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HOMOMORPHISMS OF MATRIX SEMIGROUPS

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Abstract

In this work we study non-degenerate homomorphisms from the multiplicative semigroup of all n-by-n matrices over a field to the semigroup of m-by-m matrices over the same field.

A general introduction is given in the first chapter. In the second chapter we first state our main question and give some examples. Further we characterize all homomorphisms from the multiplicative semigroup of all n-by-n matrices over an arbitrary field to the field and all non-degenerate homomorphisms from the multiplicative semigroup of all n-by-n matrices over an arbitrary field to a semigroup of m-by-m matrices over the same field, if $m \leq n$.

In the third chapter we characterize all non-degenerate homomorphisms from the multiplicative semigroup of all 2-by-2 matrices over an arbitrary field to the semigroup of 3-by-3 matrices over the same field. If the characteristic of the field is not equal to 2 then we have two possibilities. Either it is a symmetric square, combined with a field homomorphism used entrywise and a matrix conjugation, or a direct sum of the identity and the determinant, combined with a field homomorphism, a homomorphism of the multiplicative semigroup of the field and a matrix conjugation. In the characteristic 2 a symmetric square gives rise to two different homomorphisms and we get three possibilities. In the case of the field of real numbers every irreducible non-

degenerate homomorphism is a matrix conjugation of the symmetric square.

In the fourth chapter we study non-degenerate irreducible homomorphisms from the multiplicative semigroup of all 2-by-2 matrices over an algebraically closed field of characteristic zero to the semigroup of m-by-m matrices over the same field. If such a homomorphism maps a cyclic unipotent to a cyclic unipotent, it is the composition of a symmetric power, a field homomorphism used entrywise, and a matrix conjugation. In the case $m = 4$ we characterize all non-degenerate irreducible homomorphisms.

In the fifth chapter we prove that every non-degenerate homomorphism from the multiplicative semigroup of all n-by-n matrices over an algebraically closed field of characteristic zero to the semigroup of $(n+1)$-by-$(n+1)$ matrices over the same field when $n \geq 3$ is reducible and that every non-degenerate homomorphism from the multiplicative semigroup of all 3-by-3 matrices over an algebraically closed field of characteristic zero to the semigroup of 5-by-5 matrices over the same field is reducible.

Keywords: matrix semigroup, semigroup homomorphism, multiplicative map, irreducibility.

Math. Subj. Class. (2000): 08A35, 15A30, 15A69, 20G05

Povzetek

V delu študiramo nedegenerirane homomorfizme iz multiplikativne polgrupe vseh $n \times n$ matrik nad komutativnim obsegom v polgrupo $m \times m$ matrik nad istim obsegom.

Naj bo $\mathbb{F}$ poljuben komutativen obseg in n naravno število. Označimo z $\mathcal{M}_n(\mathbb{F})$ množico vseh $n \times n$ matrik z elementi v $\mathbb{F}$. Množica $\mathcal{M}_n(\mathbb{F})$ je polgrupa za operacijo množenja matrik. Vprašanje, s katerim se ukvarjamo, se glasi: Kakšni so homomorfizmi polgrup $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, torej preslikave, ki zadoščajo enačbi

$$\varphi(AB) = \varphi(A)\varphi(B)$$

za vse matrike $A, B \in \mathcal{M}_n(\mathbb{F})$?

Homomorfizem polgrup $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ je *nerazcepen*, če je njegova slika nerazcepna polgrupa, torej kot množica matrik nima skupnega invariantnega podprostora.

Primeri:

1. Matrična konjugacija: Če je $S \in \mathcal{M}_n(\mathbb{F})$ obrnljiva matrika, potem je preslikava $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$, ki je definirana s predpisom

$$\varphi(A) = SAS^{-1},$$

homomorfizem polgrupe.

2. Konstanta: Če je $E \in \mathcal{M}_m(\mathbb{F})$ idempotentna matrika, torej če zadošča enačbi $E^2 = E$, potem je preslikava $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, ki je definirana s predpisom

$$\varphi(A) = E,$$

homomorfizem polgrup.

3. Homomorfizem komutativnega obsega, uporabljen po elementih: Naj bo $f : \mathbb{F} \to \mathbb{F}$ homomorfizem komutativnega obsega. Za poljubno matriko $A = [a_{ij}]_{i,j=1}^n \in \mathcal{M}_n(\mathbb{F})$ definirajmo

$$\varphi(A) = \hat{f}(A) = [f(a_{ij})]_{i,j=1}^n.$$

Preslikava $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ je homomorfizem polgrupe.

4. Degenerani homomorfizmi: Naj bo $\varphi' : GL_n(\mathbb{F}) \to GL_m(\mathbb{F})$ homomorfizem grup. Definirajmo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ takole: Če je $\det A = 0$, vzamemo $\varphi(A) = 0$, sicer pa $\varphi(A) = \varphi'(A)$. Očitno je φ homomorfizem polgrup. Take homomorfizme imenujemo *degenerirani*. Ker so homomorfizmi grup $\varphi' : GL_n(\mathbb{F}) \to GL_m(\mathbb{F})$ poznani, se omejimo na nedegenerane homomorfizme.

5. Direktna vsota: Za poljubna homomorfizma polgrup $\varphi' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'}(\mathbb{F})$ in $\varphi'' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m''}(\mathbb{F})$ definirajmo preslikavo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'+m''}(\mathbb{F})$ s predpisom

$$\varphi(A) = \varphi'(A) \oplus \varphi''(A)$$

za vsako matriko $A \in \mathcal{M}_n(\mathbb{F})$. Preslikava φ je spet homomorfizem polgrup, ki je vedno razcepen.

6. Zunanja potenca: Naj bo $k \leq n$ naravno število. Vektorski prostor $\mathbb{F}^{\binom{n}{k}}$ je izomorfen zunanji potenci $\wedge^k \mathbb{F}^n$ vseh antisimetričnih tenzorjev stopnje k.

Če je $\mathcal{E} = \{e_1, e_2, ..., e_n\}$ baza prostora $\mathbb{F}^n$, potem je

$$\mathcal{E}' = \{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}, 1 \leq i_1 < i_2 < ... < i_k \leq n\}$$

baza prostora $\wedge^k \mathbb{F}^n$. Če matrika $A \in \mathcal{M}_n(\mathbb{F})$ predstavlja linearno preslikavo prostora $\mathbb{F}^n$, potem $\wedge^k A \in \mathcal{M}_{\binom{n}{k}}(\mathbb{F})$ predstavlja linearno preslikavo, ki deluje na tenzorjih stopnje k takole:

$$(\wedge^k A)(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}) = Ae_{i_1} \wedge Ae_{i_2} \wedge ... \wedge Ae_{i_k}.$$

Elementi matrike $\wedge^k A$ so $k \times k$ minorji matrike A. Preslikava $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n}{k}}(\mathbb{F})$, definirana s predpisom

$$\varphi(A) = \wedge^k A,$$

je homomorfizem polgrup. Če je k enak $n-1$, potem preslikava φ slika iz $\mathcal{M}_n(\mathbb{F})$ v $\mathcal{M}_n(\mathbb{F})$. V tem primeru je $\varphi(A) = \wedge^{n-1} A = \mathrm{Cof}(A)$ matrika kofaktorjev matrike A.

7. Simetrična potenca: Naj bo k naravno število. Vektorski prostor $\mathbb{F}^{\binom{n+k-1}{k}}$ je izomorfen simetrični potenci $\mathrm{Sym}^k \mathbb{F}^n$ vseh simetričnih tenzorjev stopnje k. Če je $\mathcal{E} = \{e_1, e_2, ..., e_n\}$ baza prostora $\mathbb{F}^n$, potem je

$$\mathcal{E}' = \{e_{i_1} \vee e_{i_2} \vee ... \vee e_{i_k}; 1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n\}$$

baza prostora $\mathrm{Sym}^k \mathbb{F}^n$. Če matrika $A \in \mathcal{M}_n(\mathbb{F})$ predstavlja linearno preslikavo prostora $\mathbb{F}^n$, potem $\mathrm{Sym}^k A \in \mathcal{M}_{\binom{n+k-1}{k}}(\mathbb{F})$ predstavlja linearno preslikavo, ki deluje na tenzorjih stopnje k takole:

$$(\mathrm{Sym}^k A)(e_{i_1} \vee e_{i_2} \vee ... \vee e_{i_k}) = Ae_{i_1} \vee Ae_{i_2} \vee ... \vee Ae_{i_k}.$$

Preslikava $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n+k-1}{k}}(\mathbb{F})$, definirana s predpisom

$$\varphi(A) = \mathrm{Sym}^k A,$$

je homomorfizem polgrup. Za $n = 2$ in $k = 2$, dobimo

$$\mathrm{Sym}^2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}$$

8. Tenzorski produkt: Vektorski prostor $\mathbb{F}^{m'm''}$ je izomorfen tenzorskemu produktu $\mathbb{F}^{m'} \otimes \mathbb{F}^{m''}$. Za poljubna homomorfizma polgrup $\varphi' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'}(\mathbb{F})$ in $\varphi'' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m''}(\mathbb{F})$ definirajmo preslikavo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'm''}(\mathbb{F})$ s predpisom

$$\varphi(A) = \varphi'(A) \otimes \varphi''(A)$$

za vsako matriko $A \in \mathcal{M}_n(\mathbb{F})$. Preslikava φ je spet homomorfizem polgrup. Tenzorski produkt $\varphi(A) = A \otimes A$ lahko napišemo kot direktno vsoto zunanje potence in simetrične potence

$$A \otimes A = A \wedge A \oplus \mathrm{Sym}^2 A.$$

9. Kombinacije zgornjih primerov.

V delu najprej karakteriziramo vse homomorfizme iz matrične polgrupe $\mathcal{M}_n(\mathbb{F})$ v obseg $\mathbb{F}$ kot multiplikativno polgrupo. To je dobro znan rezultat.

Trditev 1 *Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ homomorfizem polgrup. Potem obstaja homomorfizem multiplikativne polgrupe $f : \mathbb{F} \to \mathbb{F}$, za katerega velja*

$$\varphi(A) = f(\det A)$$

za vsako matriko $A \in \mathcal{M}_n(\mathbb{F})$.

Glavni rezultat drugega poglavja je karakterizacija vseh nedegeneriranih homomorfizmov polgrup $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, kjer je $m \leq n$. Izrek je bil dokazan že leta 1966 v [17] za obseg kompleksnih števil in leta 1969 v [13] za poljuben komutativen kolobar brez deliteljev niča.

Izrek 2 *Naj bo $\mathbb{F}$ poljuben komutativen obseg. Denimo, da za naravni števili n in m velja $n \geq 2$ in $m \leq n$. Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ nedegeneriran homomorfizem polgrup, za katerega velja $\varphi(0) = 0$ in $\varphi(I) = I$. Potem je $m = n$ in φ ima naslednjo obliko:*

$$\varphi(A) = S\hat{f}(A)S^{-1},$$

ali

$$\varphi(A) = S\hat{f}(\mathrm{Cof}(A))S^{-1},$$

kjer je $f : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega in $S \in \mathcal{M}_n(\mathbb{F})$ obrnljiva matrika.

V tretjem poglavju najprej pokažemo, da smemo brez škode za splošnost predpostaviti, da $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ preslika 0 v 0 in identiteto v identiteto.

Lema 3 *Naj bo $\mathbb{F}$ poljuben komutativen obseg in $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ homomorfizem polgrup. Potem ima φ obliko*

$$\varphi(A) = S(\varphi_0(A) \oplus E)S^{-1},$$

kjer je

- *$\varphi_0 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_k(\mathbb{F})$ homomorfizem polgrup, za katerega velja $\varphi_0(0) = 0$ in $\varphi_0(I) = I$,*
- *$E \in \mathcal{M}_{m-k}(\mathbb{F})$ je idempotent in*
- *$S \in \mathcal{M}_m(\mathbb{F})$ obrnljiva matrika.*

Če je $k = 0$, potem $\varphi_0(A)$ ne nastopa, če pa je $k = m$, potem E ne nastopa.

Glavni rezultat tretjega poglavja je karakterizacija homomorfizmov iz polgrupe 2×2 matrik v polgrupo 3×3 matrik.

Izrek 4 *Naj bo $\mathbb{F}$ poljuben komutativen obseg in $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$ nedegeneriran homomorfizem polgrup, za katerega velja $\varphi(0) = 0$ in $\varphi(I) = I$. Če je* char $\mathbb{F} \neq 2$, *potem ima φ eno od naslednjih oblik:*

(a)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & f(b) & 0 \\ f(c) & f(d) & 0 \\ 0 & 0 & g(ad-bc) \end{bmatrix} S^{-1},$$

kjer je $f : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega, $g : \mathbb{F} \to \mathbb{F}$ homomorfizem multiplikativne polgrupe $(\mathbb{F}, \cdot)$, za katerega velja $g(0) = 0$, $g(1) = 1$, in $S \in \mathcal{M}_3(\mathbb{F})$ obrnljiva matrika, ali

(b)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix} S^{-1},$$

kjer je $h : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega in $S \in \mathcal{M}_3(\mathbb{F})$ obrnljiva matrika.
Če je char $\mathbb{F} = 2$, *potem ima φ obliko (a), (b) ali*

(c)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & 0 & h(b^2) \\ h(ac) & h(ad+bc) & h(bd) \\ h(c^2) & 0 & h(d^2) \end{bmatrix} S^{-1},$$

kjer je $h : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega in $S \in \mathcal{M}_3(\mathbb{F})$ obrnljiva matrika.

