^2)h(\xi^2) \quad (\xi \in \partial D),$$

the product of two functions from $\mathcal{E}_{\mathbb{R}}^{m,\alpha}$ gives a function in $C_{\mathbb{R}}^{m,\alpha}(\partial D)$, and the product of two functions of the form $g + iHg$, $g \in \mathcal{E}_{\mathbb{R}}^{m,\alpha}$, gives a function in $A^{m,\alpha}(\partial D)$.

The spaces we will most often consider are the finite products of the spaces $C_{\mathbb{R}}^{0,\alpha}(\partial D)$ and $\mathcal{E}_{\mathbb{R}}^{0,\alpha}$. A product with n factors will be denoted by $\mathcal{E}_\sigma$, where σ is an n-vector with 0's and 1's as its entries. The entry 0 on the j-th place represents the space $C_{\mathbb{R}}^{0,\alpha}(\partial D)$ as the j-th factor and the entry 1 on the j-th place means that the j-th factor is the space $\mathcal{E}_{\mathbb{R}}^{0,\alpha}$. By analogy we also define the spaces $\mathcal{A}_\sigma$ which are the products of finitely many copies of $A^{0,\alpha}(\partial D)$ and $\mathcal{A}^{0,\alpha}$.

We extend the definition of the Hilbert transform in a natural way (componentwise) to the space $\mathcal{E}_\sigma$. We denote the extension by T_σ. It is a bounded linear map from $\mathcal{E}_\sigma$ into itself and it has the property that the vector function $v + iT_\sigma v$ belongs to the space $\mathcal{A}_\sigma$ for every $v \in \mathcal{E}_\sigma$. We also define the map

$$H_\sigma : \mathcal{E}_\sigma \longrightarrow (C_{\mathbb{R}}^{0,\alpha}(\partial D))^n$$

which is defined as the identity map on each factor $C^{0,\alpha}_{\mathbb{R}}(\partial D)$, and is defined as the map (2) on each factor $\mathcal{E}^{0,\alpha}_{\mathbb{R}}$.

3. Maximal real bundles over the circle

Let L be a maximal real subspace of $\mathbb{C}^n$, i.e., its real dimension is n and $L \oplus iL = \mathbb{C}^n$. To any such maximal real subspace L one can associate an $\mathbb{R}$-linear map R_L on $\mathbb{C}^n$, called the reflection about L, given by

$$z = x + i\tilde{x} \longmapsto x - i\tilde{x} \qquad (x, \tilde{x} \in L) \ ,$$

where $z = x + i\tilde{x}$ is the unique decomposition of z into the sum of vectors from L and iL. The mapping

$$R_L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

is an $\mathbb{R}$-linear automorphism of $\mathbb{C}^n$ which is also $\mathbb{C}$-antilinear, i.e., $R_L(iv) = -iR_L(v)$ for every $v \in \mathbb{C}^n$. The reflection about the maximal real subspace $\mathbb{R}^n \subseteq \mathbb{C}^n$ will be denoted by R_o. Note that in the standard notation R_o is just the ordinary conjugation on $\mathbb{C}^n$ and that for any $n \times n$ complex matrix A the following identity holds

$$\overline{A} = R_o A R_o \ .$$

LEMMA 1. *Let L be a maximal real subspace of $\mathbb{C}^n$ and let $x_1, \ldots, x_n$ be any set of vectors spanning L. Let $A := [x_1, \ldots, x_n]$ be the matrix whose columns are the given vectors x_j, $j = 1, \ldots, n$, and let $B := A\overline{A^{-1}}$. Then*

$$B = R_L R_0 \ .$$

Moreover, the matrix B does not depend on the basis of L, i.e., B remains the same even if a different basis for L is selected, and

$$\overline{B} = B^{-1} \ , \qquad |\mathrm{det} B| = 1 \ .$$

Remark. In the above lemma $n \times n$ matrices A and B are identified with $\mathbb{C}$-linear automorphisms of $\mathbb{C}^n$ in the standard basis.

Proof. Observe that A is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ which maps $\mathbb{R}^n$ onto L. Consider the following composition of automorphisms of $\mathbb{C}^n$

$$S := R_o A^{-1} R_L A \ .$$

Then S is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ which equals to the identity on $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a maximal real subspace of $\mathbb{C}^n$, S is the identity on $\mathbb{C}^n$, and hence

$$R_o A^{-1} = A^{-1} R_L \ .$$

Finally, since

$$B = A\overline{A^{-1}} = A R_o A^{-1} R_o \ ,$$

we get

$$B = A A^{-1} R_L R_o = R_L R_o \ .$$

The rest is obvious. ■

The following definition is taken from [**Glo1**], see also [**For2**].

DEFINITION 1. *Let $L = \{L_\xi; \xi \in \partial D\}$ be a real rank n subbundle of the product bundle $\partial D \times \mathbb{C}^n$ of class $C^{0,\alpha}$. If for each $\xi \in \partial D$ the fiber L_ξ is a maximal real subspace of $\mathbb{C}^n$, the bundle L is called maximal real.*

Example. A very important example of a maximal real bundle over ∂D one gets in the following case. Let M be C^2 maximal real submanifold of $\mathbb{C}^n$ and let $p : \partial D \to M$ be a C^2 closed curve in M. Then the pull-back bundle $p^*(TM)$, where TM is the tangent bundle of the submanifold M, is a maximal real bundle over ∂D of rank n.

It is known, see [**Vek1**], that for every closed path B in $Gl(n, \mathbb{C})$ of class $C^{0,\alpha}$ one can find holomorphic matrix functions

$$F^+ : \overline{D} \longrightarrow Gl(n, \mathbb{C}) \ , \qquad F^- : \overline{\mathbb{C}} \setminus D \longrightarrow Gl(n, \mathbb{C})$$

of class $C^{0,\alpha}$ and n integers $k_1 \geq k_2 \geq \cdots \geq k_n$ such that

$$B = F^+(\xi)\Lambda(\xi)F^-(\xi) \quad (\xi \in \partial D),$$

where

$$\Lambda(\xi) := \begin{pmatrix} \xi^{k_1} & 0 & \dots & \dots & 0 \\ 0 & \xi^{k_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \xi^{k_n} \end{pmatrix} .$$

The matrix Λ will be called the *characteristic matrix* of the path B. One can prove that under the condition $k_1 \geq \cdots \geq k_n$, the characteristic matrix Λ does not depend on the factorization of the matrix function B of the above form, see

[**Vek1**], [**Glo1**], [**Cla-Goh**] for more details. The integers $k_1, \ldots, k_n$ are called the *partial indices* of the path B, and their sum

$$k := k_1 + \cdots + k_n$$

is called the *total index* of the matrix function B.

DEFINITION 2. *Let L be a maximal real bundle over the unit circle ∂D. The partial indices of the $Gl(n, \mathbb{C})$ closed path*

$$B_L : \xi \longmapsto R_{L_\xi} R_o \tag{3}$$

of class $C^{0,\alpha}$, are called the partial indices of the bundle L and their sum is called the total index of L.

Remarks.

1. Observe that Definition 2 makes sense even if the bundle L is not trivial.
2. The total index of a closed path p on a maximal real submanifold $M \subseteq \mathbb{C}^n$ is also called the Maslov index of p.
3. As we will see, in the case where all partial indices satisfy the condition $k_j \geq -1,\ j = 1, \ldots, n$, the characteristic matrix $\Lambda(\xi)$ carries all important information about the bundle L, see also [**Glo1**], [**Oh1**].

Although Globevnik in [**Glo1**] works only with the trivial bundles over the circle ∂D, Lemma 5.1 in [**Glo1**] still applies and one can conclude

LEMMA 2. *The $C^{0,\alpha}$ closed path in $Gl(n,\mathbb{C})$*

$$B_L : \xi \longmapsto R_{L_\xi} R_o \qquad (\xi \in \partial D)$$

can be decomposed in the form

$$B_L(\xi) = \Theta(\xi)\Lambda(\xi)\overline{\Theta(\xi)^{-1}} \qquad (\xi \in \partial D) \ ,$$

where the map $\Theta : \overline{D} \longrightarrow Gl(n,\mathbb{C})$ is of class $C^{0,\alpha}$ and holomorphic on D, i.e., the $n \times n$ matrix Θ is in $A^{0,\alpha}(D)^{n\times n}$.

Let the total index k be an even integer. Then one can split the matrix Λ as

$$\Lambda(\xi) = Q(\xi)\overline{Q(\xi)^{-1}} \ \ (\xi \in \partial D) \ ,$$

where $Q(\xi)$ is a closed real analytic path in $Gl(n,\mathbb{C})$. See [**Glo1**] for details. Fix $\xi \in \partial D$ and select any basis $x_1, \ldots, x_n$ of the fiber L_ξ. Let $A := [x_1, \ldots, x_n]$ be the matrix whose columns are the vectors x_j's. Then

$$B_L(\xi) = A\overline{A^{-1}} = \Theta(\xi)Q(\xi)\overline{(\Theta(\xi)Q(\xi))^{-1}}$$

and so

$$A^{-1}\Theta(\xi)Q(\xi) = \overline{A^{-1}\Theta(\xi)Q(\xi)} \ .$$

Thus the invertible $n \times n$ matrix

$$U := A^{-1}\Theta(\xi)Q(\xi)$$

is real and therefore the columns of the matrix $\Theta(\xi)Q(\xi) = AU$ span the fiber L_ξ for each $\xi \in \partial D$. Together with a Globevnik's observation, see also [**For2**], one concludes

COROLLARY 1. *A maximal real bundle L over ∂D is trivial if and only if its total index is an even integer.*

According to [**Bot-Tu**], every real vector bundle over ∂D of rank n is either trivial or isomorphic to the direct sum of a trivial bundle of rank $n-1$ and the Möbius bundle. Since the trivial bundle case was discussed in details in [**Glo1**], one would only have to consider the non-trivial bundle case. But since our approach to the problem does not "see" the difference between the trivial bundle case and the non-trivial bundle case, we will still consider both cases.

Let L be a rank n maximal real $C^{0,\alpha}$ vector bundle over ∂D. Let $k_1 \geq k_2 \geq \cdots \geq k_n$ be its partial indices and let

$$\Lambda(\xi) := \begin{pmatrix} \xi^{k_1} & 0 & \dots & \dots & 0 \\ 0 & \xi^{k_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \xi^{k_n} \end{pmatrix} .$$

As we already know the $C^{0,\alpha}$ closed path in $Gl(n,\mathbb{C})$

$$B_L : \xi \longmapsto R_{L_\xi} R_o \qquad (\xi \in \partial D)$$

can be decomposed in the form

$$B_L(\xi) = \Theta(\xi)\Lambda(\xi)\overline{\Theta(\xi)^{-1}} \qquad (\xi \in \partial D)$$

for some $\Theta : \overline{D} \to Gl(n,\mathbb{C})$ of class $C^{0,\alpha}$ and holomorphic on D. The characteristic matrix Λ can be decomposed further as

$$\Lambda = \Lambda_o^2 = \Lambda_o \overline{\Lambda_o^{-1}} \ ,$$

where

$$\Lambda_o(\xi) := \begin{pmatrix} \xi^{\frac{k_1}{2}} & 0 & \dots & \dots & 0 \\ 0 & \xi^{\frac{k_2}{2}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \xi^{\frac{k_n}{2}} \end{pmatrix} .$$

Here $\xi^{\frac{k}{2}}$ stands for ξ^m if $k = 2m$ and for $\xi^m r(\xi)$ if $k = 2m+1$. We will refer to Λ_o as the square root of the characteristic matrix Λ and we say that the matrix function

$$\xi \longmapsto \Theta(\xi)\Lambda_o(\xi) \qquad (\xi \in \partial D)$$

represents the *normal form* of the bundle L . To the $C^{0,\alpha}$ closed path B_L in $Gl(n, \mathbb{C})$ we also associate the corresponding Banach space $\mathcal{E}_\sigma$ we will work with, see section 2 for the definition. The n-vector σ is defined as

$$\sigma := (k_1 \bmod 2, \dots, k_n \bmod 2) \ .$$

COROLLARY 2. *If all partial indices of a maximal real bundle L are non-negative, then there exists an $n \times n$ matrix function $A(\xi)$, $\xi \in \partial D$, with the rows from the space $\mathcal{A}_\sigma$ and such that its columns $X_1(\xi), \dots, X_n(\xi)$ span the fiber L_ξ for every $\xi \in \partial D$.*

Remark. For $\xi = 1$ the above statement still makes sense in terms of the limits (1) when $\xi \neq 1$ approaches to 1.

4. Some computations

Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{C}^n$. Let L be a maximal real subspace of $\mathbb{C}^n$. By iL we denote the maximal real subspace of vectors of the form $iv,\ v\in L$, and by $L^\perp$ the maximal real subspace of vectors perpendicular to L, i.e., a vector $u\in\mathbb{C}^n$ is perpendicular to L if and only if $\mathrm{Re}\langle u,v\rangle=0$ for every $v\in L$. We recall that R_L denotes the $\mathbb{C}$-antilinear reflection about the maximal real subspace L and that the matrix B_L is given as the product $R_L R_o$.

LEMMA 3.

a) *Let L be a maximal real subspace of $\mathbb{C}^n$. Then*

$$R_{iL}=-R_L \qquad , \qquad B_{iL}=-B_L$$

and

$$R_{L^\perp}=-\overline{R_L^t}=R_L^* \qquad , \qquad B_{L^\perp}=-B_L^t\ .$$

b) *Let $L=\{L_\xi;\xi\in\partial D\}$ be a maximal real bundle over the circle ∂D. Then the following holds :*

1. *The partial indices of the bundles L and iL are the same.*
2. *The bundles L, iL and $L^\perp$ are trivial if and only if one of them is trivial.*

Proof. Let A_L, A_{iL} and $A_{L^\perp}$ denote matrices whose columns span the subspaces L, iL and $L^\perp$, respectively. Then

$$A_{iL} = iA_L \quad \text{and} \quad \mathrm{Re}(A_L^* A_{L^\perp}) = 0 \ .$$

Hence

$$B_{iL} = A_{iL}\overline{A_{iL}^{-1}} = -B_L$$

and

$$B_{L^\perp} = A_{L^\perp}\overline{A_{L^\perp}^{-1}} = -\overline{(A_L^t)^{-1}}A_L^t\overline{A_{L^\perp}A_{L^\perp}^{-1}} = -B_L^t \ .$$

Part (a) is proved. Part (b) is now a trivial consequence of part (a) and Corollary 1. ∎

Remark and an example. One should notice that the indices of the normal bundle $L^\perp$ are not always the same as the indices of L, e.g., if the matrix function B_L is

$$\begin{pmatrix} -\xi & 1 \\ 0 & \overline{\xi} \end{pmatrix} ,$$

then its partial indices are 1 and -1, but on the other hand the partial indices of B_L^t are all 0. Of course, one also has to check that $\overline{B_L} = B_L^{-1}$. ∎

LEMMA 4. *Let L_o be any (trivial or non-trivial) maximal real bundle of rank n over the circle ∂D and let $A_o(\xi)$, $\xi \in \partial D$, be a $C^{0,\alpha}$ path in $GL(n, \mathbb{C})$ which represents the normal form of the bundle L_o.*

1. *Let L be a maximal real bundle of rank $n+1$ whose fibers are spanned by the columns of the matrix*

$$A = \begin{pmatrix} g & 0 \\ v & A_o \end{pmatrix} ,$$

where g is a nonzero function from the space $\mathcal{E}_{k_o \mathrm{mod} 2}$ with the winding number $\frac{k_o}{2}$, $k_o \in \mathbb{Z}$, and v is any vector function from $(\mathcal{E}_{k_o \mathrm{mod} 2})^n$. If $k_1, \ldots, k_n$ are the partial indices of the bundle L_o and if $k_j - k_o \geq -1$, $j = 1, \ldots, n$, then the partial indices of the bundle L are

$$k_o, k_1, \ldots, k_n \ .$$

2. *Let L be a maximal real bundle of rank $n+1$ whose fibers are spanned by the columns of the matrix*

$$A(\xi) = \begin{pmatrix} i\xi & 0 \\ \xi v(\xi) & A_o(\xi) \end{pmatrix} ,$$

where v is a vector function from the space $(A^{0,\alpha}(\partial D))^n$. If $k_1, \ldots, k_n$ are the partial indices of the bundle L_o, then the partial indices of the bundle L are

$$2, k_1, \ldots, k_n \ .$$

Proof.

1. Let $B_o = A_o \overline{A_o^{-1}}$. Then

$$B_L = A\overline{A^{-1}} = \begin{pmatrix} g/\overline{g} & 0 \\ (v - B_o \overline{v})/\overline{g} & B_o \end{pmatrix} .$$

Once we find a solution $(a, b) \in (A^{0,\alpha}(\partial D))^{n+1}$ of the equation

$$(4) \qquad B(\xi) \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} = \xi^{k_o} \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} \quad (\xi \in \partial D)$$

such that the function a extends as a nonzero holomorphic function on D the first part of the lemma will be proved.

Since the winding number $W(g)$ is $\frac{k_o}{2}$, the function g can be written in the form

$$g(\xi) = p(\xi)\xi^{k_o/2}e^{h(\xi)} \quad (\xi \in \partial D) \ ,$$

where p is a positive function of class $C^{0,\alpha}$ and h belongs to the space $A^{0,\alpha}(\partial D)$. Let $a := e^h$. Then the first equation in (4) is solved and the second equation has the form

$$(5) \qquad \frac{1}{g}(v - B_o\overline{v})e^{\overline{h}} + B_o\overline{b} = \xi^{k_o} b \ .$$

Let $\Phi\Lambda\overline{\Phi^{-1}}$ be the normal splitting of the path B_o and let

$$\Phi^{-1}v = \nu \quad , \quad \Phi^{-1}b = \beta \ .$$

Multiplying (5) by Φ^{-1} from the left-hand side yields

$$\frac{1}{g}(\nu - \Lambda\overline{\nu})e^{\overline{h}} + \Lambda\overline{\beta} = \xi^{k_o}\beta \ .$$

Thus for each $j = 1, \ldots, n$ we have the equation

$$\frac{1}{p}\xi^{k_o/2}(\nu_j - \xi^{k_j}\overline{\nu_j}) + \xi^{k_j}\overline{\beta_j} = \xi^{k_o}\beta_j \ .$$

After dividing by $\xi^{k_j/2}$ and rearranging the terms, the problem we are trying to solve is to find holomorphic functions $\beta_j \in A^{0,\alpha}(\partial D)$, $j = 1, \ldots, n$, such that

$$\mathrm{Im}(\overline{\xi}^{(k_j-k_o)/2}\beta_j(\xi) - \frac{1}{p(\xi)}\overline{\xi}^{k_j/2}\nu_j(\xi)) = 0 \quad (\xi \in \partial D) .$$

But this problem is equivalent to the problem of finding real functions u_j, $j = 1, \ldots, n$, from the space $\mathcal{E}_{(k_j-k_o)\mathrm{mod}2}$ such that the function

$$\beta_j(\xi) = \xi^{(k_j-k_o)/2}u_j(\xi) + \frac{1}{p(\xi)}\overline{\xi}^{k_o/2}\nu_j(\xi)$$

extends holomorphically to D. Since $k_j - k_o \geq -1,\ j = 1, \ldots, n$, such functions is indeed possible to find, namely, let

$$u_j := iT(\frac{1}{p(\xi)}\overline{\xi}^{k_j/2}\nu_j(\xi)) ,$$

where T is the Hilbert transform on the space $\mathcal{E}_{(k_j-k_o)\mathrm{mod}2}$.**2.** Let $B_o(\xi) = A_o(\xi)\overline{A_o(\xi)^{-1}}$, $\xi \in \partial D$. Then

$$B_L(\xi) = A(\xi)\overline{A(\xi)^{-1}} = \begin{pmatrix} -\xi^2 & 0 \\ i(\xi^2 v(\xi) - B_o(\xi)\overline{v(\xi)}) & B_o(\xi) \end{pmatrix} \quad (\xi \in \partial D) .$$

In this form it is easy to check that the vectors

$$\Phi^+(\xi) = \begin{pmatrix} i \\ v(\xi) \end{pmatrix} \quad \text{and} \quad \Phi^-(\xi) = \xi^{-2}\begin{pmatrix} -i \\ \overline{v(\xi)} \end{pmatrix} \quad (\xi \in \partial D)$$

solve the Hilbert boundary value problem

$$\Phi^+(\xi) = B(\xi)\Phi^-(\xi) \quad (\xi \in \partial D) ,$$

and that 2 is the order of zero of the function Φ^- at the infinity. ■

Example. The following example shows that in the case (1) of Lemma 4 one really needs some assumptions on the partial indices of the matrix function A_o and the winding number of the function g.

If A is the matrix

$$\begin{pmatrix} \xi & 0 \\ i & 1 \end{pmatrix} ,$$

then the matrix $B_L = A\overline{A^{-1}}$ is

$$\begin{pmatrix} \xi^2 & 0 \\ 2i\xi & 1 \end{pmatrix}$$

and one can easily check that 2 is not one of its partial indices. Moreover, both partial indices are 1. ■

The next lemma shows that a fiber preserving diffeomorphism of $\overline{D} \times \mathbb{C}^n$ which is holomorphic on each fiber and biholomorphic as a mapping from $D \times \mathbb{C}^n$ into itself does not change partial indices of a maximal real fibration over ∂D.

LEMMA 5. *Let*

$$\Phi \;:\; \overline{D} \times \mathbb{C}^n \longrightarrow \overline{D} \times \mathbb{C}^n$$

$$\Phi \;:\; (\xi, z) \longmapsto (\xi, \phi(\xi, z))$$

be a $C^{0,\alpha}(\overline{D}, C^1(\mathbb{C}^n))$ fiber preserving diffeomorphism of $\overline{D} \times \mathbb{C}^n$ such that the function $\phi(\xi, \cdot)$ is holomorphic for each $\xi \in \overline{D}$ and the mapping Φ is a biholomorphism of $D \times \mathbb{C}^n$. Let L be a maximal real bundle over ∂D. Then the partial indices of the maximal real bundles L and $\widetilde{L}$,

$$\widetilde{L}_\xi := D_z\Phi(\xi, 0) L_\xi \quad (\xi \in \partial D) ,$$

are the same.

