

Chiral covers of hypermaps

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Abstract

Generalising a conjecture of Singerman, it is shown that there are orientably regular chiral hypermaps (equivalently regular chiral *dessins d'enfants*) of every non-spherical type. The proof uses the representation theory of automorphism groups of Riemann surfaces acting on homology and on various spaces of differentials. Some examples are given.

Keywords: Chiral hypermap, chiral map, harmonic differential, homology group.

Math. Subj. Class.: 05C10, 05E18, 14H57, 20B25, 20C15, 30F10, 30F30

1 Introduction

In 1992, in an unpublished preprint [17], Singerman conjectured that if $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{2}$ there is an orientably regular chiral map of type $\{m, n\}$. Conder, Hucíková, Nedela and Širáň have announced a proof of this in [4] (see also [19, §3.4]). The aim of this note is to prove a similar but more general result for hypermaps (equivalently *dessins d'enfants* in Grothendieck's terminology [10]) of non-spherical type. Whereas most constructions of maps or hypermaps, including those in [4], involve group theory or combinatorics, this construction is mainly based on analysis (specifically, spaces of differentials on Riemann surfaces) and representation theory (the action of automorphism groups on such spaces and on associated homology groups). For background on the first topic see [9, 11], for the second see [7], and for applications of the second topic to the first see [16]. It is hoped that such techniques may find further applications in this area.

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2 Chiral covers

Orientably regular hypermaps \mathcal{H} of type (l, m, n) correspond to torsion-free normal subgroups N of the triangle group $\Delta = \Delta(l, m, n)$; here N is an orientable surface group of genus g equal to that of \mathcal{H} , and \mathcal{H} has *spherical type* if $g = 0$, or equivalently $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$. The orientation-preserving automorphism group $G := \text{Aut}^+\mathcal{H}$ of \mathcal{H} is isomorphic to Δ/N . We say that \mathcal{H} is *reflexible* if N is normal in the extended triangle group Δ^* of type (l, m, n) , in which case \mathcal{H} has full automorphism group $A := \text{Aut } \mathcal{H} \cong \Delta^*/N$; otherwise \mathcal{H} and its mirror image $\overline{\mathcal{H}}$ form a *chiral pair*, corresponding to N and its other conjugate in Δ^* .

Theorem 2.1. *If \mathcal{H} is a finite orientably regular hypermap of non-spherical type, then \mathcal{H} is covered by infinitely many finite orientably regular chiral hypermaps of that same type.*

Proof. Let Δ and Δ^* be the triangle group and extended triangle group of the given type (l, m, n) , so \mathcal{H} , which has genus $g > 0$, corresponds to a torsion-free normal subgroup N of finite index in Δ with $G := \text{Aut}^+\mathcal{H} \cong \Delta/N$.

Case 1. Suppose that \mathcal{H} is chiral, so N is not normal in Δ^* . Choose any prime p not dividing $|G|$: by a theorem in Euclid’s *Elements*, there are infinitely many such primes. Let $N_p = N'N^p$, the group generated by the commutators and p -th powers in N ; this is a torsion-free subgroup which is characteristic in N and hence normal in Δ , so it corresponds to a finite orientably regular hypermap \mathcal{H}_p of type (l, m, n) , which covers \mathcal{H} since $N_p \leq N$. Now $|N : N_p| = p^{2g}$, which is coprime to $|\Delta : N|$, so N/N_p is a Sylow p -subgroup of Δ/N_p ; being normal, it is the unique Sylow p -subgroup, so it is a characteristic subgroup of Δ/N_p . If N_p were normal in Δ^* then N/N_p would be normal in Δ^*/N_p , and hence N would be normal in Δ^* , against our assumption. Thus N_p is not normal in Δ^* , so \mathcal{H}_p is chiral. As a smooth p^{2g} -sheeted covering of \mathcal{H} it has genus $p^{2g}(g - 1) + 1$.

