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On Jacobian group and complexity of the I-graph $I(n, k, l)$ through Chebyshev polynomials

Ilya A. Mednykh *

Sobolev Institute of Mathematics, Koptyuga 4, Novosibirsk, 630090, Russia
Novosibirsk State University, Pirogova 1, Novosibirsk, 630090, Russia

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Abstract

We consider a family of I-graphs $I(n, k, l)$, which is a generalization of the class of generalized Petersen graphs. In the present paper, we provide a new method for counting Jacobian group of the I-graph $I(n, k, l)$. We show that the minimum number of generators of $\mathrm{Jac}(I(n, k, l))$ is at least two and at most $2k + 2l - 1$. Also, we obtain a closed formula for the number of spanning trees of $I(n, k, l)$ in terms of Chebyshev polynomials. We investigate some arithmetical properties of this number and its asymptotic behaviour.

Keywords: Spanning tree, Jacobian group, I-graph, Petersen graph, Chebyshev polynomial.

Math. Subj. Class.: 05C30, 39A10

1 Introduction

The notion of the Jacobian group of a graph, which is also known as the Picard group, the critical group, and the dollar or sandpile group, was independently introduced by many authors ([1, 2, 4, 9]). This notion arises as a discrete version of the Jacobian in the classical theory of Riemann surfaces. It also admits a natural interpretation in various areas of physics, coding theory, and financial mathematics. The Jacobian group is an important algebraic invariant of a finite graph. In particular, its order coincides with the number of spanning trees of the graph, which is known for some simplest graphs, such as the wheel,

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E-mail address: ilyamednykh@mail.ru (Ilya A. Mednykh)

fan, prism, ladder, and Möbius ladder [6], grids [23], lattices [25], prism and anti-prism [26]. At the same time, the structure of the Jacobian is known only in particular cases [4, 7, 9, 17, 20, 21] and [22]. We mention that the number of spanning trees for circulant graphs is expressed is terms of the Chebyshev polynomials; it was found in [8, 27], and [28]. We show that similar results are also true for the I-graph $I(n, k, l)$.

The generalized Petersen graph $GP(n, k)$ has vertex set and edge set given by

$$\begin{aligned} V(GP(n,k)) &= \{u_i, v_i \mid i = 1, 2, \ldots, n\} \\ E(GP(n,k)) &= \{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+k} \mid i = 1, 2, \ldots, n\}, \end{aligned}$$

where the subscripts are expressed as integers modulo n. The classical Petersen graph is $GP(5, 2)$. The family of generalized Petersen graphs is a subset of so-called I-graphs ([3, 14]). The I-graph $I(n, k, l)$ is a graph of the following structure

$$\begin{aligned} V(I(n,k,l)) &= \{u_i, v_i \mid i = 1, 2, \ldots, n\} \\ E(I(n,k,l)) &= \{u_i u_{i+l},\ u_i v_i,\ v_i v_{i+k} \mid i = 1, 2, \ldots, n\}. \end{aligned}$$

where all subscripts are given modulo n.

Since $I(n, k, l) = I(n, l, k)$ we will usually assume that $k \leq l$. In this paper we will deal with 3-valent graphs only. This means that in the case of even n and $l = n/2$ the graph under consideration has multiple edges. The graph $I(n, l, k)$ is connected if and only if $\gcd(n, k, l) = 1$. If $\gcd(n, k, l) = m > 1$, then $I(n, k, l)$ is a union of m copies of the graph $I(n/m, k/m, l/m)$. If $m = 1$ and $\gcd(k, l) = d$, then the graphs $I(n, k, l)$ and $I(n, k/d, l/d)$ are isomorphic [5, 16, 24]. In the case of $l = 1$ it easy to see that the graph $I(n, k, 1)$ coincides with the generalized Petersen graph $GP(n, k)$. The number of spanning trees and the structure of Jacobian group for the generalized Petersen graph were investigated in [19]. The spectrum of the I-graph was found in [11]. Even though the number of spanning trees of a given graph can be computed through eigenvalues of its Laplacian matrix, it is not easy to find the number of spanning trees for $I(n, k, l)$ using them. In this paper, we obtained a closed formula for the number of spanning trees for $I(n, k, l)$, investigate some arithmetical properties of this number and provide its asymptotic behavior. Also, we suggest an effective way for calculating Jacobian of $I(n, k, l)$ and find sharp upper and lower bounds for the rank of $\mathrm{Jac}(I(n, k, l))$.

2 Basic definitions and preliminary facts

Consider a connected finite graph G, allowed to have multiple edges but without loops. We endow each edge of G with the two possible directions. Since G has no loops, this operation is well defined. Let $O = O(G)$ be the set of directed edges of G. Given $e \in O(G)$, we denote its initial and terminal vertices by $s(e)$ and $t(e)$, respectively. Recall that a closed directed path in G is a sequence of directed edges $e_i \in O(G)$, $i = 1, \ldots, n$ such that $t(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$ and $t(e_n) = s(e_1)$.

Following [1] and [2], *the Jacobian group*, or simply *Jacobian* $\mathrm{Jac}(G)$ of a graph G is defined as the (maximal) Abelian group generated by flows $\omega(e), e \in O(G)$, obeying the following two Kirchhoff laws:

K_1: the flow through each vertex of G vanishes, that is $\sum_{e\in O, t(e)=x} \omega(e) = 0$ for all $x \in V(G)$;

K_2: the flow along each closed directed path W in G vanishes, that is $\sum_{e\in W}\omega(e)=0$.

Equivalent definitions of the group $\mathrm{Jac}(G)$ can be found in papers [1, 2, 4, 9, 12, 18, 20].

We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u,v\in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A=A(G)=\{a_{uv}\}_{u,v\in V(G)}$, called *the adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v\in V(G)$ is defined by $d(v)=\sum_u a_{uv}$. Let $D=D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv}=d(v)$. Matrix $L=L(G)=D(G)-A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

Recall [20] the following useful relation between the structure of the Laplacian matrix and the Jacobian of a graph G. Consider the Laplacian $L(G)$ as a homomorphism $\mathbb{Z}^{|V|}\to\mathbb{Z}^{|V|}$, where $|V|=|V(G)|$ is the number of vertices in G. The cokernel $\mathrm{coker}(L(G))=\mathbb{Z}^{|V|}/\,\mathrm{im}(L(G))$ — is an Abelian group. Let

$$\mathrm{coker}(L(G))\cong\mathbb{Z}_{d_1}\oplus\mathbb{Z}_{d_2}\oplus\cdots\oplus\mathbb{Z}_{d_{|V|}}$$

be its Smith normal form satisfying the conditions $d_i\big|d_{i+1}$, $(1\le i\le|V|)$. If the graph is connected, then the groups $\mathbb{Z}_{d_1},\mathbb{Z}_{d_2},\ldots,\mathbb{Z}_{d_{|V|-1}}$ — are finite, and $\mathbb{Z}_{d_{|V|}}=\mathbb{Z}$. In this case,

$$\mathrm{Jac}(G)\cong\mathbb{Z}_{t_1}\oplus\mathbb{Z}_{t_2}\oplus\cdots\oplus\mathbb{Z}_{d_{|V|-1}}$$

is the Jacobian of the graph G. In other words, $\mathrm{Jac}(G)$ is isomorphic to the torsion subgroup of the cokernel $\mathrm{coker}(L(G))$.

Let M be an integer $n\times n$ matrix, then we can interpret M as a homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^n$. In this interpretation M has a kernel $\ker M$, an image $\mathrm{im}\,M$, and a cokernel $\mathrm{coker}\,M=\mathbb{Z}^n/\,\mathrm{im}\,M$. We emphasize that $\mathrm{coker}\,M$ of the matrix M is completely determined by its Smith normal form.

In what follows, by I_n we denote the identity matrix of order n.