Proof. Let

$$P(\xi) := D_z\Phi(\xi, 0) \quad (\xi \in \overline{D}) \ .$$

The $C^{0,\alpha}$ matrix function P is holomorphic on D and, since Φ is a diffeomorphisms of $\overline{D} \times \mathbb{C}^n$, invertible on $\overline{D}$. Also,

$$B_{\widetilde{L}}(\xi) = P(\xi) B_L(\xi) \overline{P^{-1}(\xi)} \quad (\xi \in \partial D) \ ,$$

where B_L and $B_{\widetilde{L}}$ are the corresponding $Gl(n, \mathbb{C})$ closed paths (2) of the bundles L and $\widetilde{L}$, respectively. Since as soon as the partial indices are ordered, the characteristic matrix $\widetilde{\Lambda}$ of the path $B_{\widetilde{L}}$ does not depend on the factorization of $B_{\widetilde{L}}$ of the form

$$B_{\widetilde{L}} = F^+ \widetilde{\Lambda} F^- \ ,$$

where

$$F^+ : \overline{D} \longrightarrow Gl(n, \mathbb{C}) \quad \text{and} \quad F^- : \overline{\mathbb{C}} \setminus D \longrightarrow Gl(n, \mathbb{C})$$

are holomorphic, the proof of the lemma is completed. ■

5. Perturbation by analytic discs

The problem we consider in this section is the following.

Problem. Given a smooth map

$$\xi \longmapsto M(\xi) \quad (\xi \in \partial D) \ ,$$

where each $M(\xi)$, $\xi \in \partial D$, is a maximal real submanifold of $\mathbb{C}^n$, and a smooth map

$$p : \partial D \longrightarrow \mathbb{C}^n$$

such that $p(\xi) \in M(\xi)$ for each $\xi \in \partial D$, find all smooth maps $\varphi : \overline{D} \to \mathbb{C}^n$, holomorphic on D, which are close to the zero map and satisfy the condition

$$(p + \varphi)(\xi) \in M(\xi) \qquad (\xi \in \partial D) \ .$$

This problem was also considered by Globevnik, see [**Glo1**, Problem 1.2], for the orientable bundle case, and by Forstnerič in $\mathbb{C}^2$, [**For2**]. See also the paper [**Oh1**] by Y.-G. Oh. The arguments we use in this section closely follow those used by Globevnik in [**Glo1**] and Forstnerič in [**For2**], and so not every detail will be given.

The smoothness of the Problem will be $C^{0,\alpha}$ for some fixed $\alpha \in (0,1)$. That is :

a) The map $p : \partial D \to \mathbb{C}^n$ is of class $C^{0,\alpha}$.

b) For each $\xi_o \in \partial D$ there a neighbourhood $U_{\xi_o} \subseteq \partial D$ of ξ_o, there is an open ball $B_{\xi_o} \subseteq \mathbb{C}^n$ centered at the origin and maps $\rho_1^{\xi_o}, \dots, \rho_n^{\xi_o}$ from the space $C^{0,\alpha}(U_{\xi_o}, C^2(B_{\xi_o}))$ such that for each $\xi \in U_{\xi_o}$ we have

   1. $M(\xi) \cap (p(\xi) + B_{\xi_o}) = \{z \in p(\xi) + B_{\xi_o}; \rho_j^{\xi_o}(\xi, z - p(\xi)) = 0, \ j = 1, \dots, n\}$,
   2. $\rho_j^{\xi_o}(\xi, 0) = 0, \ j = 1, \dots, n$,
   3. $\overline{\partial}_z \rho_1^{\xi_o}(\xi, z) \wedge \cdots \wedge \overline{\partial}_z \rho_n^{\xi_o}(\xi, z) \neq 0$ for all $z \in B_{\xi_o}$.

An object of the above form will be called a $C^{0,\alpha}$ *maximal real fibration over the unit circle* ∂D *with* C^2 *fibers.*

Obviously each maximal real fibration over ∂D induces a maximal real vector bundle over ∂D, i.e., the bundle

$$\bigcup_{\xi\in\partial D}\{\xi\}\times T_{p(\xi)}M(\xi)\ ,$$

and hence it makes sense to talk about the partial indices and the total index of a maximal real fibration over ∂D. We define them as the indices of the corresponding maximal real vector bundle.

Let $B:\partial D\to GL(n,\mathbb{C})$ be the corresponding $C^{0,\alpha}$ closed path in $GL(n,\mathbb{C})$ defined by the map (3) which factors as

$$B(\xi)=\Phi(\xi)\Lambda(\xi)\overline{\Phi^{-1}(\xi)}=A_o(\xi)\overline{A_o^{-1}(\xi)}\qquad(\xi\in\partial D)\ , \tag{6}$$

where $A_o(\xi)$ stands for $\Phi(\xi)\Lambda_o(\xi)$. The $n\times n$ matrix $A_o(\xi)$ has the property that for each $\xi\in\partial D$ its columns span the tangent space $T_{p(\xi)}M(\xi)$ and that its rows belong to the space $\mathcal{E}_\sigma$.

One would also like to get a set of defining functions for the family $M(\xi)$ which would reflect the splitting (6).

LEMMA 6. *There exist an* $r_o>0$ *and functions*

$$\rho_j^o\in C^{0,\alpha}_{\mathbb{R}}(\partial D,C^2(B_{r_o}))\qquad(1\le j\le n)$$

such that for every odd partial index k_j the function ρ_j^o has the property

$$\rho_j^o(-\xi, z) = -\rho_j^o(\xi, z) \qquad ((\xi, z) \in \partial D \times B_{r_o}) \ ,$$

and such that for the functions

$$\rho_j(\xi, z) := \begin{cases} \rho_j^o(r(\xi), z) \ ; & k_j \text{ is odd} \ , \\ \rho_j^o(\xi, z) \ ; & k_j \text{ is even} \ , \end{cases}$$

the following holds

a) $M(\xi) \cap (p(\xi) + B_{r_o}) = \{z \in p(\xi) + B_{r_o}; \rho_j(\xi, z - p(\xi)) = 0, j = 1, \ldots, n\}$,

b) $\overline{\partial}_z \rho_1 \wedge \cdots \wedge \overline{\partial}_z \rho_n \neq 0$ *on* $\partial D \times B_{r_o}$.

Proof. For each point $\theta_o \in [0, 2\pi]$ it is easy, using the definition of a maximal real fibration over ∂D and some linear algebra, to find a neighbourhood $\mathcal{U}_{\theta_o} \subseteq \mathbb{R}$ of θ_o and functions $\rho_{\theta_o}^1, \ldots, \rho_{\theta_o}^n$ from $C^{0,\alpha}(exp(\mathcal{U}_{\theta_o}), C^2(B_{r_o}))$ such that for each $\xi \in exp(\mathcal{U}_{\theta_o})$ we have

$$M(\xi) \cap (p(\xi) + B_{r_o}) = \{z \in p(\xi) + B_{r_o}; \rho_{\theta_o}^j(\xi, z - p(\xi)) = 0, j = 1, \ldots, n\}$$

and

$$\nabla_z \rho_{\theta_o}(\xi, 0) = 2\frac{\partial \rho_{\theta_o}}{\partial \overline{z}}(\xi, 0) := 2(iA_o^{-1}(\xi))^* = -2i(\Phi^{-1})^* \Lambda_o \ ,$$

where $\rho_{\theta_o} = (\rho_{\theta_o}^1, \ldots, \rho_{\theta_o}^n)$. Here $exp(\mathcal{U}_{\theta_o})$ denotes the open set $\{e^{i\theta}; \theta \in \mathcal{U}_{\theta_o}\} \subseteq \partial D$. By the compactness we select a finite subcover $\{\mathcal{U}_j\}$ of the interval $[0, 2\pi]$. We may even assume, without loss of generality, that each of the points 0 and 2π is covered only once, and that for the sets of defining functions ρ_0 and $\rho_{2\pi}$

one has

$$\rho_0^j(e^{i\theta}, z) = \rho_{2\pi}^j(e^{i\theta}, z)$$

for every j such that the partial index k_j is an even integer and

$$\rho_0^j(e^{i\theta}, z) + \rho_{2\pi}^j(e^{i\theta}, z) = 0$$

for every j such that k_j is an odd integer. Let $\{\chi_j\}$ be a smooth partition of unity on $[0, 2\pi]$ subordinated to the cover $\{\mathcal{U}_j\}$. We define

$$\rho(e^{i\theta}, z) := \sum_j \chi_j(\theta)\rho_{\theta_j}(e^{i\theta}, z) \ .$$

For $r_o > 0$ small enough the above function ρ satisfies the required properties. Of course, the properties of the component functions ρ_j for such subscripts j that the partial index k_j is an odd integer, follow from the fact that the j-th column of the matrix Λ_o changes its sign when the argument $\arg(\xi)$ runs from 0 to 2π. Finally, for a subscript j such that the partial index k_j is an even integer, we define the function ρ_j^o to be the function ρ_j, and for a subscript j such that k_j is an odd integer we define the function ρ_j^o as $H_\xi\rho(\cdot, z)$. Here H is the map (2) defined in section 2. ∎

Using the vector notation we define

$$F : \mathcal{E}_\sigma \times \mathcal{E}_\sigma \longrightarrow \mathcal{E}_\sigma$$

as

$$F(u, v) := \rho(\cdot, A_o(u + i(v + iT_\sigma v))) \ .$$

Observe that F is well defined and that is of class C^1, see [**Glo1**, Lemma 6.1]. Observe also that if a pair $(u_o, v_o) \in \mathcal{E}_\sigma \times \mathcal{E}_\sigma$ solves the equation

$$F(u, v) = 0 \ ,$$

then the boundary of the disc $p + A_o(u_o + i(v_o + iT_\sigma v_o))$ lies in the given maximal real fibration, i.e., for each $\xi \in \partial D$ we have

$$p(\xi) + A_o(\xi)(u_o(\xi) + i(v_o(\xi) + i(T_\sigma v_o)(\xi))) \in M(\xi) \ .$$

The rest is completely standard. First one should use the implicit mapping theorem in Banach spaces, [**Car**], for the mapping F and the space $\mathcal{E}_\sigma$ to get a parametrization of all, not necessary holomorphic, nearby discs with boundaries in the maximal real fibration $\{M(\xi)\}_{\xi \in \partial D}$.

Let $\overline{\partial}\rho$ denote the matrix whose columns are the coefficients of the $(0,1)$ forms $\overline{\partial}\rho_1, \ldots, \overline{\partial}\rho_n$. For each $\xi \in \partial D$ we have

$$\overline{\partial}\rho(\xi, 0)^* A_o(\xi) = iI_n \ ,$$

where I_n denotes the $n \times n$ identity matrix. Then the partial derivative of F with respect to v at the point $(0,0) \in \mathcal{E}_\sigma \times \mathcal{E}_\sigma$, applied to a function $\nu \in \mathcal{E}_\sigma$, is in the matrix notation given by

$$((D_g F(0,0))\nu)(\xi) = 2\mathrm{Re}(\overline{\partial}\rho(\xi, 0)^* A_o(\xi)(i\nu(\xi) - (T_\sigma \nu)(\xi))) = -2\nu(\xi)$$

for every $\xi \in \partial D$. Hence the partial derivative of the mapping F with respect to variable v is an invertible linear map from the space $\mathcal{E}_\sigma$ into itself. So the implicit mapping theorem in the Banach spaces gives neighbourhoods V_1 and V_2 of $0 \in \mathcal{E}_\sigma$, and a unique C^1 mapping

$$\varphi : V_1 \longrightarrow V_2$$

such that on $V_1 \times V_2$

$$F(u,v) = 0 \quad \text{if and only if} \quad v = \varphi(u) \ .$$

Finally one would like to select from the above family of all possible closed $C^{0,\alpha}$ curves in the maximal real fibration $\{M(\xi)\}_{\xi \in \partial D}$ near p, those which bound a sum

$$p + \text{analytic disc} \ .$$

This is the point where one can effectively use the normal form of the maximal real bundle $p^*(T_{p(\xi)}M(\xi))$, $\xi \in \partial D$, over the circle ∂D. And this is also the point where one should assume that all partial indices of $p^*(TM)$ are greater or equal to -1. For the final argument one should first observe that for the case when all partial indices of the maximal real bundle $p^*(TM)$ are greater or equal to -1, the vector function

$$A_o(v + iT_\sigma v)$$

extends holomorphically to D for every $v \in \mathcal{E}_\sigma$. For the nonnegative partial indices this follows immediately but for the partial indices which equal to -1 the above claim follows from the fact that for any odd partial index k_j the function $v_j + iTv_j$ is of the form $r(\xi)g_j^o(\xi)$ for some function $g_j^o \in A^{0,\alpha}(\partial D)$.

So the condition for the vector function

$$\xi \longmapsto A_o(u + i(v + iT_\sigma v))(\xi) \qquad (\xi \in \partial D)$$

to extend holomorphically to D is in the case $k_j \geq -1,\ j = 1, \ldots, n$, equivalent to the condition that the vector function

$$\xi \longmapsto A_o(\xi)u(\xi) \qquad (\xi \in \partial D)$$

extends holomorphically to D. To detect all possible $u's$ which have the above property one has to find all vector functions $a \in (A^{0,\alpha}(\partial D))^n$ such that on ∂D

$$\Lambda \overline{a} = a \ ,$$

i.e., for all $j = 1, \ldots, n$

$$\xi^{k_j} \overline{a_j(\xi)} = a_j(\xi) \qquad (\xi \in \partial D) \ .$$

For each partial index $k_j = -1$ there is only one solution of the above equation, namely, $a_j = 0$. For $k_j \geq 0$ one gets $k_j + 1$ dimensional parameter family of solutions. A parametrization Ψ of all functions $u \in \mathcal{E}_\sigma$ such that the vector function $A_o u$ extends holomorphically to D is for each component function u_j given by

$$\Psi_j(t_0, \ldots, t_{k_j})(\xi) := t_0 + \mathrm{Re}(\sum_{s=1}^{k_j/2} (t_{2s-1} + it_{2s})\xi^s) \tag{7}$$

in the case the partial index k_j is an even integer and by

$$\Psi_j(t_0, \ldots, t_{k_j})(\xi) := \mathrm{Re}(\sum_{s=0}^{(k_j-1)/2} (t_{2s} + it_{2s+1})r(\xi)^{2s+1}) \tag{8}$$

in the case k_j is an odd integer. See [**Glo1**] for more details. Hence, altogether one has

$$\sum_{k_j \geq 0} (k_j + 1) = \sum_{k_j \geq 0} k_j + \sum_{k_j \geq 0} 1 = \sum_{k_j \geq 0} k_j - \sum_{k_j < 0} 1 + n = k + n$$

parameter family of solutions of the Problem.

THEOREM 1. *Let $M(\xi) \subseteq \mathbb{C}^n$, $\xi \in \partial D$, be a $C^{0,\alpha}$ maximal real fibration over the unit circle ∂D with C^2 fibers and let*

$$p : \partial D \longrightarrow \mathbb{C}^n$$

be a $C^{0,\alpha}$ closed path in $\mathbb{C}^n$ such that

$$p(\xi) \in M(\xi) \qquad (\xi \in \partial D)\ .$$

If all partial indices of the maximal real fibration $M(\xi)$, $\xi \in \partial D$, along the path p are greater or equal to -1 and the total index of p is k, then there is an open neighbourhood U of $0 \in \mathbb{R}^{n+k}$, an open neighbourhood W of p in $(C^{0,\alpha}(\partial D))^n$ and a map

$$\Psi : U \longrightarrow (C^{0,\alpha}(\partial D))^n$$

of class C^1 such that

1) $\Psi(0) = p$,
2) *for each $t \in U \subseteq \mathbb{R}^{n+k}$ the map $\Psi(t) - p$ extends holomorphically to D,*
3) $\Psi(t_1) \neq \Psi(t_2)$ *for $t_1 \neq t_2$,*

4) *if* $\widetilde{p} \in W$ *satisfies the condition* $\widetilde{p}(\xi) \in M(\xi),\ \xi \in \partial D$, *and is such that* $\widetilde{p} - p$ *extends holomorphically to* D, *then there is* $t \in U$ *such that* $\Psi(t) = \widetilde{p}$.

Remark 1. The above theorem was first proved by Forstnerič in [**For2**] in the case where the ambient space is $\mathbb{C}^2$. Next, it was proved by Globevnik [**Glo1**] in $\mathbb{C}^n$ for the case where the pull-back bundle $p^*(TM)$ is trivial and where all partial indices are nonnegative. Later I observed that Forstnerič's and Globevnik's tehnique also works in the case when the pull-back bundle is nontrivial but the indices are still nonnegative. In the meantime the Problem was, although in a little bit different context, solved by Y-G.Oh [**Oh1**] who noticed that the partial indices can be even taken to be greater or equal to -1. The above theorem thus also includes his observation but in the context of the Forstnerič's and Globevnik's tehnique to tackle the Problem. For more on the history of the Problem, partial results and applications one should also check the papers by Alexander [**Ale2**], Bedford [**Bed1, Bed2**], Bedford and Gaveau [**Bed-Gav**], Eliashberg [**Eli**], Gromov [**Gro**].

Remark 2. As in [**Glo1**, Theorem 7.1] one could also add to the above theorem perturbations of a maximal real fibration over the unit circle and the condition that in the case where all partial indices are greater or equal to 1, the set of the centers $\Psi(t)(0) - p(0),\ t \in U$, contains an open set in $\mathbb{C}^n$.

6. Analytic varieties over the disc

Let M be a maximal real submanifold of $\mathbb{C}^n$. The problem we consider in this section is a very special case of the following general question :

Given M and $V \subseteq \mathbb{C}^n \setminus M$, a purely one-dimensional analytic subvariety with boundary in M, can one "find and describe" all purely one-dimensional analytic subsets in $\mathbb{C}^n \setminus M$ that are near V ?

Let $n = 2$ and let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be the projection on the first coordinate plane $\pi : (\xi, z) \mapsto \xi$. Let $\mathcal{T} \subseteq \partial D \times \mathbb{C}$ be a compact, connected, totally real two-dimensional submanifold of $\mathbb{C}^2$ of class C^2 such that for each $\xi \in \partial D$ the fiber

$$\mathcal{T}_\xi = \{z \in \mathbb{C}; (\xi, z) \in \mathcal{T}\}$$

is the union of q, $q \in \mathbb{N}$, simple closed curves in $\mathbb{C}$ whose polynomial hulls are pairwise disjoint. Observe that $\mathcal{T}$ is a totally real embedded torus in $\mathbb{C}^2$.

In our setting $M \subseteq \mathbb{C}^2$ will be a finite disjoint union of totally real tori fibered over the unit circle $\partial D \subseteq \mathbb{C} \times \{0\} \subseteq \mathbb{C}^2$, i.e.,

$$M = \bigcup_{j=1}^{k} \mathcal{T}_j$$

where $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ are pairwise disjoint totally real tori in $\partial D \times \mathbb{C}$. Thus each fiber

$$M_\xi = \{z \in \mathbb{C}; (\xi, z) \in M\}$$

is a finite pairwise disjoint union of $q = q_1 + \cdots + q_k$ Jordan curves $J_\xi^1, \ldots, J_\xi^q$.

For V, a purely one-dimensional analytic subset of $\mathbb{C}^2 \setminus M$, we assume that is a q-sheeted analytic variety over D, i.e.,

a) $V \subseteq D \times \mathbb{C}$ is given by a Weierstrass polynomial of degree q

$$V = \{(\xi, z) \in D \times \mathbb{C}; z^q + a_1(\xi) z^{q-1} + \cdots + a_q(\xi) = 0\} ,$$

where $a_1, a_2, \ldots, a_q$ are in the disc algebra $A(D)$,

b) $\overline{V} \cap J_\xi^j \neq \emptyset$ for every $j = 1, 2, \ldots, k$ and every $\xi \in \partial D$.

Observe that by a theorem of Čirka, [**Čir**], $a_1, \ldots, a_q$ are also in $A^{2-0}(D)$.

Given M and V as above one can construct a compact connected maximal real manifold $\widetilde{M}$ in $\mathbb{C}^{q+1}$ and an analytic disc $\widetilde{V}$ with boundary in $\widetilde{M}$ in the following way :

Each fiber $M \cap \pi^{-1}(\xi)$, $\xi \in \partial D$, is the disjoint union of q Jordan curves J_ξ^j, $j = 1, \ldots, q$. Let $\mathcal{T}_\xi := J_\xi^1 \times \cdots \times J_\xi^q \subseteq \mathbb{C}^q$. We define the map

$$\begin{aligned} \Phi : \mathbb{C}^q &\longrightarrow \mathbb{C}^q \\ (z_1, \ldots, z_q) &\longmapsto (s_1, \ldots, s_q) \end{aligned}$$

where

$$s_p = \frac{1}{p} \sum_{j=1}^{q} z_j^p \qquad (p = 1, \ldots, q) .$$

Observe that Φ is a proper map from $\mathbb{C}^q$ into itself and that its Jacobian determinant is

$$J_\Phi(z) = \det(D\Phi(z)) = \prod_{j>t} (z_j - z_t) \; .$$

By the assumptions on the manifold M, i.e., the curves J^j_ξ, $j = 1, \ldots, q$, are pairwise disjoint, the q-dimensional tori $\mathcal{T}_\xi$, $\xi \in \partial D$, do not intersect the branch locus of the map Φ, i.e., the set of points in $\mathbb{C}^q$ where $J_\Phi(z) = 0$. Moreover, by the same reason and since the coefficients of a polynomial uniquely, up to the order, determine its zeros, Φ is injective on each $\mathcal{T}_\xi$. The image of $\mathcal{T}_\xi$ under Φ is denoted by $\widetilde{\mathcal{T}}_\xi$ and is a maximal real q-torus in $\mathbb{C}^q$ of class C^2.