Case 2. Suppose that \mathcal{H} is reflexible, so that N is normal in Δ^* . Let \mathcal{S} be the compact Riemann surface \mathcal{U}/N canonically associated with \mathcal{H} , where $\mathcal{U} = \mathbb{C}$ or \mathbb{H} as $g = 1$ or $g > 1$, so that the elements of G and of $A \setminus G$ induce conformal and anticonformal automorphisms of \mathcal{S} . The harmonic differentials on \mathcal{S} form a complex vector space $V = H^1(\mathcal{S}, \mathbb{C})$ of dimension $2g$ affording a representation ρ of A . This space admits a G -invariant direct sum decomposition $V^+ \oplus V^-$, where V^+ and V^- are the g -dimensional spaces of holomorphic and antiholomorphic differentials on \mathcal{S} , affording complex conjugate representations ρ^+ and ρ^- of G . The elements of $A \setminus G$ transpose holomorphic and antiholomorphic differentials, and hence transpose the subspaces V^+ and V^- of V .

Integration around closed paths allows one to identify the first homology group

$$H_1(\mathcal{S}, \mathbb{C}) = H_1(\mathcal{S}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

of \mathcal{S} with the dual space V^* of V , so it affords the dual (or contragredient) representation ρ^* of A , which is equivalent to the complex conjugate representation $\bar{\rho}$ since A is finite. Since $\rho|_G$ is the sum of two complex conjugate representations, $\bar{\rho}|_G$ is equivalent to $\rho|_G$. Thus $H_1(\mathcal{S}, \mathbb{C})$ also decomposes as a direct sum $H_1^+ \oplus H_1^-$ of two g -dimensional G -invariant subspaces affording complex conjugate representations ρ^+ and ρ^- of G , and these are transposed by elements of $A \setminus G$.

For any prime p , let $N_p := N'N^p$; this is a characteristic subgroup of N and hence a normal subgroup of Δ^* . Let $M_p := N/N_p$. Since $N \cong \pi_1\mathcal{S}$ we have $N/N' \cong H_1(\mathcal{S}, \mathbb{Z}) \cong$

\mathbb{Z}^{2g} and hence

$$M_p \cong H_1(\mathcal{S}, \mathbb{Z})/pH_1(\mathcal{S}, \mathbb{Z}) \cong H_1(\mathcal{S}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong H_1(\mathcal{S}, \mathbb{F}_p) \cong \mathbb{F}_p^{2g}$$

where \mathbb{F}_p is the field of p elements. Indeed, these are isomorphisms of $\mathbb{F}_p A$ -modules, with the natural action of A on homology corresponding to the induced action of Δ^*/N by conjugation on M_p .

Let e be the exponent of A . By a theorem of Dirichlet, there are infinitely many primes $p \equiv 1 \pmod{e}$. Each such prime p is coprime to $|A|$, so Maschke’s Theorem holds for representations of A over \mathbb{F}_p ; moreover, since \mathbb{F}_p contains a full set of e -th roots of unity, \mathbb{F}_p is a splitting field for A [7, Corollary 70.24], so the representation theory of A over \mathbb{F}_p is ‘the same’ as that over \mathbb{C} , and this also applies to any subgroup of A . Specifically, the representation ρ_p of A on M_p can be regarded as the reduction mod (p) of its representation $\bar{\rho} \sim \rho$ on $V^* = H_1(\mathcal{S}, \mathbb{C})$ with respect to a suitable basis for V^* , with the same decomposition as a direct sum of absolutely irreducible subspaces. In particular, M_p has a G -invariant decomposition $M_p^+ \oplus M_p^-$, with M_p^+ and M_p^- affording g -dimensional representations of G , and the elements of $A \setminus G$ transpose the two direct factors. (See [16, §3] for full details concerning these representations of G ; the extension to A is straightforward.)