We call an $n\times n$ matrix *circulant,* and denote it by $\mathrm{circ}(a_0,a_1,\ldots,a_{n-1})$ if it is of the form

$$\mathrm{circ}(a_0,a_1,\ldots,a_{n-1})=\begin{pmatrix}a_0 & a_1 & a_2 & \ldots & a_{n-1}\\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_1 & a_2 & a_3 & \ldots & a_0\end{pmatrix}.$$

Recall [10] that the eigenvalues of matrix $C=\mathrm{circ}(a_0,a_1,\ldots,a_{n-1})$ are given by the following simple formulas $\lambda_j=p(\varepsilon_n^j)$, $j=0,1,\ldots,n-1$ where $p(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$ and ε_n is the order n primitive root of the unity. Moreover, the circulant matrix $C=p(T)$, where $T=\mathrm{circ}(0,1,0,\ldots,0)$ is the matrix representation of the shift operator $T\colon(x_0,x_1,\ldots,x_{n-2},x_{n-1})\to(x_1,x_2,\ldots,x_{n-1},x_0)$.

By [15, Lemma 2.1] the $2n\times 2n$ adjacency matrix of the I-graph $I(n,k,l)$ has the following block form

$$A(I(n,k,l))=\begin{pmatrix}C_n^k & I_n\\ I_n & C_n^l\end{pmatrix},$$

where C_n^k is the $n\times n$ circulant matrix of the form

$$C_n^k=\mathrm{circ}(\underbrace{0,\ldots,0}_{k\text{ times}},1,\underbrace{0,\ldots,0}_{n-2k-1},1,\underbrace{0,\ldots,0}_{k-1\text{ times}}).$$

Denote by $L = L(I(n,k,l))$ the Laplacian of $I(n,k,l)$. Since the graph $I(n,k,l)$ is three-valent, we have

$$L = 3I_{2n} - A(I(n,k,l)) = \begin{pmatrix} 3I_n - C_n^k & -I_n \\ -I_n & 3I_n - C_n^l \end{pmatrix}.$$

3 Cokernels of linear operators

Let $P(z)$ be a bimonic integer Laurent polynomial. That is $P(z) = z^p + a_1 z^{p+1} + \cdots + a_{s-1} z^{p+s-1} + z^{p+s}$ for some integers $p, a_1, a_2, \ldots, a_{s-1}$ and some positive integer s. Introduce the following companion matrix $\mathcal{A}$ for the polynomial $P(z)$:

$$\mathcal{A} = \left(\frac{0 \mid I_{s-1}}{-1, -a_1, \ldots, -a_{s-1}} \right),$$

where I_{s-1} is the identity $(s-1) \times (s-1)$ matrix. We will use the following properties of $\mathcal{A}$. Note that $\det \mathcal{A} = (-1)^s$. Hence $\mathcal{A}$ is invertible and inverse matrix $\mathcal{A}^{-1}$ is also integer matrix. The characteristic polynomial of $\mathcal{A}$ coincides with $z^{-p}P(z)$.

Let $\mathbb{A} = \langle \alpha_j,\, j \in \mathbb{Z} \rangle$ be a free Abelian group freely generated by elements $\alpha_j,\, j \in \mathbb{Z}$. Each element of $\mathbb{A}$ is a linear combination $\sum_j c_j \alpha_j$ with integer coefficients c_j.

Define the shift operator $T\colon \mathbb{A} \to \mathbb{A}$ as a $\mathbb{Z}$-linear operator acting on generators of $\mathbb{A}$ by the rule $T\colon \alpha_j \to \alpha_{j+1}, j \in \mathbb{Z}$. Then T is an endomorphism of $\mathbb{A}$. Let $P(z)$ be an arbitrary Laurent polynomial with integer coefficients, then $A = P(T)$ is also an endomorphism of $\mathbb{A}$. Since A is a linear combination of powers of T, the action of A on generators α_j can be given by the infinite set of linear transformations $A\colon \alpha_j \to \sum_i a_{i,j}\alpha_i,\ j \in \mathbb{Z}$. Here all sums under consideration are finite. We set $\beta_j = \sum_i a_{i,j}\alpha_i$. Then $\operatorname{im} A$ is a subgroup of $\mathbb{A}$ generated by $\beta_j,\ j \in \mathbb{Z}$. Hence, $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$ is an abstract Abelian group $\langle x_i, i \in \mathbb{Z} \mid \sum_i a_{i,j}x_i = 0,\ j \in \mathbb{Z} \rangle$ generated by $x_i, i \in \mathbb{Z}$ with the set of defining relations $\sum_i a_{i,j}x_i = 0,\ j \in \mathbb{Z}$. Here x_j are images of α_j under the canonical homomorphism $\mathbb{A} \to \mathbb{A}/\operatorname{im} A$. Since T and $A = P(T)$ commute, subgroup $\operatorname{im} A$ is invariant under the action of T. Hence, the actions of T and A are well defined on the factor group $\mathbb{A}/\operatorname{im} A$ and are given by $T\colon x_j \to x_{j+1}$ and $A\colon x_j \to \sum_i a_{i,j}x_i$ respectively.

This allows to present the group $\mathbb{A}/\operatorname{im} A$ as follows $\langle x_i,\ i \in \mathbb{Z} \mid P(T)x_j = 0,\ j \in \mathbb{Z} \rangle$. In a similar way, given a set $P_1(z), P_2(z), \ldots, P_s(z)$ of Laurent polynomials with integer coefficients, one can define the group $\langle x_i,\ i \in \mathbb{Z} \mid P_1(T)x_j = 0, P_2(T)x_j = 0, \ldots, P_s(T)x_j = 0,\ j \in \mathbb{Z} \rangle$.

We will use the following lemma.

Lemma 3.1. *Let $T\colon \mathbb{A} \to \mathbb{A}$ be the shift operator. Consider endomorphisms A and B of the group $\mathbb{A}$ given by the formulas $A = P(T)$, $B = Q(T)$, where $P(z)$ and $Q(z)$ are Laurent polynomials with integer coefficients. Then $B\colon \mathbb{A} \to \mathbb{A}$ induces an endomorphism $B|_{\operatorname{coker} A}$ of the group $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$ defined by $B|_{\operatorname{coker} A}(\alpha + \operatorname{im} A) = B(\alpha) + \operatorname{im} A$, $\alpha \in \mathbb{A}$. Furthermore*

$$\langle x_i,\ i \in \mathbb{Z} \mid A(T)x_j = 0,\ B(T)x_j = 0,\ j \in \mathbb{Z} \rangle \cong \operatorname{coker} A/\operatorname{im}(B|_{\operatorname{coker} A}) \cong \operatorname{coker}(B|_{\operatorname{coker} A}).$$

Proof. The images $\operatorname{im} A$ and $\operatorname{im} B$ are subgroups in $\mathbb{A}$. Denote by $\langle \operatorname{im} A, \operatorname{im} B \rangle$ the subgroup generated by elements of $\operatorname{im} A$ and $\operatorname{im} B$. Since $P(z)$ and $Q(z)$ are Laurent polynomials, the operators $A = P(T)$ and $B = Q(T)$ do commute. Hence, subgroup $\operatorname{im} A$

is invariant under endomorphism B. Indeed for any $y = Ax \in \operatorname{im} A$, we have $By = B(Ax) = A(Bx) \in \operatorname{im} A$. This means that $B\colon \mathbb{A} \to \mathbb{A}$ induces an endomorphism of the group $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$. We denote this endomorphism by $B|_{\operatorname{coker} A}$. We note that the Abelian group $\langle x_i,\, i \in \mathbb{Z} \mid A(T)x_j = 0,\, B(T)x_j = 0,\, j \in \mathbb{Z}\rangle$ is naturally isomorphic to $\mathbb{A}/\langle \operatorname{im} A, \operatorname{im} B\rangle$. So we have

$$\mathbb{A}/\langle \operatorname{im} A, \operatorname{im} B\rangle \cong (\mathbb{A}/\operatorname{im} A)/\operatorname{im}(B|_{\operatorname{coker} A}) \cong \operatorname{coker} A/\operatorname{im}(B|_{\operatorname{coker} A}) \cong \operatorname{coker}(B|_{\operatorname{coker} A}).$$

The lemma is proved. □

4 Jacobian group for the I-graph $I(n,k,l)$

In this section we prove one of the main results of the paper. We start in the following theorem.