We define

$$\widetilde{M} := \bigcup_{\xi \in \partial D} \{\xi\} \times \widetilde{\mathcal{T}}_\xi \; .$$

Since locally, over some arc $I \subseteq \partial D$, $\widetilde{M}$ is given as the image of $\cup_{\xi \in I} \{\xi\} \times \mathcal{T}_\xi$ under the map

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^q & \longrightarrow & \mathbb{C} \times \mathbb{C}^q \\ (\xi, z) & \longmapsto & (\xi, \Phi(z)) \end{array}$$

and since the component functions of the maps Φ_ξ are symmetric in their arguments, $\widetilde{M}$ is a maximal real compact connected manifold in $\mathbb{C}^{q+1}$ fibered over ∂D of class C^2.

By an assumption on V the points in the fiber $\overline{V} \cap \pi^{-1}(\xi)$ represent exactly one point in $\mathcal{T}_\xi$. The mapping Φ maps each of them into $\widetilde{\mathcal{T}}_\xi$. Since V is given by a Weierstrass polynomial and since Φ_ξ is given by elementary symmetric functions of its arguments, one concludes that the boundary of V is mapped into the boundary of an analytic disc in $\mathbb{C}^{q+1}$ attached to $\widetilde{M}$. Moreover, this disc is given as the graph of an analytic disc in $\mathbb{C}^q$ which takes $\xi \in \partial D$ into $\widetilde{\mathcal{T}}_\xi$.

One also observes that the varieties over D with boundaries in M which satisfy the above conditions are in one to one correspondence with the graphs of the analytic discs in $\mathbb{C}^q$ which map $\xi \in \partial D$ into $\widetilde{\mathcal{T}}_\xi$. Namely, a basis of symmetric functions in q arguments $z_1, \ldots, z_q$ can be given either with the functions $s_1, \ldots, s_q$ or with the symmetric functions which one gets through the Vieta formulae. So the problem of finding and describing all analytic varieties with boundaries in M near to the given analytic variety V with boundary in M is now translated to the problem of finding all analytic discs with boundaries in $\widetilde{M}$ close to the given one.

Remark and an example. Although the fibers $\widetilde{\mathcal{T}}_\xi$ of the manifold $\widetilde{M}$ are maximal real q-dimensional tori, the manifold $\widetilde{M}$ itself is not necessary a $q+1$-dimensional torus as the following example shows :

Let the manifold M be given by

$$M := \{(\xi, z) \in \partial D \times \mathbb{C}; |z^2 - \xi| = \frac{1}{2}\}$$

and the variety V by the equation $z^2 - \xi = \frac{1}{2}$. Then the pull-back of the tangent bundle of the 3-dimensional totally real manifold $\widetilde{M} \subseteq \mathbb{C}^3$ along the graph of the associated analytic disc is not trivial. Thus $\widetilde{M}$ is not a 3-torus. ■

We denote by $p : \partial D \longrightarrow \mathbb{C}^q$ the corresponding analytic disc whose graph has boundary in $\widetilde{M}$ and by p' its derivative. Fix $\xi_o \in \partial D$. Since the curves J_ξ^j, $j = 1, \ldots, q$, $\xi \in \partial D$, are pairwise disjoint, one can, locally near ξ_o, order the roots of the Weierstrass polynomial defining the variety V. Let $\alpha_1(\xi) \in J_\xi^1, \ldots, \alpha_q(\xi) \in J_\xi^q$ denote its roots. Also, to each root $\alpha_j(\xi)$, $j = 1, \ldots, q$, $\xi \in \partial D$, there corresponds a unique unit outer normal $\tau_j(\xi)$ to the curve $J_\xi^j \subseteq \mathbb{C}$ at the point $\alpha_j(\xi)$. Using local parametrization of M, definition of the manifold $\widetilde{M}$, and the above notation, a matrix $\widetilde{A}$ whose columns span $T_{(\xi,a(\xi))}\widetilde{M}$ can be written as

$$\widetilde{A} = \begin{pmatrix} i\xi & 0 \\ i\xi p'(\xi) & i\alpha(\xi)\tau(\xi) \end{pmatrix} ,$$

where

$$\alpha(\xi) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1(\xi) & \alpha_2(\xi) & \ldots & \alpha_q(\xi) \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^{q-1}(\xi) & \ldots & \ldots & \alpha_q^{q-1}(\xi) \end{pmatrix}$$

and

$$\tau(\xi) = \begin{pmatrix} \tau_1(\xi) & 0 & \ldots & 0 \\ 0 & \tau_2(\xi) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \tau_q(\xi) \end{pmatrix} .$$

Thus if one defines $B(\xi) := \alpha(\xi)\tau(\xi)\overline{(\alpha(\xi)\tau(\xi))^{-1}}$ and $\widetilde{B}(\xi) := \widetilde{A}\overline{\widetilde{A}^{-1}}$, $\xi \in \partial D$, then the proof of Lemma 4 part 2 implies that one of the indices of the

$Gl(q+1,\mathbb{C})$ closed path $\widetilde{B}$ equals 2 and the remaining q indices are given by the $Gl(q,\mathbb{C})$ closed path B.

COROLLARY 3. *The total index k_o of the closed path B is given by*

$$k_o = 2W(\prod_{j=1}^{q} \tau_j) + W(\prod_{j>t} (\alpha_j - \alpha_t)^2)$$

where $W(g)$ denotes the winding number of a function $g \in C(\partial D)$ with no zeros.

Proof. The total index k_o is also given as the winding number of the determinant $\det(B)$. Using the special form of the matrix B and the fact that the matrix α is the Van der Mond matrix the corollary follows immediately. One should also recall that the winding number of the product of two functions equals to the sum of the winding numbers of its factors. ∎

Results from the previous section and [**Glo1**] now imply

PROPOSITION 1. *If all partial indices of the $Gl(q,\mathbb{C})$ closed path B are*

a) *(Existence) greater or equal to -1, then near V there is a k_o+q+3 parameter family of analytic varieties with boundaries in M. Moreover, if each partial index of B is at least 1, then the family of analytic varieties with boundaries in M that are near V contains an open subset of $\mathbb{C}^2$.*

b) *(Nonexistence) negative, then there is a neighbourhood of $\overline{V}$ in $\mathbb{C}^2$ such that in this neighbourhood the variety V is the only analytic variety with boundary in M.*

COROLLARY 4. *A necessary condition that all partial indices of the $Gl(q,\mathbb{C})$ closed path B are positive, and thus the union of the family of analytic varieties with boundaries in M close to the variety V contains an open subset of $\mathbb{C}^2$, is*

$$2W(\prod_{j=1}^{q}\tau_j) + W(\prod_{j>t}(\alpha_j-\alpha_t)^2) \geq q \ .$$

Proof. A necessary condition that a family of nearby varieties with boundaries in M contains an open set of $\mathbb{C}^2$ is that all partial indices are greater or equal to 1. So the total index k_o has to be greater or equal to q. ■

Example. Let the manifold M be given by

$$M := \{(\xi,z)\in\partial D\times\mathbb{C};|z^2-\xi^2| = \frac{1}{2}\}$$

and the variety V by the equation $z^2-\xi^2=\frac{1}{2}$. The normals to the fibers of M at the boundary of V are given by

$$\tau_j(\xi) = \overline{\alpha_j(\xi)} \qquad (j=1,2) \ .$$

Therefore $W(\tau_j) = -1$, $j = 1,2$, and $W(\alpha_2-\alpha_1) = 1$, and our necessary condition to have a lot of nearby analytic varieties with the boundary in M fails. One can also calculate the partial indices for this example. They are 2, 0 and -2. ■

7. Minimal discs with free boundaries

Let $M\subseteq\mathbb{C}^n$ be a C^2 manifold and let $p:\overline{D}\to\mathbb{C}^n$ be a disc of class $C^{1,\alpha}$ with boundary in M, i.e., $p:\partial D\to M$. By the energy of the disc p we mean the Dirichlet integral of p

$$E(p) := \frac{1}{2}\int_D |\nabla p|^2 dxdy \ .$$

The maps which minimize the energy functional are of a special interest in the Riemannian geometry, namely, any map with boundary in a submanifold M which minimizes energy E in a certain homotopy class $[p(\partial D)] \in G \subseteq \pi_1(M)$, also minimizes the area functional

$$A(p) := \int_D \left| \frac{\partial p}{\partial x} \bigwedge \frac{\partial p}{\partial y} \right| dxdy$$

in G, see [**Nit**] and [**Ye2**] for more details. The advantage of the energy functional E with respect to the area functional A is that the first is only conformal invariant but the latter is invariant under any diffeomorphic change of coordinates.

Let (r, θ) denote the polar coordinates on the unit disc D. It is well known, see e.g., [**Jäg**], [**Lew**], [**Ye2**], that any solution p of the Euler-Lagrange equations for the energy functional E with free boundary in the manifold M must satisfy the following conditions:

a) All component functions of the mapping p are harmonic in D, and

b) $\frac{\partial p}{\partial r}(\xi) \perp T_{p(\xi)}M$ for every $\xi \in \partial D$.

In our case, where the ambient space is $\mathbb{C}^n$, it is quite easy to check the above statement:

Let p_t, $t \in (-1, 1)$, denote a one-parameter family of maps from the unit disc

$D \subseteq \mathbb{C}$ into $\mathbb{C}^n$ which all take the unit circle ∂D into the submanifold M. Let

$$\frac{\partial}{\partial t} p_t \Big|_{t=0} = \eta \ .$$

Observe that the vector η is tangent to M along $p_o(\partial D)$. For every such variation η Stokes' theorem for the pair $(D, \partial D)$ implies

$$\frac{d}{dt} E(p_t) \Big|_{t=0} = -\int_D \triangle p_o \cdot \eta \, dxdy + \int_{\partial D} \frac{\partial p_o}{\partial r} \cdot \eta \, d\theta \ .$$

A necessary condition on a disc p_o with boundary in M to be an energy functional minimizer with free boundary in M is the vanishing of the first variation of E at p_o, i.e.,

$$\frac{d}{dt} E(p_t) \Big|_{t=0} = 0 \ ,$$

and the above statement on energy functional minimizers easily follows.

Since $p : \partial D \to M$ we also have

$$\frac{\partial p}{\partial \theta}(\xi) \in T_{p(\xi)} M$$

on ∂D. Combining both conditions we conclude

$$\frac{\partial p}{\partial r}(\xi) \perp \frac{\partial p}{\partial \theta}(\xi) \qquad (\xi \in \partial D) \ .$$

Since p is harmonic on the unit disk D it can be written in the form

$$p = f + \overline{g}$$

for some holomorphic vector functions f and g from $(A^{1,\alpha}(\overline{D}))^n$. The condition on p to be minimal can be written as

$$\mathrm{Re} \langle \frac{\partial p}{\partial r}(\xi), \frac{\partial p}{\partial \theta}(\xi) \rangle = 0$$

and so

$$\mathrm{Re}\langle \xi f'(\xi) + \overline{\xi g'(\xi)}, i\xi f'(\xi) - i\overline{\xi g'(\xi)}\rangle = 0 \qquad (\xi \in \partial D)\ .$$

A short calculation shows

$$\mathrm{Im}\xi^2\langle f'(\xi), \overline{g'(\xi)}\rangle = 0$$

for every $\xi \in \partial D$. But the function

$$F(\xi) = \xi^2\langle f'(\xi), \overline{g'(\xi)}\rangle = 0 \qquad (\xi \in \partial D)$$

extends holomorphically to D and so $F(\xi)$ is a constant function. Since $F(0) = 0$, one concludes that

$$\langle f'(\xi), \overline{g'(\xi)}\rangle = \sum_{j=1}^{n} f_j'(\xi) g_j'(\xi) = 0$$

for every $\xi \in \partial D$.

COROLLARY 5. *Let p be a complex function $p : \overline{D} \to \mathbb{C}$ of the class $C^{1,\alpha}$. If either of the discs*

$$F, G : \overline{D} \longrightarrow \mathbb{C}^2\ ,$$

where

$$\begin{aligned} F(z) &= (z, p(z))\ , \\ G(z) &= (\overline{z}, p(z))\ , \end{aligned}$$

is minimal for some submanifold M in $\mathbb{C}^2$, then the function p is either holomorphic or antiholomorphic. In particular, M can be a totally real torus in $\mathbb{C}^2$ fibered over the unit circle ∂D.

Proof. Let us assume that the disc $F(z) = (z, p(z))$ is minimal for some submanifold M. Then the corresponding holomorphic discs f and g are of the form

$$f(z) = (z, a(z)) \quad \text{and} \quad g(z) = (0, b(z))$$

for some holomorphic functions a and b from $A^{1,\alpha}(D)$. The above argument then implies

$$a'(z)b'(z) = 0$$

for every $z \in \overline{D}$. Thus at least one of the functions a and b has to be a constant function. In the case the disc $G(z) = (\overline{z}, p(z))$ is minimal, the proof is similar. ■

We would like to use the special normal form of the pull-back bundle to investigate the energy functional minimizers with boundaries in a Lagrangian submanifold $M \subseteq \mathbb{C}^n$ of class C^2.

PROPOSITION 2. *Let L be a maximal real n-dimensional vector bundle over ∂D of class $C^{0,\alpha}$ such that every fiber L_ξ, $\xi \in \partial D$, is a Lagrangian subspace of $\mathbb{C}^n$, and let B_L be the $C^{0,\alpha}$ closed path in $Gl(n, \mathbb{C})$ which represents L. Then :*

a) $B_L^* = B_L^{-1}$ *so the matrix $B_L(\xi)$ is unitary for every $\xi \in \partial D$.*

b) $B_L^t = B_L$, *and conversely, if for a maximal real bundle L over ∂D one has $B_L^t = B_L$, then the bundle L is Lagrangian, i.e., each fiber is a Lagrangian subspace of $\mathbb{C}^n$.*

c) *Let n_o denote the number of partial indices of the bundle L which are greater or equal to -1. Then there are $n \times n_o$ and $n \times (n - n_o)$ dimensional matrix functions X and Y, respectively, with entries from the space $A^{0,\alpha}(\partial D)$ such that the columns of the matrix function*

$$[X, \overline{Y}]\Lambda_o$$

span the fiber L_ξ for each $\xi \in \partial D$. Moreover, $X^t Y = 0$ on ∂D. Here Λ_o is the square root of the corresponding characteristic matrix Λ.

Remark. In the case n_o is n (resp. 0) the matrix Y (resp. X) does not exist.

Proof. Since the fibers of the bundle L are Lagrangian subspaces of $\mathbb{C}^n$, the bundles iL and $L^\perp$ are the same. Lemma 3 implies $B_L^t = B_L$ and then also, by Lemma 1, $B_L^* = B_L^{-1}$. This proves (a) and the first part of (b). To prove the reverse implication of part (b) one should observe that if A is any $n \times n$ matrix whose columns span L_ξ for some $\xi \in \partial D$, then

$$A\overline{A^{-1}} = B_L(\xi) = B_L^t(\xi) = \overline{A^{-1}}^t A^t = (A^*)^{-1} A^t \ .$$

So

$$A^* A = A^t \overline{A} = \overline{A^* A}$$

and the matrix $A^* A$ is a real $n \times n$ matrix. Hence

$$\mathrm{Re}((iA)^* A) = 0$$

and so the columns of the matrix iA are perpendicular to the columns of the matrix A. Part (b) is proved.

To prove part (c) we use the factorization of the matrix function B_L,

$$B_L = \Phi\Lambda\overline{\Phi^{-1}} \ ,$$

for some map $\Phi : \overline{D} \to Gl(n, \mathbb{C})$ from $A^{0,\alpha}(D)^{n\times n}$. We define $\Psi := (\Phi^{-1})^*$ and by part (b) we also have

$$B_L = \Psi\Lambda\overline{\Psi^{-1}} \ .$$

Let

$$\Phi = [\Phi_1, \Phi_2] \quad \text{and} \quad \Psi = [\Psi_1, \Psi_2]$$

be the block notation of the matrix functions Φ and Ψ such that the matrices Φ_1 and Ψ_1 have dimensions $n \times n_o$. Since

$$\Phi^t\overline{\Psi} = \Phi^t(\Phi^{-1})^t = I_n \ ,$$

we get

$$\begin{bmatrix} \Phi_1^t \\ \Phi_2^t \end{bmatrix} [\overline{\Psi_1}, \overline{\Psi_2}] = I_n$$

and so

$$\Phi_1^t\overline{\Psi_2} = 0 \ .$$

This means that the columns of the matrix Φ_1 are orthogonal to the columns of the matrix Ψ_2, and so the matrix

$$[\Phi_1, \Psi_2]$$

is invertible. We set $X := \Phi_1$ and $Y := \overline{\Psi_2}$ and the proof is finished. ■

The proof of part (c) of the above proposition also implies

COROLLARY 6. *Let L be a Lagrangian vector bundle over ∂D of class $C^{0,\alpha}$, let B_L be the corresponding $C^{0,\alpha}$ closed path in $Gl(n,\mathbb{C})$ and let Λ be its characteristic matrix. Then the characteristic matrix of the $Gl(n,\mathbb{C})$ closed path*

$$\xi \longmapsto B_L(\overline{\xi})$$

is $\overline{\Lambda}$.

Remark. Statement (c) of the above proposition does not hold for an arbitrary maximal real vector bundle over the ∂D. As a counterexample one can take the bundle whose matrix B is given by

$$\begin{pmatrix} -\xi^2 & 1 \\ 0 & \overline{\xi}^2 \end{pmatrix} .$$

It is easy to check that the partial indices of B are 2 and -2 and that there are no nontrivial holomorphic functions $a, b \in A^{0,\alpha}(\partial D)$ which satisfy the equation

$$\begin{pmatrix} -\xi^2 & 1 \\ 0 & \overline{\xi}^2 \end{pmatrix} \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} = \overline{\xi}^2 \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} .$$

■

Henceforth M will denote a C^2 Lagrangian submanifold in $\mathbb{C}^n$ and $p : \overline{D} \to \mathbb{C}^n$ will be an energy functional stationary disc of class $C^{1,\alpha}$ with boundary in M, i.e., the first variation of the energy functional E at p is 0. We recall that the pair (r, θ) represents the polar coordinates on D. Then one has

$$\xi \frac{\partial p}{\partial z}(\xi) = \frac{1}{2}\left(\frac{\partial p}{\partial r}(\xi) + i\frac{\partial p}{\partial \theta}(\xi)\right)$$

and

$$\overline{\xi}\frac{\overline{\partial} p}{\partial \overline{z}}(\xi) = \frac{1}{2}\left(\frac{\partial p}{\partial r}(\xi) - i\frac{\partial p}{\partial \theta}(\xi)\right)$$

for every $\xi \in \partial D$. Since the disc p is attached to the manifold M, the vector $\frac{\partial p}{\partial \theta}(\xi)$ is tangent to M at $p(\xi)$ for each $\xi \in \partial D$. Since p is minimal, the vector $\frac{\partial p}{\partial r}(\xi)$ is perpendicular to M at $p(\xi)$ for each $\xi \in \partial D$.

Let A_o denote the matrix constructed in part (c) of Proposition 2. Since M is a Lagrangian submanifold of $\mathbb{C}^n$ we get

$$\mathrm{Re}(iA^* \frac{\partial p}{\partial \theta}(\xi)) = 0 \ ,$$

and since the first variation of the energy functional E at p vanishes, we have

$$\mathrm{Re}(A^* \frac{\partial p}{\partial r}(\xi)) = 0 \ .$$

Thus for each $\xi \in \partial D$ we get

$$\mathrm{Re}(\overline{\xi} A_o^*(\xi)\frac{\partial p}{\partial \overline{z}}(\xi)) = 0 \quad \text{and} \quad \mathrm{Re}(\xi A_o^*(\xi)\frac{\partial p}{\partial z}(\xi)) = 0 \ .$$

Since $p = f + \overline{g}$ for some vectors f, g from $A^{1,\alpha}(\partial D)^n$, we have

$$\mathrm{Re}(\xi A_o^t(\xi) g'(\xi)) = 0 \quad \text{and} \quad \mathrm{Re}(\xi A_o^*(\xi) f'(\xi)) = 0 \ . \tag{9}$$

Since also $A_o = [X, \overline{Y}]\Lambda_o$ where the matrices X and Y have holomorphic extensions into D and are "orthogonal" to each other, i.e., $Y^t X = 0$, we first conclude that

$$X^t g' = 0 \quad \text{and} \quad Y^t f' = 0 \ ,$$

and finally

$$f' = Xh \quad \text{and} \quad g' = Yk$$

for some vector functions $h \in (A^{0,\alpha}(D))^{n_o}$ and $k \in (A^{0,\alpha}(D))^{n-n_o}$. For these results we used only one part of each equation in (9). The rest, together with the above equalities, implies

$$\mathrm{Re}(\xi \overline{\Lambda_o^+} X^* X h) = 0 \quad \text{and} \quad \mathrm{Re}(\xi \overline{\Lambda_o^-} Y^* Y k) = 0 \ ,$$

where Λ_o^+ and Λ_o^- are the $n_o \times n_o$ and the $(n - n_o) \times (n - n_o)$ dimensional matrices, respectively, such that

$$\Lambda_o = \begin{pmatrix} \Lambda_o^+ & 0 \\ 0 & \Lambda_o^- \end{pmatrix} \ .$$

Since

$$\mathrm{Re}(-iA_o^* A_o) = 0$$

on ∂D, the matrices

$$\overline{\Lambda_o^+} X^* X \Lambda_o^+ \quad \text{and} \quad \overline{\Lambda_o^-} Y^* Y \Lambda_o^-$$

are real and invertible. Thus

$$\mathrm{Re}(\xi \overline{\Lambda_o^+(\xi)} h(\xi)) = 0 \quad \text{and} \quad \mathrm{Re}(\xi \overline{\Lambda_o^-(\xi)} k(\xi)) = 0$$

for every $\xi \in \partial D$. This proves the following theorem.