Če je char $\mathbb{F} = 2$, sta primera (b) in (c) bistveno različna: Matrike v sliki φ imajo v primeru (b) natanko en skupen netrivialen invarianten podprostor, ki je dimenzije 2. Po drugi strani pa imajo v primeru (c) skupen invarianten podprostor dimenzije 1.

Preslikava φ je *popolnoma razcepna*, če ima vsak invarianten podprostor slike φ invarianten komplement. Naslednje trditve so preproste posledice Izreka 4.

Posledica 5 *Naj bo* $\mathbb{F}$ *komutativen obseg s* char $\mathbb{F} \neq 2$. *Vsak nedegeneriran homomorfizem polgrup* $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$ *je popolnoma razcepen.*

Posledica 6 *Naj bo* $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$ *nerazcepen nedegeneriran homomorfizem polgrup. Potem je* char $\mathbb{F} \neq 2$ *in*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix} S^{-1},$$

kjer je $h : \mathbb{F} \to \mathbb{F}$ *homomorfizem obsega in* $S \in \mathcal{M}_3(\mathbb{F})$ *obrnljiva matrika.*

V obsegu realnih števil $\mathbb{R}$ je edini neničelni homomorfizem identiteta.

Posledica 7 *Naj bo* $\varphi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ *nerazcepen nedegeneriran homomorfizem polgrup. Potem je*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix} S^{-1},$$

kjer je $S \in \mathcal{M}_3(\mathbb{R})$ *obrnljiva matrika.*

Edina zvezna homomorfizma obsega kompleksnih števil $\mathbb{C}$ sta identiteta in kompleksna konjugacija.

Posledica 8 *Naj bo* $\varphi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_3(\mathbb{C})$ *zvezen nerazcepen nedegeneriran homomorfizem polgrup. Potem je*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix} S^{-1},$$

kjer je $h : \mathbb{C} \to \mathbb{C}$ *identiteta ali kompleksna konjugacija in* $S \in \mathcal{M}_3(\mathbb{C})$ *obrnljiva matrika.*

V četrtem in petem poglavju se omejimo na primer, ko je komutativni obseg $\mathbb{F}$ algebraično zaprt in ima karakteristiko nič. Obravnavamo samo nerazcepne homomorfizme. Pogosto uporabljamo naslednjo trditev, ki je posledica Burnsidovega izreka.

Trditev 9 *Denimo, da je $\mathbb{F}$ algebraično zaprt obseg s karakteristiko nič. Naj bo $n \geq 2$ in $\mathcal{S}$ polgrupa v $\mathcal{M}_n(\mathbb{F})$. Če obstaja neničelen linearen funkcional f na $\mathcal{M}_n(\mathbb{F})$, ki je enak nič na $\mathcal{S}$, potem je polgrupa $\mathcal{S}$ razcepna.*

Na začetku četrtega poglavja pokažemo, da nerazcepen nedegeneriran homomorfizem polgrup $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ preslika matrike ranga 1 v matrike ranga 1.

Vsako $n \times n$ matriko razdelimo v 3×3 bločno strukturo, kjer je srednji blok velikosti $(n-2) \times (n-2)$. Torej je

$$\begin{bmatrix} a & b & \cdots & c & d \\ e & * & \cdots & * & f \\ \vdots & \vdots & & \vdots & \vdots \\ g & * & \cdots & * & h \\ i & j & \cdots & k & l \end{bmatrix} = \begin{bmatrix} a & x & d \\ y & T & z \\ i & w & l \end{bmatrix}$$

kjer je T matrika velikosti $(n-2) \times (n-2)$.

Lema 10 *Naj bo $n \geq 3$ in $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ nerazcepen nedegeneriran homomorfizem polgrup. Glede na zgornjo dekompozicijo ima φ naslednjo obliko:*

- *če je $a, b, c \neq 0$, potem je*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) =$$

$$= S \begin{bmatrix} f(a) & x^T G(a) E G(b) & f(b) \\ G(c) E G(a) y & G(c) E G(a) C G(\frac{b}{a}) & G(d) E G(b) y \\ f(c) & x^T G(c) E G(d) & f(d) \end{bmatrix} S^{-1},$$

kjer smo s C označili matriko

$$yx^T + VEG(\frac{ad}{bc} - 1)V;$$

- *če je $b \neq 0$, potem je*

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & x^T G(a)EG(b) & f(b) \\ 0 & G(\frac{d}{b})VG(b)EG(a)E & G(d)EG(b)y \\ 0 & 0 & f(d) \end{bmatrix} S^{-1},$$

- *sicer pa*

$$\varphi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & 0 & 0 \\ 0 & EG(a)EG(d) & 0 \\ 0 & 0 & f(d) \end{bmatrix} S^{-1},$$

kjer sta $f : \mathbb{F} \to \mathbb{F}$ in $G : \mathbb{F} \to \mathcal{M}_{n-2}(\mathbb{F})$ homomorfizma polgrup, $x, y \in \mathbb{F}^{n-2}$ neničelna vektorja, $S \in \mathcal{M}_n(\mathbb{F})$ obrnljiva matrika, $E \in \mathcal{M}_{n-2}(\mathbb{F})$ matrika z lastnostjo $E^2 = I$ in $V \in \mathcal{M}_{n-2}(\mathbb{F})$ matrika s spektrom enakim $\{1\}$.

Zgornja tehnična lema nam pomaga pokazati naslednji izrek, ki je karakterizacija homomorfizmov iz polgrupe 2×2 matrik v polgrupo 4×4 matrik.

Izrek 11 *Naj bo $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_4(\mathbb{F})$ nerazcepen nedegeneriran homomorfizem polgrup. Potem ima φ eno od naslednjih oblik:*

(a)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S\hat{g} \begin{bmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & a^2d + 2abc & 2abd + b^2c & 3b^2d \\ 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{bmatrix} S^{-1} =$$

$$= S\hat{g}(\mathrm{Sym}^3 A)S^{-1},$$

kjer je $g : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega in $S \in \mathcal{M}_4(\mathbb{F})$ obrnljiva matrika, ali

(b)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} g(a)h(a) & g(a)h(b) & g(b)h(a) & g(b)h(b) \\ g(a)h(c) & g(a)h(d) & g(b)h(c) & g(b)h(d) \\ g(c)h(a) & g(c)h(b) & g(d)h(a) & g(d)h(b) \\ g(c)h(c) & g(c)h(d) & g(d)h(c) & g(d)h(d) \end{bmatrix} S^{-1} =$$

$$= S(\hat{g}(A) \otimes \hat{h}(A))S^{-1},$$

kjer sta $g, h : \mathbb{F} \to \mathbb{F}$ *različna homomorfizma komutativnega obsega in* $S \in \mathcal{M}_4(\mathbb{F})$ *obrnljiva matrika.*

Če v primeru (b) velja $g = h$, potem je homomorfizem φ razcepen, ker je $A \otimes A \cong (A \vee A) \oplus (A \wedge A)$; sicer je φ nerazcepen.

Posledica 12 *Naj bo* $\varphi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *zvezen nerazcepen nedegeneriran homomorfizem polgrup. Potem je bodisi*

$$\varphi(A) = S\hat{g}(\mathrm{Sym}^3 A)S^{-1},$$

kjer je $g : \mathbb{C} \to \mathbb{C}$ *identiteta ali kompleksna konjugacija in* $S \in \mathcal{M}_4(\mathbb{C})$ *obrnljiva matrika, bodisi*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} a\bar{a} & a\bar{b} & b\bar{a} & b\bar{b} \\ a\bar{c} & a\bar{d} & b\bar{c} & b\bar{d} \\ c\bar{a} & c\bar{b} & d\bar{a} & d\bar{b} \\ c\bar{c} & c\bar{d} & d\bar{c} & d\bar{d} \end{bmatrix} S^{-1},$$

kjer je $S \in \mathcal{M}_4(\mathbb{C})$ *obrnljiva matrika.*

Matrika $A \in \mathcal{M}_n(\mathbb{F})$ je *unipotentna*, če je njen spekter enak $\{1\}$. Matrika $A \in \mathcal{M}_n(\mathbb{F})$ je *ciklična*, če ima ciklični vektor; to je tak vektor $x \in \mathbb{F}^n$ za katerega množica $\{x, Ax, A^2x, ..., A^{n-1}x\}$ napenja ves prostor $\mathbb{F}^n$. Vsaka ciklična unipotentna matrika v $\mathcal{M}_n(\mathbb{F})$ je podobna matriki

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Izrek 13 *Naj bo $n \geq 3$ in $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ nerazcepen nedegeneriran homomorfizem polgrup, ki preslika ciklični unipotent v ciklični unipotent. Potem je*

$$\varphi(A) = S\hat{g}(\mathrm{Sym}^{n-1}A)S^{-1},$$

kjer je $g : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega in $S \in \mathcal{M}_n(\mathbb{F})$ obrnljiva matrika.

V petem poglavju obravnavamo primer, ko je dimenzija matrik v polgrupi iz katere slikamo vsaj 3.

Trditev 14 *Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ homomorfizem polgrup, ki preslika 0 v 0 in identiteto v identiteto. Naj bo*

$$k = \min\{\mathrm{rang}A; \varphi(A) \neq 0\}.$$

Potem je

$$\binom{n}{k} \leq m.$$

Če je rang $A =$ rang B, *potem je* rang $\varphi(A) =$ rang $\varphi(B)$.

Trditev 15 *Denimo, da je $n \geq 3$ in $m < 2n$. Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ nedegeneriran homomorfizem polgrup, ki preslika 0 v 0 in identiteto v identiteto. Denimo, da φ slika matrike ranga 1 v matrike ranga 1. Potem φ slika matrike ranga 2 v matrike ranga 2.*

Naslednja trditev je očitna za $n = 3$ in $m < 6$. Dokažemo jo še za večje vrednosti n.

Trditev 16 *Denimo, da je $n > 4$ in $m < 2n$ ali pa $n = 4$ in $m \leq 5$. Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ nedegeneriran homomorfizem polgrup, ki preslika 0 v 0 in identiteto v identiteto. Potem imamo dve možnosti:*

(a) če je rang $A = 1$, *potem je* rang $\varphi(A) = 1$, *in če je* rang $A = 2$, *potem je* rang $\varphi(A) = 2$, *ali*

(b) če je rang $A < n - 1$, *potem je* $\varphi(A) = 0$, *in če je* rang $A = n - 1$, *potem je* rang $\varphi(A) = 1$.

Naslednji dve trditvi obravnavata možnosti, ki nam jih da Trditev 16.

Trditev 17 *Denimo, da je $n \geq 2$ in $m \geq n$. Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ nedegeneriran homomorfizem polgrup, ki preslika 0 v 0 in identiteto v identiteto. Naj φ slika matrike ranga 1 v matrike ranga 1 in matrike ranga 2 v matrike ranga 2. Potem je*

$$\varphi(A) = S \begin{bmatrix} \hat{f}(A) & * \\ * & * \end{bmatrix} S^{-1},$$

kjer je $f : \mathbb{F} \to \mathbb{F}$ homomorfizem obsega in $S \in \mathcal{M}_m(\mathbb{F})$ obrnljiva matrika.

Trditev 18 *Denimo, da je $n \geq 3$ in $m \geq n$. Naj bo $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ nedegeneriran homomorfizem polgrup, ki preslika 0 v 0 in identiteto v identiteto. Naj φ slika matrike ranga manjšega kot $n - 1$ v 0 in matrike ranga $n - 1$ v matrike ranga 1. Potem je*

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & * \\ * & * \end{bmatrix} S^{-1},$$

kjer je $f : \mathbb{F} \to \mathbb{F}$ homomorfizem multiplikativne polgrupe $(\mathbb{F}, \cdot)$ in $S \in \mathcal{M}_m(\mathbb{F})$ obrnljiva matrika.

Naslednja izreka sta osrednja rezultata petega poglavja.

Izrek 19 *Naj bo* $n \geq 3$. *Vsak nedegeneriran homomorfizem polgrup* $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{n+1}(\mathbb{F})$ *je razcepen.*

Izrek 20 *Naj bo* $m = 4$ *ali* $m = 5$. *Vsak nedegeneriran homomorfizem polgrup* $\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ *je razcepen.*

Na koncu dodamo še nekaj primerov:
Obstajata dva bistveno različna nedegenerirana nerazcepna homomorfizma polgrup $\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F})$: simetrični kvadrat

$$\varphi(A) = \mathrm{Sym}^2 A$$

in simetrični kvadrat zunanje potence

$$\varphi(A) = \mathrm{Sym}^2(A \wedge A).$$

Obstaja nedegeneriran nerazcepen homomorfizem polgrup $\varphi : \mathcal{M}_4(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F})$, to je zunanja potenca

$$\varphi(A) = A \wedge A.$$

Ključne besede: matrična polgrupa, homomorfizem polgrup, multiplikativna preslikava, nerazcepnost.