Theorem 4.1. *Let $L = L(I(n,k,l))$ be the Laplacian of a connected I-graph $I(n,k,l)$. Then*

$$\operatorname{coker} L \cong \operatorname{coker}(\mathcal{A}^n - I),$$

where $\mathcal{A}$ is $2(k+l) \times 2(k+l)$ companion matrix for the Laurent polynomial

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1.$$

Proof. Let L be the Laplacian matrix of the graph $I(n,k,l)$. Then, as it was mentioned above, L is a $2n \times 2n$ matrix of the form

$$L = \begin{pmatrix} 3I_n - C_n^k & -I_n \\ -I_n & 3I_n - C_n^l \end{pmatrix},$$

where $C_n^k = \operatorname{circ}(\underbrace{0,\ldots,0}_{k \text{ times}},1,0,\ldots,0,1,\underbrace{0,\ldots,0}_{k-1 \text{ times}})$.

Consider L as a $\mathbb{Z}-$linear operator $L\colon \mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$. In this case, $\operatorname{coker}(L)$ is an abstract Abelian group generated by elements $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ satisfying the system of linear equations $3x_j - x_{j-k} - x_{j+k} - y_j = 0, 3y_j - y_{j-l} - y_{j+l} - x_j = 0$ for any $j = 1,\ldots,n$. Here the indices are considered modulo n. By the property mentioned in Section 2, the Jacobian of the graph $I(n,k,l)$ is isomorphic to the finite part of cokernel of the operator L.

To study the structure of $\operatorname{coker}(L)$ we extend the list of generators to the two bi-infinite sequences of elements $(x_j)_{j\in\mathbb{Z}}$ and $(y_j)_{j\in\mathbb{Z}}$ setting $x_{j+mn} = x_j$ and $y_{j+mn} = y_j$ for any $m \in \mathbb{Z}$. Then we have the following representation for cokernel of L:

$$\operatorname{coker}(L) = \langle x_i, y_i, i \in \mathbb{Z} \mid 3x_j - x_{j+k} - x_{j-k} - y_j = 0, \\ 3y_j - y_{j+l} - y_{j-l} - x_j = 0, x_{j+n} = x_j, y_{j+n} = y_j, j \in \mathbb{Z}\rangle.$$

Let T be the shift operator defined by the rule $T\colon x_j \to x_{j+1},\, y_j \to y_{j+1},\, j \in \mathbb{Z}$. Consider the operator $P(T)$ defined by $P(T) = (3 - T^k - T^{-k})(3 - T^l - T^{-l}) - 1$. We use the operator notation from Section 3 to represent the cokernel of L. Then we have

$$\begin{aligned}\operatorname{coker}(L) &= \langle x_i,\, y_i, i \in \mathbb{Z} \mid (3 - T^k - T^{-k})x_j = y_j, (3 - T^l - T^{-l})y_j = x_j,\\ &\qquad T^n x_j = x_j, T^n y_j = y_j, j \in \mathbb{Z}\rangle\\ &= \langle x_i,\, i \in \mathbb{Z} \mid (3 - T^l - T^{-l})(3 - T^k - T^{-k})x_j = x_j, T^n x_j = x_j,\, j \in \mathbb{Z}\rangle\\ &= \langle x_i,\, i \in \mathbb{Z} \mid ((3 - T^k - T^{-k})(3 - T^l - T^{-l}) - 1)x_j = 0,\\ &\qquad (T^n - 1)x_j = 0,\, j \in \mathbb{Z}\rangle\\ &= \langle x_i,\, i \in \mathbb{Z} \mid P(T)x_j = 0, (T^n - 1)x_j = 0, j \in \mathbb{Z}\rangle.\end{aligned}$$

To finish the proof, we apply Lemma 3.1 to the operators $A = P(T)$ and $B = Q(T) = T^n - 1$.

Since the Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ is bimonic, it can be represented in the form $P(z) = z^{-k-l} + a_1 z^{-k-l+1} + \cdots + a_{2k+2l-1} z^{k+l-1} + z^{k+l}$, where $a_1, a_2, \ldots, a_{2k+2l-1}$ are integers. Then the corresponding companion matrix $\mathcal{A}$ is

$$\left(\begin{array}{c|c} 0 & I_{2k+2l-1} \\ \hline \multicolumn{2}{c}{-1, -a_1, \ldots, -a_{2k+2l-1}} \end{array}\right).$$

It is easy to see that $\det \mathcal{A} = 1$ and its inverse $\mathcal{A}^{-1}$ is also integer matrix.

For convenience we set $s = 2k + 2l$ to be the size of matrix $\mathcal{A}$.

Note that for any $j \in \mathbb{Z}$ the relations $P(T)x_j = 0$ can be rewritten as $x_{j+s} = -x_j - a_1 x_{j+1} - \cdots - a_{s-1} x_{j+s-1}$. Let $\mathbf{x}_j = (x_{j+1}, x_{j+2}, \ldots, x_{j+s})^t$ be s-tuple of generators $x_{j+1}, x_{j+2}, \ldots, x_{j+s}$. Then the relation $P(T)x_j = 0$ is equivalent to $\mathbf{x}_j = \mathcal{A}\,\mathbf{x}_{j-1}$. Hence, we have $\mathbf{x}_1 = \mathcal{A}\,\mathbf{x}_0$ and $\mathbf{x}_{-1} = \mathcal{A}^{-1}\,\mathbf{x}_0$, where $\mathbf{x}_0 = (x_1, x_2, \ldots, x_s)^t$. So, $\mathbf{x}_j = \mathcal{A}^j\,\mathbf{x}_0$ for any $j \in \mathbb{Z}$. Conversely, the latter implies $\mathbf{x}_j = \mathcal{A}\,\mathbf{x}_{j-1}$ and, as a consequence, $P(T)x_j = 0$ for all $j \in \mathbb{Z}$.

Consider $\operatorname{coker} A = \mathbb{A}/\operatorname{im} A$ as an abstract Abelian group with the following representation $\langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z}\rangle$.

Our present aim is to show that $\operatorname{coker} A \cong \mathbb{Z}^s$. We have

$$\begin{aligned}\operatorname{coker} A &= \langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z}\rangle\\ &= \langle x_j, j \in \mathbb{Z} \mid x_\ell + a_1 x_{\ell+1} + \cdots + a_{s-1} x_{\ell+s-1} + x_{\ell+s} = 0, \ell \in \mathbb{Z}\rangle\\ &= \langle x_j, j \in \mathbb{Z} \mid (x_{\ell+1}, x_{\ell+2}, \ldots, x_{\ell+s})^t = \mathcal{A}(x_\ell, x_{\ell+1}, \ldots, x_{\ell+s-1})^t, \ell \in \mathbb{Z}\rangle\\ &= \langle x_j, j \in \mathbb{Z} \mid (x_{\ell+1}, x_{\ell+2}, \ldots, x_{\ell+s})^t = \mathcal{A}^\ell (x_1, x_2, \ldots, x_s)^t, \ell \in \mathbb{Z}\rangle\\ &= \langle x_1, x_2, \ldots, x_s \mid \emptyset\rangle \cong \mathbb{Z}^s.\end{aligned}$$