THEOREM 2. *Let M be a C^2 Lagrangian submanifold in $\mathbb{C}^n$ and let $p = f + \overline{g}$, $f, g \in (A^{1,\alpha}(\partial D))^n$, be the energy functional stationary disc of class $C^{1,\alpha}$ with boundary in M. Let n_o denote the number of partial indices of the path $p(\partial D) \subseteq M$ which are greater or equal to -1, and let X and Y be the $n \times n_o$ and $n \times (n - n_o)$ dimensional holomorphic matrices given by part (c) of Proposition 2. Then there exist vector functions $h \in (A^{0,\alpha}(\partial D))^{n_o}$ and $k \in (A^{0,\alpha}(\partial D))^{n-n_o}$ such that*

$$\mathrm{Re}(\xi\overline{\Lambda_o^+(\xi)}h(\xi)) = 0 \quad \text{and} \quad \mathrm{Re}(\xi\overline{\Lambda_o^-(\xi)}k(\xi)) = 0 \quad (\xi \in \partial D)\ ,$$

and

$$f' = Xh \quad \text{and} \quad g' = Yk\ .$$

Remark. In the case n_o equals n (resp. 0) one part of the above conlusion is empty, namely, the matrix Y (resp. X) does not exist.

As a simple consequence one has the following

COROLLARY 7. *(Hypothesis as above.)*

1) *If all partial indices of the pull-back bundle $p^*(TM)$ are greater or equal to -1, then p is a holomorphic disc.*
2) *If all partial indices of the pull-back bundle $p^*(TM)$ are less or equal to 1, then p is an antiholomorphic disc.*

8. Stationary discs

Let $\partial\Omega_\xi,\ \xi \in \partial D$, be a $C^{1,\alpha}$ family of strongly pseudoconvex C^4 hypersurfaces in $\mathbb{C}^n$, i.e., there exists a $C^{1,\alpha}(\partial D, C^4(\mathbb{C}^n))$ function $\rho = \rho(\xi, z)$ on $\partial D \times \mathbb{C}^n$ such that for every $\xi \in \partial D$

a) the hypersurface $\partial\Omega_\xi$ equals to the set $\{z \in \mathbb{C}^n; \rho(\xi, z) = 0, \partial_z\rho(\xi, z) \neq 0\}$ and

b) the domain $\{z \in \mathbb{C}^n; \rho(\xi, z) < 0\}$ is strongly pseudoconvex.

Let

$$f : D \longrightarrow \mathbb{C}^n$$

be a holomorphic map of class $C^{1,\alpha}$ such that

$$f(\xi) \in \partial\Omega_\xi \quad (\xi \in \partial D)\ .$$

We will call such f a holomorphic disc in $\mathbb{C}^n$ with boundary in the family of strongly pseudoconvex hypersurfaces $\{\partial\Omega_\xi\}_{\xi \in \partial D}$. For any such mapping f we define

$$\mu_f(\xi) := \frac{\partial\rho}{\partial z}(\xi, f(\xi))\ .$$

The following definition seems to be a natural extension of the definition in [**Lem**], see also [**Slo4**] and [**Pan1**].

DEFINITION 3. *The disc f is said to be stationary if and only if there exists a $C^{1,\alpha}$ positive function p on ∂D such that the mapping*

$$\xi \longmapsto p(\xi)\mu_f(\xi) \qquad (\xi \in \partial D)$$

extends as a holomorphic mapping on D with no zeros.

Remark. In the cases considered by Lempert in [**Lem**], by Slodkowski in [**Slo4**], and by Pang in [**Pan1**] the geometry of the problem under consideration implies that if p is a positive $C^{0,\alpha}$ function on ∂D such that the mapping $\xi \mapsto p(\xi)\mu(\xi)$ extends holomorphically to D, the extension is nonzero on D.

We recall the Webster's construction in [**Web**] where he showed that the natural embedding of a C^2 hypersurface $\Sigma \subseteq \mathbb{C}^n$ into $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ via the map

$$\Psi : z \longmapsto (z, T_z^{\mathbb{C}}\Sigma) \qquad (z \in \Sigma) \tag{10}$$

is maximal real near a point $(z_o, T_{z_o}^{\mathbb{C}}\Sigma)$ if and only if the Levi form of Σ at z_o is nondegenerate. Thus the image of a strongly pseudoconvex hypersurface Σ under this natural embedding is always a maximal real submanifold of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$. Using the natural duality, i.e., the space of complex hyperplanes in $\mathbb{C}^n$ is naturally biholomorphic to the space of complex lines in $\mathbb{C}^n$, observe that every stationary disc f with boundary in the family of strongly pseudoconvex hypersurfaces $\{\partial\Omega_\xi\}_{\xi\in\partial D}$, induces an analytic disc

$$\xi \longmapsto (f(\xi), [\mu_f(\xi)])$$

in $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ attached to the maximal real fibration $\{\Psi(\partial\Omega_\xi)\}_{\xi\in\partial D}$. So it makes sense to talk about the partial indices of a stationary map, i.e., we define the partial indices of a stationary map as the partial indices of the induced map. Notice that to each stationary map f we associate $2n-1$ partial indices.

LEMMA 7. *Let $h : D \longrightarrow \mathbb{C}^n$ be a holomorphic disc of class C^k, $k \geq 1$, such that*

$$h(\xi) \neq 0 \qquad (\xi \in \overline{D}) \ .$$

Then there exist holomorphic discs $h^2, h^3, \ldots, h^n$ of class C^k, $k \geq 1$, such that

$$\det(h(\xi), h^2(\xi), \ldots, h^n(\xi)) = 1 \qquad (\xi \in \overline{D}) \ .$$

In particular, the vectors $h(\xi), h^2(\xi), \ldots, h^n(\xi)$ are linearly independent for every $\xi \in \overline{D}$.

Remark. The lemma was inspired by Proposition 9 in [**Lem**].

Proof. We will prove the lemma by induction on the dimension n. For $n = 1$ the claim is trivial. For $n = 2$ the lemma follows from the fact that since the component functions f and g of the mapping h, i.e., $h = (f, g)$, have no common zeros and since the space $A^k(D)$ of the k times differentiable holomorphic functions on $\overline{D}$ is a Banach algebra with a unit where the holomorphic polynomials are dense, then the characterization of the maximal closed ideals in $A^k(D)$ as in the proof of Theorem 18.18 in [**Rud1**] implies that there exist holomorphic functions F and G of class C^k such that

$$f(\xi)F(\xi) + g(\xi)G(\xi) = 1 \quad (\xi \in \overline{D}) .$$

Then the mapping

$$h_2(\xi) := (-G(\xi), F(\xi)) \quad (\xi \in \overline{D})$$

is such that $\det(h(\xi), h_2(\xi)) = 1$ for every $\xi \in \overline{D}$.

Let us assume the lemma for $n \geq 2$ and we will prove it for $n + 1$. Let $h : D \to \mathbb{C}^{n+1}$ be a holomorphic disc of class C^k with no zeros on $\overline{D}$. After a linear change of coordinates in $\mathbb{C}^n$ one may assume that the first n component functions of the mapping h have no common zeros on $\overline{D}$. This follows since h can also be considered as a C^k mapping from $\overline{D}$ into $\mathbb{C}P^n$ and therefore it can not be surjective. Let $h = (g, h_{n+1})$ where g stands for the first n components of the mapping h. By the inductive assumption one can find $g_2, \ldots, g_n$ analytic mappings from D into $\mathbb{C}^n$ of class C^k on $\overline{D}$ such that

$$\det(g, g_2, \ldots, g_n) = 1$$

on $\overline{D}$. Now the mappings

$$h_j := (g_j, 0) \quad (j = 2, \ldots, n)$$

and

$$h_{n+1} := (0, \ldots, 0, 1)$$

prove the lemma for the dimension $n + 1$. ∎

Henceforth let

$$f : D \longrightarrow \mathbb{C}^n$$

be a stationary map for the family of strongly pseudoconvex hypersurfaces $\partial\Omega_\xi$, $\xi \in \partial D$, in $\mathbb{C}^n$. Therefore there exists a positive function p on ∂D of class $C^{1,\alpha}$ such that $p(\xi)\mu_f(\xi)$, $\xi \in \partial D$, can be extended as a nonzero holomorphic mapping on the disc D. The above lemma for $k = 1$ implies that the function p is essentially unique, i.e., if p_1 and p_2 are two real functions on ∂D such that

$$p_1\mu_f \quad \text{and} \quad p_2\mu_f$$

extend as holomorphic mappings on D with no zeros, then there exists a positive constant a such that

$$p_2(\xi) = ap_1(\xi) \quad (\xi \in \partial D) \ .$$

Observe that here we really need an assumption on the zeros of the maps $p_i\mu_f$, $i = 1, 2$, namely, e.g., the function

$$\xi \longmapsto \mathrm{Re}(\xi + 2)\xi \quad (\xi \in \partial D)$$

extends as a holomorhic function on D but the real function $p(\xi) = \mathrm{Re}(\xi + 2)$ is not constant.

We normalize the map $p(\xi)\mu_f(\xi)$, $\xi \in \partial D$, so that at the point $\xi = 0$ the length of the extended holomorphic vector is 1. The so obtained map we call, following Pang [**Pan1**] (although our definition differs a little bit from his), the dual map of f, and we denote it by $\widetilde{f}$. Since $\widetilde{f}$ has no zeros on $\overline{D}$, one can use the above lemma to get a C^1 holomorphic frame $\widetilde{f}, h_2, \ldots, h_n$ over $\overline{D}$.

We define two fiber preserving diffeomorphisms of $\overline{D} \times \mathbb{C}^n$ which are holomorphic on each fiber and biholomorphic as mappings from $D \times \mathbb{C}^n$ into itself. Such a diffeomorphism does not change partial indices of a $C^{0,\alpha}$ closed path in a family of totally real submanifolds, Lemma 5. The first one is

$$\Phi_1 : (\xi, z) \longmapsto (\xi, z - f(\xi))$$

and the inverse of the second one is

$$\Phi_2^{-1} : (\xi, z) \longmapsto (\xi, \widetilde{f}(\xi) z_1 + \sum_{j=2}^{n} h_j(\xi) z_j) \ .$$

The composition $\Phi := \Phi_2 \circ \Phi_1$ is then a C^1 fiber preserving diffeomorphism such that in the new coordinates, i.e., after applying Φ, the stationary disc f and its dual $\widetilde{f}$ have extremely simple form, namely,

$$f(\xi) = 0 \quad \text{and} \quad \widetilde{f}(\xi) = (1, 0, \ldots, 0) \quad (\xi \in \partial D) \ .$$

We still denote the defining function of the family of strongly pseudoconvex hypersurfaces $\partial\Omega_\xi$, $\xi \in \partial D$, by $\rho = \rho(\xi, z)$ and we may assume, without loss of generality, that $\frac{\partial \rho}{\partial z_1}(\xi, 0) = 1$ for every $\xi \in \partial D$. Since the hypersurfaces $\partial\Omega_\xi$ are strongly pseudoconvex for each $\xi \in \partial D$, the complex Hessian of the function $\rho(\xi, \cdot)$ is positive definite when restricted to the maximal complex tangent space of $\partial\Omega_\xi$ at the point 0,

$$T_0^{\mathbb{C}} \partial\Omega_\xi = \{z \in \mathbb{C}^n; z_1 = 0\} = \mathbb{C}^{n-1} \ ,$$

i.e., the $(1,1)$ minor $L_o(\xi)$ of the complex Hessian of $\rho(\xi,\cdot)$ represents the Levi form of the hypersurface $\partial\Omega_\xi$ at 0. Thus the $(n-1)\times(n-1)$ matrix function of class $C^{0,\alpha}$ on ∂D

$$\xi \longmapsto L_o(\xi)$$

satisfies the theorem [**Lem**, Théorème B] and so there exists an $(n-1)\times(n-1)$ matrix function K,

$$K : \overline{D} \longrightarrow GL(n-1,\mathbb{C}) \ ,$$

of the same smoothness and such that

a) K is holomorphic on D,

b) $K^*(\xi)K(\xi) = L_o(\xi) \quad (\xi \in \partial D)$.

After using another fiber preserving diffeomorphism on $\overline{D}\times\mathbb{C}^n$,

$$(\xi, z_1, z') \longmapsto (\xi, z_1, K^{-1}(\xi)z') \ ,$$

where z' stands for $(z_2,\dots,z_n)$, one may assume that

$$f(\xi) = 0 \ , \qquad \widetilde{f}(\xi) = (1,0,\dots,0) \ , \qquad L_o(\xi) = Id \quad (\xi \in \partial D) \ .$$

We will now compute the total index of a stationary map. Later on we will apply the same kind of computation to find all partial indices of a stationary disc under some geometric assumptions on the family $\partial\Omega_\xi,\ \xi\in\partial D$.

We denote by I_n the $n \times n$ identity matrix and by J_n the $(n-1) \times n$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} .$$

The other matrices we need are

$$L(\xi) := \left(\frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(\xi, 0) \right)_{i,j=1}^n \quad , \quad H(\xi) := \left(\frac{\partial^2 \rho}{\partial z_i \partial z_j}(\xi, 0) \right)_{i,j=1}^n$$

and

$$H_o := J_n H J_n^t \ .$$

Since the derivative $\rho_{z_1}(\xi, 0)$ equals 1 for each $\xi \in \partial D$, the mapping

$$\begin{aligned} \Psi_\xi : \mathbb{C}^n &\longrightarrow \mathbb{C}^{2n-1} \\ \Psi_\xi(z) &= (z, \frac{\rho_{z_2}(\xi, z)}{\rho_{z_1}(\xi, z)}, \dots, \frac{\rho_{z_n}(\xi, z)}{\rho_{z_1}(\xi, z)}) \end{aligned}$$

is well defined in a neighbourhood of $\partial D \times \{0\}$. Notice that Ψ_ξ restricted to $\partial \Omega_\xi$ is just the Webster's map (10) written in the local coordinates. A short computation shows

$$\partial_z \Psi_\xi(0) = \begin{pmatrix} I_n \\ J_n H \end{pmatrix} \quad , \quad \partial_{\overline{z}} \Psi_\xi(0) = \begin{pmatrix} 0 \\ J_n L \end{pmatrix} .$$

Since the columns of the matrix

$$(ie_1, J_n^t, iJ_n^t)$$

span the tangent space to $\partial \Omega_\xi$ at the point 0 and since for every $v \in \mathbb{C}^n$ we have

$$D_z \Psi_\xi(0) v = \partial_z \Psi_\xi(0) v + \partial_{\overline{z}} \Psi_\xi(0) \overline{v} \ ,$$

the columns of the matrix

$$\begin{pmatrix} ie_1 & J_n^t & iJ_n^t \\ i(J_nHe_1 - J_nLe_1) & J_nHJ_n^t + J_nLJ_n^t & i(J_nHJ_n^t - J_nLJ_n^t) \end{pmatrix}$$

span the tangent space to $\Psi_\xi(\partial\Omega_\xi)$ at the point $(0,0)$. Using our notation we can simplify the above matrix as

$$\begin{pmatrix} ie_1 & J_n^t & iJ_n^t \\ iJ_nHe_1 & H_o + I_{n-1} & i(H_o - I_{n-1}) \end{pmatrix} . \tag{11}$$

We compute the determinant of the above matrix by expending along the first row. We get that the determinant of (11) equals to

$$i \det \begin{pmatrix} I_{n-1} & iI_{n-1} \\ H_o + I_{n-1} & i(H_o - I_{n-1}) \end{pmatrix} .$$

We multiply by i the first column and subtract it from the second to get

$$i \det \begin{pmatrix} I_{n-1} & 0 \\ H_o + I_{n-1} & -2iI_{n-1} \end{pmatrix} .$$

Thus the determinant of the matrix (11) equals to

$$i \det(-2iL_o) = i \det(-2iI_{n-1}) = (-2)^{n-1} i^n$$

and we proved the following proposition.

PROPOSITION 3. *The total index of a stationary disc f with boundary in a family of strongly pseudoconvex domains in $\mathbb{C}^n$ is 0.*

To compute the partial indices of this path in the family of maximal real submanifolds $\{\Psi(\partial\Omega_\xi)\}_{\xi\in\partial D}$ one first has to find the inverse of the $(2n-2)\times(2n-2)$

matrix

$$A_o := \begin{pmatrix} I_{n-1} & iI_{n-1} \\ H_o + I_{n-1} & i(H_o - I_{n-1}) \end{pmatrix} .$$

The inverse is

$$A_o^{-1} := \frac{1}{2} \begin{pmatrix} I_{n-1} - H_o & I_{n-1} \\ -i(H_o + I_{n-1}) & iI_{n-1} \end{pmatrix}$$

and so the matrix $B_o := A_o \overline{A_o^{-1}}$ is

$$\begin{pmatrix} -\overline{H_o} & I_{n-1} \\ I_{n-1} - H_o\overline{H_o} & H_o \end{pmatrix} .$$

We recall Definitions 3.7 and 3.9 from [**Pan1**] which in our context are

DEFINITION 4. *The family of hypersurfaces $\partial\Omega_\xi$, $\xi \in \partial D$, is strongly convex along $f(\partial D)$ if and only if the real quadratic form on $\mathbb{C}^{n-1}$*

$$v \longmapsto |v|^2 + \mathbf{Re}(H_o v \cdot v)$$

is strongly positive definite. We also say that the family $\partial\Omega_\xi$, $\xi \in \partial D$, is strongly convexifiable along a stationary disc f if there exists a fiber preserving biholomorphism of $\overline{D} \times \mathbb{C}^n$ such that in the new coordinates the family $\{\partial\Omega_\xi\}_{\xi\in\partial D}$ is strongly convex along $f(\partial D)$.

Remark. Observe that the condition on a family of hypersurfaces to be strongly convex along $f(\partial D)$ is slightly weaker than the strict geometric convexity of hypersurfaces $\partial\Omega_\xi$ at $f(\xi)$ for each ξ. For example, let the hypersurface $\partial\Omega_\xi$ for each $\xi \in \partial D$ be defined by the equation

$$\rho(z_1, z_2) := 2\mathrm{Re}(z_1) + |z_2|^2 + 2\mathrm{Re}(z_1^2) .$$

Then, at the point $(0,0)$, $L_o = 1$ and $H_o = 0$ for every $\xi \in \partial D$ and so this family of hypersurfaces is strongly convex along the disc $f(\xi) = 0$, $\xi \in \partial D$. On the other hand

$$\rho(t + i\sqrt{|t| + 2t^2}, 0) = 2(t - |t|) - 2t^2 < 0$$

for any real number $t \in \mathbb{R}$. Hence the domain Ω_ξ lies on both sides of the hyperplane $\mathrm{Re}(z_1) = 0$ and can not be convex.

PROPOSITION 4. *If $\partial\Omega_\xi$, $\xi \in \partial D$, is a family of strongly pseudoconvex hypersurfaces in $\mathbb{C}^n$ which is strongly convexifiable along a stationary disc f, then all partial indices of f are* 0.

COROLLARY 8. *If all hypersurfaces $\partial\Omega_\xi$, $\xi \in \partial D$, are strictly geometrically convex, i.e., the real Hessians of their defining functions are positive definite, then all partial indices of any stationary disc of this fibration are equal to* 0.

Remark. This was the situation studied by Slodkowski in [**Slo4**] and Lempert in [**Lem**].

Proof.(Proposition) Since the sum of the indices of a stationary map is 0, it is enough to prove that there are no positive partial indices. Let us assume that there exists a positive partial index k_o of the path B_o. By the definition of partial indices there exist holomorphic discs a and b in $\mathbb{C}^{n-1}$, with no common zeros on $\overline{D}$, of class $C^{0,\alpha}$, and such that on ∂D

$$B_o(\xi)\begin{pmatrix}\overline{a(\xi)}\\ \overline{b(\xi)}\end{pmatrix} = \xi^{k_o}\begin{pmatrix}a(\xi)\\ b(\xi)\end{pmatrix} \qquad (\xi \in \partial D)\ .$$

So for every $\xi \in \partial D$

$$-\overline{H_o(\xi)a(\xi)} + \overline{b(\xi)} = \xi^{k_o}a(\xi)\ .$$

We conjugate the above identity and dot it by a to get

$$-H_o(\xi)a(\xi)\cdot a(\xi) + b(\xi)\cdot a(\xi) = \overline{\xi}^{k_o}\overline{a(\xi)}\cdot a(\xi)\ .$$

Multiplication by ξ^{k_o} and taking the real parts of the equation yield the following pointwise equation on ∂D

$$\text{(12)} \qquad \operatorname{Re}(\xi^{k_o}a(\xi)\cdot b(\xi)) = |a|^2 + \operatorname{Re}(H_o(\xi^{k_o/2}a(\xi))\cdot(\xi^{k_o/2}a(\xi)))\ .$$

Since the family of hypersurfaces $\partial\Omega_\xi$, $\xi \in \partial D$, is strongly convex along $f(\partial D)$, the right hand side of (12) is positive for every fixed $\xi \in \partial D$. On the other hand the function $\operatorname{Re}(\xi^{k_o}a(\xi)\cdot b(\xi))$, $\xi \in D$, is harmonic on D and equals 0 at the point 0. The mean value property for harmonic functions gives a contradiction to the assumption that there exists a positive partial index of the path B_o. Lemma 4 completes the proof of the proposition. ■

Remark. One can easily observe that the proof of the above proposition also works in the case where the fibers of the fibration $\partial\Omega_\xi$, $\xi \in \partial D$, are only convex for each $\xi \in \partial D$ and strongly convex on a subset with a positive Lebesgue measure.

One can observe that in the above proof only the upper part of the matrix B_o was used to derive a contradiction. Next lemma tells us that this was not just a coincidence.