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Chapter 1

Introduction

Let S be a set and $\circ : S \times S \to S$ a binary operation on S. Then $(S, \circ)$ is a *semigroup*, if the operation $\circ$ is associative. Let $(S_1, \circ)$ and $(S_2, \circ)$ be two semigroups. A mapping $\varphi : S_1 \to S_2$ is a *homomorphism* of semigroups, if it preserves the operation $\circ$,

$$\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$$

for all $a, b \in S_1$. Let $\mathbb{F}$ be an arbitrary field and n be an integer. Denote by $\mathcal{M}_n(\mathbb{F})$ the set of all n-by-n matrices with entries in $\mathbb{F}$. Then $\mathcal{M}_n(\mathbb{F})$ is a semigroup under the multiplication of matrices. In this work we study homomorphisms of these semigroups and try to classify them.

The question of classification of semigroup homomorphisms is quite old and it may be difficult. Let us look first at a simple example. Let $(\mathbb{R}, +)$ be the additive semigroup of real numbers. A semigroup homomorphism $f : \mathbb{R} \to \mathbb{R}$ satisfies Cauchy's functional equation

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. This equation has some simple solutions

$$f(x) = cx \quad \text{for all} \quad x \in \mathbb{R},$$

where c is a real constant. All other solutions are quite wild. The graph of each solution of Cauchy's equation which is not of this form is everywhere dense in the plane $\mathbb{R}^2$. The equation was solved by Hamel in [10] a hundred years ago. He proved that there exists a subset H of $\mathbb{R}$ such that every real number x can be expressed in a unique way in the form

$$x = \sum_{k=1}^{n} r_k h_k,$$

where $h_k \in H$ and r_k are rational. The general solution of Cauchy's equation is given by choosing the values of f arbitrary on H and defining

$$f(x) = f\left(\sum_{k=1}^{n} r_k h_k\right) = \sum_{k=1}^{n} r_k f(h_k).$$

The set of real numbers is also a multiplicative semigroup. Its homomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfy Cauchy's power equation

$$g(xy) = g(x)g(y)$$

Every solution of this equation is of the form

$$g(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}$$

or

$$g(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}$$

or

$$g(0) = 0 \quad \text{and} \quad g(x) = e^{f(\log(|x|))} \quad \text{for all} \quad x \neq 0,$$

or

$$g(0) = 0 \quad \text{and} \quad g(x) = \text{sign} x e^{f(\log(|x|))} \quad \text{for all} \quad x \neq 0,$$

where f is a solution of (additive) Cauchy's equation.

Let us now move on to matrices. The set of all matrices $\mathcal{M}_n(\mathbb{F})$ is an algebra. It is well-known that every automorphism of this algebra is inner. More precisely, every bijective linear map $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ satisfying $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$ has the form

$$\varphi(A) = SAS^{-1}$$

where $S \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix.

The above theorem is usually derived as a straightforward consequence of the Noether-Skolem theorem (see [6], p. 93, theorem 3.14), an easy proof can be find in [36]. It can also be improved. Every non-zero endomorphism of the algebra $\mathcal{M}_n(\mathbb{F})$ is inner. Indeed, the kernel of an endomorphism is an ideal in $\mathcal{M}_n(\mathbb{F})$. The algebra $\mathcal{M}_n(\mathbb{F})$ is simple, i. e., there are no non-trivial two-sided ideals in $\mathcal{M}_n(\mathbb{F})$. So, if $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is a non-zero endomorphism, it must be injective and thus, automatically bijective.

A more general approach is to consider $\mathcal{M}_n(\mathbb{F})$ only as a ring. If $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism, we can apply it entrywise on matrices, to obtain a ring homomorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$,

$$\varphi(A) = \hat{f}(A) = [f(a_{ij})]_{i,j=1}^n.$$

Here we have the following result: every bijective additive map $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ satisfying $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$ has the form

$$\varphi(A) = S\hat{f}(A)S^{-1}$$

where $S \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix and $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism.

Recall that a map $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is called an *anti-automorphism* of the algebra $\mathcal{M}_n(\mathbb{F})$ if it is bijective, linear, and satisfies $\varphi(AB) = \varphi(B)\varphi(A)$

for all $A, B \in \mathcal{M}_n(\mathbb{F})$. The transposition map $A \mapsto A^T$ is an example of such a map. It is a straightforward consequence of the theorem on algebra automorphisms, that every anti-automorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ has the form

$$\varphi(A) = SA^T S^{-1},$$

where $S \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix.

A map $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is called a *Jordan automorphism* of the algebra $\mathcal{M}_n(\mathbb{F})$ if it is bijective, linear, and satisfies $\varphi(A^2) = \varphi(A)^2$ for every $A \in \mathcal{M}_n(\mathbb{F})$. It follows from [12] and [37] that every Jordan automorphism of $\mathcal{M}_n(\mathbb{F})$, $\mathrm{char}\mathbb{F} \neq 2$ is either an automorphism or an anti-automorphism. Thus every Jordan automorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$, $\mathrm{char}\mathbb{F} \neq 2$ has the form

$$\varphi(A) = SAS^{-1} \text{ for all } A \in \mathcal{M}_n(\mathbb{F}),$$

or

$$\varphi(A) = SA^T S^{-1} \text{ for all } A \in \mathcal{M}_n(\mathbb{F}),$$

where $S \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix.

The next step from ring endomorphisms is to omit the additivity assumption and consider multiplicative maps on matrix algebras, thus homomorphisms of matrix semigroups. One way to get a semigroup homomorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is to take a group homomorphism $\varphi' : GL_n(\mathbb{F}) \to GL_m(\mathbb{F})$ and trivially extend it to all matrices taking $\varphi(A) = 0$ for every A with $\det A = 0$. This trivial extensions are called *degenerate*. A group homomorphism $\varphi : GL_n(\mathbb{F}) \to GL_m(\mathbb{F})$ can be viewed as a representation of a full matrix group $GL_n(\mathbb{F})$ in $GL_m(\mathbb{F})$. The theory of group representations is well-developed but highly non-trivial. In the case of the field of complex numbers $\mathbb{C}$ the problem of representations of $GL_n(\mathbb{F})$ was solved by Schur in 1901 (see [33]), today the proof is based on the Weyl theory of representations of semisimple Lie groups (see for example [39], page 115-136 or [8], page 231). In the case of finite fields the problem is covered by the theory of representations of finite groups (for example [34]). The theory for infinite fields of an arbitrary characteristic can be found in [9]. Other literature on group representations includes [2], [26], [29] and [41].

We will give here the description of differentiable representations of the full complex matrix group $GL_n(\mathbb{C})$. Denote $G = GL_n(\mathbb{C}), V = \mathbb{C}^n$ and choose $\mathbf{a} = (a_1, a_2, ..., a_{n-1}, a_n)$ an n-tuple of integers satisfying $a_1 \geq 0, a_2 \geq 0, ..., a_{n-1} \geq 0$ and a_n arbitrary. Let

$$\Psi_{\mathbf{a}} : G \to \mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes ... \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V)$$

be defined by

$$\Psi_{\mathbf{a}}(A) = \mathrm{Sym}^{a_1} A \otimes \mathrm{Sym}^{a_2}(\wedge^2 A) \otimes ... \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} A) \cdot (\mathrm{det} A)^{a_n}$$

for every $A \in G$. Representation $\Psi_{\mathbf{a}}$ is not irreducible, so let $\Phi_{\mathbf{a}}$ be an irreducible subrepresentation of $\Psi_{\mathbf{a}}$ generated by vector

$$v = (\vee^{a_1}(e_1)) \otimes (\vee^{a_2}(e_1 \wedge e_2)) \otimes ... \otimes (\vee^{a_{n-1}}(e_1 \wedge ... \wedge e_{n-1}))$$

where $\{e_1, e_2, ..., e_n\}$ is a basis for V. Vector v is a highest weight vector of the representations $\Psi_{\mathbf{a}}$ and $\Phi_{\mathbf{a}}$. Every (differentiable) irreducible complex representation of G is isomorphic to $\Phi_{\mathbf{a}}$ for a unique index $\mathbf{a} = (a_1, a_2, ..., a_{n-1}, a_n)$ with $a_1, ..., a_{n-1} \geq 0$. For more details see [8].

The problem of homomorphisms $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is solved for $m \leq n$ in [13] (see Theorem 2.2). The problem for $n = 1$ of homomorphisms $\varphi : \mathbb{C} \to \mathcal{M}_m(\mathbb{C})$ in the field of complex numbers is solved in [28].

Beside multiplicative maps on full matrix algebras we may be interested also in maps that are multiplicative with respect to Jordan product or Lie product. Namely, $\mathcal{M}_n(\mathbb{F})$ can be equipped with other products like Lie product $[A, B] = AB - BA$, or Jordan product $A \circ B = AB + BA$. Maps that are multiplicative with respect to Lie or Jordan product are maps satisfying the following equations

$$\varphi(AB - BA) = \varphi(A)\varphi(B) - \varphi(B)\varphi(A),$$

$$\varphi(AB + BA) = \varphi(A)\varphi(B) + \varphi(B)\varphi(A),$$

for all $A, B \in \mathcal{M}_n(\mathbb{F})$. A related problem is to characterize maps that are multiplicative with respect to Jordan triple product, i. e. maps satisfying

$$\varphi(ABA) = \varphi(A)\varphi(B)\varphi(A),$$

for all $A, B \in \mathcal{M}_n(\mathbb{F})$.

Next, instead of considering maps that are multiplicative with respect to one of the above products on the full matrix algebra we can consider such maps on any subset that is closed under this product. For example, we can ask what is the general form of maps acting on upper triangular matrices that are multiplicative with respect to one of the above products. The set of all symmetric matrices and the set of all complex hermitian matrices are closed under Jordan product and under Jordan triple product, while the set of skew symmetric matrices and the set of skew hermitian matrices are closed under Lie product. So, we can study Jordan multiplicative and Lie multiplicative maps on these sets. Further, we can try to solve this kind of problems on matrices over commutative rings or division algebras or on operator algebras over a Banach space.

There has been a lot of work done on these questions in recent years and lots of them are still open. We will state here some results; others can be found in [4], [11], [18], [20], [21], [23], [24], [25], [30], [38] and [40].

In [7] Dolinar proved that every bijective map $\varphi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ that is multiplicative under Lie product has one of the following forms

$$\varphi(A) = S\hat{f}(A)S^{-1} + g(A)I \ \text{ for all } \ A \in \mathcal{M}_n(\mathbb{C}),$$

or

$$\varphi(A) = -S\hat{f}(A^T)S^{-1} + g(A)I \ \text{ for all } \ A \in \mathcal{M}_n(\mathbb{C}),$$

where $f : \mathbb{C} \to \mathbb{C}$ is a field homomorphism $S \in \mathcal{M}_m(\mathbb{F})$ is an invertible matrix and $g : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ a function satisfying $g(C) = 0$ for every trace zero matrix C.

Cheung in [3] studied the following problem: If G is a multiplicative semigroup of $\mathcal{M}_n(\mathbb{C})$ and if $f : G \to \mathbb{C}$ is a function, then $\Phi(f)$ denotes the set of all multiplicative maps $\varphi : G \to \mathcal{M}_k(\mathbb{C})$, for some k, such that the $(1,1)$-entry of $\varphi(A)$ is $f(A)$, for every $A \in G$. The set $\Phi(f)$ is nonempty for a variety of functions f, including linear functionals on $\mathcal{M}_n(\mathbb{C})$. Further, if $f, g : G \to \mathbb{C}$ and neither $\Phi(f)$ nor $\Phi(g)$ is empty, then one can describe all multiplicative maps $\tau : G \to \mathcal{M}_n(\mathbb{C})$ such that $f(A) = g(\tau(A))$, for every $A \in G$.

Cao and Zhang in [5] dealt with the semigroup of upper triangular matrices $T_n(R)$ over a ring R. They proved that if $n \geq 2$ and R is a semiprime ring or a ring in which all idempotents are central, then $\varphi : T_n(R) \to T_n(R)$ is a multiplicative semigroup automorphism if and only if there exist a nonsingular matrix S in $T_n(R)$ and a ring automorphism f of R such that $\varphi(A) = S\hat{f}(A)S^{-1}$ for all $A \in T_n(R)$.

Let X and Y be complex Banach spaces with $\dim X \geq 3$, and let $\mathcal{A} \subset B(X)$

and $\mathcal{B} \subset B(Y)$ be standard operator algebras (that is, algebras of bounded linear operators that contain all finite-rank operators). Šemrl in [35] described multiplicative bijective maps of $\mathcal{A}$ onto $\mathcal{B}$, while Molnar in [22] dealt with a problem of bijective maps multiplicative under Jordan triple product. It is proved that such a map is necessarily linear or conjugate-linear in the case when X is infinite-dimensional. Lu in [19] proved that if $\mathcal{A}$ is a standard operator algebra on a Banach space X with $\dim X > 1$, R any ring, and $\varphi : \mathcal{A} \to R$ a Jordan multiplicative bijective map, then ϕ is either a ring isomorphism or a ring anti-isomorphism.