Now we describe the action of the endomorphism $B|_{\operatorname{coker} A}$ on the $\operatorname{coker} A$. Since the operators $A = P(T)$ and T commute, the action $T|_{\operatorname{coker} A} \colon x_j \to x_{j+1}$, $j \in \mathbb{Z}$ on the $\operatorname{coker} A$ is well defined. First of all, we describe the action of $T|_{\operatorname{coker} A}$ on the set of generators $x_1, x_2, \ldots, x_s$. For any $i = 1, \ldots, s - 1$, we have $T|_{\operatorname{coker}}(x_i) = x_{i+1}$ and $T|_{\operatorname{coker} A}(x_s) = x_{s+1} = -x_1 - a_1 x_2 - \cdots - a_{s-2} x_{s-1} - a_{s-1} x_s$. Hence, the action of $T|_{\operatorname{coker} A}$ on the $\operatorname{coker} A$ is given by the matrix $\mathcal{A}$. Considering $\mathcal{A}$ as an endomorphism of the $\operatorname{coker} A$, we can write $T|_{\operatorname{coker} A} = \mathcal{A}$. Finally, $B|_{\operatorname{coker} A} = Q(T|_{\operatorname{coker} A}) = Q(\mathcal{A})$. Applying Lemma 3.1, we finish the proof of the theorem. □

Corollary 4.2. *The Jacobian group* $\operatorname{Jac}(I(n, k, l))$ *of a connected* I*-graph* $I(n, k, l)$ *is isomorphic to the torsion subgroup of* $\operatorname{coker}(\mathcal{A}^n - I)$*, where* $\mathcal{A}$ *is the companion matrix for the Laurent polynomial* $(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.

The Corollary 4.2 gives a simple way to find Jacobian group $\mathrm{Jac}(I(n,k,l))$ for small values of k, l and sufficiently large numbers n. The numerical results are given in the Tables 2 and 3.

5 Counting the number of spanning trees for the I-graph $I(n,k,l)$

In what follows, we always assume that the numbers k and l are relatively prime. To get the result for an arbitrary connected I-graph $I(n,k,l)$ with $\gcd(n,k,l)=1$ and $\gcd(k,l)=d>1$ we observe that $I(n,k,l)$ is isomorphic to $I(n,k',l')$, where the numbers $k'=k/d$ and $l'=l/d$ are relatively prime.

Theorem 5.1. *The number of spanning trees of the I-graph $I(n,k,l)$ is given by the formula*

$$\tau_{k,l}(n)=(-1)^{(n-1)(k+l)}n\prod_{s=1}^{k+l-1}\frac{T_n(w_s)-1}{w_s-1},$$

where w_s, $s=1,2,\ldots,k+l-1$ are roots of the order $k+l-1$ algebraic equation

$$\frac{(3-2T_k(w))(3-2T_l(w))-1}{w-1}=0,$$

and $T_j(w)$ is the Chebyshev polynomial of the first kind.

Proof. By the celebrated Kirchhoff theorem, the number of spanning trees $\tau_{k,l}(n)$ is equal to the product of nonzero eigenvalues of the Laplacian of a graph $I(n,k,l)$ divided by the number of its vertices $2n$. To investigate the spectrum of Laplacian matrix we note that matrix $C_n^k=T^k+T^{-k}$, where $T=\mathrm{circ}(0,1,\ldots,0)$ is the $n\times n$ shift operator. The latter equality easily follows from the identity $T^n=I_n$. Hence,

$$L=\begin{pmatrix}3I_n-T^k-T^{-k} & -I_n\\ -I_n & 3I_n-T^l-T^{-l}\end{pmatrix}.$$

The eigenvalues of circulant matrix T are ε_n^j, where $\varepsilon_n=e^{\frac{2\pi i}{n}}$. Since all eigenvalues of T are distinct, the matrix T is conjugate to the diagonal matrix $\mathbb{T}=\mathrm{diag}(1,\varepsilon_n,\ldots,\varepsilon_n^{n-1})$, where diagonal entries of $\mathrm{diag}(1,\varepsilon_n,\ldots,\varepsilon_n^{n-1})$ are $1,\varepsilon_n,\ldots,\varepsilon_n^{n-1}$. To find spectrum of L, without loss of generality, one can assume that $T=\mathbb{T}$. Then the blocks of L are diagonal matrices. This essentially simplifies the problem of finding eigenvalues of L. Indeed, let λ be an eigenvalue of L and $(x,y)=(x_1,\ldots,x_n,y_1,\ldots,y_n)$ be the corresponding eigenvector. Then we have the following system of equations

$$\begin{cases}(3I_n-T^k-T^{-k})x-y=\lambda x\\ -x+(3I_n-T^l-T^{-l})y=\lambda y\end{cases}.$$

From here we conclude that $y=(3I_n-T^k-T^{-k})x-\lambda x=((3-\lambda)I_n-T^k-T^{-k})x$. Substituting y in the second equation, we have $(((3-\lambda)I_n-T^l-T^{-l})((3-\lambda)I_n-T^k-T^{-k})-1)x=0$.

Recall the matrices under consideration are diagonal and the $(j+1,j+1)$-th entry of T is equal to ε_n^j. Therefore, we have $((3-\lambda-\varepsilon_n^{jk}-\varepsilon_n^{-jk})(3-\lambda-\varepsilon_n^{jl}-\varepsilon_n^{-jl})-1)x_{j+1}=0$ and $y_{j+1}=(3-\lambda-\varepsilon_n^{jl}-\varepsilon_n^{-jl})x_{j+1}$.

So, for any $j = 0, \dots, n-1$ the matrix L has two eigenvalues, say $\lambda_{1,j}$ and $\lambda_{2,j}$ satisfying the quadratic equation $(3 - \lambda - \varepsilon_n^{jk} - \varepsilon_n^{-jk})(3 - \lambda - \varepsilon_n^{jl} - \varepsilon_n^{-jl}) - 1 = 0$. The corresponding eigenvectors are (x, y), where

$$x = \mathbf{e}_{j+1} = (0, \dots, \underbrace{1}_{(j+1)\text{-th}}, \dots, 0) \text{ and}$$
$$y = (3 - \lambda - T^k - T^{-k})\mathbf{e}_{j+1}.$$

In particular, if $j = 0$ for $\lambda_{1,0}, \lambda_{2,0}$ we have $(1-\lambda)(1-\lambda) - 1 = \lambda(\lambda - 2) = 0$. That is, $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Since $\lambda_{1,j}$ and $\lambda_{2,j}$ are roots of the same quadratic equation, we obtain $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j)$, where $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$.

Now we have

$$\tau_{k,l}(n) = \frac{1}{2n}\lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = \frac{1}{n}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = \frac{1}{n}\prod_{j=1}^{n-1}P(\varepsilon_n^j).$$

To continue we need the following lemma.

Lemma 5.2. *The following identity holds*

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1 = (3 - 2T_k(w))(3 - 2T_l(w)) - 1,$$

where $T_k(w)$ is the Chebyshev polynomial of the first kind and $w = \frac{1}{2}(z + z^{-1})$. Moreover, if k and l are relatively prime then all roots of the Laurent polynomial

$$(3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$$

counted with multiplicities are $1, 1, z_1, 1/z_1, \dots, z_{k+l-1}, 1/z_{k+l-1}$, where we have $|z_s| \neq 1$, $s = 1, 2, \dots, k+l-1$. So, the right-hand polynomial has the roots $1, w_1, \dots, w_{k+l-1}$, where $w_s \neq 1$ for all $s = 1, 2, \dots, k+l-1$.