LEMMA 8. *Let a and b be two $C^{0,\alpha}$ holomorphic discs in $\mathbb{C}^{n-1}$ such that*

$$-\overline{H_o(\xi)a(\xi)} + \overline{b(\xi)} = \xi^{k_o} a(\xi) \quad (\xi \in \partial D) \ . \tag{13}$$

Then the pair (a, b) solves the equation

$$B_o(\xi) \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} = \xi^{k_o} \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} \qquad (\xi \in \partial D) \ .$$

Proof. We recall that

$$B_o = \begin{pmatrix} -\overline{H_o} & I_{n-1} \\ I_{n-1} - H_o\overline{H_o} & H_o \end{pmatrix} \ .$$

We would like to show that the discs a and b also solve the equation

$$(I_{n-1} - H_o(\xi)\overline{H_o(\xi)})\overline{a(\xi)} + H_o(\xi)\overline{b(\xi)} = \xi^{k_o} b(\xi) \qquad (\xi \in \partial D) \ . \tag{14}$$

We rewrite (14) as

$$H_o(\xi)(-\overline{H_o(\xi)a(\xi)} + \overline{b(\xi)} - \xi^{k_o} a(\xi)) = \xi^{k_o}(-H_o(\xi)a(\xi) + b(\xi) - \overline{\xi}^{k_o}\overline{a(\xi)})$$

and observe that the part in the parenthesis on the right hand side is exactly the expression which one gets by conjugating the equation (13). Thus both sides are zero, and the lemma is proved. ∎

By a theorem of Vekua [**Vek1**] we know that if k_j is a partial index of a $Gl(n, \mathbb{C})$ path B , then $-k_j$ is a partial index of the path $(B^{-1})^t$. This fact follows quite easily once one knows that the set of partial indices of a $Gl(n,\mathbb{C})$ path B is invariant with respect to the factorization of B into the form $F^+\Lambda F^-$, where F^+ and F^- are invertible holomorphic matrices on $\overline{D}$ and $\mathbb{C}^*\setminus D$, respectively. In our case, where

$$B_o = \begin{pmatrix} -\overline{H_o} & I_{n-1} \\ I_{n-1} - H_o\overline{H_o} & H_o \end{pmatrix} ,$$

we have that $\overline{B_o} = B_o^{-1}$ and so

$$(B_o^{-1})^t = B_o^* = \begin{pmatrix} -H_o & I_{n-1} - H_o\overline{H_o} \\ I_{n-1} & \overline{H_o} \end{pmatrix} ,$$

where we also used the property that $H_o^t = H_o$. Thus if a pair (a,b) of two $C^{0,\alpha}$ holomorphic discs solves the equation

$$B_o(\xi)\begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} = \xi^{k_o}\begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} \quad (\xi \in \partial D)$$

for some $k_o \in \mathbb{Z}$, then the pair $(-ib, ia)$ solves the equation

$$B_o^*(\xi)\begin{pmatrix} \overline{-ib(\xi)} \\ \overline{ia(\xi)} \end{pmatrix} = \xi^{k_o}\begin{pmatrix} -ib(\xi) \\ ia(\xi) \end{pmatrix} \quad (\xi \in \partial D) .$$

Hence the partial indices of the matrices B_o and B_o^* are the same and we proved the following

PROPOSITION 5. *If k_o is a partial index of the $Gl(2n-2,\mathbb{C})$ path B_o, then $-k_o$ is also a partial index of the path B_o.*

We recall the definition of a non-degenerate stationary disc from [**Pan1**] but in a modified form. See [**Pan1**] for more details.

DEFINITION 5. *A stationary disc f is said to be non-degenerate if the equation*

$$\overline{\beta(\xi)} + \xi^2 H_o(\xi)\beta(\xi) = \xi\gamma(\xi) \quad (\xi \in \partial D)$$

has only the trivial solution in the space $(A^{0,\alpha}(\partial D))^{n-1}$, i.e., a pair of vector functions β and γ from the space $(A^{0,\alpha}(\partial D))^{n-1}$ solves the above equation if and only if $\beta = \gamma = 0$.

PROPOSITION 6. *The only possible partial indices of a non-degenerate stationary disc are $0, 1$ and -1.*

Remark. Note that by Theorem 1 and by an observation by Slodkowski, [**Slo4**], this proposition immediately implies Theorem 4.8 from [**Pan1**].

Proof. Let (a, b) be a nontrivial pair of functions from $(A^{0,\alpha}(\partial D))^{n-1}$ which solves the problem

$$B_o(\xi) \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} = \xi^{k_o} \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} \quad (\xi \in \partial D)$$

for some $k_o \in \mathbb{N}$. Then, after the multiplication by $\overline{\xi}^2$, the first $n-1$ equations can be rewritten as

$$\overline{\xi^2 b(\xi)} = \xi^{k_o - 2} a(\xi) + \overline{\xi^2 H_o(\xi) a(\xi)} \quad (\xi \in \partial D) \ .$$

We consider two cases.

1. $k_o = 2m+2$ for some integer $m \geq 0$. The multiplication by $\overline{\xi}^m$ yields the equation

$$\overline{\xi^{m+2}b(\xi)} = (\xi^m a(\xi)) + \overline{\xi^2 H_o(\xi)(\xi^m a(\xi))}$$

and the non-degeneracy of the stationary disc implies

$$a = b = 0 \ ,$$

a contradiction.

2. $k_o = 2m+3$ for some $m \geq 0$. Then the multiplication by $\overline{\xi}^m$ gives

$$\overline{\xi^{m+2}b(\xi)} = \xi(\xi^m a(\xi)) + \overline{\xi^2 H_o(\xi)(\xi^m a(\xi))} \quad (\xi \in \partial D) \ .$$

Let $\beta(\xi) := \xi^m a(\xi)$ and $\gamma(\xi) := \xi^{m+1}b(\xi)$. Then the above equation has the form

$$\overline{\xi\gamma(\xi)} = \xi\beta(\xi) + \overline{\xi^2 H_o(\xi)\beta(\xi)} \ .$$

After the multiplication by $\overline{\xi}$ we also have

$$\overline{\xi^2\gamma(\xi)} = \beta(\xi) + \overline{\xi^2 H_o(\xi)(\xi\beta(\xi))} \ .$$

Adding both identities yields

$$\overline{\xi(\xi+1)g(\xi)} = (\xi+1)f(\xi) + \overline{\xi^2 H_o(\xi)((\xi+1)f(\xi))} \ .$$

Thus, by the non-degeneracy of the stationary disc, the functions β and γ are identically 0 and so also

$$a = b = 0 \ ,$$

a contradiction. Lemma 4 finishes the proof of the proposition. ■

9. Examples and counterexamples

Example 1. The first example shows that there exist a real analytic family of real analytic hypersurfaces in $\partial D \times \mathbb{C}^n$ with strongly pseudoconvex fibers and a family of corresponding stationary analytic discs such that the partial indices associated to each stationary disc change for some isolated values of the parameter.

Let

$$\rho_\xi^t(z_1, z_2) = 2\mathrm{Re}(z_1) + |z_1|^2 + |z_2|^2 - t\mathrm{Re}(\overline{\xi}^N z_2^2) \qquad ((t,\xi) \in \mathbb{R} \times \partial D)$$

be a two parameter family of strongly plurisubharmonic functions on $\mathbb{C}^2$. For each pair $(t,\xi) \in \mathbb{R} \times \partial D$ the function ρ_ξ^t defines in a neighbourhood U_ξ^t of the point $(0,0)$ a strongly pseudoconvex hypersurface Σ_ξ^t given by the equation $\rho_\xi^t(z) = 0$. If one restricts the parameter $t \in \mathbb{R}$ on a compact subset $I \subseteq \mathbb{R}$, the neighbourhoods U_ξ^t, $t \in I, \xi \in \partial D$, can be chosen uniformly.

It is clear that

$$\rho_\xi^t(0,0) = 0$$

and

$$\partial_z \rho_\xi^t(0,0) = (1,0)\ .$$

Let M_ξ^t denote the maximal real submanifold of the complex manifold $\mathbb{C}^2 \times \mathbb{C}P^1$ which one gets as the image of the strongly pseudoconvex hypersurface Σ_ξ^t by the mapping

$$\Psi : z \longmapsto (z, [\partial \rho_\xi^t(z)]) \ .$$

See the section on the stationary discs and [**Web**].

Claim. The partial indices of the closed curve

$$\xi \longmapsto (0,0,[1,0]) \qquad (\xi \in \partial D) \tag{15}$$

are

a) all 0 for $t \neq \pm 1$,

b) 0, N and $-N$ for $t = \pm 1$.

Proof. A computation in the coordinate chart $V = \{(z,[w]) \in \mathbb{C}^2 \times \mathbb{C}P^1; w_1 \neq 0\}$, similar to the one in the proof of the fact that the total index of a stationary disc is always 0, gives a matrix $A_t(\xi)$ whose columns span the tangent space to the maximal real submanifold M_ξ^t at the point $(0,0,[1,0])$, i.e., at the point $(0,0,0)$ in the coordinates,

$$A_t(\xi) := \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t\overline{\xi}^N + 1 & -i(t\overline{\xi}^N + 1) \end{pmatrix} \qquad (\xi \in \partial D) \ .$$

So we get

$$B_t(\xi) = A_t(\xi)\overline{A_t(\xi)^{-1}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t\xi^N & 1 \\ 0 & 1-t^2 & -t\overline{\xi}^N \end{pmatrix} \qquad (\xi \in \partial D) \ .$$

By Lemma 4 one of the partial indices is 0 and the rest are given as the partial indices of the 2×2 matrix

$$B_t^o(\xi) := \begin{pmatrix} t\xi^N & 1 \\ 1-t^2 & -t\overline{\xi}^N \end{pmatrix} \qquad (\xi \in \partial D) \ .$$

Since the determinant of the matrix $B_t^o(\xi)$ is identically equal to -1, the sum of the partial indices is 0. Assume that one of the partial indices is positive, say $k_o \in \mathbb{N}$. Then there exists a nonzero pair (a, b) of holomorphic functions on D, real analytic up to the boundary and such that on the boundary for every $\xi \in \partial D$ one has

$$B_t^o(\xi) \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} = \xi^{k_o} \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix} \ . \tag{16}$$

Thus

$$(1-t^2)\overline{a}(\xi) - t\overline{\xi}^N \overline{b}(\xi) = \xi^{k_o} b(\xi) \ . \tag{17}$$

But the left hand side of (17) extends as an antiholomorphic function on D. So

$$\xi^{k_o} b(\xi) = \text{constant} \qquad (\xi \in \partial D) \ .$$

Since by our assumption $k_o > 0$, the constant has to be 0, and thus

$$b = 0 \ .$$

Going back to the equation (17) one gets

$$(1-t^2)\overline{a(\xi)} = 0 \qquad (\xi \in \partial D) \ .$$

Thus if $t \neq \pm 1$, the function a has to be identically 0, which gives a contradiction to the assumption $k_o > 0$ and so in the case $t \neq \pm 1$ all partial indices of the curve (15) are 0. For the case $t = 1$ one can check that one of the partial indices is N, namely, the pair of holomorphic functions $(1, 0)$ solves the equation (16) for $k = N$. The pair of functions $(i, 0)$ shows the same for $t = -1$. Of course, the second partial index is $-N$. ■

Example 2. Examples 2 and 3 together with Example 1 will show that the so called continuity method for describing the polynomial hull of a general hypersurface in $\partial D \times \mathbb{C}^n$, $n > 1$, with strongly pseudoconvex fibers fails. This is in contrast with the case $n = 1$, where the continuity method was successfully used by Forstnerič, [**For3**], to describe the polynomial hull of a totally real torus fibered over ∂D. We will find an isotopy of hypersurfaces Σ_t, $t \in [0, 1]$, in $\partial D \times \mathbb{C}^2$ with strongly pseudoconvex fibers which starts at a hypersurface Σ_0 in $\partial D \times \mathbb{C}^2$ whose fibers are Euclidean spheres in $\mathbb{C}^2$, is strictly decreasing in the sense that Σ_t is included in the domain bounded by the hypersurface Σ_τ for all τ, $\tau < t$, and ends with a hypersurface Σ_1 in $\partial D \times \mathbb{C}^n$ with the property that its polynomial hull is nontrivial but there is no graph of a bounded analytic disc with boundary almost everywhere in the hypersurface Σ_1. See also Example 3.

Let γ be a smooth arc in $\mathbb{R}^2 \subseteq \mathbb{C}^2$ and let f be any smooth nonnegative function on $\mathbb{R}^2$ such that

a) the zero set of f and the zero set of the gradient ∇f are both equal to γ and

b) there exists an $r_o > 0$ such that $f(x_1, x_2) = x_1^2 + x_2^2$ for $x_1^2 + x_2^2 \geq r_o^2$.

Here the coordinates in $\mathbb{R}^2 \subseteq \mathbb{C}^2$ are x_1, x_2 and the coordinates in $\mathbb{C}^2$ are $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. For $\lambda > 0$ we define

$$\rho_\lambda(z_1, z_2) = f(x_1, x_2) + \lambda(y_1^2 + y_2^2) \ .$$

Then

a) the zero set of ρ_λ and the zero set of $\nabla\rho_\lambda$ are both equal to the arc γ and

b) the Levi form of the function ρ_λ is

$$L(\rho_\lambda) := \frac{1}{4}\begin{pmatrix} f_{x_1x_1} + 2\lambda & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} + 2\lambda \end{pmatrix} \ ,$$

where the notation $f_{x_ix_j}$ stands for the second partial derivative of the function f with respect to x_i and x_j, $i, j = 1, 2$.

So if λ is large enough, the function ρ_λ is strictly plurisubharmonic on $\mathbb{C}^2$. We fix such a λ and denote the function ρ_λ by ρ.

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function whose support is contained in the interval $[-1, (r_0 + 2)^2]$ and which equals 1 on the interval $[0, (r_o + 1)^2]$. Also, let g be a smooth nonnegative function on $\mathbb{R}$ such that

1) $g(x) = 0$ for $x \leq r_o^2$,
2) $g'(x) > 0$ and $g''(x) \geq 0$ for $x > r_o^2$,
3) $\rho(z)\chi'(|z|^2) + g'(|z|^2) \geq 0$ for every $z \in \mathbb{C}^2$.

For $\varepsilon \in (0, 1)$ we define

$$\widetilde{\rho}_\varepsilon(z) = \varepsilon\chi(|z|^2)\rho(z) + g(|z|^2) \quad (z \in \mathbb{C}^2) \ .$$

If ε is small enough, the function $\widetilde{\rho}$ is strictly plurisubharmonic on $\mathbb{C}^2$ and its zero set is the arc γ. We fix such an ε and denote the corresponding function by $\widetilde{\rho}$.

Claim. The zero set of the gradient $\nabla\widetilde{\rho}$ is the arc γ.

Proof. Let $z^o = (x_1^o + iy_1^o, x_2^o + iy_2^o)$ be a point where the gradient $\nabla\widetilde{\rho}$ is zero. We consider the following three cases :

1. $|z^o| < r_o$. Then $\widetilde{\rho} = \varepsilon\rho$ in a neighbourhood of the point z^o and thus $z^o \in \gamma$.

2. $|z^o| > r_o + 2$. Then $\widetilde{\rho}(z) = g(|z|^2)$ in a neighbourhood of the point z^o. Since $g'(x) > 0$ for $x > r_o^2$, we get a contradiction.

3. $r_o \leq |z^o| \leq r_o + 2$. The y components of the gradient $\nabla\widetilde{\rho}$, i.e., the derivatives of $\widetilde{\rho}$ with respect to y_1 and y_2 at the point z equal to

$$\frac{\partial\rho}{\partial y_j}(z) = 2(\lambda\varepsilon\chi(|z|^2) + \varepsilon\rho(z)\chi'(|z|^2) + g'(|z|^2))y_j \qquad (j = 1, 2) \ .$$

Therefore, if $\nabla\widetilde{\rho}(z^o) = 0$, one concludes that since

(18) $\lambda\varepsilon\chi(|z|^2) + \varepsilon\rho(z)\chi'(|z|^2) + g'(|z|^2) > \varepsilon(\rho(z)\chi'(|z|^2) + g'(|z|^2)) \geq 0$

on $\mathbb{C}^2$, it follows

$$y_1^o = y_2^o = 0 \ .$$

This, together with the fact that $|z^o| \geq r_o$ and our initial assumption (b) on the function f, implies

$$f_{x_1}(x_1^o, x_2^o) = 2x_1^o \quad \text{and} \quad f_{x_2}(x_1^o, x_2^o) = 2x_2^o \ .$$

The x components, i.e., the derivatives with respect to x_1 and x_2 variables, of the equation $\nabla\widetilde{\rho}(z^o) = 0$, together with (18) give

$$x_1^o = x_2^o = 0 \ .$$

Thus also the assumption $r_o \leq |z^o| \leq r_o + 2$ leads to a contradiction and the claim is proved. ■.

Thus for every simple arc γ in $\mathbb{R}^2 \subseteq \mathbb{C}^2$ we found a smooth parameter family of strictly pseudoconvex hypersurfaces Σ_t, $t \in [0,1]$, in $\mathbb{C}^2$ which starts at γ, it is strictly increasing in the sense that for each pair of parameters $t < \tau$ the hypersurface Σ_t is included in the interior of the domain bounded by Σ_τ and which ends at some large Euclidean sphere.

Remark 1. If one is given a smooth family of simple arcs γ_ξ, $\xi \in \partial D$, in $\mathbb{R}^2 \subseteq \mathbb{C}^2$, then one can choose a smooth family of smooth functions f_ξ, $\xi \in \partial D$, satisfying the conditions (a) and (b) for each $\xi \in \partial D$. Since the set of parameters is compact, the functions χ and g and the constants λ and ε can be chosen uniformly, i.e., independent of the parameter $\xi \in \partial D$.

Remark 2. The above construction can be applied to any arc γ in $\mathbb{C}^2$ for which there exists an automorphism Φ of $\mathbb{C}^2$ such that $\Phi(\gamma) \subseteq \mathbb{R}^2$.

We consider the following family of arcs in $\mathbb{R}^2 \subseteq \mathbb{C}^2$. Let γ_1 be the semicircle in $\mathbb{R}^2$ given by the equation

$$x_1^2 + x_2^2 = 1 \ , \qquad x_2 \geq 0 \ .$$

For each $\xi \in \partial D$ we denote by R_ξ the map

$$R_\xi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

defined by

$$R_\xi(z_1, z_2) := (\xi z_1, z_2) \ .$$

Observe that R_ξ is a linear isomorphism of $\mathbb{C}^2$.

For $\xi \in \partial D$ such that $0 \leq \arg(\xi) \leq \frac{\pi}{2}$ or $\frac{3\pi}{2} \leq \arg(\xi) \leq 2\pi$ let

$$\gamma_\xi := \gamma_1 \ .$$

For the parameters $\xi \in \partial D$ such that $\frac{\pi}{2} < \arg(\xi) < \frac{3\pi}{2}$ we smoothly perturb the initial arc γ_1 to get arcs γ_ξ such that they do not pass through the point $(0, 1)$ but they still pass through the points $(1, 0)$ and $(-1, 0)$. For instance, for $\xi = e^{is}$ one may take γ_ξ to be defined by the equation

$$(1 - \varrho(s))^2 x_1^2 + x_2^2 = (1 - \varrho(s))^2 \ , \qquad x_2 \geq 0 \ ,$$

where $\varrho : \mathbb{R} \to [0, 1)$ is any smooth function whose support is the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$. We define

$$\widetilde{\gamma}_\xi := R_{\sqrt{\xi}}(\gamma_\xi) \ .$$

Here by $\sqrt{\xi}$ we mean the principal branch of the square root, i.e., $\sqrt{1} = 1$. Since we have $\gamma_\xi = \gamma_1$ in a neighbourhood of $\xi = 1$ and since the arc γ_1 is symmetric with respect to the x_2-axis, the family of arcs $\widetilde{\gamma}_\xi,\ \xi \in \partial D$, is smooth. Using our initial construction for one arc $\gamma \subseteq \mathbb{R}^2$ and Remarks 1 and 2, one gets a smooth family of hypersurfaces $\Sigma_t,\ t > 0$, in $\partial D \times \mathbb{C}^2$ such that for each t all their fibers are strongly pseudoconvex and for t large enough all the fibers of the hypersurface Σ_t are equal to a sphere of a fixed radius $\sqrt{t}$ centered at the point $(0, 0)$. Also, for every pair $t, \tau \in \mathbb{R}^+,\ t < \tau$, the hypersurface Σ_t is included in the domain bounded by Σ_τ.

Remark. Observe that by a theorem of Docquier and Grauert [**Doc-Gra**] the above properties of the isotopy $\Sigma_t,\ t > 0$, assure that the closures of the fibers of the domain bounded by Σ_t remain polynomially convex for each time t.

To finish our example we first observe that since

$$(\sqrt{\xi}, 0), (-\sqrt{\xi}, 0) \in \widetilde{\gamma}_\xi \qquad (\xi \in \partial D)\ ,$$

the polynomial hull of Σ_t contains the point $(0, 0, 0)$ for all $t > 0$. Finally we prove the following claim.

Claim. For $t > 0$ small enough there is no graph of a bounded analytic mapping $F : D \to \mathbb{C}^2$ whose boundary is almost everywhere with respect to the Lebesgue measure on ∂D contained in the closure of the domain bounded by the hypersurface $\Sigma_t \subseteq \partial D \times \mathbb{C}^2$.

Proof. We first prove the claim for

$$\Sigma_o := \bigcup_{\xi \in \partial D} \{\xi\} \times \widetilde{\gamma}_\xi \ .$$

Once this is proved the normal family argument and the above remark finish the proof of the claim.