Chapter 2

State of the art

In this chapter we first state our main question and give some examples. We characterize all homomorphisms from the multiplicative semigroup of all n-by-n matrices over an arbitrary field to the field and all non-degenerate homomorphisms from the multiplicative semigroup of all n-by-n matrices over an arbitrary field to a semigroup of m-by-m matrices over the same field, if $m \leq n$.

2.1 Question and examples

Let $\mathbb{F}$ be an arbitrary field and let n be an integer. Denote by $\mathcal{M}_n(\mathbb{F})$ the set of all n-by-n matrices with entries in $\mathbb{F}$. Then $\mathcal{M}_n(\mathbb{F})$ is a semigroup under the multiplication of matrices.

Question. What are semigroup homomorphisms $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, i. e. maps satisfying the equation

$$\varphi(AB) = \varphi(A)\varphi(B)$$

for all matrices $A, B \in \mathcal{M}_n(\mathbb{F})$?

Sometimes we will be interested only in irreducible homomorphisms. A semigroup homomorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is *irreducible* if the image of φ is an irreducible semigroup i. e. it has no proper non-trivial invariant subspace of $\mathbb{F}^m$ when it is viewed as a set of matrices acting on vector space $\mathbb{F}^m$.

We first give some examples.

Examples:

1. Matrix conjugation: Let $S \in \mathcal{M}_n(\mathbb{F})$ be an invertible matrix. Then $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$

$$\varphi(A) = SAS^{-1}$$

is a semigroup homomorphism.

2. Constant: Assume $E \in \mathcal{M}_m(\mathbb{F})$ is an idempotent, i. e. a matrix satisfying equation $E^2 = E$. Then $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$

$$\varphi(A) = E$$

is a semigroup homomorphism. We would like to avoid such trivial homomorphisms. Lemma 3.1 tells us the following: Let $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ be a semigroup homomorphism. Then φ has the form

$$\varphi(A) = S(\varphi_0(A) \oplus E)S^{-1},$$

where $\varphi_0 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_k(\mathbb{F})$ is a semigroup homomorphism with $\varphi_0(0) = 0$, $\varphi_0(I) = I$, $E \in \mathcal{M}_{m-k}(\mathbb{F})$ is idempotent and $S \in \mathcal{M}_m(\mathbb{F})$ is an invertible matrix. Here either k or $m - k$ may be 0, i. e. either $\varphi_0(A)$ or E may be absent. So we may assume that a semigroup homomorphism maps 0 to 0 and identity to identity. If we assume that φ is irreducible, this is automatically true.

3. Field homomorphism used entrywise: Let $f : \mathbb{F} \to \mathbb{F}$ be a field homomorphism. If $A = [a_{ij}]_{i,j=1}^n \in \mathcal{M}_n(\mathbb{F})$ is an arbitrary matrix, define

$$\varphi(A) = \hat{f}(A) = [f(a_{ij})]_{i,j=1}^n.$$

Then $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is a semigroup homomorphism.

4. Degenerate homomorphisms: If $\varphi' : GL_n(\mathbb{F}) \to GL_m(\mathbb{F})$ is a group homomorphism, define $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$: if $\det A = 0$ take $\varphi(A) = 0$ and if $\det A \neq 0$ take $\varphi(A) = \varphi'(A)$. It is obvious that φ is a semigroup homomorphism. Such homomorphisms are called *degenerate*. Since group homomorphisms $\varphi' : GL_n(\mathbb{F}) \to GL_m(\mathbb{F})$ are known (see chapter 1) we will restrict ourselves to non-degenerate homomorphisms.

5. Direct sum: If $\varphi' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'}(\mathbb{F})$ and $\varphi'' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m''}(\mathbb{F})$ are two semigroup homomorphisms, define $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'+m''}(\mathbb{F})$,

$$\varphi(A) = \varphi'(A) \oplus \varphi''(A)$$

for every matrix $A \in \mathcal{M}_n(\mathbb{F})$. Map φ is again a semigroup homomorphism, which is always reducible.

6. Exterior power: Let $k \leq n$ be an integer. The vector space $\mathbb{F}^{\binom{n}{k}}$ is isomorphic to the exterior power $\wedge^k \mathbb{F}^n$ of all antisymmetric k-tensors. If

$$\mathcal{E} = \{e_1, e_2, ..., e_n\}$$

is a basis of $\mathbb{F}^n$, then

$$\mathcal{E}' = \{e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}, 1 \leq i_1 < i_2 < ... < i_k \leq n\}$$

is a basis of $\wedge^k \mathbb{F}^n$. If a matrix $A \in \mathcal{M}_n(\mathbb{F})$ represents a linear mapping of $\mathbb{F}^n$, then $\wedge^k A \in \mathcal{M}_{\binom{n}{k}}(\mathbb{F})$ represents a linear mapping, acting on k-tensors as follows:

$$(\wedge^k A)(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}) = Ae_{i_1} \wedge Ae_{i_2} \wedge ... \wedge Ae_{i_k}.$$

The entries of the matrix $\wedge^k A$ are all k-by-k minors of the matrix A. It is a direct calculation to prove that

$$(\wedge^k A)(\wedge^k B) = \wedge^k (AB)$$

So $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n}{k}}(\mathbb{F})$, defined as

$$\varphi(A) = \wedge^k A$$

is a semigroup homomorphism. If k equals $n-1$, then $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$. In this case $\varphi(A) = \wedge^{n-1} A = \mathrm{Cof}(A)$ is the so called cofactor matrix of all $(n-1)$-by-$(n-1)$ minors of matrix A.

7. Symmetric power: Let k be an integer. The vector space $\mathbb{F}^{\binom{n+k-1}{k}}$ is isomorphic to the symmetric power $\mathrm{Sym}^k\mathbb{F}^n$ of all symmetric k-tensors. If

$$\mathcal{E} = \{e_1, e_2, ..., e_n\}$$

is a basis of $\mathbb{F}^n$, then

$$\mathcal{E}' = \{e_{i_1} \vee e_{i_2} \vee ... \vee e_{i_k}; 1 \le i_1 \le i_2 \le ... \le i_k \le n\}$$

is a basis of $\mathrm{Sym}^k\mathbb{F}^n$. If a matrix $A \in \mathcal{M}_n(\mathbb{F})$ represents a linear mapping of $\mathbb{F}^n$, then $\mathrm{Sym}^k A \in \mathcal{M}_{\binom{n+k-1}{k}}(\mathbb{F})$ represents a linear mapping, acting on k-tensors as follows:

$$(\mathrm{Sym}^k A)(e_{i_1} \vee e_{i_2} \vee ... \vee e_{i_k}) = Ae_{i_1} \vee Ae_{i_2} \vee ... \vee Ae_{i_k}.$$

It is again a direct calculation to prove that

$$(\mathrm{Sym}^k A)(\mathrm{Sym}^k B) = \mathrm{Sym}^k (AB)$$

So $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{\binom{n+k-1}{k}}(\mathbb{F})$, defined as

$$\varphi(A) = \mathrm{Sym}^k A$$

is a semigroup homomorphism. In a special case $n = 2$ we have

$$\mathcal{E}' = \{e_{(1)}, ..., e_{(k+1)}\}$$

where

$$e_{(i)} = (\vee^{k+1-i} e_1) \vee (\vee^{i-1} e_2).$$

So, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$(\mathrm{Sym}^k A) e_{(i)} = (\vee^{k+1-i}(ae_1 + ce_2)) \vee (\vee^{i-1}(be_1 + de_2)).$$

Thus

$$\mathrm{Sym}^k \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$= \left[\sum_{s=\max\{0,k+2-i-j\}}^{\min\{k+1-i,k+1-j\}} \binom{k+1-i}{s} \binom{i-1}{k+1-j-s} a^s b^{k+1-j-s} c^{k+1-i-s} d^{i+j+s-k-2} \right]_{i,j=1}^{k+1}.$$

If also $k = 2$, we have

$$\mathrm{Sym}^2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}$$

8. Tensor product: The vector space $\mathbb{F}^{m'm''}$ is isomorphic to tensor product $\mathbb{F}^{m'} \otimes \mathbb{F}^{m''}$. If $\varphi' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'}(\mathbb{F})$ and $\varphi'' : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m''}(\mathbb{F})$ are two semigroup homomorphisms, define $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m'm''}(\mathbb{F})$ by

$$\varphi(A) = \varphi'(A) \otimes \varphi''(A)$$

for every matrix $A \in \mathcal{M}_n(\mathbb{F})$. Map φ is again a semigroup homomorphism. The tensor product $\varphi(A) = A \otimes A$ can be written as a direct sum of the exterior power and the symmetric power

$$A \otimes A = A \wedge A \oplus \mathrm{Sym}^2 A.$$

If the factors are not equal, the tensor product may be irreducible (see for example case (b) of Theorem 4.3).

9. Compositions of the above

If $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is an non-degenerate irreducible homomorphism, these examples show that φ can be a tensor product, an exterior power or a symmetric power combined with field homomorphisms used entrywise and matrix conjugation. In all these examples m is not arbitrary; it has a special form, depending on n. We will show that under some additional assumptions for small n and m this is all that we can get.

2.2 Case $m = 1$

We will now characterize homomorphisms from the matrix semigroup $\mathcal{M}_n(\mathbb{F})$ to the field $\mathbb{F}$ as a multiplicative semigroup. It is a well known result. Our proof is due to A. Jafarian and H. Radjavi.

Proposition 2.1 *Let $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ be a semigroup homomorphism. Then there exists a homomorphism $f : \mathbb{F} \to \mathbb{F}$ of the multiplicative semigroup $(\mathbb{F}, \cdot)$ such that*

$$\varphi(A) = f(\det A)$$

for every $A \in \mathcal{M}_n(\mathbb{F})$.

Proof. If $\varphi(A) = 0$ for all $A \in \mathcal{M}_n(\mathbb{F})$, take $f = 0$, if $\varphi(A) = 1$ for all A, take $f = 1$. So assume otherwise. Then it is clear that invertible matrices have nonzero images and that $\varphi(A) = 0$ whenever A is singular. Next define $f : \mathbb{F} \to \mathbb{F}$ by

$$f(x) = \varphi\left(\begin{bmatrix} x & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right).$$

It follows that f is a semigroup homomorphism. Since the relation $\varphi(A) = f(\det A)$ trivially holds for singular A, we must only verify it for invertible matrices A. Now every such A can be expressed as

$$A = \begin{bmatrix} \det A & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} A_1$$

with $\det A_1 = 1$. By [31] every matrix with determinant 1 is a product of simple involutions, that is matrices $E \in \mathcal{M}_n(\mathbb{F})$ with $E^2 = I$ and $\mathrm{rank}(E - I) = 1$. So we have

$$A_1 = E_1 E_2 ... E_k$$

If $\mathrm{char} F = 2$, then every simple involution is similar to

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since $\varphi(E_i)$ is also an involution in $\mathbb{F}$, we have $\varphi(E_i) = 1$ for all i and consequently $\varphi(A_1) = 1$. If $\mathrm{char} F \neq 2$, then every simple involution is similar to

$$\begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since $\det E_i = -1$, the number of involutions k must be even. As before, $\varphi(E_i)$ is an involution in $\mathbb{F}$, so we have $\varphi(E_i) = \pm 1$ for all i. We observe that E_i is similar to E_j for all i and j, and since similar matrices have equal images under φ, either $\varphi(E_i) = 1$ for all i or $\varphi(E_i) = -1$ for all i. In either case

$$\varphi(A_1) = \varphi(E_1)\varphi(E_2)...\varphi(E_k) = (\pm 1)^k = 1$$

Now

$$\varphi(A) = \varphi\left(\begin{bmatrix} \det A & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right)\varphi(A_1) = f(\det A)$$

as desired. $\square$

2.3 Case $m \leq n$

The main result in this chapter is characterization of all non-degenerate semigroup homomorphisms $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$, where $m \leq n$. The result was proved in [17] for the field of complex numbers and in [13] for an arbitrary integral domain.

Theorem 2.2 *Let $\mathbb{F}$ be a field. Assume that integers n, m satisfy $n \geq 2$ and $m \leq n$. Let $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ be a semigroup homomorphism, which is non-degenerate and has the properties $\varphi(0) = 0$ and $\varphi(I) = I$. Then $m = n$ and φ has one of the following forms:*
(a)

$$\varphi(A) = S\hat{f}(A)S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism, and $S \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix, or

(b)

$$\varphi(A) = S\hat{f}(\mathrm{Cof}(A))S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ *is a field homomorphism, and* $S \in \mathcal{M}_n(\mathbb{F})$ *is an invertible matrix.*

We will later (especially in chapter 5) extend the proof of this theorem to more general setting. We will give the proof in Section 5.2.

Chapter 3

Homomorphisms from dimension two to three

In this chapter we characterize all non-degenerate homomorphisms from the multiplicative semigroup of all 2-by-2 matrices over an arbitrary field to the semigroup of 3-by-3 matrices over the same field. If the characteristic of the field is not equal to 2 then we have two possibilities. Either it is a symmetric square, combined with a field homomorphism used entrywise and a matrix conjugation, or a direct sum of the identity and the determinant, combined with a field homomorphism, a homomorphism of the multiplicative semigroup of the field and a matrix conjugation. In the characteristic 2 a symmetric square gives rise to two different homomorphisms and we get three possibilities. In the case of the field of real numbers every irreducible non-degenerate homomorphism is a matrix conjugation of the symmetric square.