Proof. Let us substitute $z = e^{i\varphi}$. It is easy to see that $w = \frac{1}{2}(z + z^{-1}) = \cos\varphi$, so we have $T_k(w) = \cos(k \arccos w) = \cos(k\varphi)$. Then the first statement of the lemma is equivalent to the following trigonometric identity

$$(3 - 2\cos(k\varphi))(3 - 2\cos(l\varphi)) - 1 = (3 - 2T_k(w))(3 - 2T_l(w)) - 1.$$

To prove the second statement of the lemma we suppose that the Laurent polynomial $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$ has a root z_0 such that $|z_0| = 1$. Then $z_0 = e^{i\varphi_0}$, $\varphi_0 \in \mathbb{R}$. Now we have $(3 - 2\cos(k\varphi_0))(3 - 2\cos(l\varphi_0)) - 1 = 0$. Since $3 - 2\cos(k\varphi_0) \geq 1$ and $3 - 2\cos(l\varphi_0) \geq 1$ the equations holds if and only if $\cos(k\varphi_0) = 1$ and $\cos(l\varphi_0) = 1$. So $k\varphi_0 = 2\pi s_0$ and $l\varphi_0 = 2\pi t_0$ for some integer s_0 and t_0. As k and l are relatively prime, so there exist two integers p and q such that $kp + ql = 1$. Hence $\varphi_0 = \varphi_0(kp + lq) = 2\pi(ps_0 + qt_0) \in 2\pi\mathbb{Z}$. As a result $z_0 = e^{i\varphi_0} = 1$. Now we have to show that the multiplicity of the root $z_0 = 1$ is 2. Indeed, $P(1) = P'(1) = 0$ and $P''(1) = -2(k^2 + l^2) \neq 0$. □

Let us set $H(z) = \prod_{s=1}^{m}(z - z_s)(z - z_s^{-1})$, where $m = k + l - 1$ and z_s are roots of $P(z)$ different from 1. Then by Lemma 5.2, we have $P(z) = \frac{(z-1)^2}{z^{k+l}}H(z)$.

Lemma 5.3. *Let $H(z)=\prod_{s=1}^{m}(z-z_s)(z-z_s^{-1})$ and $H(1)\neq 0$. Then*

$$\prod_{j=1}^{n-1}H(\varepsilon_n^j)=\prod_{s=1}^{m}\frac{T_n(w_s)-1}{w_s-1},$$

where $w_s=\frac{1}{2}(z_s+z_s^{-1})$, $s=1,\ldots,m$ and $T_n(x)$ is the Chebyshev polynomial of the first kind.

Proof. It is easy to check that $\prod_{j=1}^{n-1}(z-\varepsilon_n^j)=\frac{z^n-1}{z-1}$ if $z\neq 1$. Also we note that $\frac{1}{2}(z^n+z^{-n})=T_n(\frac{1}{2}(z+z^{-1}))$. By the substitution $z=e^{i\,\varphi}$, the latter follows from the evident identity $\cos(n\varphi)=T_n(\cos\varphi)$. Then we have

$$\begin{aligned}\prod_{j=1}^{n-1}H(\varepsilon_n^j)&=\prod_{j=1}^{n-1}\prod_{s=1}^{m}(\varepsilon_n^j-z_s)(\varepsilon_n^j-z_s^{-1})\\&=\prod_{s=1}^{m}\prod_{j=1}^{n-1}(z_s-\varepsilon_n^j)(z_s^{-1}-\varepsilon_n^j)\\&=\prod_{s=1}^{m}\frac{z_s^n-1}{z_s-1}\,\frac{z_s^{-n}-1}{z_s^{-1}-1}=\prod_{s=1}^{m}\frac{T_n(w_s)-1}{w_s-1}.\end{aligned}$$

□

Note that $\prod_{j=1}^{n-1}(1-\varepsilon_n^j)=\lim\limits_{z\to 1}\prod_{j=1}^{n-1}(z-\varepsilon_n^j)=\lim\limits_{z\to 1}\frac{z^n-1}{z-1}=n$ and $\prod_{j=1}^{n-1}\varepsilon_n^j=(-1)^{n-1}$. As a result, taking into account Lemma 5.2 and Lemma 5.3, we obtain

$$\begin{aligned}\tau_{k,l}(n)&=\frac{1}{n}\prod_{j=1}^{n-1}P(\varepsilon_n^j)=\frac{1}{n}\prod_{j=1}^{n-1}\frac{(\varepsilon_n^j-1)^2}{(\varepsilon_n^j)^{k+l}}H(\varepsilon_n^j)\\&=\frac{(-1)^{(n-1)(k+l)}n^2}{n}\prod_{j=1}^{n-1}H(\varepsilon_n^j)\\&=(-1)^{(n-1)(k+l)}n\prod_{s=1}^{k+l-1}\frac{T_n(w_s)-1}{w_s-1}.\end{aligned}$$

□

Corollary 5.4. $\tau_{k,l}(n)=n\left|\prod_{s=1}^{k+l-1}U_{n-1}\left(\sqrt{\frac{1+w_s}{2}}\right)\right|^2$, *where w_s, $s=1,2,\ldots,k$ are the same as in Theorem 5.1 and $U_{n-1}(w)$ is the Chebyshev polynomial of the second kind.*

Proof. Follows from the identity $\frac{T_n(w)-1}{w-1}=U_{n-1}^2\left(\sqrt{\frac{1+w}{2}}\right)$. □

The following theorem appeared after fruitful discussion with professor D. Lorenzini.

Theorem 5.5. *Let $\tau(n)=\tau_{k,l}(n)$ be the number of spanning trees of the graph $I(n,k,l)$. Then there exist an integer sequence $a(n)=a_{k,l}(n)$, $n\in\mathbb{N}$ such that*

1° $\tau(n)=n\,a^2(n)$ *when n is odd,*

2° $\tau(n)=6n\,a^2(n)$ *when n is even and $k+l$ is even,*

3° $\tau(n)=n\,a^2(n)$ *when n is even and $k+l$ is odd.*

Proof. Recall that all nonzero eigenvalues are given by the list $\{\lambda_{2,0},\, \lambda_{1,j}, \lambda_{2,j},\, j = 1,\ldots,n-1\}$. By the Kirchhoff theorem we have $2n\tau(n) = \lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$.

Since $\lambda_{2,0} = 2$, we have $n\tau(n) = \prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$. We note that $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j) = P(\varepsilon_n^{n-j}) = \lambda_{1,n-j}\lambda_{2,n-j}$. So, we get $n\tau(n) = (\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j})^2$ if n is odd and $n\tau(n) = \lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}(\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$, if n is even. The value $\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}} = P(-1) = (3-2(-1)^k)(3-2(-1)^l)-1$ is equal to 4 if k and l are of different parity and 24 if both k and l are odd. The case when both k and l are even is impossible, since k and l are relatively prime.

The graph $I(n,k,l)$ admits a cyclic group of automorphisms isomorphic to $\mathbb{Z}_n$ which acts freely on the set of spanning trees. Therefore, the value $\tau(n)$ is a multiple of n. So $\frac{\tau(n)}{n}$ is an integer. Hence

1° $\frac{\tau(n)}{n} = \left(\frac{\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}}{n}\right)^2$ when n is odd,

2° $\frac{\tau(n)}{n} = 6\left(\frac{2\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}}{n}\right)^2$ when n is even and $k+l$ is even,

3° $\frac{\tau(n)}{n} = \left(\frac{2\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}}{n}\right)^2$ when n is even and $k+l$ is odd.

Each algebraic number $\lambda_{i,j}$ comes into both products $\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}$ and $\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}$ with all its Galois conjugate elements. Therefore, both products are integer numbers. From here we conclude that in equalities 1°, 2° and 3° the value that is squared is a rational number. Because $\frac{\tau(n)}{n}$ is integer and 6 is a squarefree, all these rational numbers are integer. Setting $a(n) = \frac{\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}}{n}$ if n is odd and $a(n) = \frac{2\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}}{n}$ if n is even, we finish the proof of the theorem. □

From now on, we aim to estimate the minimum number of generators for the Jacobian of I-graph $I(n,k,l)$.