Let us assume that there is an analytic mapping

$$(f, g) : D \longrightarrow \mathbb{C}^2$$

such that

$$(f(\xi), g(\xi)) \in \widetilde{\gamma}_\xi \quad (\text{a.e. } \xi \in \partial D) \ .$$

Therefore the imaginary part of the function g is almost everywhere 0 on ∂D and thus g is a constant function, i.e., there is a real number $a \in [0, 1]$ such that $g(\xi) = a$ for every $\xi \in \partial D$. Since the arcs $\widetilde{\gamma}_\xi$ for $\frac{\pi}{2} < \arg(\xi) < \frac{3\pi}{2}$ do not pass through the point $(0, 1)$ the constant a has to be less than 1. But then

$$f(\xi)^2 = (1 - a^2)\xi \quad (\text{a.e. } \xi \in \partial D) \ ,$$

which leads to a contradiction. ■

Example 3. In this example we will construct a smooth family $\Sigma_t, \ t \in [0, 1]$, of smooth hypersurfaces in $\partial D \times \mathbb{C}^2$ similar to the one in the Example 2, i.e., $\Sigma_0 = \partial D \times S^{2n-1}(R)$ for some $R > 0$, the family is strictly decreasing in the sense that Σ_t is included in the domain bounded by $\Sigma_\tau, \ \tau < t$, and all the fibers $\Sigma_t \cap (\{\xi\} \times \mathbb{C}^n)$, $\xi \in \partial D$, are strongly pseudoconvex for each value of the parameter t, but this time we will also have a fixed neighbourhood $\partial D \times B(\varepsilon_o)$ of $\partial D \times \{(0,1)\}$ included in every domain bounded by Σ_t, $t \in [0,1]$, and there will be a point in the polynomial hull of Σ_1 that can not be reached by the graphs of bounded analytic discs with boundaries almost everywhere in Σ_1.

Let $\gamma \subseteq \mathbb{R}^2 \subseteq \mathbb{C}^2$ be the arc

$$x_1^2 + x_2^2 = 1 \quad , \quad x_2 \geq 0$$

as in the Example 2. Let $X_1 := \gamma$ and let

$$X_\xi := R_{\sqrt{\xi}} X_1 \ .$$

Since again

$$(\sqrt{\xi}, 0), (-\sqrt{\xi}, 0) \in X_\xi \quad (\xi \in \partial D) \ ,$$

it is obvious that the polynomial hull of

$$X := \bigcup_{\xi \in \partial D} \{\xi\} \times X_\xi$$

contains the point $(0,0,0)$.

Claim. There is no graph of a bounded analytic disc $F : D \to \mathbb{C}^2$ whose boundary is almost everywhere with respect to the Lebesgue measure on ∂D contained in X and which passes through the point $(0,0,0)$.

Proof. Assume that there is an analytic disc $F = (f, g)$ whose graph has boundary almost everywhere contained in X and such that $F(0) = (0,0)$. This implies, similarly as in the Example 2, that

$$g(\xi) = 0 \quad (\xi \in \overline{D}) \ .$$

Thus

$$f^2(\xi) = \xi \qquad (\text{a.e. } \xi \in \partial D) \ ,$$

a contradiction. ■

The rest is similar to the Example 2 and thus omitted.

Remark. Examples 2 and 3 were inspired by the example by Helton and Merino in [**Hel-Mer**] where they constructed a connected and simply connected fibration over the unit circle ∂D with a nontrivial polynomial hull and such that there exists no graph of an analytic disc with boundary in the fibration.

10. CR-vector bundles

We begin with a definition.

DEFINITION 6. *Let* $L = \{L_\xi \subseteq \mathbb{C}^{m+n}; \xi \in \partial D\}$ *be a real vector bundle over* ∂D *of class* $C^{0,\alpha}$. *If for each* $\xi \in \partial D$ *the fiber* L_ξ *is a real vector subspace of CR-dimension* m, *the bundle* L *is called a CR-bundle of CR-dimension* m *over the unit circle* ∂D. *If, in addition, for each* $\xi \in \partial D$ *the fiber* L_ξ *is a generating subspace of* $\mathbb{C}^{m+n}$, *i.e.,* $L_\xi + iL_\xi = \mathbb{C}^{n+m}$, $\xi \in \partial D$, *then the bundle* L *is called a generating CR-bundle over* ∂D.

Remarks.

1. Every maximal real bundle over ∂D is a generating CR-bundle with CR-dimension 0.
2. To each CR-bundle L over the unit circle one can associate a complex m-dimensional vector subbundle $L^{\mathbb{C}} \subseteq L$ which is just the bundle of the maximal complex subspaces of the bundle L, i.e., for each $\xi \in \partial D$ the fiber $L_\xi^{\mathbb{C}}$ equals to $L_\xi \cap iL_\xi$.

LEMMA 9. *Let V be a C^1 complex vector bundle over ∂D such that for each $\xi \in \partial D$ the fiber V_ξ is an m dimensional complex subspace of $\mathbb{C}^{m+n}$. Then there exists a linear change of coordinates in $\mathbb{C}^{m+n}$ such that in the new coordinates each fiber V_ξ projects isomorphically onto $\mathbb{C}^m \times \{0\} \subseteq \mathbb{C}^{m+n}$. Moreover, the set of invertible $(m+n) \times (m+n)$ matrices satisfying this property is open and dense in $Gl(m+n, \mathbb{C})$.*

Proof. We denote by $\mathcal{G}$ the set of invertible $(m+n)\times(m+n)$ complex matrices having the above property. Clearly $\mathcal{G}$ is open in $Gl(m+n, \mathbb{C})$. So, to prove the lemma, we have to show that the complement of $\mathcal{G}$ in $Gl(m+n, \mathbb{C})$ has no interior.

Fix $\xi_o \in \partial D$. Let A_{ξ_o} be any $(m+n)\times m$ matrix such that its columns form a basis of the fiber V_{ξ_o}. We define the mapping

$$\Phi_{\xi_o} : Gl(m+n, \mathbb{C}) \longrightarrow \mathbb{C}$$

by

$$\Phi_{\xi_o}(U) = \det([I_m, 0]UA_{\xi_o})$$

where $[I_m, 0]$ is an $m \times (m+n)$ matrix which has the identity matrix in its first m columns and the 0 matrix in its last n columns. The mapping Φ_{ξ_o} depends on the matrix A_{ξ_o}, i.e., on the basis of the fiber V_{ξ_o}, but the set

$$\mathcal{U}_{\xi_o} := \Phi_{\xi_o}^{-1}(0)$$

does not. The equation

$$\Phi_{\xi_o}(U) = 0$$

is algebraic and so $\mathcal{U}_{\xi_0}$ is an algebraic subset of $Gl(m+n, \mathbb{C})$. Hence, locally the set $\mathcal{U}_{\xi_o}$ has finite $2((m+n)^2 - 1)$ dimensional Hausdorff measure.

Let

$$\mathcal{U} := \bigcup_{\xi \in \partial D} \{\xi\} \times \mathcal{U}_\xi \subseteq \partial D \times Gl(m+n, \mathbb{C}) \ .$$

Then, locally again, the $2(m+n)^2 - 1$ dimensional Hausdorff measure of the set $\mathcal{U}$ is finite and so for every compact set $K \subseteq Gl(m+n, \mathbb{C})$ we have

$$\mathcal{H}_{2(m+n)^2-1}(\pi(\mathcal{U}) \cap K) < \infty$$

where π is the projection

$$\pi : \partial D \times Gl(m+n, \mathbb{C}) \longrightarrow Gl(m+n, \mathbb{C}) \ .$$

Since $\pi(\mathcal{U})$ is exactly the complement of the set $\mathcal{G}$, the lemma is proved. ■

Let $\Sigma \subseteq \mathbb{C}^{m+n}$ be a generating CR-subspace of CR-dimension m such that its maximal complex subspace $\Sigma^{\mathbb{C}}$ projects isomorphically onto $\mathbb{C}^m \times \{0\}$.

LEMMA 10. *The subspace*

$$S := \Sigma \cap (\{0\} \times \mathbb{C}^n)$$

is a maximal real subspace of $\{0\} \times \mathbb{C}^n$ *and is the only subspace of* $\{0\} \times \mathbb{C}^n$ *for which*

$$\Sigma = \Sigma^{\mathbb{C}} \oplus S \ .$$

Proof. We denote by $\pi : \mathbb{C}^{m+n} \to \mathbb{C}^m \times \{0\}$ the orthogonal projection onto $\mathbb{C}^m \times \{0\}$. Since π projects $\Sigma^{\mathbb{C}}$ isomorphically onto $\mathbb{C}^m \times \{0\}$, we conclude that for every $x \in \Sigma$ there exists exactly one vector $v \in \Sigma^{\mathbb{C}}$ such that $\pi(x) = \pi(v)$. Hence the vector $x - v \in \Sigma$ is in the kernel of the projection π, i.e., $x - v$ is in $\{0\} \times \mathbb{C}^n$. Therefore $x - v$ is in S. The assumption on the projection π also implies

$$\Sigma^{\mathbb{C}} \cap (\{0\} \times \mathbb{C}^n) = \{0\} \ , \tag{19}$$

and so S is a totally real subspace of $\{0\} \times \mathbb{C}^n$ for which

$$\Sigma = \Sigma^{\mathbb{C}} \oplus S \ .$$

Finally, the subspace Σ is a generating CR-subspace of $\mathbb{C}^{m+n}$ and thus S is a maximal real subspace of $\{0\} \times \mathbb{C}^n$. The uniqueness follows from (19). ■

Let $L \subseteq \partial D \times \mathbb{C}^{m+n}$ be a generating CR-bundle of the class $C^{0,\alpha}$ over the unit circle and of CR-dimension m. We assume that each fiber $L_\xi^{\mathbb{C}}$, $\xi \in \partial D$, projects isomorphically onto $\mathbb{C}^m \times \{0\} \subseteq \mathbb{C}^{m+n}$. By Lemma 9 this assumption can always be realized in the case where the bundle L is of class C^1. The above Lemma 10 implies that there is a unique maximal real bundle $\mathcal{L} \subseteq \partial D \times \mathbb{C}^n$ such that for each $\xi \in \partial D$ we have

$$L_\xi = L_\xi^{\mathbb{C}} \oplus \mathcal{L}_\xi \ .$$

DEFINITION 7. *Let $L \subseteq \partial D \times \mathbb{C}^{m+n}$ be a generating CR-bundle over ∂D of CR-dimension m whose fibers project isomorphically onto $\mathbb{C}^m \times \{0\}$. We define the partial indices and the total index of the bundle L as the partial indices and the total index of the maximal real bundle $\mathcal{L} \subseteq \partial D \times \mathbb{C}^n$.*

We fix $\xi_o \in \partial D$. Let $N(\xi_o)$ be any $(m+n) \times n$ matrix whose columns span the real orthogonal complement $L_{\xi_o}^{\perp}$. Then the equations of the fibers L_{ξ_o} and $L_{\xi_o}^{\mathbb{C}}$ are

$$\mathrm{Re}(N^*(\xi_o) \begin{bmatrix} z \\ w \end{bmatrix}) = 0 \quad \text{ and } \quad N^*(\xi_o) \begin{bmatrix} z \\ w \end{bmatrix} = 0 \ ,$$

respectively. Here $z \in \mathbb{C}^m$ and $w \in \mathbb{C}^n$. Since we are assuming that each fiber of the bundle $L^{\mathbb{C}}$ projects isomorphically onto $\mathbb{C}^m \times \{0\} \subseteq \mathbb{C}^{m+n}$, the matrix $N(\xi_o)$ can be written in the following block form

$$N(\xi_o) = \begin{bmatrix} G_o(\xi_o) \\ N_o(\xi_o) \end{bmatrix} \ ,$$

where $N_o(\xi_o)$ is an invertible $n \times n$ complex matrix. The definition of the bundle $\mathcal{L}$ immediately implies that $\mathcal{L}_{\xi_o}$ is given by the equations

$$\mathrm{Re}(N_o^*(\xi_o)w) = 0 \ .$$

Therefore the columns of the matrix $N_o(\xi_o)$ span the real orthogonal space $\mathcal{L}_{\xi_o}^{\perp} \subseteq \mathbb{C}^n$.

Let $k_1, k_2, \ldots, k_n$ be the partial indices of the bundle $\mathcal{L}$ and let $\Lambda(\xi)$ be its characteristic matrix. Then there exists an $n \times n$ invertible holomorphic matrix function Θ_o on $\overline{D}$ such that the columns of the matrix function

$$A_o(\xi) := \Theta_o(\xi)\Lambda_o(\xi) \quad (\xi \in \partial D)$$

span the fibers of the maximal real bundle $\mathcal{L}$. Here Λ_o denotes the square root of the characteristic matrix Λ. Once A_o is fixed, there is naturally given basis of the bundle $L^{\perp}$, namely, there is an $(m+n) \times n$ matrix function $N(\xi),\ \xi \in \partial D$, whose rows are from the space $\mathcal{E}_{\sigma}$, whose columns for each $\xi \in \partial D$ span $L_{\xi}^{\perp}$, and such that

$$N_o^* = iA_o^{-1} \ .$$

Let $F : \overline{D} \longrightarrow \mathbb{C}^{m+n}$ be an analytic disc with boundary in the generating CR-bundle L. With respect to the splitting of the space $\mathbb{C}^{m+n}$ the mapping F is written as (f, g), where f and g are holomorphic maps into $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Since for each $\xi \in \partial D$ the matrix function $A_o(\xi)$ is invertible, the mapping g can be written in a unique way in the form

$$g = A_o(u + i(v + iT_\sigma v)) \qquad (u, v \in \mathcal{E}_\sigma) \ .$$

Since F has boundary in the bundle L, the discs f and g satisfy the equation

$$\mathrm{Re}(G_o^*(\xi)f(\xi) + N_o^*(\xi)g(\xi)) = 0 \quad (\xi \in \partial D) \ .$$

Hence, since $N_o^* A_o$ equals to iI_n,

$$\mathrm{Re}(G_o^*(\xi)f(\xi) + i(u(\xi) - (T_\sigma v)(\xi)) - v(\xi)) = 0 \quad (\xi \in \partial D)$$

and the mapping v is given by the equation

$$v = \mathrm{Re}(G_o^* f) \ . \tag{20}$$

If the partial indices of the bundle $\mathcal{L}$ are all greater or equal to -1, then the product

$$A_o(v + iT_\sigma v)$$

extends holomorphically to D and so, given a holomorphic disc f in $\mathbb{C}^m$, the equation (20) is also a sufficient condition for the existence of a holomorphic disc g in $\mathbb{C}^n$ such that the disc $F = (f, g)$ has boundary in the bundle L. Even more, in this case one can find an explicit parametrization of all holomorphic discs attached to the bundle L with the parameter space $\mathbb{R}^{n+k} \times (A^{0,\alpha}(\partial D))^m$, where k is the total index of the bundle $\mathcal{L}$. Namely, for each holomorphic vector $f \in (A^{0,\alpha}(\partial D))^m$ and for each real vector function $u \in \mathcal{E}_\sigma$ such that

$\Lambda_o u$ extends holomorphically to D, there exists exactly one holomorphic disc $F = (f, g)$ attached to L.

Before we consider the nonlinear case let us make a few remarks.

1. Since our choice of a change of coordinates in $\mathbb{C}^{m+n}$ involves quite a lot of freedom, it looks like that one could get, using a different change of coordinates, also a different set of attached discs to the bundle L. It is quite easy to construct an example, e.g., the real normal bundle $L^{\perp} \subseteq \partial D \times \mathbb{C}^2$ is given by the matrix $N^t(\xi) := [\overline{\xi}, \overline{\xi}^2]$, where different linear changes of coordinates result in different sets of partial indices of the associated bundle $\mathcal{L}$. But, as already the above argument shows, as soon as all partial indices of the bundle $\mathcal{L}$ are greater or equal to -1, we know how to parametrize all holomorphic discs in $\mathbb{C}^{m+n}$ with boundaries in L. Also, the following simple lemma is true.

LEMMA 11. *The Banach spaces $X = (A^{0,\alpha}(\partial D))^m$ and $\mathbb{R}^2 \times X$ are naturally isomorphic.*

Proof. We define

$$\Psi : \mathbb{R}^2 \times X \longrightarrow X$$

as

$$\Psi(s, t, f)(\xi) := (s + it) + \xi f(\xi) \quad (\xi \in \partial D) \ .$$

It is easy to verify that Ψ is one to one and onto bounded linear map. ■

Since the parameter space of holomorphic discs attached to L is $\mathbb{R}^{n+k} \times (A^{0,\alpha}(\partial D))^m$, we get that as soon as the CR-dimension of the bundle L is at least 1, the set of parameters is isomorphic either to $(A^{0,\alpha}(\partial D))^m$ or to $\mathbb{R} \times (A^{0,\alpha}(\partial D))^m$, depending on the codimension n and the orientability of the bundle L. Observe that L is orientable if and only if the bundle $\mathcal{L}$ is orientable and thus L is orientable if and only if the total index k is an even integer.

Moreover, in the example of the CR-bundle $L \subseteq \partial D \times \mathbb{C}^2$, where its real normal bundle $L^{\perp}$ is given by the matrix $N^t(\xi) = [\overline{\xi}^2, \xi^2]$, one can see that it can also happen that a certain linear change of coordinates can produce only positive partial indices but some other only negative partial indices. Therefore, to work on general CR-manifolds, we will have to assume that there exists a linear change of coordinates in $\mathbb{C}^{m+n}$ such that in the new coordinates each fiber of the maximal complex tangent bundle along a certain curve projects isomorphically onto $\mathbb{C}^m \times \{0\}$ and the corresponding partial indices are all greater or equal to -1. Observe that in the case of positive CR-dimension the condition that all partial indices are negative does not necessary imply, as in the case of maximal real bundles, that there is no nearby analytic discs attached to L. See the next remark.

2. The set of discs attached to a generating CR-bundle of a positive CR-dimension is always parametrized by an infinite dimensional Banach space. Even in the case where all partial indices of the associated bundle $\mathcal{L}$ are negative we will find a subspace of finite codimension in $(A^{0,\alpha}(\partial D))^m$ which is in one to one correspondence with the analytic discs with boundaries in L.

As it was already seen above, a necessary condition for a disc (f,g) in $\mathbb{C}^{m+n}$ to be attached to the bundle L is

$$v = \mathrm{Re}(G_o^* f) \ .$$

To get all holomorphic discs attached to L the function v should be such that there exists a mapping $u \in \mathcal{E}_\sigma$ such that

$$A_o(u + i(v + iT_\sigma v))$$

is the boundary value of a holomorphic disc in $\mathbb{C}^n$. For each partial index $k_j \geq -1$ the dimension of the corresponding set of parameters is $k_j + 1$. But if $k_j < -1$, then the function u_j has to be chosen to be identically 0 and the sufficient condition on v_j to generate a holomorphic disc is that the Fourier coefficients $\widehat{v_j}(0), \widehat{v_j}(1), \ldots, \widehat{v_j}([\frac{|k_j|}{2}] - 1)$ are all equal to 0. Here $[x]$, $x \in \mathbb{R}$, stands for the greatest integer less or equal to x. This condition is equivalent to the condition

$$F_{js}(f) = \int_0^{2\pi} e^{is\theta} \mathrm{Re}(\omega_j(\theta) \cdot f(\theta))\, d\theta = 0 \ ,$$

for $s = 0, 1, \ldots, \frac{1}{2}|k_j| - 1$ in the case k_j is an even integer and for

$$s = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{1}{2}k_j - 1$$

in the case k_j is an odd integer. Here ω_j stands for the j-th row of the matrix G_o^*. Since the linear functionals F_{js} are continuos on the space $(A^{0,\alpha}(\partial D))^m$ in the case k_j is an even integer and on the space $(\mathcal{A}^{0,\alpha})^m$ in the case k_j is an odd integer, the claim is proved.

3. The following example shows that in the nonlinear case some assumptions on the partial indices are really needed. We already know that this can happen in the maximal real case, but when the CR-dimension is positive the difference can be even more striking. Namely, although in the linear model the set of solutions is always parametrized by an infinite dimensional vector space, it can happen that the set of local nearby solutions on a CR-manifold is only finite dimensional.

Example. Let

$$M_\xi := \{(z,w) \in \mathbb{C}^2; \mathrm{Im}(\xi w) = |z|^2\} \quad (\xi \in \partial D) \ .$$

Then the disc

$$\xi \longmapsto (0,0) \quad (\xi \in \partial D)$$

is the only analytic disc with boundary in the fibration $\{M_\xi\}_{\xi\in\partial D}$.

Proof. Let (f,g) be an analytic disc with boundary in the fibration $\{M_\xi\}_{\xi\in\partial D}$. Then

$$\mathrm{Im}(\xi g(\xi)) = |f(\xi)|^2 \quad (\xi \in \partial D) \ .$$

But

$$0 = \int_0^{2\pi} \mathrm{Im}(\xi g(\xi)) d\theta = \int_0^{2\pi} |f(\xi)|^2 d\theta$$

and so $f = 0$. But then

$$\mathrm{Im}(\xi g(\xi)) = 0$$

on ∂D and so also $g = 0$. ■

Observe that in the above example the matrix $A(\xi)$ equals to $(0, i\overline{\xi})$ and thus the only partial index is -2.