3.1 Preliminaries

We will first show that there is no loss of generality if we assume that a semigroup homomorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ maps 0 to 0 and the identity to the identity.

Lemma 3.1 *Let* $\mathbb{F}$ *be a field and* $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ *a semigroup homomorphism. Then* φ *has the form*

$$\varphi(A) = S(\varphi_0(A) \oplus E)S^{-1},$$

where $\varphi_0 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_k(\mathbb{F})$ *is a semigroup homomorphism with* $\varphi_0(0) = 0$, $\varphi_0(I) = I$, $E \in \mathcal{M}_{m-k}(\mathbb{F})$ *is idempotent and* $S \in \mathcal{M}_m(\mathbb{F})$ *is an invertible matrix. Here either* k *or* $m - k$ *may be* 0*, i. e. either* $\varphi_0(A)$ *or* E *may be absent.*

Proof. Since 0 and I are two commuting idempotents with $0I = 0$, $\varphi(0)$ and $\varphi(I)$ are also two commuting idempotents with $\varphi(0)\varphi(I) = \varphi(0)$. So they have the form

$$\varphi(0) = S(0_k \oplus I_l \oplus 0_{m-l-k})S^{-1}$$

and

$$\varphi(I) = S(I_k \oplus I_l \oplus 0_{m-l-k})S^{-1},$$

where $0_s, I_s \in \mathcal{M}_s(\mathbb{F})$ and S is an invertible matrix. For any matrix $A \in \mathcal{M}_n(\mathbb{F})$ the matrix $\varphi(A)$ commutes with $\varphi(0)$ and $\varphi(I)$, so it has the form

$$\varphi(A) = S(A_1 \oplus A_2 \oplus A_3)S^{-1}.$$

Since $A0 = 0$ and $AI = A$ we have $A_2 I_l = I_l$ and $A_3 0_{m-l-k} = A_3$, so $A_2 = I_l$ and $A_3 = 0_{m-l-k}$. Writing $\varphi_0(A) := A_1$ we obtain the asserted form, since φ_0 is obviously a semigroup homomorphism. $\square$

In the proof of our main result we need the following proposition which is proved in [13].

Proposition 3.2 *Let $\mathbb{F}$ be a field and $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_2(\mathbb{F})$ a semigroup homomorphism, which is non-degenerate and has the properties $\varphi(0) = 0$ and $\varphi(I) = I$. Then φ has the form*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_2(\mathbb{F})$ is an invertible matrix.

The following proposition is a special case of Proposition 2.1 for $n = 2$.

Proposition 3.3 *Let $\mathbb{F}$ be a field and $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathbb{F}$ a semigroup homomorphism. Then φ has the form*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = h(ad - bc),$$

where $h : \mathbb{F} \to \mathbb{F}$ is a homomorphism of the multiplicative semigroup $(\mathbb{F}, \cdot)$.

3.2 Main result

The main result of this chapter is the following:

Theorem 3.4 *Let $\mathbb{F}$ be a field and $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$ a semigroup homomorphism, which is non-degenerate and has the properties $\varphi(0) = 0$ and $\varphi(I) = I$. If $\operatorname{char} \mathbb{F} \neq 2$ then φ has one of the following forms:*

(a)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & f(b) & 0 \\ f(c) & f(d) & 0 \\ 0 & 0 & g(ad - bc) \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism, $g : \mathbb{F} \to \mathbb{F}$ is a homomorphism of the multiplicative semigroup $(\mathbb{F}, \cdot)$ with $g(0) = 0$, $g(1) = 1$ and $S \in \mathcal{M}_3(\mathbb{F})$ is an invertible matrix,

(b)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix} S^{-1},$$

where $h : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_3(\mathbb{F})$ is an invertible matrix.

If char $\mathbb{F} = 2$ *then φ has one of the forms (a), (b) or*

(c)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & 0 & h(b^2) \\ h(ac) & h(ad+bc) & h(bd) \\ h(c^2) & 0 & h(d^2) \end{bmatrix} S^{-1},$$

where $h : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_3(\mathbb{F})$ is an invertible matrix.

Remark. If char $\mathbb{F} = 2$, the cases (b) and (c) are essentially different: The image of φ in the case (b) has exactly one non-trivial invariant subspace in common, which has dimension 2. On the other hand, in the case (c) the image of φ has an invariant subspace of dimension 1 in common.

Proof. Let us denote by E_{ij} the matrix which has 1 in the i-th row and the j-th column, and 0 elsewhere. We will divide the proof into several steps.

Step 1. Without loss of generality we may assume that $\varphi(E_{12}) = E_{13}$ and $\varphi(E_{21}) = E_{31}$. Then $\varphi(E_{11}) = E_{11}$ and $\varphi(E_{22}) = E_{33}$.

Proof: Matrix E_{12} is nilpotent of order 2, so $\varphi(E_{12})$ must be nilpotent of order at most 2. Let us suppose that $\varphi(E_{12}) = 0$. If $A \in \mathcal{M}_2(\mathbb{F})$ is any non-invertible matrix, it has rank at most 1 and we can write it as $A = PE_{12}Q$. So $\varphi(A) = \varphi(P)\varphi(E_{12})\varphi(Q) = 0$ and φ is degenerate. Thus $\varphi(E_{12})$ must be nonzero and we can write it as $\varphi(E_{12}) = xy^T$ where x, y are two column vectors in $\mathbb{F}^3$ and $y^T x = 0$. Similarly we obtain $\varphi(E_{21}) = uv^T$ where $v^T u = 0$. Since

$E_{12}E_{21}E_{12} = E_{12}$, we have

$$xy^T uv^T xy^T = xy^T,$$

so $y^T u \cdot v^T x = 1$. With no loss of generality we may assume that $y^T u = v^T x = 1$. Let us choose a vector $z \in \mathbb{F}^3$ orthogonal to v and y, i. e. $v^T z = y^T z = 0$. Then $\{x, z, u\}$ is a basis of $\mathbb{F}^3$. In this basis $\varphi(E_{12})$ has the matrix E_{13} and $\varphi(E_{21})$ has the matrix E_{31}. So without loss of generality we may assume that $\varphi(E_{12}) = E_{13}$ and $\varphi(E_{21}) = E_{31}$. Then

$$\varphi(E_{11}) = \varphi(E_{12}E_{21}) = E_{13}E_{31} = E_{11}$$

and similarly $\varphi(E_{22}) = E_{33}$.

Step 2. $\varphi(aI)$ has the form $f(a)(E_{11} + E_{33}) + g(a)E_{22}$ where $f, g : \mathbb{F} \to \mathbb{F}$ are semigroup homomorphisms with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$.

Proof: Matrix aI commutes with E_{12} and E_{21}, so $\varphi(aI)$ commutes with E_{13} and E_{31} and we obtain the asserted form.

Step 3. Homomorphism φ has the form

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} f(a) & * & f(b) \\ * & * & * \\ f(c) & * & f(d) \end{bmatrix}.$$

Proof: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an arbitrary matrix, we have

$$E_{11}\varphi(A)E_{11} = \varphi(E_{11}AE_{11}) = \varphi(aE_{11}) = \varphi(aI)E_{11} = f(a)E_{11},$$

so the element in the first row and the first column of $\varphi(A)$ must be $f(a)$. We argue similarly for the other corners.

Step 4. If A is upper-right (resp. upper-left, lower-right, lower-left) triangular, then $\varphi(A)$ is upper-right (resp. upper-left, lower-right, lower-left) triangular. If A is diagonal, then $\varphi(A)$ is diagonal. A similar result holds for counter-diagonal A.

Proof: Let

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

Then

$$\varphi(A)E_{11} = \varphi(AE_{11}) = \varphi(aE_{11}) = f(a)E_{11}$$

and

$$E_{33}\varphi(A) = \varphi(E_{22}A) = \varphi(dE_{22}) = f(d)E_{33}$$

so the first column of $\varphi(A)$ must be $[f(a), 0, 0]^T$ and the last row must be $[0, 0, f(d)]$. Thus $\varphi(A)$ is upper-right triangular. Similarly we prove the other cases.

Step 5. If $f(a) \neq g(a)$ for some $a \in \mathbb{F}$, then

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} f(a) & 0 & f(b) \\ 0 & h(ad-bc) & 0 \\ f(c) & 0 & f(d) \end{bmatrix},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism, $h : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism, so we are in the case (a) of the Theorem.

Proof: Matrix aI commutes with every $A \in \mathcal{M}_2(\mathbb{F})$, so $\varphi(aI) = f(a)(E_{11}+E_{33}) + g(a)E_{22}$ commutes with $\varphi(A)$. Since $f(a) \neq g(a)$, $\varphi(A)$ has the form

$$\begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix}.$$

Thus

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} f(a) & 0 & f(b) \\ 0 & s(a,b,c,d) & 0 \\ f(c) & 0 & f(d) \end{bmatrix}.$$

So homomorphism φ is a direct sum of two semigroup homomorphisms $\varphi_1 : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_2(\mathbb{F})$ and $\varphi_2 : \mathcal{M}_2(\mathbb{F}) \to \mathbb{F}$ where

$$\varphi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix}$$

and

$$\varphi_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = s(a,b,c,d).$$

Now, f is a field homomorphism by Proposition 3.2 and $s(a,b,c,d)$ has the form $h(ad-bc)$ by Proposition 3.3.

From now on we will assume that $f(a) = g(a)$ for every $a \in \mathbb{F}$. So $\varphi(aI) = f(a)I$.

Step 6. If $\det A = 1$, then $\det\varphi(A) = 1$. Furthermore, $f(-1) = 1$ and

$$\varphi(E_{12} - E_{21}) = E_{13} - E_{22} + E_{31}.$$

Proof: Let $\varphi_1 : \mathcal{M}_2(\mathbb{F}) \to \mathbb{F}$ be the semigroup homomorphism $\varphi_1(A) = \det\varphi(A)$. By Proposition 3.3 it has the form $\varphi_1(A) = h(\det A)$. So, if $\det A = 1$, then $\det\varphi(A) = 1$. Now, $\det(-I) = 1$, so $\det\varphi(-I) = f(-1)^3 = 1$, thus $f(-1) = 1$. By step 4 $\varphi(E_{12} - E_{21})$ has the form $E_{13} + uE_{22} + E_{31}$. By the determinant condition we obtain $u = -1$.

From step 7 to step 14 we assume that char $\mathbb{F} \neq 2$.

Step 7. Without loss of generality we may assume

$$\varphi(E_{11} + E_{12}) = E_{11} + E_{12} + E_{13}, \qquad \varphi(E_{11} + E_{21}) = E_{11} + 2E_{21} + E_{31},$$

$$\varphi(E_{21} + E_{22}) = E_{31} + E_{32} + E_{33}, \qquad \varphi(E_{12} + E_{22}) = E_{13} + 2E_{23} + E_{33}.$$

Proof: Every matrix of rank one has the form $A = PE_{12}Q$ with P, Q invertible. So its image has the form $\varphi(A) = \varphi(P)E_{13}\varphi(Q)$. Thus every matrix of rank 1 is sent to a matrix of rank 1. So the matrix $\varphi(E_{11}+E_{12})$ has rank 1. Since it is upper triangular, we have

$$\varphi(E_{11}+E_{12}) = E_{11} + xE_{12} + E_{13}.$$

Similarly

$$\varphi(E_{11}+E_{21}) = E_{11} + yE_{21} + E_{31},$$

$$\varphi(E_{21}+E_{22}) = E_{31} + zE_{32} + E_{33}, \qquad \varphi(E_{12}+E_{22}) = E_{13} + tE_{23} + E_{33}.$$

Now,

$$\begin{bmatrix} 1 & x & 1 \\ y & xy & y \\ 1 & x & 1 \end{bmatrix} = \varphi\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & z & 1 \\ t & zt & t \\ 1 & z & 1 \end{bmatrix},$$

so $x = z$ and $y = t$. Furthermore,

$$\varphi(0) = \varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2-xy & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $xy = 2$. Since char $\mathbb{F} \neq 2$, both x and y are nonzero. If we take

$$\varphi'(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \varphi(A) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain

$$\varphi'(E_{11}+E_{12}) = E_{11} + E_{12} + E_{13}.$$

Homomorphism φ' has all the properties we have proved for φ. So without loss of generality we may assume $x = 1$ and thus $y = 2$. (Actually we have multiplied the vector z from step 1 by a scalar, so we have chosen its length which was arbitrary in step 1.)

Step 8. $\varphi(E_{11}-E_{22}) = E_{11}-E_{22}+E_{33}$ and $\varphi(E_{12}+E_{21}) = E_{13}+E_{22}+E_{31}$.