Lemma 5.6. *For any given I-graph $I(n,k,l)$ the number of spanning trees $\tau(n)$ satisfies the inequality $\tau(n) \geq n^3$.*

Proof. Recall that for any $j = 0,\ldots,n-1$, the Laplacian matrix L of $I(n,k,l)$ has two eigenvalues, say $\lambda_{1,j}$ and $\lambda_{2,j}$, which are roots of the quadratic equation $Q_j(\lambda) = (3-\lambda-\varepsilon_n^{jk}-\varepsilon_n^{-jk})(3-\lambda-\varepsilon_n^{jl}-\varepsilon_n^{-jl})-1 = 0$. So, $\lambda_{1,j}\lambda_{2,j} = (3-\varepsilon_n^{jk}-\varepsilon_n^{-jk})(3-\varepsilon_n^{jl}-\varepsilon_n^{-jl})-1 = P(\varepsilon_n^j)$. Note that $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Furthermore $\{\lambda_{1,j}, \lambda_{2,j} \mid j = 0,\ldots,n-1\}$ is the set of all eigenvalues of L. The Kirchhoff theorem states the following

$$2n\,\tau_{k,l}(n) = 2n\,\tau(n) = \lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = 2\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}.$$

Hence $n\tau(n) = \prod_{j=1}^{n-1}P(\varepsilon_n^j)$, where $P(\varepsilon_n^j) = (3-2\cos(\frac{2jk\pi}{n}))(3-2\cos(\frac{2jl\pi}{n}))-1$.

It is easy to prove the following trigonometric identity

$$\Big(3-2\cos\Big(\frac{2jk\pi}{n}\Big)\Big)\Big(3-2\cos\Big(\frac{2jl\pi}{n}\Big)\Big)-1=$$
$$4\sin^2\Big(\frac{jk\pi}{n}\Big)+4\sin^2\Big(\frac{jl\pi}{n}\Big)+16\sin^2\Big(\frac{jk\pi}{n}\Big)\sin^2\Big(\frac{jl\pi}{n}\Big).$$

Connectedness of I-graph implies $\gcd(n,k,l)=1$. It may happen that $\gcd(n,k)=m\neq 1$ and $\gcd(n,l)=m'\neq 1$. We will use the notation $n=m\,q=m'q'$, $k=p\,m$, $l=p'm'$. We introduce three sets, J, J_k and J_l in the following way

$$J=\{1,2,\dots,n-1\},$$
$$J_k=\{j\mid j=d\,q,\ d=1,\dots,m-1\}\text{ and}$$
$$J_l=\{j\mid j=d'\,q',\ d'=1,\dots,m'-1\}.$$

If $j\in J_k$ then $\sin(\frac{j\,k\,\pi}{n})=0$ and if $j\in J_l$ then $\sin(\frac{j\,l\,\pi}{n})=0$. We note that J_k and J_l do not intersect. Otherwise, for $j\in J_k\cap J_l$ we have $\lambda_{1,j}\lambda_{2,j}=P(\varepsilon_n^j)=0$. Then at least one of the eigenvalues $\lambda_{1,j}$ and $\lambda_{2,j}$ is equal to zero. This leads to contradiction, as we have the unique zero eigenvalue $\lambda_{1,0}=0$. Now we are going to find a low bound for $\tau(n)$. As $n\,\tau(n)=\prod_{j=1}^{n-1}P(\varepsilon_n^j)$ we evaluate the product

$$\prod_{j=1}^{n-1}P(\varepsilon_n^j)=\prod_{j=1}^{n-1}\Big(4\sin^2\Big(\frac{jk\pi}{n}\Big)+4\sin^2\Big(\frac{jl\pi}{n}\Big)+16\sin^2\Big(\frac{jk\pi}{n}\Big)\sin^2\Big(\frac{jl\pi}{n}\Big)\Big)$$
$$\geq\prod_{j\in J_k}4\sin^2\Big(\frac{jl\pi}{n}\Big)\prod_{j\in J_l}4\sin^2\Big(\frac{jk\pi}{n}\Big)\prod_{j\in J\setminus(J_k\cup J_l)}16\sin^2\Big(\frac{jk\pi}{n}\Big)\sin^2\Big(\frac{jl\pi}{n}\Big)$$
$$=\prod_{j\in J\setminus J_k}4\sin^2\Big(\frac{jk\pi}{n}\Big)\prod_{j\in J\setminus J_l}4\sin^2\Big(\frac{jl\pi}{n}\Big).$$

Now we analyze individual component of the product. We make use of the following simple identity $\cos(\frac{2jp\pi}{q})=\cos(\frac{2(j+q)p\pi}{q})$.

$$\prod_{j\in J\setminus J_k}4\sin^2\Big(\frac{jk\pi}{n}\Big)=\prod_{j\in J\setminus J_k}\Big(2-2\cos\Big(\frac{2jk\pi}{n}\Big)\Big)=\prod_{j\in J\setminus J_k}\Big(2-2\cos\Big(\frac{2jmp\pi}{mq}\Big)\Big)$$
$$=\prod_{j\in J\setminus J_k}\Big(2-2\cos\Big(\frac{2jp\pi}{q}\Big)\Big)=\prod_{j=1}^{q-1}\Big(2-2\cos\Big(\frac{2jp\pi}{q}\Big)\Big)^m.$$

The Chebyshev polynomial $T_q(x)=\cos(q\arccos(x))$ has the following property. The roots of the equation $T_q(x)-1=0$ are $\cos(\frac{2j\pi}{q})$, $j=0,1,\dots,q-1$. Since the leading coefficient of $T_q(x)$ is 2^{q-1}, for $x\neq 1$ we have the identity

$$\prod_{j=1}^{q-1}\Big(2x-2\cos\Big(\frac{2j\pi}{q}\Big)\Big)=\frac{T_q(x)-1}{x-1}.$$

As p and q are relatively prime we obtain

$$\prod_{j=1}^{q-1}\Big(2-2\cos\Big(\frac{2jp\pi}{q}\Big)\Big)^m=\prod_{j=1}^{q-1}\Big(2-2\cos\Big(\frac{2j\pi}{q}\Big)\Big)^m=$$
$$\Big(\lim_{x\to 1}\frac{T_q(x)-1}{x-1}\Big)^m=(q^2)^m=\Big(\frac{n}{m}\Big)^{2m}.$$

Hence

$$\prod_{j\in J\setminus J_k}4\sin^2\Big(\frac{jk\pi}{n}\Big)=\Big(\frac{n}{m}\Big)^{2m}.$$

In a similar way we obtain

$$\prod_{j\in J\setminus J_l}4\sin^2\Big(\frac{jl\pi}{n}\Big)=\Big(\frac{n}{m'}\Big)^{2m'}.$$

To get the final result we use the following trivial inequality. For any integers $a\geq 2$ and $b\geq 2$ we have $a^b\geq ab$. Since $q=n/m\geq 2$ and $q'=n/m'\geq 2$, we conclude

$$n\,\tau(n)=\prod_{j=1}^{n-1}P(\varepsilon_n^j)\geq\Big(\frac{n}{m}\Big)^{2m}\Big(\frac{n}{m'}\Big)^{2m'}\geq n^2n^2=n^4. \qquad\square$$

Using Lemma 5.6, one can show the following theorem.

Theorem 5.7. *For any given I-graph $I(n,k,l)$ the minimum number of generators for Jacobian* $\operatorname{Jac}(I(n,k,l))$ *is at least* 2 *and at most* $2k+2l-1$.

Proof. The upper bound for the number of generators follows from Theorem 4.1. Indeed, by this theorem the group $\operatorname{coker}(L(I(n,k,l))\cong\operatorname{Jac}(I(n,k,l))\oplus\mathbb{Z}$ is generated by $2k+2l$ elements. One of these generators is needed to generate the infinite cyclic group $\mathbb{Z}$. Hence $\operatorname{Jac}(I(n,k,l))$ is generated by $2k+2l-1$ elements.