The inverse images N_p^+ and N_p^- of M_p^+ and M_p^- in N are torsion-free normal subgroups of finite index p^g in Δ , so let \mathcal{H}_p^+ and \mathcal{H}_p^- be the corresponding finite orientably regular hypermaps of type (l, m, n) . These cover \mathcal{H} , and are non-isomorphic as oriented hypermaps since $N_p^+ \neq N_p^-$. Elements of $\Delta^* \setminus \Delta$ transpose N_p^+ and N_p^- by conjugation, so \mathcal{H}_p^+ and \mathcal{H}_p^- form a chiral pair. As smooth p^g -sheeted coverings of \mathcal{H} , they have genus $(g - 1)p^g + 1$. □

Corollary 2.2. *There exist infinitely many orientably regular chiral hypermaps of each non-spherical type.*

Proof. Being residually finite, the triangle group of the given type has a normal subgroup of finite index which contains no non-identity powers of the canonical generators, and is therefore torsion-free. Applying Theorem 2.1 to the corresponding orientably regular hypermap gives the required chiral hypermaps. □

Remark 2.3. In many cases, the condition $p \equiv 1 \pmod{e}$ in case 2 is unduly restrictive. It guarantees that every representation of A or G over \mathbb{C} decomposes in the same way when regarded as a representation over \mathbb{F}_p , whereas we are interested in just one representation, namely ρ . There are cases, illustrated in the following examples, where ρ decomposes in the required way, yielding a chiral pair, for certain primes $p \not\equiv 1 \pmod{e}$.

Remark 2.4. Theorem 2.1 and Corollary 2.2 can be reinterpreted as statements about dessins d’enfants by replacing the phrase ‘orientably regular hypermap’ with the equivalent ‘regular dessin’. However, their proofs make no essential use of the fact that Δ and Δ^* are triangle groups, or equivalently that the Riemann surface \mathcal{S} is quasiplatonic, so they yield more general results concerning coverings of compact Riemann surfaces. It is hoped to explore these in a later paper.

Remark 2.5. The method used in case 1 of the proof of Theorem 2.1 is sometimes called the ‘Macbeath trick’, since it was used by Macbeath [15] to produce an infinite sequence of Hurwitz groups of the form $\Delta/N'N^m$, with Δ of type $(2, 3, 7)$ and $G = \Delta/N \cong L_2(7)$. (See §4 for background, and for more details concerning this example.) The resulting

hypermaps (maps of type $\{3, 7\}$) are all regular, whereas here, in cases 1 and 2 of Theorem 2.1, Macbeath’s method is modified to produce chiral hypermaps.

Remark 2.6. It is *not* claimed that there are chiral hypermaps of every non-spherical genus. Indeed, Conder’s lists of chiral maps and hypermaps [3] show that there are none of genus 2, 5 or 23. It would be interesting to characterise the genera with this property. Conder, Širáň and Tucker [5] have shown that there are no chiral maps of genus $g = p+1$ for primes p such that $p - 1$ is not divisible by 3, 5 or 8, but finding a similar result for hypermaps would seem to be more difficult.

3 An example in genus 2

There is a unique orientably regular hypermap \mathcal{H} of genus 2 and type $(8, 2, 8)$. This is a map of type $\{8, 8\}$, arising from an epimorphism $\Delta = \Delta(8, 2, 8) \rightarrow G = C_8$ with kernel $N = \Delta'$. By its uniqueness \mathcal{H} is reflexible, with $A \cong \Delta^*/N \cong D_8$; it is R2.6 in Conder’s list of regular maps [3], and is the first entry in [6, Table 9].

One can construct \mathcal{H} as a map by identifying opposite sides of an octagon, so that \mathcal{H} has one vertex, four edges and one face. Going around the boundary of the octagon gives a presentation

$$N = \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} = 1 \rangle.$$

The images of a, b, c, d in N/N' form a basis for the homology group $V^* = H_1(\mathcal{S}, \mathbb{C})$ of the underlying surface \mathcal{S} . The automorphism group A of \mathcal{H} is induced by the isometries of the octagon: thus $G = \langle r \rangle$ and $A = \langle r, s \rangle$ where r is a rotation through $\pi/4$ and s is a reflection; these are represented on homology by the matrices

$$\begin{pmatrix} & 1 & & \\ & & 1 & \\ -1 & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