Proof: We have

$$\varphi(E_{11}-E_{22}) = E_{11}+vE_{22}+E_{33},$$

so

$$E_{11}+vE_{12}+E_{13} = (E_{11}+E_{12}+E_{13})(E_{11}+vE_{22}+E_{33}) =$$
$$= \varphi((E_{11}+E_{12})(E_{11}-E_{22})) = \varphi((E_{11}+E_{12})(E_{21}-E_{12})) =$$
$$= (E_{11}+E_{12}+E_{13})(E_{13}-E_{22}+E_{31}) = E_{11}-E_{12}+E_{13}.$$

Thus $v=-1$. Now,

$$\varphi(E_{12}+E_{21}) = \varphi((E_{21}-E_{12})(E_{11}-E_{22})) = E_{13}+E_{22}+E_{31}.$$

Step 9.

$$\varphi\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} \quad \text{and} \quad \varphi\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}.$$

Proof: We have

$$\varphi\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & u & 1\\ 0 & v & w\\ 0 & 0 & 1\end{bmatrix}.$$

Since $\det\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = 1$, v must be 1. Furthermore,

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 1 & 1 & 1\end{bmatrix} = \varphi\left(\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\right) =$$
$$= \varphi\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 1 & 1\\ w & w & w\\ 1 & 1 & 1\end{bmatrix},$$

so $w = 2$. Similarly we prove $u = 1$ and the other equality.

Step 10. Mapping $f : \mathbb{F} \to \mathbb{F}$ has the form $f(a) = (h(a))^2$, where $h : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism.

Proof: We have

$$\varphi(aE_{11} + E_{22}) = f(a)E_{11} + h(a)E_{22} + E_{33},$$

where $h : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism. Now,

$$f(a)I = \varphi(aI) = \varphi((aE_{11} + E_{22})(E_{12} + E_{21})(aE_{11} + E_{22})(E_{12} + E_{21})) =$$

$$= (f(a)E_{11} + h(a)E_{22} + E_{33})(E_{13} + E_{22} + E_{31})\cdot$$

$$\cdot(f(a)E_{11} + h(a)E_{22} + E_{33})(E_{13} + E_{22} + E_{31}) =$$

$$= f(a)E_{11} + h(a)^2E_{22} + f(a)E_{33}.$$

So $f(a) = h(a)^2 = h(a^2)$.

Step 11. $\varphi(aE_{11} + bE_{22}) = h(a^2)E_{11} + h(ab)E_{22} + h(b^2)E_{33}$ and $\varphi(aE_{12} + bE_{21}) = h(a^2)E_{13} + h(ab)E_{22} + h(b^2)E_{31}$.

Proof: If $b \neq 0$, we have

$$\varphi(aE_{11} + bE_{22}) = \varphi(bI(\frac{a}{b}E_{11} + E_{22})) =$$

$$f(b)f(\frac{a}{b})E_{11} + f(b)h(\frac{a}{b})E_{22} + f(b)E_{33} = h(a^2)E_{11} + h(ab)E_{22} + h(b^2)E_{33}$$

and

$$\varphi(aE_{12}+bE_{21}) = \varphi((aE_{11}+bE_{22})(E_{12}+E_{21})) = h(a^2)E_{13}+h(ab)E_{22}+h(b^2)E_{31}.$$

Step 12. Mapping $h : \mathbb{F} \to \mathbb{F}$ is a field homomorphism.

Proof: We have to prove that h is additive.

$$\begin{bmatrix} h(a^2) & h(a(a+b)) & h((a+b)^2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a+b \end{bmatrix}\right) =$$

$$= \varphi\left(\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) =$$

$$= \begin{bmatrix} h(a^2) & h(a^2)+h(ab) & h(a^2)+2h(ab)+h(b^2)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

So $h(a(a+b)) = h(a^2)+h(ab)$. If $a \neq 0$, it follows that $h(a+b) = h(a)+h(b)$.

Step 13.

$$\varphi\left(\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}\right) = \begin{bmatrix} h(a^2) & h(ab) & h(b^2)\\ 0 & h(ad) & h(2bd)\\ 0 & 0 & h(d^2)\end{bmatrix}$$

and

$$\varphi\left(\begin{bmatrix}a & 0\\ c & d\end{bmatrix}\right) = \begin{bmatrix} h(a^2) & 0 & 0\\ h(2ac) & h(ad) & 0\\ h(c^2) & h(cd) & h(d^2)\end{bmatrix}.$$

Proof: If $b \neq 0$, we have

$$\varphi\left(\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}\right) = \varphi\left(\begin{bmatrix}1 & 0\\ 0 & d/b\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\right) =$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & h(d/b) & 0\\ 0 & 0 & h(d^2/b^2)\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}h(a^2) & 0 & 0\\ 0 & h(ab) & 0\\ 0 & 0 & h(b^2)\end{bmatrix} =$$

$$= \begin{bmatrix} h(a^2) & h(ab) & h(b^2)\\ 0 & h(ad) & h(2bd)\\ 0 & 0 & h(d^2)\end{bmatrix}.$$

Similarly we prove the other equality.

Step 14.

$$\varphi\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \begin{bmatrix} h(a^2) & h(ab) & h(b^2)\\ h(2ac) & h(ad+bc) & h(2bd)\\ h(c^2) & h(cd) & h(d^2)\end{bmatrix},$$

so we are in case (b) of the Theorem.

Proof: If $a \neq 0$, we have

$$\varphi\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \varphi\left(\begin{bmatrix}a & 0\\ c & d-\frac{bc}{a}\end{bmatrix}\begin{bmatrix}1 & \frac{b}{a}\\ 0 & 1\end{bmatrix}\right) =$$

$$= \begin{bmatrix} h(a^2) & 0 & 0 \\ h(2ac) & h(ad-bc) & 0 \\ h(c^2) & h(cd-\frac{bc^2}{a}) & h((d-\frac{bc}{a})^2) \end{bmatrix} \begin{bmatrix} 1 & h(\frac{b}{a}) & h(\frac{b^2}{a^2}) \\ 0 & 1 & h(\frac{2b}{a}) \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix}.$$

If $a = 0$ and $d \neq 0$, then

$$\varphi\left(\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} -\frac{bc}{d} & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} h(\frac{b^2c^2}{d^2}) & h(-\frac{b^2c}{d}) & h(b^2) \\ 0 & h(-bc) & h(2bd) \\ 0 & 0 & h(d^2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ h(\frac{2c}{d}) & 1 & 0 \\ h(\frac{c^2}{d^2}) & h(\frac{c}{d}) & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & h(b^2) \\ 0 & h(bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix}.$$

The case $a = d = 0$ we have already proved in step 11.

Step 15. If char $\mathbb{F} = 2$, then either

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ 0 & h(ad+bc) & 0 \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix}$$

or

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} h(a^2) & 0 & h(b^2) \\ h(ac) & h(ad+bc) & h(bd) \\ h(c^2) & 0 & h(d^2) \end{bmatrix}$$

or

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} h(a^2) & 0 & h(b^2) \\ 0 & h(ad-bc) & 0 \\ h(c^2) & 0 & h(d^2) \end{bmatrix},$$

where $h : \mathbb{F} \to \mathbb{F}$ is a field homomorphism, so we are in the cases (b), (c) or (a) of the Theorem.

Proof: We do the same as in step 7 and obtain $xy = 2 = 0$. If $x \neq 0$, we may assume with no loss of generality that $x = 1$ and then $y = 0 = 2$. Then everything is the same as in steps 8 - 14 and we obtain the first possibility. If $y \neq 0$, then we may assume with no loss of generality that $y = 1$ and then $x = 0 = 2$. In this case all the matrices in steps 8 - 14 are just transposed and we obtain the second possibility. If both x and y are 0, we obtain

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as in step 9 and

$$\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

by the determinant condition. The semigroup $\mathcal{M}_2(\mathbb{F})$ is generated by diagonal matrices and the three matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

as we saw in steps 13, 14. So we obtain

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} f(a) & 0 & f(b) \\ 0 & h(ad-bc) & 0 \\ f(c) & 0 & f(d) \end{bmatrix}.$$

Now $\varphi(aI) = f(a)I$ gives that $f(a) = h(a^2)$. Since f is additive by Proposition 3.2 and char $\mathbb{F} = 2$, we have

$$(h(a+b))^2 = h((a+b)^2) = f(a+b) = f(a) + f(b) =$$

$$= h(a)^2 + h(b)^2 = (h(a) + h(b))^2,$$

so h is additive as well. □

3.3 Corollaries

A matrix semigroup homomorphism φ is reducible if the image of φ has a nontrivial invariant subspace. We say that φ is *completely reducible* if every invariant subspace of the image of φ has an invariant complement.

Corollary 3.5 *Let $\mathbb{F}$ be a field with* char $\mathbb{F} \neq 2$. *Every non-degenerate semigroup homomorphism $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$ is completely reducible.*

Corollary 3.6 *Let $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$ be an irreducible non-degenerate semigroup homomorphism. Then* char $\mathbb{F} \neq 2$ *and*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix} S^{-1},$$

where $h : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_3(\mathbb{F})$ is an invertible matrix.

If $\mathbb{F}$ is the field of real numbers $\mathbb{R}$, then the only nonzero field homomorphism of $\mathbb{F}$ is the identity (see [1], page 57). This implies

Corollary 3.7 *Let $\varphi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$ be an irreducible non-degenerate semigroup homomorphism. Then*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix} S^{-1},$$

where $S \in \mathcal{M}_3(\mathbb{R})$ is an invertible matrix.

If $\mathbb{F}$ is the field of complex numbers $\mathbb{C}$ we may be interested only in continuous semigroup homomorphism $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_3(\mathbb{F})$. Then semigroup or field homomorphisms $f, g, h : \mathbb{F} \to \mathbb{F}$ in the Theorem 3.4 must be continuous. The only continuous field homomorphisms of $\mathbb{C}$ are the identity and the complex conjugation (see [1], page 53).

Corollary 3.8 *Let $\varphi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_3(\mathbb{C})$ be a continuous irreducible non-degenerate semigroup homomorphism. Then φ has the form*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} h(a^2) & h(ab) & h(b^2) \\ h(2ac) & h(ad+bc) & h(2bd) \\ h(c^2) & h(cd) & h(d^2) \end{bmatrix} S^{-1},$$

where $h : \mathbb{C} \to \mathbb{C}$ *is the identity or the complex conjugation and* $S \in \mathcal{M}_3(\mathbb{C})$ *is an invertible matrix.*

Chapter 4

More on homomorphisms from dimension two

In this chapter we study non-degenerate irreducible homomorphisms from the multiplicative semigroup of all 2-by-2 matrices over an algebraically closed field of characteristic zero to the semigroup of n-by-n matrices over the same field. If such a homomorphism maps a cyclic unipotent to a cyclic unipotent, it is the composition of a symmetric power, a field homomorphism used entrywise, and a matrix conjugation. In the case $n = 4$ we characterize all non-degenerate irreducible homomorphisms.

From now on we will assume that the field $\mathbb{F}$ has characteristic zero and is algebraically closed. We have seen in chapter 3 that from dimension two to three we get a different result in characteristic 2 from other characteristics. The situation is similar when going from dimension two to higher dimensions: the results will depend on whether the characteristic zero or finite and small. We will also restrict ourselves to irreducible homomorphisms. We will often use the following proposition which is a consequence of a theorem of Burnside. It is proved in [32], page 27.

Proposition 4.1 *Assume $\mathbb{F}$ is an algebraically closed field of characteristic* *zero. Let $n \geq 2$ and $\mathcal{S}$ be a semigroup in $\mathcal{M}_n(\mathbb{F})$. If there exists a nonzero linear functional f on $\mathcal{M}_n(\mathbb{F})$ which vanishes on $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

4.1 Preserving rank 1

We first show that every irreducible non-degenerate $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ maps rank 1 matrices to rank 1 matrices.

Proposition 4.2 *Let $n \geq 2$ and $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ be a semigroup homomorphism, which is irreducible and non-degenerate. Then* $\operatorname{rank} \varphi(A) = 1$ *whenever* $\operatorname{rank} A = 1$.

Proof. Since φ is irreducible, it maps 0 to 0 and the identity to the identity. (see Lemma 3.1). So it maps invertible matrices to invertible matrices. It also maps scalar matrices to scalar matrices, because $\varphi(aI)$ commutes with every matrix in the image of φ, which is irreducible. If the rank of a matrix A is equal to the rank of B, then there exist invertible matrices P, Q such that $A = PBQ$. So the rank of $\varphi(A)$ is equal to the rank of $\varphi(B)$. Thus it suffices to show that the rank of

$$B = \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

is 1. First of all, B is nonzero, since φ is non-degenerate. The matrix B is square-zero, so we have block decomposition

$$B = S \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1}$$

Here the first two blocks are of the same size k and the third block may be absent. Let us write

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} S^{-1}.$$

Now

$$S \begin{bmatrix} 0 & A_{21} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1} =$$

$$= S \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1} =$$

$$= \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) =$$

$$= \varphi\left(\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \right) = \varphi(cI) S \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1}.$$

So the matrix A_{21} is scalar for every $a, b, c, d \in \mathbb{F}$. Thus if $k > 1$, φ is reducible (Proposition 4.1). □

4.2 A technical lemma

Let us divide every n-by-n matrix into 3-by-3 block structure where the middle block is $(n-2)$-by-$(n-2)$. So

$$\begin{bmatrix} a & b & \cdots & c & d \\ e & * & \cdots & * & f \\ \vdots & \vdots & & \vdots & \vdots \\ g & * & \cdots & * & h \\ i & j & \cdots & k & l \end{bmatrix} = \begin{bmatrix} a & x & d \\ y & T & z \\ i & w & l \end{bmatrix}$$

where T is a $(n-2)$-by-$(n-2)$ matrix.