One can also define a 4-dimensional submanifold of $\mathbb{C}^3$ of CR-dimension 1 with a similar property. Let

$$M := \bigcup_{\xi \in \partial D} \{\xi\} \times M_\xi \ .$$

Then any holomorphic disc with boundary in M and close to the disc

$$\xi \longmapsto (\xi, 0, 0) \quad (\xi \in \partial D)$$

is of the form

$$\xi \longmapsto (a(\xi), 0, 0) \quad (\xi \in \partial D) \ ,$$

where a is an automorphism of the unit disc close to the identity. Thus the family of such discs is 3-dimensional. ■

We consider now the nonlinear case. Let $\{M(\xi)\}_{\xi \in \partial D}$ be a family of generating CR-submanifolds of CR-dimension m in $\mathbb{C}^{m+n}$ and let

$$p : \partial D \longrightarrow \mathbb{C}^{m+n}$$

be a map of class $C^{0,\alpha}$ such that

$$p(\xi) \in M(\xi) \quad (\xi \in \partial D) \ .$$

We say that the family $\{M(\xi)\}_{\xi\in\partial D}$ is a $C^{0,\alpha}$ generating CR-fibration over the unit circle ∂D with C^2 fibers if for each $\xi_o \in \partial D$ there are a neighbourhood $U_{\xi_o} \subseteq \partial D$ of ξ_o, an open ball $B_{\xi_o} \subseteq \mathbb{C}^{m+n}$ centered at the origin and maps $\rho_1^{\xi_o}, \ldots, \rho_n^{\xi_o}$ from the space $C^{0,\alpha}(U_{\xi_o}, C^2(B_{\xi_o}))$ such that for every $\xi \in U_{\xi_o}$

1. the CR-submanifold $M(\xi) \cap (p(\xi) + B_{\xi_o})$ equals to $\{(z,w) \in p(\xi) + B_{\xi_o}; \rho_j^{\xi_o}(\xi, (z,w) - p(\xi)) = 0,\ j = 1, \ldots, n\}$,
2. $\rho_j^{\xi_o}(\xi, 0, 0) = 0,\ j = 1, \ldots, n$, and
3. $\overline{\partial}_{z,w}\rho_1^{\xi_o}(\xi, z, w) \wedge \cdots \wedge \overline{\partial}_{z,w}\rho_n^{\xi_o}(\xi, z, w) \neq 0$ on B_{ξ_o}.

THEOREM 3. *Let $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, be a $C^{0,\alpha}$ generating CR-fibration over the unit circle ∂D with C^2 fibers and CR-dimension m. Let*

$$p : \partial D \longrightarrow \mathbb{C}^{m+n}$$

be a $C^{0,\alpha}$ closed path in $\mathbb{C}^{m+n}$ such that

$$p(\xi) \in M(\xi) \qquad (\xi \in \partial D) .$$

Assume that there exists a linear change of coordinates in $\mathbb{C}^{m+n}$ such that in the new coordinate system all maximal complex subspaces of the generating CR-bundle

$$L := \bigcup_{\xi\in\partial D} \{\xi\} \times T_{p(\xi)}M(\xi)$$

project isomorphically onto the subspace $\mathbb{C}^m \times \{0\}$. Assume also that all partial indices of the corresponding maximal real bundle $\mathcal{L} \subseteq \partial D \times \mathbb{C}^n$ are greater or equal to -1 and that the total index is k. Then there are an open neighbourhood

U of $0 \in \mathbb{R}^{n+k}$, an open neighbourhood V of the function 0 in $(A^{0,\alpha}(\partial D))^m$, an open neighbourhood W of p in $(C^{0,\alpha}(\partial D))^{m+n}$ and a map

$$\Psi : U \times V \longrightarrow W$$

of class C^1 such that

1) *$\Psi(0,0) = p$,*
2) *for each $(t,f) \in U \times V$ the map $\widetilde{p} := \Psi(t,f) - p$ extends holomorphically to D and is such that $\widetilde{p}(\xi) \in M(\xi)$ for each $\xi \in \partial D$,*
3) *$\Psi(t_1, f) \neq \Psi(t_2, f)$ for $t_1 \neq t_2$ from the neighbourhood U and any $f \in V$,*
4) *if $\widetilde{p} \in W$ satisfies the condition $\widetilde{p}(\xi) \in M(\xi)$, $\xi \in \partial D$, and is such that $\widetilde{p} - p$ extends holomorphically to D, then there are $t \in U$ and $f \in V$ such that $\Psi(t,f) = \widetilde{p}$.*

Proof. Since we are assuming that all maximal complex subspaces project isomorphically onto the subspace $\mathbb{C}^m \times \{0\}$, one can, using the same construction as in Lemma 6, find a set

$$\rho(\xi, z, w) = (\rho_1(\xi, z, w), \ldots, \rho_n(\xi, z, w))$$

of "global" defining functions for the fibration $\{M(\xi)\}_{\xi \in \partial D}$, i.e., there exist an $r_o > 0$ and functions

$$\rho_j^o \in C_{\mathbb{R}}^{0,\alpha}(\partial D, C^2(B_{r_o})) \quad (1 \leq j \leq n)$$

such that for every odd partial index k_j the function ρ_j^o has the property

$$\rho_j^o(-\xi, z, w) = -\rho_j^o(\xi, z, w) \quad ((\xi, z, w) \in \partial D \times B_{r_o}) \ ,$$

and such that for the functions

$$\rho_j(\xi, z) := \begin{cases} \rho_j^o(r(\xi), z, w); & k_j \text{ is odd} \\ \rho_j^o(\xi, z, w); & k_j \text{ is even} \end{cases}$$

the following holds

a) $M(\xi) \cap (p(\xi) + B_{r_o}) = \{(z, w) \in p(\xi) + B_{r_o}; \rho_j(\xi, (z, w) - p(\xi)) = 0; j = 1, \ldots, n\}$,

b) $\overline{\partial}_w \rho_1 \wedge \ldots \overline{\partial}_w \rho_n \neq 0$ on $\partial D \times B_{r_o}$.

One may also assume that for each $\xi \in \partial D$ one has

$$(\overline{\partial}_w \rho)^*(\xi, 0, 0) = N_o^*(\xi) \ .$$

We define

$$\Psi : (A^{0,\alpha}(\partial D))^m \times \mathcal{E}_\sigma \times \mathcal{E}_\sigma \longrightarrow \mathcal{E}_\sigma$$

by

$$\Psi(f, u, v)(\xi) := \rho(\xi, f(\xi), A_o(u + i(v + iT_\sigma v))(\xi)) \quad (\xi \in \partial D) \ .$$

Then for every $v \in \mathcal{E}_\sigma$ and $\xi \in \partial D$ we have

$$(D_v \Psi(0, 0, 0)v)(\xi) = 2\text{Re}(\partial_w \rho(\xi, 0, 0) A_o(\xi) i(v(\xi) + i(T_\sigma v)(\xi))) = -2v(\xi)$$

and thus the partial derivative of the mapping Ψ with respect to variable v is an invertible linear map from the space $\mathcal{E}_\sigma$ into itself. By the implicit mapping theorem one can find a neighbourhood V of the zero function in $(A^{0,\alpha}(\partial D))^m$, neighbourhoods $\widetilde{W}$ and $\widetilde{U}$ of the zero function in $\mathcal{E}_\sigma$ and a unique mapping

$$\psi : \widetilde{U} \times V \longrightarrow \widetilde{W}$$

such that a triple $(f, u, v) \in V \times \widetilde{U} \times \widetilde{W}$ solves the equation

$$\Psi(f, u, v) = 0 \tag{21}$$

if and only if $v = \psi(u, f)$. Finally one would like to select from the above family of all possible $C^{0,\alpha}$ closed curves in the CR-fibration $\{M(\xi)\}_{\xi \in \partial D}$ near p those which bound a sum

$$p + \text{analytic disc }.$$

The rest of the argument is the same as in the proof of Theorem 1. At this point one should assume that all partial indices of the maximal real bundle $\mathcal{L}$ are greater or equal to -1. In this case the vector function

$$A_o(v + iT_\sigma v)$$

extends holomorphically to D. This follows from the fact that for any odd partial index k_j the function $v_j + iTv_j$ is of the form $r(\xi)g_j^o(\xi)$ for some function $g_j^o \in A^{0,\alpha}(\partial D)$. We recall that $r(\xi)$ represents the principal branch of the square root $\sqrt{\xi}$. So the condition on the vector function

$$\xi \longmapsto A_o(u + i(v + iT_\sigma v))(\xi) \quad (\xi \in \partial D)$$

to extend holomorphically into D is in the case where $k_j \geq -1,\ j = 1, \ldots, n$, equivalent to the condition that the vector function

$$\xi \longmapsto A_o(\xi)u(\xi) \quad (\xi \in \partial D)$$

extends holomorphically to D. To find all such functions $u \in \mathcal{E}_\sigma$ one has to find all vector functions $a \in (A^{0,\alpha}(\partial D))^n$ such that on ∂D

$$\Lambda \overline{a} = a \ ,$$

i.e., for all $j = 1, \ldots, n$

$$\xi^{k_j} \overline{a_j(\xi)} = a_j(\xi) \quad (\xi \in \partial D) \ .$$

For any partial index $k_j = -1$ the only solution of the above equation is $a_j = 0$ and for $k_j \geq 0$ one has a $k_j + 1$ dimensional parameter family of solutions. Hence, altogether one gets a $k + n$ parameter family of solutions. ■

The rest of this section was inspired by the work [**Bao-Rot-Tre**] by Baouendi, Rothschild and Trepreau. See also the paper [**Tum**] by Tumanov for some related results and definitions.

We recall the definition of the conormal bundle of a CR-submanifold $M \subseteq \mathbb{C}^N$ as given in [**Bao-Rot-Tre**]. We identify the complex bundle $\Lambda^{1,0}\mathbb{C}^N$ of $(1,0)$ forms on $\mathbb{C}^N$ with the real cotangent bundle $T^*\mathbb{C}^N$ as follows. To a real 1-form $\Gamma = \sum c_j dz_j + \overline{c_j} d\overline{z}_j$ on $\mathbb{C}^N$ we associate the complex $(1,0)$ form $\gamma = 2i \sum c_j dz_j$

so that the pairings between the vectors and covectors are related by the identity

$$\langle \Gamma, X \rangle = \mathrm{Im}\langle \gamma, X \rangle$$

for all $X \in T_z\mathbb{C}^N$. Under this identification, the fiber of the conormal bundle $\Sigma(M)$ on a CR-submanifold M at the point $p \in M$ is given by

$$\Sigma_p(M) = \{\gamma \in \Lambda^{1,0}\mathbb{C}^N; \mathrm{Im}\langle \gamma, X \rangle = 0, X \in T_pM\} \ .$$

If the manifold M is generating, then the conormal bundle can be naturally identified with the characteristic bundle $(T^{\mathbb{C}}M)^{\perp}$ of the CR-structure on M. If locally, near some point $p \in M$, the submanifold M is generating and is given by the set of equations $\rho = (\rho_1, \ldots, \rho_n) = 0$, then the fiber of the conormal bundle over the point p is given by

$$\Sigma_p(M) = \{is^t\partial\rho(p) = i\sum_j s_j \frac{\partial \rho_j}{\partial z}(p); s_j \in \mathbb{R}, 1 \leq j \leq n\} \ .$$

From now on let $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, be a generating CR-fibration over the unit circle ∂D of class $C^{0,\alpha}$ with C^2 fibers and with CR-dimension m. Let

$$p : \partial D \longrightarrow \mathbb{C}^{m+n}$$

be a $C^{0,\alpha}$ curve such that

$$p(\xi) \in M(\xi) \quad (\xi \in \partial D) \ .$$

Let V_p be the set of all holomorphic discs $c(\xi) = (c_1(\xi), \ldots, c_{m+n}(\xi))$ of class $C^{0,\alpha}$ such that for each $\xi \in \partial D$ the $(1,0)$ form

$$\sum_{j=1}^{m+n} c_j(\xi)\, dz_j \tag{22}$$

belongs to the space $\Sigma_{p(\xi)} M(\xi)$. For each $\xi \in \partial D$ we denote by $V_p(\xi) \subseteq \Sigma_{p(\xi)} M(\xi)$ the subset consisting of all such forms (22),

$$V_p(\xi) = \{\gamma \in \Sigma_{p(\xi)} M(\xi); \gamma = \sum c_j(\xi) dz_j, c \in V_p\} \; .$$

Clearly $V_p(\xi)$ is a real linear subspace of $\Sigma_{p(\xi)} M(\xi)$.

Henceforth we will assume that the coordinates in $\mathbb{C}^{m+n}$ can be chosen so that each maximal complex subspace of the tangent space $T_{p(\xi)} M(\xi)$, $\xi \in \partial D$, projects isomorphically onto $\mathbb{C}^m \times \{0\}$. We recall that this is always possible in the case the fibration $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, is of at least class C^1,i.e., the defining functions of the fibration belong to the space $C^1(\partial D, C^2(B_{r_o}))$ for some $r_o > 0$, and the closed path p is of class C^1. We also recall that for each $\xi \in \partial D$ the columns of the matrix function

$$N(\xi) = \begin{bmatrix} G_o(\xi) \\ N_o(\xi) \end{bmatrix} \; ,$$

span the fiber of the normal bundle of the submanifold $M(\xi)$ at the point $p(\xi)$.

The following characterization of elements of $V_p(\xi)$, $\xi \in \partial D$, is immediate, see also [**Bao-Rot-Tre**], Proposition 3.6.

PROPOSITION 7. *A covector* $\gamma = \sum c_j dz_j \in T^*_{p(\xi_o)}M(\xi_o)$ *belongs to the subspace* $V_p(\xi_o)$ *if and only if there is a real function* $s = (s_1, \ldots, s_n) \in \mathcal{E}_\sigma$ *such that*

1. $c^t = [c_1, \ldots, c_{m+n}] = is^t[G_o^*, N_o^*](\xi_o)$ *and*
2. *the covector function* $s^t[G_o^*, N_o^*]$ *extends holomorphically to* D.

Remark. For any real vector function $s \in \mathcal{E}_\sigma$ for which property (2) holds we will say that it *generates* an element from V_p.

COROLLARY 9. *If all partial indices of the associated maximal real fibration* $\mathcal{L}$ *are greater or equal to 1, then* $V_p = \{0\}$ *and so each of the subspaces* $V_p(\xi)$, $\xi \in \partial D$, *is trivial.*

Proof.(Corollary) Let A_o denote the matrix function whose columns for every $\xi \in \partial D$ span the fibers of $\mathcal{L}$. Then $A_o = \Theta\Lambda_o$, where Θ is an invertible holomorphic matrix on $\overline{D}$ and Λ_o is the square root of the characteristic matrix Λ of the maximal real vector bundle $\mathcal{L}$. Then

$$N_o^* = \overline{\Lambda_o}\Theta^{-1}$$

and a necessary condition to get an element from V_p is that there exists a real function $s \in \mathcal{E}_\sigma$ such that

$$s^t\overline{\Lambda_o}$$

extends holomorphically to D. But since all partial indices of Λ are greater or equal to 1, one concludes that s has to be 0. ■

Since our method gives all nearby analytic discs of class $C^{0,\alpha}$ attached to the CR-fibration $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, only in the case when all partial indices of the associated maximal real bundle are greater or equal to -1, this will be the case we will consider from now on. In this case we have already proved, Theorem 3, that the family of all nearby holomorphic discs, i.e., all holomorphic discs $F \in (A^{0,\alpha}(\partial D))^{m+n}$ with the property that the disc $p+F$ is attached to the fibration $M(\xi)$, $\xi \in \partial D$, forms a Banach submanifold $\mathcal{A}$ of the Banach space $(A^{0,\alpha}(\partial D))^{m+n}$. In the case where the CR-dimension of the fibration is 0, this submanifold is of finite real dimension $n+k$, where k is the total index of the fibration, but in the case of positive CR-dimension we get an infinite dimensional submanifold. Also, differentiation of the equation (21) with respect to u and f at the point $(0,0)$ yields

$$(D_u\psi)v = 0 \quad \text{ and } \quad (D_f\psi)f = \mathrm{Re}(G_o^* f) \ .$$

We recall that $v = \psi(u,f)$ is the solution function of the equation (21) which we got using the implicit mapping theorem in an appropriate Banach space. Thus all vectors of the tangent space $T_0\mathcal{A}$ to the submanifold $\mathcal{A}$ at the point 0 are of the form

$$(f, A_o(u + i(v + iT_\sigma v))) \ ,$$

where $f \in (A^{0,\alpha}(\partial D))^m$, and the functions $u, v \in \mathcal{E}_\sigma$ are such that $v = \mathrm{Re}(G_o^* f)$ and $\Lambda_o u$ extends holomorphically to D.

Remark. In the case considered by Baouendi, Rothschild and Trepreau in [**Bao-Rot-Tre**] one works only in a neighbourhood of a point on a given CR-submanifold and so all partial indices of any nearby holomorphic disc attached to the manifold are 0. It is easy to see that all partial indices of a constant map are 0. On the other hand, this condition is stable under small perturbations of the disc, see [**Vek2**].

PROPOSITION 8. *The dimension of the subspace* $V_p(\xi) \subseteq \Sigma_{p(\xi)}(M(\xi))$ *does not depend on* $\xi \in \partial D$*, i.e., it is the same for every* $\xi \in \partial D$*.*

Remark. This proposition extends the Proposition 3.6 from [**Bao-Rot-Tre**].

Proof. We split the space $\mathbb{R}^n$ into three subspaces

$$\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_o} \oplus \mathbb{R}^{n_{-1}} ,$$

where n_1 is the number of positive partial indices, n_o is the number of partial indices which equal to 0, and n_{-1} is the number of partial indices which equal to -1. With respect to this splitting we denote the coordinates on $\mathbb{R}^n$ by (q, y, t).

We recall that every element of the space V_p is of the form

$$is^t[G_o^*, N_o^*]$$

for some real function $s \in \mathcal{E}_\sigma$. We also recall that $N_o^* = iA_o^{-1}$. Since the first n_1 partial indices are positive, any real vector function s from the space $\mathcal{E}_\sigma$ which generates an element in V_p, must have, by the same argument as in the proof of Corollary 9, the first n_1 coordinate functions identically equal to 0. Because of this reason, and to simplify the notation, we will assume, and we can do so without loss of generality, that $n_1 = 0$.

Each element of V_p is now generated by a real function of the form

$$(y, \mathrm{Re}(\omega r(\xi)))$$

where $y \in \mathbb{R}^{n_o}$, $\omega \in \mathbb{C}^{n-1}$ and $r(\xi)$ is the principal branch of the square root. Let k_o be the dimension of the space $V_p(1)$. We will prove that for each $\xi \in \partial D$ the dimension of the space $V_p(\xi)$ is also k_o. Since for each $\xi_o \in \partial D$ there exists an automorphism of the unit disc D which takes 1 to 1 and ξ_o to -1, it is enough to prove the above claim for $\xi_o = -1$.

Let $(y_j, \mathrm{Re}(\omega_j r(\xi)))$, $j = 1, \ldots, k_o$, be a set of real functions on ∂D which for each j generate an element of V_p, and such that the real vectors

$$(y_j, \mathrm{Re}(\omega_j)) \quad (j = 1, \ldots, k_o)$$

are linearly independent. If also the set of vectors

$$(y_j, \mathrm{Re}(i\omega_j)) \quad (j = 1, \ldots, k_o)$$

is linearly independent, the claim is already proved and we are done. Let us assume now that this is not the case and that these vectors are not linearly independent. Then there are real numbers $\lambda_1, \ldots, \lambda_{k_o}$, not all equal to 0, such that

$$\sum_{j=1}^{k_o} \lambda_j y_j = 0$$

and

$$\sum_{j=1}^{k_o} \lambda_j \mathrm{Re}(i\omega_j) = 0\ .$$

The second equation is equivalent to

$$\sum_{j=1}^{k_o} \lambda_j \omega_j = t$$

for some real vector t from $\mathbb{R}^{n-1}$. The way how t is defined immediately implies that $t \neq 0$ and that the real vector function

$$(0, \mathrm{Re}(tr(\xi)))$$

generates an element from V_p. Since t is a real vector, both functions

$$(0, \mathrm{Re}(tr(\xi))) \quad \text{and} \quad (0, \mathrm{Re}(itr(\xi)))$$

generate an element from the space V_p. This follows from the following claim.

Claim. Let $f = u + iv$, $u, v \in (\mathcal{E}_{\mathbb{R}}^{0,\alpha})^n$, be a vector function such that the function

$$\xi \longmapsto \mathrm{Re}(r(\xi)) f(\xi) \quad (\xi \in \partial D)$$

extends holomorphically into D. Then $f \in (\mathcal{A}^{0,\alpha})^n$. In particular, also the function

$$\xi \longmapsto \mathrm{Re}(ir(\xi)) f(\xi) \quad (\xi \in \partial D)$$

extends holomorphically into D.

Proof.(Claim) Since the function

$$\xi \longmapsto \mathrm{Re}(r(\xi))f(\xi) \quad (\xi \in \partial D)$$

extends holomorphically into D, all its negative Fourier coefficients have to vanish. This implies that for every $j \in \mathbb{N}$ we have

$$\widehat{f}(-j) + \widehat{f}(-j-1) = 0 \ .$$

Since we also have

$$\lim_{j\to\infty} \widehat{f}(-j) = 0 \ ,$$

we conclude that all negative Fourier coefficients of the function f are 0 and the claim is proved. ■

Also, since not all real numbers $\lambda_j,\ j = 1, \ldots, k_o$, are 0, we may assume, without loss of generality, that $\lambda_1 \neq 0$. We repeat the above argument on the set of real functions $(y_j, \mathrm{Re}(\omega_j r(\xi))),\ j = 2, \ldots, k_o$, and the vector function $(0, \mathrm{Re}(itr(\xi)))$. If at $\xi = -1$ these vectors are still linearly dependent, one can find real numbers $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{k_o}$, not all equal to 0, such that

$$\widetilde{\lambda}_1 it + \sum_{j=2}^{k_o} \widetilde{\lambda}_j \omega_j = t_1$$

for some nonzero real vector $t_1 \in \mathbb{R}^{n-1}$. We observe that it can not happen that $\widetilde{\lambda}_2 = \cdots = \widetilde{\lambda}_{k_o} = 0$ since the vectors t and t_1 are real. We also observe that t and t_1 are linearly independent vectors. Repeating the above argument we either stop at the j^{th} step, $j < k_o$, or we produce k_o linearly independent vectors which span $V_p(-1)$. ■

So we can define the defect of the closed curve p in a generating CR-fibration over ∂D with partial indices greater or equal to -1 in the same way as Baouendi, Rothschild and Trepreau do in [**Bao-Rot-Tre**]. See Definition 3.5 and Proposition 3.6 in [**Bao-Rot-Tre**]. See also [**Tum**].

DEFINITION 8. *The defect* $\mathrm{def}(p)$ *of the curve* p *is defined as the dimension of the real vector spaces* $V_p(\xi),\ \xi \in \partial D$.