To get the lower bound we use Lemma 5.6. Let us suppose that $\operatorname{Jac}(I(n,k,l))$ is generated by one element. Then it is the cyclic group of order $\tau(n)$. Denote by D be a product of all distinct nonzero eigenvalues of $I(n,k,l)$. By Proposition 2.6 from [20] the order of each element of $\operatorname{Jac}(I(n,k,l))$ is divisor of D. Hence, $\tau(n)$ is divisor of D and we have inequality $D\geq\tau(n)$. By the Kirchhoff theorem we have $2n\tau(n)=\lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$. We note that all algebraic numbers $\lambda_{i,j}$ comes into product together with its Galois conjugate, so $2n\tau(n)$ is a multiple of D. In particular $2n\tau(n)\geq D$.

From the proof of Theorem 5.5 we have $n\tau(n)=(\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j})^2$ if n is odd and $n\tau(n)=\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}(\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$ if n is even. Moreover, the value $\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}$ is equal to 4 if k and l are of different parity and 24 if both k and l are odd. The case when both k and l are even is impossible as k and l are relatively prime.

Now, we have $4n\tau(n)=(2\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j})^2$ if n is odd. Again, all algebraic numbers $\lambda_{i,j}$ comes into the product $\rho=2\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}$ together with its Galois conjugate. Therefore, the product ρ is an integer number and contains all distinct nonzero eigenvalues. Hence ρ is a multiple of D. So we obtain $4n\tau(n)=\rho^2\geq D^2\geq\tau(n)^2$.

Also we get $4n\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\tau(n) = (2\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j})^2$ if n is even. By a similar argument, taking into account the inequality $24 \geq \lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}$ we obtain $96n\tau(n) \geq 4n\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\tau(n) \geq D^2 \geq \tau(n)^2$.

As result, by Lemma 5.6 we have $4n \geq \tau(n) \geq n^3$ if n is odd and $96n \geq \tau(n) \geq n^3$ if n is even. For $n \geq 10$ this is impossible. So, the rank of $\mathrm{Jac}(I(n,k,l))$ is at least two for all $n \geq 10$. For n less than 10 this statement can be proved by direct calculation. □

For graphs $I(4,2,3)$ and $I(6,3,4)$, the Jacobian group $\mathrm{Jac}(I(n,k,l))$ is generated by 2 elements. The upper bound $2k+2l-1$ for the minimum number of generators of $\mathrm{Jac}(I(n,k,l))$ is attained for graph $I(34,2,3)$ and $I(170,3,4)$. See Tables 2 and 3 in Section 7.

So the lower bound 2 and the upper bound $2k+2l-1$ for the minimum number of generators of $\mathrm{Jac}(I(n,k,l))$ are sharp.

6 Asymptotic for the number of spanning trees

The asymptotic for the number of spanning trees of the graph $I(n,k,l)$ is given in the following theorem.

Theorem 6.1. *Let $P(z) = (3 - z^k - z^{-k})(3 - z^l - z^{-l}) - 1$. Suppose that k and l are relatively prime and set $A_{k,l} = \prod_{P(z)=0,\,|z|>1}|z|$. Then the number $\tau_{k,l}(n)$ of spanning trees of the graph $I(n,k,l)$ has the asymptotic*

$$\tau_{k,l}(n) \sim \frac{n}{k^2+l^2}A_{k,l}^n,\ n \to \infty.$$

Proof. By Theorem 5.1 we have

$$\tau_{k,l}(n) = (-1)^{(n-1)(k+l)}n\prod_{s=1}^{k+l-1}\frac{T_n(w_s)-1}{w_s-1},$$

where w_s, $s = 1,2,\ldots,k+l-1$ are roots of the polynomial

$$Q(w) = \frac{(3-2T_k(w))(3-2T_l(w))-1}{w-1}.$$

So we obtain

$$\tau_{k,l}(n) = n\prod_{s=1}^{k+l-1}\left|\frac{T_n(w_s)-1}{w_s-1}\right| = n\prod_{s=1}^{k+l-1}|T_n(w_s)-1|\Big/\prod_{s=1}^{k+l-1}|w_s-1|.$$

By Lemma 5.2 we have $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$, where the z_s and $1/z_s$ are roots of the polynomial $P(z)$ with the property $|z_s| \neq 1$, $s = 1,2,\ldots,k+l-1$. Replacing z_s by $1/z_s$, if it is necessary, we can assume that all $|z_s| > 1$ for all $s = 1,2,\ldots,k+l-1$. Then $T_n(w_s) \sim \frac{1}{2}z_s^n$ as n tends to ∞. So $|T_n(w_s)-1| \sim \frac{1}{2}|z_s|^n$ as $n \to \infty$. Hence

$$\prod_{s=1}^{k+l-1}|T_n(w_s)-1| \sim \frac{1}{2^{k+l-1}}\prod_{s=1}^{k+l-1}|z_s|^n = \frac{1}{2^{k+l-1}}\prod_{P(z)=0,\,|z|>1}|z|^n = \frac{1}{2^{k+l-1}}A_{k,l}^n.$$

Now we directly evaluate the quantity $\prod_{s=1}^{k+l-1}|w_s-1|$. We note that

$$Q(w)=a_0w^{k+l-1}+a_1w^{k+l-2}+\cdots+a_{k+l-2}w+a_{k+l-1}$$

is an integer polynomial with the leading coefficient $a_0=2^{k+l}$. From here we obtain

$$\prod_{s=1}^{k+l-1}|w_s-1|=\prod_{s=1}^{k+l-1}|1-w_s|=\left|\frac{1}{a_0}Q(1)\right|=\frac{2(k^2+l^2)}{2^{k+l}}=\frac{k^2+l^2}{2^{k+l-1}}.$$

Indeed,

$$\begin{aligned}Q(1)&=\lim_{w\to1}\frac{(3-2T_k(w))(3-2T_l(w))-1}{w-1}\\&=-2T_k'(1)(3-2T_l(1))-2T_l'(1)(3-2T_k(1))\\&=-2kU_{k-1}(1)(3-2T_l(1))-2lU_{l-1}(1)(3-2T_k(1))=-2(k^2+l^2)\end{aligned}$$

and $a_0=2^{k+l}$.

In order to get the statement of the theorem we combine the above mentioned results. Then

$$\tau_{k,l}(n)\sim n\,\frac{A_{k,l}^n}{2^{k+l-1}}\Big/\frac{k^2+l^2}{2^{k+l-1}}=\frac{n}{k^2+l^2}A_{k,l}^n\text{ as }n\to\infty.\qquad\square$$

Remark 6.2. It was noted by professor A. Yu. Vesnin that constant $A_{k,l}$ coincides with the Mahler measure of Laurent polynomial $P(z)=(3-z^k-z^{-k})(3-z^l-z^{-l})-1$. It gives a simple way to evaluate $A_{k,l}$ using the following formula

$$A_{k,l}=\exp\left(\int_0^1\log|P(e^{2\pi it})|\mathrm{d}t\right).$$

See, for example, [13, p. 6] for the proof.

The numerical values for $A_{k,l}$, where k and l are relatively prime numbers $1\le k\le l\le 9$ will be given in Table 1 in the Section 7.

7 Examples and tables

7.1 Examples

1° The Prism graph $I(n,1,1)$. We have the following asymptotic

$$\tau_{1,1}(n)=n(T_n(2)-1)\sim\frac{n}{2}(2+\sqrt{3})^n,\ n\to\infty.$$

2° The generalized Petersen graph $GP(n,2)=I(n,1,2)$. The the number of spanning trees (see [19]) behaves like $\tau_{1,2}(n)\sim\frac{n}{5}A_{1,2}^n,\ n\to\infty$, where

$$A_{1,2}=\frac{7+\sqrt{5}+\sqrt{38+14\sqrt{5}}}{4}\cong 4.39026.$$

3° The smallest proper I-graph $I(n,2,3)$ has the following asymptotic for the number of spanning trees

$$\tau_{2,3}(n) \sim \frac{n}{13} A_{2,3}^n,\ n \to \infty.$$

Here $A_{2,3} \cong 4.84199$ is a suitable root of the algebraic equation

$$1 - 7x + 13x^2 - 35x^3 + 161x^4 - 287x^5 + 241x^6 - 371x^7 + 577x^8 - 371x^9 + 241x^{10} - 287x^{11} + 161x^{12} - 35x^{13} + 13x^{14} - 7x^{15} + x^{16} = 0.$$

Here is the table for asymptotic constants $A_{k,l}$ for relatively prime numbers $1 \le k \le l \le 9$.