Lemma 4.3 *Let $n \geq 3$ and $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ be a semigroup homomorphism, which is irreducible and non-degenerate. Then it has the following form with respect to the above decomposition:*

- *if $a, b, c \neq 0$ and d is arbitrary then*

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) =$$

$$= S \begin{bmatrix} f(a) & x^T G(a)EG(b) & f(b) \\ G(c)EG(a)y & G(c)EG(a)CG(\frac{b}{a}) & G(d)EG(b)y \\ f(c) & x^T G(c)EG(d) & f(d) \end{bmatrix} S^{-1},$$

where

$$C = yx^T + VEG(\frac{ad}{bc} - 1)V;$$

- *if* $b \neq 0$ *and* a, d *are arbitrary then*

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & x^T G(a)EG(b) & f(b) \\ 0 & G(\frac{d}{b})VG(b)EG(a)E & G(d)EG(b)y \\ 0 & 0 & f(d) \end{bmatrix} S^{-1};$$

- *otherwise*

$$\varphi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = S \begin{bmatrix} f(a) & 0 & 0 \\ 0 & EG(a)EG(d) & 0 \\ 0 & 0 & f(d) \end{bmatrix} S^{-1};$$

where $f : \mathbb{F} \to \mathbb{F}$ *and* $G : \mathbb{F} \to \mathcal{M}_{n-2}(\mathbb{F})$ *are semigroup homomorphisms,* $x, y \in \mathbb{F}^{n-2}$ *are nonzero vectors,* $E, V \in \mathcal{M}_{n-2}(\mathbb{F})$ *are matrices with* $E^2 = I$ *and the spectrum of* V *equal to* $\{1\}$*, and* $S \in \mathcal{M}_n(\mathbb{F})$ *is an invertible matrix.*

Proof. Let us denote by E_{ij} the matrix which has 1 in the i-th row and the j-th column, and 0 elsewhere. We will divide the proof into several steps.

Step 1. Without loss of generality we may assume that $\varphi(E_{12}) = E_{1n}$ and $\varphi(E_{21}) = E_{n1}$. Then $\varphi(E_{11}) = E_{11}$ and $\varphi(E_{22}) = E_{nn}$.

Proof: The matrix E_{12} is nilpotent of rank 1, so $\varphi(E_{12})$ must be nilpotent of rank 1. So $\varphi(E_{12}) = uv^T$ where u, v are two column vectors in $\mathbb{F}^n$ and $v^T u = 0$. Similarly we obtain $\varphi(E_{21}) = zt^T$ where $t^T z = 0$. Since $E_{12}E_{21}E_{12} = E_{12}$, we have

$$uv^T zt^T uv^T = uv^T,$$

so $v^T z \cdot t^T u = 1$. With no loss of generality we may assume that $v^T z = t^T u = 1$. Let us choose linearly independent vectors $w_1, ...w_{n-2} \in \mathbb{F}^n$ orthogonal to v and

t, i. e. $v^T w_i = t^T w_i = 0$ for every i. Then $\{u, w_1, ...w_{n-2}, z\}$ is a basis of $\mathbb{F}^n$. In this basis $\varphi(E_{12})$ has the matrix E_{1n} and $\varphi(E_{21})$ has the matrix E_{n1}. So without loss of generality we may assume that $\varphi(E_{12}) = E_{1n}$ and $\varphi(E_{21}) = E_{n1}$. Then

$$\varphi(E_{11}) = \varphi(E_{12}E_{21}) = E_{1n}E_{n1} = E_{11}$$

and similarly $\varphi(E_{22}) = E_{nn}$.

Step 2. φ maps aI to $f(a)I$ where $f : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism with $f(0) = 0$ and $f(1) = 1$.

Proof: The matrix aI commutes with every matrix in $\mathcal{M}_2(\mathbb{F})$, so $\varphi(aI)$ commutes with every matrix in the image of φ. But the image of φ is irreducible, so $\varphi(aI)$ is a scalar matrix of the form $f(a)I$. The mapping f is obviously a semigroup homomorphism.

Step 3. The homomorphism φ has the form

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} f(a) & * & f(b) \\ * & * & * \\ f(c) & * & f(d) \end{bmatrix}.$$

Proof: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an arbitrary matrix, we have

$$E_{11}\varphi(A)E_{11} = \varphi(E_{11}AE_{11}) = \varphi(aE_{11}) = \varphi(aI)E_{11} = f(a)E_{11},$$

so the element in the first row and the first column of $\varphi(A)$ must be $f(a)$. We argue similarly for the other corners.

Step 4. If A is upper-right (resp. upper-left, lower-right, lower-left) triangular, then $\varphi(A)$ is block upper-right (resp. upper-left, lower-right, lower-left) triangular with respect to the defined decomposition. If A is diagonal, then $\varphi(A)$ is block diagonal.

Proof: Let

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

Then

$$\varphi(A)E_{11} = \varphi(AE_{11}) = \varphi(aE_{11}) = f(a)E_{11}$$

and

$$E_{nn}\varphi(A) = \varphi(E_{22}A) = \varphi(dE_{22}) = f(d)E_{nn}$$

so the first column of $\varphi(A)$ must be $[f(a), 0, ..., 0]^T$ and the last row must be $[0, 0, ..., f(d)]$. Thus $\varphi(A)$ is block upper-right triangular. Similarly we prove the other cases.

Step 5.

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & x^T & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & y \\ 0 & 0 & 1 \end{bmatrix},$$

$$\varphi\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & x^T & 1 \end{bmatrix}, \qquad \varphi\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ y & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Proof: The matrix

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

has rank 1. Since it is upper triangular, we have

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & x^T & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly

$$\varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & y \\ 0 & 0 & 1 \end{bmatrix},$$

$$\varphi\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & z^T & 1 \end{bmatrix}, \qquad \varphi\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ t & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & x^T & 1 \\ y & yx^T & y \\ 1 & x^T & 1 \end{bmatrix} = \varphi\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & z^T & 1 \\ t & tz^T & t \\ 1 & z^T & 1 \end{bmatrix},$$

so $x = z$ and $y = t$.

Step 6.

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & x^T & 1 \\ 0 & V & y \\ 0 & 0 & 1 \end{bmatrix}$$

where the spectrum of V is $\{1\}$,

$$\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & E & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where $E^2 = I$, and

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & G(a) & 0 \\ 0 & 0 & f(a) \end{bmatrix}$$

where $G : \mathbb{F} \to \mathcal{M}_{n-2}(\mathbb{F})$ is semigroup homomorphism.

Proof: The matrix

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

has the form

$$\begin{bmatrix} 1 & u^T & 1 \\ 0 & V & w \\ 0 & 0 & 1 \end{bmatrix}$$

by step 4. Since

$$E_{11}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

we have $u = x$ and similarly $v = y$. Because the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is similar to its square, also V is similar to its square, and since it is also invertible, its spectrum is equal to $\{1\}$. $\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ and $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right)$ have the asserted form by step 4. From $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I$ it follows that $E^2 = I$ and from

$$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & ab \end{bmatrix}$$

follows the multiplicativity of G.

Step 7.

$$\varphi\left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & 0 & 0 \\ 0 & EG(a)EG(c) & 0 \\ 0 & 0 & f(c) \end{bmatrix}.$$

If $b \neq 0$ then

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & x^T G(a)EG(b) & f(b) \\ 0 & G(\frac{c}{b})VG(b)EG(a)E & G(c)EG(b)y \\ 0 & 0 & f(c) \end{bmatrix}$$

and

$$\varphi\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & 0 & 0 \\ G(b)EG(a)y & G(b)EG(a)VEG(\frac{c}{b}) & 0 \\ f(b) & x^T G(b)EG(c) & f(c) \end{bmatrix}$$

Proof: From

$$\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

we obtain the first equality. Since

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{c}{b} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

we have

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & x^T G(b)EG(a)E & f(b) \\ 0 & G(\frac{c}{b})VG(b)EG(a)E & f(b)G(\frac{c}{b})y \\ 0 & 0 & f(c) \end{bmatrix}$$

Since $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have $x^T E = x^T$ and similarly $Ey = y$. So

$$x^T G(b)EG(a)E = x^T EG(a)EG(b) = x^T G(a)EG(b)$$

and

$$f(b)G(\frac{c}{b})y = EG(b)EG(b)G(\frac{c}{b})y = EG(b)EG(c)y = G(c)EG(b)y.$$

Similarly we obtain the third equality.

Step 8. If $a, b, c \neq 0$ then

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) =$$

$$= \begin{bmatrix} f(a) & x^T G(a)EG(b) & f(b) \\ G(c)EG(a)y & G(c)EG(a)(yx^T + VEG(\frac{ad}{bc} - 1)V)G(\frac{b}{a}) & G(d)EG(b)y \\ f(c) & x^T G(c)EG(d) & f(d) \end{bmatrix}$$

Furthermore, vectors $x, y \neq 0$.

Proof: From

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we obtain the first row. Similarly we obtain the last row and the first and the last column. From the equality

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d - \frac{bc}{a} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$$

we see that the middle block is equal to

$$G(c)EG(a)yx^T G(\frac{b}{a}) + G(c)EG(a)VEG(\frac{d}{c} - \frac{b}{a})G(\frac{a}{b})VG(\frac{b}{a}) =$$

$$G(c)EG(a)(yx^T + VEG(\frac{ad}{bc} - 1)V)G(\frac{b}{a}).$$

Now, if $x = 0$ then the first row of $\varphi(A)$ is equal to $[f(a), 0, ..., 0, f(b)]$ for every $A \in \mathcal{M}_2(\mathbb{F})$. So the image of φ is a matrix semigroup where every element has 0 on the second place in the first row. Thus it is reducible (Proposition 4.1). So $x \neq 0$ and similarly $y \neq 0$. This completes the proof. $\square$

4.3 Case $n = 4$

Theorem 4.4 *Let $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_4(\mathbb{F})$ be a semigroup homomorphism, which is irreducible and non-degenerate. Then it has one of the following forms:*

(a)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S\hat{g}\begin{bmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & a^2d + 2abc & 2abd + b^2c & 3b^2d \\ 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{bmatrix} S^{-1} =$$
$$= S\hat{g}(\mathrm{Sym}^3 A)S^{-1},$$

where $g : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_4(\mathbb{F})$ is an invertible matrix,

(b)

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S\begin{bmatrix} g(a)h(a) & g(a)h(b) & g(b)h(a) & g(b)h(b) \\ g(a)h(c) & g(a)h(d) & g(b)h(c) & g(b)h(d) \\ g(c)h(a) & g(c)h(b) & g(d)h(a) & g(d)h(b) \\ g(c)h(c) & g(c)h(d) & g(d)h(c) & g(d)h(d) \end{bmatrix} S^{-1} =$$
$$= S(\hat{g}(A) \otimes \hat{h}(A))S^{-1},$$

where $g, h : \mathbb{F} \to \mathbb{F}$ are field homomorphisms with $g \neq h$ and $S \in \mathcal{M}_4(\mathbb{F})$ is an invertible matrix.

Remark: If in case (b) hold $g = h$ then φ is reducible, since $A \otimes A \cong (A \vee A) \oplus (A \wedge A)$. But if $g(a) \neq h(a)$ for at least one a, then φ is irreducible, because $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right)$ is similar to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & h(a) & 0 & 0 \\ 0 & 0 & g(a) & 0 \\ 0 & 0 & 0 & g(a)h(a) \end{bmatrix}$$

which has four distinct eigenvalues, so if φ were reducible, the possible invariant subspace would be standard, i. e. $S^{-1}\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)S$ would have a constant zero block.

Proof. Suppose that φ has the form of Lemma 4.3. Since the spectrum of the matrix V is equal to $\{1\}$ we have two possibilities: either V is similar to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ or $V = I$. In the first case we may assume without loss of generality that $V = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Case 1. $V = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Since

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2$$

for the middle parts of their images under φ holds $VG(2) = G(2)V^2$. So $G(2)$ is of the form

$$G(2) = \begin{bmatrix} \alpha & \beta \\ 0 & 2\alpha \end{bmatrix}$$

where α is nonzero. But

$$\begin{bmatrix} \alpha & \beta \\ 0 & 2\alpha \end{bmatrix} = \begin{bmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 2\alpha \end{bmatrix}\begin{bmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{bmatrix}^{-1}$$

and $\begin{bmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{bmatrix}$ commutes with V, so we may assume with no loss of generality that

$$G(2) = \begin{bmatrix} \alpha & 0 \\ 0 & 2\alpha \end{bmatrix}.$$

The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so $EG(2)E$, the middle part of its image under φ, commutes with $G(2)$ and is similar to $G(2)$, thus is either equal to $\begin{bmatrix} \alpha & 0 \\ 0 & 2\alpha \end{bmatrix}$ or equal to $\begin{bmatrix} 2\alpha & 0 \\ 0 & \alpha \end{bmatrix}$. The first case is impossible since

$$EG(2)EG(2) = \begin{bmatrix} \alpha^2 & 0 \\ 0 & 4\alpha^2 \end{bmatrix}$$

and that should be equal to $f(2)I$. Thus

$$EG(2)E = \begin{bmatrix} 2\alpha & 0 \\ 0 & \alpha \end{bmatrix}.$$

So E is of the form

$$E = \begin{bmatrix} 0 & \beta \\ 1/\beta & 0 \end{bmatrix}$$

and $f(2) = 2\alpha^2$. Let us look again at the equality

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2.$$

Writing $x^T = [x_1, x_2]$ and $y^T = [y_1, y_2]$ we have

$$\begin{bmatrix} 1 & \alpha x_1 & 2\alpha x_2 & 2\alpha^2 \\ 0 & \alpha & 4\alpha & 2\alpha^2 y_1 \\ 0 & 0 & 2\alpha & 2\alpha^2 y_2 \\ 0 & 0 & 0 & 2\alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 2x_1 & 2x_1 + 2x_2 & 2 + x_1 y_1 + x_2 y_2 \\ 0 & \alpha & 4\alpha & 2\alpha y_1 + 2\alpha y_2 \\ 0 & 0 & 2\alpha & 4\alpha y_2 \\ 0 & 0 & 0 & 2\alpha^2 \end{bmatrix}$$

so $\alpha = 2$, $x_1 = x_2$ and $y_1 = y_2$. ($y_2 = 0$ is impossible since y is eigenvector of E.) Without loss of generality we may assume $x_1 = 1$. Because

$$2 + x_1 y_1 + x_2 y_2 = f(2) = 8$$

we have $y_1 = 3$. The vector y is an eigenvector of E, so $\beta = 1$.