From now on we will restrict our discussion to the set $\mathcal{A}_*$ of holomorphic perturbations of p which leave one of the points, say $p(1)$, on the curve p fixed. But to prove that the set $\mathcal{A}_*$ is in fact a manifold, we have to assume that all partial indices of the path p are nonnegative. See the examples at the end of this section. Let $\mathcal{E}_{\sigma,*} \subseteq \mathcal{E}_\sigma$ and $(A_*^{0,\alpha}(\partial D))^m \subseteq (A^{0,\alpha}(\partial D))^m$ be the subspaces of the functions which are 0 at $\xi = 1$. Let

$$T_* : C_{\mathbb{R}}^{0,\alpha}(\partial D) \to C_{\mathbb{R}}^{0,\alpha}(\partial D)$$

be the Hilbert transform which assigns to a function $v \in C_{\mathbb{R}}^{0,\alpha}(\partial D)$ the harmonic conjugate function $\widetilde{v}$ for which $\widetilde{v}(1) = 0$. Since T_* does not preserve the subspace of odd functions in $C_{\mathbb{R}}^{0,\alpha}(\partial D)$, there is no natural way of defining an appropriate Hilbert transform on $\mathcal{E}_{\sigma,*}$.

Let k be the total index of the associated maximal real bundle $\mathcal{L} \subseteq \partial D \times \mathbb{C}^n$. Then the following lemma holds.

LEMMA 12. *$\mathcal{A}_*$ is a Banach submanifold of the manifold $\mathcal{A}$ of the infinite dimension in the case the CR-dimension of the fibration $\{M(\xi)\}_{\xi\in\partial D}$ is positive, and of real dimension k in the case of maximal real fibration over ∂D.*

Proof. We define the map

$$F : (A_*^{0,\alpha}(\partial D))^m \times \mathcal{E}_{\sigma,*} \times \mathcal{E}_{\sigma,*} \longrightarrow \mathcal{E}_{\sigma,*}$$

by

$$F(f, \widetilde{u}, v)(\xi) := \rho(\xi, f(\xi), A_o((\widetilde{u} + T_\sigma v) + i(v + iT_\sigma v))(\xi)) \quad (\xi \in \partial D) \ .$$

Here $\rho = (\rho_1, \ldots, \rho_n)$ is the set of defining functions of the fibration $\{M(\xi)\}_{\xi\in\partial D}$ along the path p. Using the implicit mapping theorem as in the proof of Theorem 3 one gets a neighbourhood $\mathcal{N}$ of the zero function in $(A_*^{0,\alpha}(\partial D))^m$, neighbourhoods $\widetilde{U}$ and V of 0 in $\mathcal{E}_{\sigma,*}$, and a unique mapping $\psi : \mathcal{N} \times \widetilde{U} \to V$ such that the triple $(f, \widetilde{u}, v) \in \mathcal{N} \times \widetilde{U} \times V$ solves the equation $F(f, \widetilde{u}, v) = 0$ if and only if $v = \psi(f, \widetilde{u})$.

As we already know a necessary and sufficient condition for any disc from the above family to be the boundary value of a holomorphic disc is that the mapping

$$A_o(\widetilde{u} + T_\sigma v)$$

extends holomorphically to D. Let $\Psi(t)$, $t \in \mathbb{R}^{n+k}$, denote the linear parametrization (7), (8) of all real functions $u \in \mathcal{E}_\sigma$ such that $A_o u$ extends holomorphically to D. Thus to extract from the above family of discs $A_o(\widetilde{u} + i\psi(f, \widetilde{u}))$ all holomorphic discs which are 0 at $\xi = 1$, we have to find all functions $\widetilde{u} \in \mathcal{E}_{\sigma,*}$ and values $t \in \mathbb{R}^{m+n}$ which solve the equations

$$\widetilde{u} + T_\sigma \psi(f, \widetilde{u}) = \Psi(t) \quad \text{and} \quad T_\sigma \psi(f, \widetilde{u})(1) = \Psi(t)(1) \ .$$

Since all partial indices are nonnegative, the $n \times (n+k)$ matrix $D_t(\Psi(t)(1))|_{t=0}$ has the maximal rank. Since we also have

$$D_{\widetilde{u}}(\widetilde{u} + T_\sigma \psi(f, \widetilde{u}))|_{f=0, \widetilde{u}=0} = Id \ ,$$

one gets, using the implicit mapping theorem again, a unique mapping μ from a neighbourhood of the point $(0,0) \in (A_*^{0,\alpha}(\mathcal{D}))^m \times \mathbb{R}^k$ into $\mathcal{E}_{\sigma,*}$ such that all small holomorphic disc (f, g) from $(A_*^{0,\alpha}(\partial D))^{m+n}$ which solve the equation

$$\rho(\xi, f(\xi), g(\xi)) = 0 \quad (\xi \in \partial D)$$

are of the form

$$(f, A_o(\mu(f, s) + i\psi(f, \mu(f, s))))$$

for a unique pair (f, s) from a neighbourhood of the point $(0,0)$ in $(A_*^{0,\alpha}(\mathcal{D}))^m \times \mathbb{R}^k$. ■

Note that any element of the tangent space $T_0\mathcal{A}_*$ is of the form

$$(f, A_o(u + i(v + iT_\sigma v)))$$

for some $f \in (A_*^{0,\alpha}(\partial D))^m$, $v \in \mathcal{E}_{\sigma,*}$ such that $v = \mathrm{Re}(G_o^* f)$, and $u \in \mathcal{E}_\sigma$ such that $A_o u$ extends holomorphically to D and for which one also has $u - T_\sigma v \in \mathcal{E}_{\sigma,*}$.

Henceforth our goal will be to reprove and to generalize Theorem 1 from [**Bao-Rot-Tre**]. In fact we can prove the same statement for an arbitrary closed path p in a generating CR-fibration $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, for which there exists a linear change of coordinates in $\mathbb{C}^{m+n}$ such that the partial indices of the corresponding maximal real bundle are all greater or equal to 0.

We recall the definition of the evaluation maps $\mathcal{F}_\xi$ defined on the manifold $\mathcal{A}_*$, see [**Bao-Rot-Tre**] for more details. See also [**Tum**]. For every $\xi \in \partial D$ and $F \in \mathcal{A}_*$ we define

$$\mathcal{F}_\xi(F) := (p + F)(\xi) \ .$$

Then for every $\xi \in \partial D$ the derivative $\mathcal{F}_\xi'(0)$ maps the tangent space $T_0\mathcal{A}_*$ into $T_{p(\xi)}M(\xi)$.

THEOREM 4. *Let p and $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, be as above. Then for each $\xi \in \partial D$, $\xi \neq 1$, one has*

$$\mathcal{F}_\xi'(0)(T_0\mathcal{A}_*) = V_p(\xi)^\perp \ . \tag{23}$$

Proof. We first prove the following partial statement, namely,

$$\mathcal{F}_\xi'(0)(T_0\mathcal{A}_*) \subseteq V_p(\xi)^\perp$$

for each $\xi \in \partial D$. To prove this claim let

$$(f, A_o(u + i(v + iT_\sigma v))) \tag{24}$$

be an arbitrary element of $T_0\mathcal{A}_*$. We recall that $f \in (A_*^{0,\alpha}(\partial D))^m$, that $u \in \mathcal{E}_\sigma$, that $v, u - T_\sigma v \in \mathcal{E}_{\sigma,*}$, and that also

$$v = \mathrm{Re}(G_o^* f) \ .$$

On the other hand let

$$u_o^t[G_o^*, N_o^*] \tag{25}$$

be an element of V_p. Here $u_o \in \mathcal{E}_\sigma$. Since both vector functions (24) and (25) extend holomorphically to D, their product also has to extend holomorphically to D. But on the other hand the multiplication of (24) and (25) yields a purely imaginary vector function

$$iu_o^t(u - T_\sigma v + \mathrm{Im}(G_o^* f)) \ .$$

Since the vector function (24) is 0 at $\xi = 1$, the claim is proved.

To prove that in the above inclusion in fact the equality holds it is enough to prove that the dimension of the space $\mathcal{F}'_\xi(0)(T_0\mathcal{A}_*)$ is $2m + n - \mathrm{def}(p)$. Since for each $\xi \in \partial D$, $\xi \neq 1$, one can find an automorphism of the unit disc D which takes 1 to 1 and ξ to -1, it is enough to prove the claim for $\xi = -1$. For this it is enough, since the set of function values $\{f(-1); f \in (A_*^{0,\alpha}(\partial D))^m\}$ already spans a $2m$-dimensional subspace, to prove that the subspace

$$\{(u - T_\sigma v)(-1); v = \mathrm{Re}(G_o^* f), f(-1) = 0, A_o u \in (A^{1,\alpha}(\partial D))^n, u - T_\sigma v \in \mathcal{E}_{\sigma,*}\}$$

has dimension $n - \mathrm{def}(p)$.

Denote by n_1 the number of positive indices and by n_o the number of indices which are equal to 0, and split the space $\mathbb{R}^n$ correspondingly. For each positive partial index k_j the set of real functions u_j such that the function $\xi \mapsto \xi^{k_j/2} u_j(\xi)$ extends holomorphically to D and $(u_j - Tv_j)(1) = 0$ is at least 1-dimensional. Thus the proof of the claim will be finished once we prove that the following subspace of $\mathbb{R}^{n_o}$

$$\{((u - T_\sigma v)(-1); v = \mathrm{Re}(G_o^* f), f(-1) = 0, u - T_\sigma v \in \mathcal{E}_{\sigma,*}\} \bigcap \mathbb{R}^{n_o}$$

has dimension $n - n_1 - \mathrm{def}(p)$.

To prove the last claim we will show that for a vector $u_o \in \mathbb{R}^{n_o}$ the condition

$$u_o^t (u - T_\sigma v)(-1) = 0$$

for every $v \in (C_{\mathbb{R}}^{0,\alpha}(\partial D))^{n_o}$ such that v is given as the last n_o component functions of $\mathrm{Re}(G_o^* f)$, $f(-1) = 0$, and every constant vector $u \in \mathbb{R}^{n_o}$ such that $(u - T_\sigma v)(1) = 0$, implies

$$(0, u_o^t)[G_o^*, N_o^*](-1) \in V_p(-1) \ .$$

This will complete the proof of (23). But since every real vector function $\widetilde{u}_o \in \mathcal{E}_\sigma$ which generates an element from V_p has the first n_1 coordinate functions identically equal to 0, see the proof of Corollary 9, it is enough to prove the statement for the case where all partial indices are 0 and $n_1 = 0$.

From here on the argument goes very much the same as the one given by Baouendi, Rothschild and Trepreau in [**Bao-Rot-Tre**].

We recall that T_* denotes the Hilbert transform on $(A^{0,\alpha}_{\mathbb{R}}(\partial D))^n$ such that for every $v \in (A^{0,\alpha}_{\mathbb{R}}(\partial D))^n$ we have

$$(T_* v)(1) = 0 \ .$$

Also, since all partial indices are 0, the vector function u is in fact a constant such that $(T_\sigma v - u)(1) = 0$ and hence

$$T_\sigma v - u = T_* v \ .$$

Let $u_o \in \mathbb{R}^{n_o}$ be a vector with the property that

$$u_o^t (T_* v)(-1) = 0$$

for every $v \in (C^{0,\alpha}_{\mathbb{R}}(\partial D))^n$ such that $v = \mathrm{Re}(G_o^* f),\ f(-1) = 0$. We recall that for $v \in C^{0,\alpha}_{\mathbb{R}}(\partial D)$ and $\xi_o \in \partial D$ one has

$$(T_* v)(\xi_o) = \mathrm{PV} \frac{i}{\pi} \int_0^{2\pi} \frac{v(\xi)(1-\xi_o)}{(\xi - 1)(\xi - \xi_o)} \xi \, d\theta \ ,$$

where ξ stands for $e^{i\theta}$. We denote the vector function $u_o^t G_o^*$ by a_o^t. Then for every nonnegative integer q, every vector $z_o \in \mathbb{C}^m$, and a function f of the form

$$f(\xi) = (\xi^2 - 1)\xi^q z_o \quad (\xi \in \partial D)$$

one has

$$(26) \qquad (u_o^t(T_*v))(-1) = \mathrm{PV}\frac{i}{\pi}\int_0^{2\pi} 2\frac{\mathrm{Re}(a_o^t(\xi)f(\xi))}{(\xi-1)(\xi+1)}\xi\,d\theta$$

$$(27) \qquad = \frac{2i}{2\pi}\int_0^{2\pi}[\xi^{q+1}a_o^t(\xi)z_o - \overline{\xi^{q+1}a_o^t(\xi)z_o}]\,d\theta\ .$$

By our assumption the integrals (26) and (27) equal 0 for every nonnegative integer q and every vector $z_o \in \mathbb{C}^m$. Since one can also take iz_o instead of z_o, one gets that

$$\frac{2i}{2\pi}\int_0^{2\pi}\xi^{q+1}a_o^t(\xi)d\theta = 0$$

for every nonnegative integer q. The above identity can be written in terms of Fourier coefficient as

$$\widehat{a_o}(-q-1) = 0 \quad (q = 0, 1, 2, \dots)$$

which immediately implies that the real vector u_o^t generates an element of V_p. The identity (23) is proved. ■

For the next theorem we have to assume more regularity on the fibration $\{M(\xi)\}_{\xi\in\partial D}$ and the closed path p. We assume now that we have a $C^{1,\alpha}$ fibration with C^3 fibers, i.e., the fibration is given by a set of real functions from the space $C^{1,\alpha}(\partial D, C^3(B_{r_o}))$, and the closed path p shall be of class $C^{1,\alpha}$. Under this conditions one can repeat the proofs of Theorem 3 and Theorem 4 in the $C^{1,\alpha}$ category. We recall the definition of the mapping $\mathcal{G}$ from [**Bao-Rot-Tre**].

Let

$$\mathcal{G} : T_0\mathcal{A}_* \longrightarrow \mathbb{C}^{m+n}$$

be defined by

$$\mathcal{G}(F) := \frac{\partial}{\partial\theta} F(e^{i\theta})\bigg|_{\theta=0} \ .$$

THEOREM 5. *Let p and $M(\xi) \subseteq \mathbb{C}^{m+n}$, $\xi \in \partial D$, be as above. Then $\mathcal{G}$ maps $T_0\mathcal{A}_*$ into $T_{p(1)}M(1)$ and*

$$\mathcal{G}(T_0\mathcal{A}_*) = V_p(1)^{\perp} \ . \tag{28}$$

Proof. We first observe that for every $F \in T_0\mathcal{A}_*$ one has

$$\mathrm{Re}([G_o^*, N_o^*]F) = 0$$

on ∂D. Differentiation with respect to θ and setting $\theta = 0$ implies that $\mathcal{G}$ maps $T_0\mathcal{A}_*$ into $T_{p(1)}M(1)$.

The proof of (28) is quite similar to the proof of (23). The inclusion

$$\mathcal{G}(T_0\mathcal{A}_*) \subseteq V_p(1)^{\perp}$$

follows as above since the product of any two functions $G \in V_p$ and $F \in T_0\mathcal{A}_*$ equals to 0,

$$G^t F = 0 \ . \tag{29}$$

Namely, the differentiation of (29) with respect to θ and setting $\theta = 0$ yields

$$G^t(1)\frac{\partial}{\partial\theta} F(e^{i\theta})\bigg|_{\theta=0} = 0 \ .$$

To prove the opposite inclusion in (28) we proceed similarly as in the proof of (23) and reduce the problem to to the case where all partial indices of the associated maximal real bundle are 0, and showing the following claim.

Claim. The vector space

$$\{ \frac{\partial}{\partial\theta}(T_* v)(e^{i\theta})\Big|_{\theta=0} ; v = \mathrm{Re}(G_o^* f), f(\xi) = (\xi-1)^2\xi^q z_o, q \in \mathbb{N}\cup\{0\}, z_o \in \mathbb{C}^m \}$$

has dimension $n - \mathrm{def}(p)$.

Proof.(Claim) We are using a similar notation as in the proof of (23). Let $u_o \in \mathbb{R}^n$ be a real n-vector which annihilates the above vector space. Also, let a_o^t be the vector $u_o^t G_o^*$. We recall that

$$\begin{aligned} u_o^t \frac{\partial}{\partial\theta}(T_* v)(e^{i\theta})|_{\theta=0} &= \frac{1}{\pi}\int_0^{2\pi} \frac{\mathrm{Re}(a_o^t(\xi) f(\xi))}{(\xi-1)^2}\xi \, d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} [a_o^t(\xi)\xi^{q+1} z_o + \overline{a_o^t(\xi)\xi^{q+1} z_o}] \, d\theta \ . \end{aligned}$$

Replacing z_o by iz_o and adding the identities one gets

$$\int_0^{2\pi} a_o^t(\xi)\xi^{q+1} \, d\theta = 0$$

for every nonnegative integer q. In terms of Fourier coefficients we have

$$\widehat{a_o}(-q-1) = 0$$

for every $q \in \mathbb{N}\cup\{0\}$. Thus the real vector u_o generates an element from V_p. This finishes the proof of the claim and so also the theorem. ■

Remarks and examples. In the case when the partial indices of the path p in the generating CR-fibration $\{M(\xi)\}_{\xi\in\partial D}$ are not all nonnegative, the conclusions in Theorems 4 and 5 are not true even if we consider only the case where all indices are greater or equal to -1. One problem, of course, occurs if the total index k happens to be negative. Then the number of free parameters is strictly less than the number of additional equations we have to satisfy. Here we give two examples in $\mathbb{C}^2$ for which $k \geq 0$ but the conclusions of Theorems 4 and 5 still do not hold.

Example 1. In this example we find a maximal real fibration in $\mathbb{C}^2$ for which the set of attached discs passing through the point $(0,0)$ does not form a manifold.

Let the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ be given by the set equations

$$\mathrm{Im}(z\overline{\xi}) = 0$$

and

$$\mathrm{Re}(wr(\xi)) = \mathrm{Re}(r(\xi))\mathrm{Re}((z\overline{\xi})^2)\ .$$

It is easy to check that the partial indices of the path $p(\xi) = 0$, $\xi \in \partial D$, are 2 and -1, and so the total index k equals 1 and the defect of the path p is 1. Hence the dimension of the spaces

$$V_p(\xi)^{\perp} \quad (\xi \in \partial D)$$

is also 1.

Claim. The family of holomorphic discs with boundaries in the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ which all pass through the point $(0,0)$ at $\xi=1$ is not a manifold.

Proof.(Claim) Let (z,w) be a holomorphic disc with boundary in the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ and such that $(z(1),w(1))=(0,0)$. Then from the first equation

$$\mathrm{Im}(z(\xi)\overline{\xi})=0$$

we get

$$z(\xi)\overline{\xi}=\omega\xi-2\mathrm{Re}(\omega)+\overline{\omega}\overline{\xi}$$

for some complex number ω. The second equation

$$\mathrm{Re}(w(\xi)r(\xi))=\mathrm{Re}(r(\xi))\mathrm{Re}((z(\xi)\overline{\xi})^2)\quad(\xi\in\partial D)$$

implies

$$\mathrm{Re}(w(\xi^2)\xi)=\mathrm{Re}(\xi)\mathrm{Re}((z(\xi^2)\overline{\xi}^2)^2)\quad(\xi\in\partial D)\ .$$

A short calculation shows that the right hand side of the last equation equals to

$$\mathrm{Re}(\omega^2\xi^5+(\omega^2-4\omega\mathrm{Re}(\omega))\xi^3+(2|\omega|^2+4(\mathrm{Re}(\omega))^2-4\omega\mathrm{Re}(\omega))\xi)\ .$$

Thus

$$w(\xi)=\omega^2\xi^2+(\omega^2-4\omega\mathrm{Re}(\omega))\xi+(2|\omega|^2+4(\mathrm{Re}(\omega))^2-4\omega\mathrm{Re}(\omega))$$

and so one must have

$$\omega^2+(\omega^2-4\omega\mathrm{Re}(\omega))+(2|\omega|^2+4(\mathrm{Re}(\omega))^2-4\omega\mathrm{Re}(\omega))=0$$

or after division by 2

$$\omega^2 - 4\omega\mathrm{Re}(\omega) + |\omega|^2 + 2(\mathrm{Re}(\omega))^2 = 0 \ .$$

If we write $\omega = x + iy$, then the imaginary part of the last equation yields

$$-2xy = 0 \ .$$

Thus the constant ω has to be either real or purely imaginary. So the set of solutions of the above equations is the union of two intersecting curves in $(A^{0,\alpha}(\partial D))^2$ and therefore not a manifold. ■

Example 2. Let the maximal real fibration in $\mathbb{C}^2$ be given by

$$\mathrm{Im}(z\overline{r(\xi)}) = 0$$

and

$$\mathrm{Im}(wr(\xi)) = \mathrm{Re}((z\overline{r(\xi)})^3) \ .$$

Then the partial indices of the closed path $p(\xi) = 0$, $\xi \in \partial D$, are 1 and -1 and so the total index is 0. Also, the defect $\mathrm{def}(p)$ is 1 and thus the dimension of the spaces $V_p^{\perp}(\xi)$, $\xi \in \partial D$, is 1. Let (z, w) be a holomorphic disc with boundary in the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ and such that $(z(1), w(1)) = (0, 0)$. Then from the first equation we get

$$z(\xi) = ia(\xi - 1)$$

for some real number a. The second equation now implies

$$\mathrm{Im}(w(\xi^2)\xi) = \mathrm{Re}((ia(\xi - \overline{\xi}))^3)$$

or

$$\mathrm{Im}(w(\xi^2)\xi) = 2a^3\mathrm{Im}(\xi^3 - 3\xi)\ .$$

Hence we get

$$w(\xi) = 2a^3(\xi - 3)$$

which can be 0 at $\xi = 1$ if only if $a = 0$. Thus we showed that the only holomorphic disc attached to the fibration $\{M(\xi)\}_{\xi\in\partial D}$ and which is passing through the point $(0, 0)$, is the zero disc p.

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