Table 1: Asymptotic constants $A_{k,l}$.

$k\backslash l$	1	2	3	4	5	6	7	8	9
1	3.7320	4.3902	4.7201	4.8954	4.9953	5.0559	5.0945	5.1203	5.1382
2		-	4.8419	-	5.0249	-	5.1033	-	5.1414
3			-	5.0054	5.0541	-	5.1137	5.1320	-
4				-	5.0802	-	5.1244	-	5.1504
5					-	5.1201	5.1346	5.1461	5.1554
6						-	5.1438	-	-
7							-	5.1589	5.1649
8								-	5.1691

7.2 The tables of Jacobians of I-graphs

Theorem 4.1 is the first step to understand the structure of the Jacobian for $I(n,k,l)$. Also, it gives a simple way for numerical calculations of $\mathrm{Jac}(I(n,k,l))$ for small values of k and l. See Tables 2 and 3.

The first example of Jacobian $\mathrm{Jac}(I(n,3,4))$ with the maximum rank 13:

$$n = 170,$$
$$\mathrm{Jac}(I(170,3,4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4^8 \oplus \mathbb{Z}_{6108} \oplus \mathbb{Z}_{30540} \oplus \mathbb{Z}_{2^2\cdot3\cdot5\cdot103\cdot509\cdot1699\cdot11593\cdot p\cdot q} \oplus \mathbb{Z}_{2^2\cdot3\cdot5\cdot17\cdot103\cdot509\cdot1699\cdot11593\cdot p\cdot q},$$

and

$$\tau_{3,4}(170) = 2^{25}\cdot3^4\cdot5^3\cdot17\cdot103^2\cdot509^4\cdot1699^2\cdot11593^2\cdot p^2\cdot q^2,$$

where $p = 16901365279286026289$ and $q = 34652587005966540929$.

Table 2: Graph $I(n, 2, 3)$.

n	$\text{Jac}(I(n, 2, 3))$	$\tau_{2,3}(n) = \lvert \text{Jac}(I(n, 2, 3)) \rvert$
4	$\mathbb{Z}_7 \oplus \mathbb{Z}_{28}$	196
5	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{95}$	1805
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{83} \oplus \mathbb{Z}_{581}$	48223
8	$\mathbb{Z}_{161} \oplus \mathbb{Z}_{1288}$	207368
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_{1558} \oplus \mathbb{Z}_{3895}$	6068410
11	$\mathbb{Z}_{1693} \oplus \mathbb{Z}_{18623}$	31528739
12	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{665} \oplus \mathbb{Z}_{7980}$	132667500
13	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{325}$	858203125
14	$\mathbb{Z}_{17513} \oplus \mathbb{Z}_{245182}$	4293872366
15	$\mathbb{Z}_{37069} \oplus \mathbb{Z}_{556035}$	20611661415
16	$\mathbb{Z}_{84847} \oplus \mathbb{Z}_{1357552}$	115184214544
17	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448
18	$\mathbb{Z}_{400843} \oplus \mathbb{Z}_{7215174}$	2892151991682
19	$\mathbb{Z}_{898243} \oplus \mathbb{Z}_{17066617}$	15329969253931
20	$\mathbb{Z}_{19}^4 \oplus \mathbb{Z}_{5453} \oplus \mathbb{Z}_{109060}$	77502443441780
21	$\mathbb{Z}_{4301807} \oplus \mathbb{Z}_{90337947}$	388616412770229
22	$\mathbb{Z}_{9536669} \oplus \mathbb{Z}_{209806718}$	2000857223542342
23	$\mathbb{Z}_{20949827} \oplus \mathbb{Z}_{481846021}$	10094590780588367
24	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{9192295} \oplus \mathbb{Z}_{220615080}$	50598972420215000
25	$\mathbb{Z}_{101468531} \oplus \mathbb{Z}_{2536713275}$	257396569582449025
26	$\mathbb{Z}_{25} \oplus \mathbb{Z}_{325} \oplus \mathbb{Z}_{8923525} \oplus \mathbb{Z}_{17847050}$	1293976099416406250
27	$\mathbb{Z}_{490309597} \oplus \mathbb{Z}_{13238359119}$	6490894524578165043
28	$\mathbb{Z}_{49} \oplus \mathbb{Z}_{154342069} \oplus \mathbb{Z}_{4321577932}$	32683062689111444092
29	$\mathbb{Z}_{2376466133} \oplus \mathbb{Z}_{68917517857}$	163780147157583236981
30	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \oplus \mathbb{Z}_{275089049} \oplus \mathbb{Z}_{8252671470}$	819549256247415262830
31	$\mathbb{Z}_{11507960491} \oplus \mathbb{Z}_{356746775221}$	4105427794534925793511
32	$\mathbb{Z}_{25318259953} \oplus \mathbb{Z}_{810184318496}$	20512457185525873990688
33	$\mathbb{Z}_{55700389051} \oplus \mathbb{Z}_{1838112838683}$	102383600234281102459833
34	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_{1915580948} \oplus \mathbb{Z}_{32564876116}$	511022336096582352633856
35	$\mathbb{Z}_{269747901677} \oplus \mathbb{Z}_{9441176558695}$	2546737566070056079431515

Table 3: Graph $I(n,3,4)$.

n	$\mathrm{Jac}(I(n,3,4))$	$\tau_{3,4}(n) = \lvert \mathrm{Jac}(I(n,3,4))\rvert$
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	2000
6	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{114}$	2166
7	$\mathbb{Z}_{71} \oplus \mathbb{Z}_{497}$	35287
8	$\mathbb{Z}_{73} \oplus \mathbb{Z}_{584}$	42632
9	$\mathbb{Z}_{289} \oplus \mathbb{Z}_{2601}$	751689
10	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{60}$	5184000
11	$\mathbb{Z}_{1541} \oplus \mathbb{Z}_{16951}$	26121491
12	$\mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_{209} \oplus \mathbb{Z}_{2508}$	63424812
13	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{1555} \oplus \mathbb{Z}_{20215}$	785858125
14	$\mathbb{Z}_{16969} \oplus \mathbb{Z}_{237566}$	4031257454
15	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{17410} \oplus \mathbb{Z}_{52230}$	18186486000
16	$\mathbb{Z}_{71321} \oplus \mathbb{Z}_{1141136}$	81386960656
17	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_{23186} \oplus \mathbb{Z}_{394162}$	584898568448
18	$\mathbb{Z}_{400843} \oplus \mathbb{Z}_{7215174}$	2892151991682
19	$\mathbb{Z}_{37} \oplus \mathbb{Z}_{37} \oplus \mathbb{Z}_{23939} \oplus \mathbb{Z}_{454841}$	14906272578931
20	$\mathbb{Z}_8 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{120} \oplus \mathbb{Z}_{79080} \oplus \mathbb{Z}_{79080}$	72042006528000
21	$\mathbb{Z}_{4487981} \oplus \mathbb{Z}_{94247601}$	422981442583581
22	$\mathbb{Z}_{10002631} \oplus \mathbb{Z}_{220057882}$	2201157792287542
23	$\mathbb{Z}_{22138559} \oplus \mathbb{Z}_{509186857}$	11272663275719063
24	$\mathbb{Z}_{187} \oplus \mathbb{Z}_{187} \oplus \mathbb{Z}_{259369} \oplus \mathbb{Z}_{6224856}$	56458663080288216
25	$\mathbb{Z}_{2114} \oplus \mathbb{Z}_{52850} \oplus \mathbb{Z}_{52850} \oplus \mathbb{Z}_{52850}$	312061332000250000

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