The Riemann surface \mathcal{S} underlying \mathcal{H} is the hyperelliptic curve

$$w^2 = z(z^4 - 1),$$

a quasiplatonic curve with Belyĭ function $\beta : (w, z) \mapsto z^4$. The vertex is at $(0, 0)$, the face-centre is at (∞, ∞) , and the edge-centres are at the four points $(0, z)$ with $z^4 = 1$, where $\beta = 0, 1, \infty$ respectively. The automorphisms r and s send points $(w, z) \in \mathcal{S}$ to $(\zeta w, iz)$ and (\bar{w}, \bar{z}) respectively, where $\zeta = \exp(2\pi i/8)$.

The differentials

$$\omega_1 = \frac{dz}{w} \quad \text{and} \quad \omega_2 = \frac{z dz}{w}$$

form a basis for V^+ [9, §III.7.5], and their conjugates form a basis for V^- . The rotation r sends ω_1 and ω_2 to $i\omega_1/\zeta = \zeta\omega_1$ and $i^2\omega_2/\zeta = \zeta^3\omega_2$, so these span 1-dimensional G -invariant subspaces E_λ on which r has eigenvalues $\lambda = \zeta$ and ζ^3 ; there is a similar decomposition for V^- , except that here the eigenvalues are ζ^{-1} and ζ^{-3} . The action of s is to transpose E_ζ with $E_{\zeta^{-1}}$, and E_{ζ^3} with $E_{\zeta^{-3}}$.

Taking the dual space and then reducing mod (p) , we see that the same applies to the A -module $M_p = H_1(\mathcal{S}, \mathbb{F}_p)$ for any prime $p \equiv 1 \pmod{8}$, with ζ now interpreted as a primitive 8th root of 1 in the splitting field \mathbb{F}_p . This gives a decomposition $M_p =$

$M_p^+ \oplus M_p^-$ with G -submodules $M_p^+ = E_\zeta \oplus E_{\zeta^3}$ and $M_p^- = E_{\zeta^{-1}} \oplus E_{\zeta^{-3}}$ transposed by s . Lifting these back to N gives subgroups N_p^+ and N_p^- corresponding to a chiral pair of maps \mathcal{H}_p^+ and \mathcal{H}_p^- of type $\{8, 8\}$ and genus $p^2 + 1$, as in case 2 of the proof of Theorem 2.1.

Remark 3.1. In this example, A has exponent 8, so the smallest prime for which this construction applies is $p = 17$, giving a chiral pair of orientably regular hypermaps of type $\{8, 8\}$ and genus 290. They correspond to entry C290.4 in Conder’s list of chiral maps [3].

Remark 3.2. This particular example also yields two chiral pairs of p -sheeted coverings of \mathcal{H} , corresponding to the four maximal submodules of M_p , each omitting one of the four 1-dimensional eigenspaces E_λ . These are orientably regular maps of type $\{8, 8\}$ and genus $p + 1$, and each pair consists of the duals of the other (see also [1, Theorem 1(a)(iii) and §3, Example (iii)] for the associated groups and Riemann surfaces). When $p = 17$ they correspond to entry C18.1 in [3].

Remark 3.3. It is interesting to see what happens if we use primes $p \not\equiv 1 \pmod{8}$ in this example, so that \mathbb{F}_p is not a splitting field for A . If $p \equiv 3 \pmod{8}$ then M_p is a direct sum of two irreducible G -submodules, with r having eigenvalues ζ and ζ^3 on one, and their inverses on the other, where ζ is now a primitive 8th root of 1 in \mathbb{F}_{p^2} ; these submodules are transposed by s , so we obtain a chiral pair of self-dual maps of type $\{8, 8\}$ and genus $p^2 + 1$; see C10.3 in [3] for the case $p = 3$, and C122.7 for $p = 11$.

The same applies if $p \equiv -3 \pmod{8}$, except that r now has eigenvalues ζ and ζ^{-3} on one submodule, and their inverses on the other; again these submodules are transposed by s , so we obtain a chiral pair of maps of type $\{8, 8\}$ and genus $p^2 + 1$, now duals of each other; see C26.1 and C170.7 in [3] for $p = 5$ and 13.

If $p \equiv -1 \pmod{8}$ then M_p is a direct sum of two irreducible G -submodules, with r having eigenvalues $\zeta^{\pm 1}$ on one and $\zeta^{\pm 3}$ on the other; these submodules are both invariant under s , so we obtain a dual pair of regular maps of type $\{8, 8\}$, rather than a chiral pair; see R50.7 in [3] for the case $p = 7$.

Finally, if $p = 2$ we again obtain no chiral maps, but there is a unique series

$$M_2 > (r - 1)M_2 > (r - 1)^2M_2 > (r - 1)^3M_2 > (r - 1)^4M_2 = 0$$

of A -submodules, giving regular maps of type $\{8, 8\}$ and genus 2, 3, 5, 9 and 17.

Remark 3.4. Further examples of chiral and regular hypermaps, arising as elementary abelian coverings of regular hypermaps of genus 2, have been found by Kazaz in [12]; this example is based on his methods (see also [13]).

4 Chiral covers of Klein’s quartic curve

Klein’s quartic curve $x^3y + y^3z + z^3x = 0$ is a compact Riemann surface \mathcal{S} of genus 3, which carries a regular map \mathcal{H} of type $\{3, 7\}$ with $G \cong L_2(7)$ and $A \cong PGL_2(7)$; see [14] for a comprehensive study of this curve. The method of proof of case 2 of Theorem 2.1, with Δ of type $(2, 3, 7)$, yields chiral pairs of maps of type $\{3, 7\}$ and genus $2p^3 + 1$ for all primes $p \equiv 1 \pmod{168}$. These are of interest because their automorphism groups, as finite quotients of Δ , are all Hurwitz groups, attaining Hurwitz’s upper bound of $84(g - 1)$ for the number of automorphisms of a compact Riemann surface of a given genus $g > 1$.

In fact, such chiral pairs exist for all primes $p \equiv 1, 2$ or $4 \pmod{7}$. These can be obtained from a classification by Cohen [2] of those Hurwitz groups which arise as abelian

coverings of G ; his methods of construction are purely algebraic, using 6×6 matrices which can be interpreted as representing generators of Δ on various homology modules. (These covers were also obtained in an earlier paper of Wohlfahrt [20], using ideals in the ring of integers of $\mathbb{Q}(\sqrt{-7})$.) The case $p = 2$ is due to Sinkov [18], giving the chiral pair C17.1 of genus 17 in [3].

In this example, V^+ and V^- are irreducible G -submodules of V , affording complex conjugate representations with characters taking the values 3, -1 , 0 and 1 on elements of orders 1, 2, 3 and 4, and $(-1 \pm \sqrt{-7})/2$ on the two classes of elements of order 7. Elkies has studied these representations, and their reduction modulo various primes, in [8, §1]. They give irreducible representations over \mathbb{F}_p for any prime p such that -7 is a quadratic residue, that is, for $p \equiv 1, 2$ or $4 \pmod{7}$; for such primes we have a G -module decomposition $M_p = M_p^+ \oplus M_p^-$, giving a chiral pair \mathcal{H}_p^+ and \mathcal{H}_p^- . However, no other primes give chiral pairs. The module M_7 is indecomposable but reducible, with a submodule of dimension 3 yielding a regular map of genus 687, and the zero submodule yielding one of genus 235299. For primes $p \equiv 3, 5$ or $6 \pmod{7}$, M_p is irreducible and again no chiral coverings arise. (Sah's statement in [16, §3(b)] is incorrect for such primes: for instance, M_3 yields only a regular map of genus 1459, corresponding to its zero submodule, and not the chiral pair of genus 55 claimed there.)

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