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so $G(a)$ is of the form

$$G(a) = \begin{bmatrix} g(a) & 0 \\ 0 & h(a) \end{bmatrix}.$$

where $g, h : \mathbb{F} \to \mathbb{F}$ are semigroup homomorphisms and $g(a)h(a) = f(a)$. So

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} f(a) & h(a)g(b) & g(a)h(b) & f(b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ 0 & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} f(a) & h(a)g(b) & g(a)h(b) & f(b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} f(a) & h(a)g(a+b) & g(a)h(a+b) & f(a+b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $f(a) + h(a)g(b) = h(a)(g(a) + g(b)) = h(a)g(a + b)$. If $a \neq 0$, we have $g(a) + g(b) = g(a + b)$, so g is additive. Furthermore,

$$f(a) + 2h(a)g(b) + g(a)h(b) = g(a)h(a + b),$$

so $h(a) + 2h(a)g(b/a) + h(b) = h(a + b)$. If also $b \neq 0$, we can interchange the role of a and b, and obtain $h(b)+2h(b)g(a/b)+h(a) = h(a+b)$. Subtracting the second equality from the first, we see that $h(a)/h(b) = g(a/b)^2$, so $h(a) = g(a^2)$ and $f(a) = g(a^3)$.

Now, if $a, b, c \neq 0$, we have

$$G(c)EG(a)(yx^T + VEG(\frac{ad}{bc} - 1)V)G(\frac{b}{a}) =$$

$$= \begin{bmatrix} g(a^2d + 2abc) & g(2abd + b^2c) \\ g(2acd + bc^2) & g(ad^2 + 2bcd) \end{bmatrix}$$

so for all a, b, c, d

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \hat{g} \begin{bmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & a^2d + 2abc & 2abd + b^2c & 3b^2d \\ 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{bmatrix}$$

and we are in case (a) of the theorem.

Case 2. $V = I$.

The mapping $G : \mathbb{F} \to \mathcal{M}_2(\mathbb{F})$ is a semigroup homomorphism, so we have three possibilities:

(i) $G(a) = g(a)I$ for every $a \in \mathbb{F}$ where $g : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism. In this case we may assume without loss of generality that $x^T = [1, 0]$. But then the first row of

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

is equal to $[f(a), g(ab), 0, f(b)]$, thus φ is reducible (Proposition 4.1), so this case is impossible.

(ii) $G(a)$ is similar to

$$g(a)\begin{bmatrix} 1 & h(a) \\ 0 & 1 \end{bmatrix}$$

for every $a \in \mathbb{F}$ where $g : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism and $h : \mathbb{F} \to \mathbb{F}$ satisfies $h(ab) = h(a) + h(b)$ with $h(a) \neq 0$ for at least one $a \in \mathbb{F}$. We may assume without loss of generality that

$$G(a) = g(a)\begin{bmatrix} 1 & h(a) \\ 0 & 1 \end{bmatrix}.$$

Let us choose an a with $h(a) \neq 0$. The matrix $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ and is similar to $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$, so

$$EG(a)E = g(a)\begin{bmatrix} 1 & -h(a) \\ 0 & 1 \end{bmatrix}.$$

Thus E is of the form

$$E = \pm\begin{bmatrix} 1 & \beta \\ 0 & -1 \end{bmatrix}.$$

If $E = \begin{bmatrix} 1 & \beta \\ 0 & -1 \end{bmatrix}$ then $y^T = [y_1, 0]$ and the first column of $\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ is equal to $[f(a), *, 0, f(c)]$; if $E = \begin{bmatrix} -1 & -\beta \\ 0 & 1 \end{bmatrix}$ then $x^T = [0, x_2]$ and the first row of $\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ is equal to $[f(a), 0, *, f(b)]$; in both cases φ is reducible.

(iii) $G(a)$ is similar to $\begin{bmatrix} h(a) & 0 \\ 0 & g(a) \end{bmatrix}$ for every $a \in \mathbb{F}$ where $g, h : \mathbb{F} \to \mathbb{F}$ are semigroup homomorphisms and $h(a) \neq g(a)$ for at least one $a \in \mathbb{F}$. We may assume without loss of generality that $G(a) = \begin{bmatrix} h(a) & 0 \\ 0 & g(a) \end{bmatrix}$. Let us choose an a with $h(a) \neq g(a)$. If $h(a) = -g(a)$, take $\sqrt{a}$ instead of a, so that $h(a) \neq \pm g(a)$. The same way as in case (a) we obtain that E is of the form

$$E = \begin{bmatrix} 0 & \beta \\ 1/\beta & 0 \end{bmatrix}$$

This matrix is diagonally similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so we may assume without loss of generality that $\beta = 1$. Since $x^T = x^T E$ and $Ey = y$ we may assume without loss of generality that $x^T = [1, 1]$ and $y^T = [y_1, y_1]$. Now,

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} f(a) & h(a)g(b) & g(a)h(b) & f(b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} f(a) & h(a)g(b) & g(a)h(b) & f(b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} f(a) & h(a)g(a+b) & g(a)h(a+b) & f(a+b) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $h(a)(g(a) + g(b)) = h(a)g(a+b)$ and $g(a)(h(a) + h(b)) = g(a)h(a+b)$ If $a \neq 0$, we have $g(a) + g(b) = g(a+b)$ and $h(a) + h(b) = h(a+b)$, so g and h are additive. From $f(2) = g(2)h(2) = 4 = 2 + 2y_1$ it follows $y_1 = 1$. Again, if $a, b, c \neq 0$, we have

$$G(c)EG(a)(yx^T + VEG(\frac{ad}{bc} - 1)V)G(\frac{b}{a}) = \begin{bmatrix} g(a)h(d) & g(b)h(c) \\ g(c)h(b) & g(d)h(a) \end{bmatrix}$$

so for all a, b, c, d

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} g(a)h(a) & g(a)h(b) & g(b)h(a) & g(b)h(b) \\ g(a)h(c) & g(a)h(d) & g(b)h(c) & g(b)h(d) \\ g(c)h(a) & g(c)h(b) & g(d)h(a) & g(d)h(b) \\ g(c)h(c) & g(c)h(d) & g(d)h(c) & g(d)h(d) \end{bmatrix}$$

and we are in the case (b) of the theorem. □

The only continuous field homomorphisms of complex numbers are the identity and the conjugation (see [1], page 53). So we have the following corollary.

Corollary 4.5 *Let $\varphi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ be a semigroup homomorphism, which is irreducible, non-degenerate and continuous. Then*

$$\varphi(A) = S\hat{g}(\mathrm{Sym}^3 A)S^{-1},$$

where $g : \mathbb{C} \to \mathbb{C}$ is the identity or complex conjugation and $S \in \mathcal{M}_4(\mathbb{C})$ is an invertible matrix, or

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = S \begin{bmatrix} a\bar{a} & a\bar{b} & b\bar{a} & b\bar{b} \\ a\bar{c} & a\bar{d} & b\bar{c} & b\bar{d} \\ c\bar{a} & c\bar{b} & d\bar{a} & d\bar{b} \\ c\bar{c} & c\bar{d} & d\bar{c} & d\bar{d} \end{bmatrix} S^{-1},$$

where $S \in \mathcal{M}_4(\mathbb{C})$ is an invertible matrix.

Proof. If φ is continuous, then so are the field homomorphisms g and h. Case (a) of the theorem gives us the first possibility. If we are in the case (b) of the theorem, then g is the identity and h is complex conjugation or the other way around. But the matrices

$$\begin{bmatrix} a\bar{a} & a\bar{b} & b\bar{a} & b\bar{b} \\ a\bar{c} & a\bar{d} & b\bar{c} & b\bar{d} \\ c\bar{a} & c\bar{b} & d\bar{a} & d\bar{b} \\ c\bar{c} & c\bar{d} & d\bar{c} & d\bar{d} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{a}a & \bar{a}b & \bar{b}a & \bar{b}b \\ \bar{a}c & \bar{a}d & \bar{b}c & \bar{b}d \\ \bar{c}a & \bar{c}b & \bar{d}a & \bar{d}b \\ \bar{c}c & \bar{c}d & \bar{d}c & \bar{d}d \end{bmatrix}$$

are simultaneously similar for all a, b, c, d, so we obtain the second possibility. □

4.4 Preserving cyclic unipotent

A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is *unipotent* if its spectrum is equal to $\{1\}$. A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is *cyclic* if it has a cyclic vector, i. e. a vector $x \in \mathbb{F}^n$ for which the set $\{x, Ax, A^2x, ..., A^{n-1}x\}$ spans all $\mathbb{F}^n$. Every cyclic unipotent in $\mathcal{M}_n(\mathbb{F})$ is similar to the matrix

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Theorem 4.6 *Let $n \geq 3$ and $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ be a semigroup homomorphism, which is irreducible, non-degenerate and maps a cyclic unipotent to a cyclic unipotent. Then*

$$\varphi(A) = S\hat{g}(\mathrm{Sym}^{n-1} A)S^{-1},$$

where $g : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix.

Proof. Without loss of generality we may assume that

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Denoting

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = [x_{ij}]_{i,j=1}^n$$

and using the equality

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2, \tag{4.1}$$

we obtain

$$x_{i+1,j} = x_{i,j-2} + 2x_{i,j-1}$$

for all $i, j = 1, 2, ..., n$, where $x_{kl} = 0$ if k or l is less than 1 or grater than n. It follows that $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)$ is upper-triangular and $x_{i+1,i+1} = 2x_{ii}$. So

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \alpha \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & 2 & * & \cdots & * \\ 0 & 0 & 4 & \ddots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2^{n-1} \end{bmatrix}.$$

We may now apply a simultaneous similarity with an upper-triangular matrix, so that

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)$$

is diagonal. This will change $\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$, but it will still remain upper-triangular. We again apply a simultaneous similarity with a diagonal matrix, so that for

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = [w_{ij}]_{i,j=1}^n$$

the entries will satisfy $w_{i,i+1} = i$. This similarity will leave $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)$ diagonal. So without loss of generality we may assume that

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & w_{13} & \cdots & w_{1,n-1} & w_{1n} \\ 0 & 1 & 2 & \ddots & w_{2,n-1} & w_{2n} \\ 0 & 0 & 1 & \ddots & w_{3,n-1} & w_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \alpha \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2^{n-1} \end{bmatrix}.$$

Let us prove that $w_{ij} = \binom{j-1}{i-1}$ by induction on $j-i$. This is true if $j = i+1$. From the equality (4.1) it follows that

$$2^{j-1} w_{ij} = 2^{i-1} \sum_{k=i}^{j} w_{ik} w_{kj}.$$

Thus we obtain using the inductive hypothesis that

$$(2^{j-i} - 2) w_{ij} = \sum_{k=i+1}^{j-1} w_{ik} w_{kj} = \sum_{k=i+1}^{j-1} \binom{k-1}{i-1} \binom{j-1}{k-1} =$$

$$= \sum_{k=i+1}^{j-1} \frac{(k-1)!\,(j-1)!}{(i-1)!\,(k-i)!\,(k-1)!\,(j-k)!} =$$

$$= \frac{(j-1)!}{(i-1)!\,(j-i)!} \sum_{k=i+1}^{j-1} \frac{(j-i)!}{(k-i)!\,(j-k)!} = \binom{j-1}{i-1} (2^{j-i} - 2).$$

Now, E_{11} commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so $\varphi(E_{11})$ is diagonal. Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} E_{11} = E_{11}$ it follows that $\varphi(E_{11}) = E_{11}$. Similarly we prove $\varphi(E_{22}) = E_{nn}$. So the conclusion of step 1 of Lemma 4.3 holds, and thus all the steps of Lemma 4.3 also hold. It means that $\alpha = 1$ and $f(2) = 2^{n-1}$.

From

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

it follows that

$$\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} 2^{n-1} & 0 & 0 & \cdots & 0 \\ 0 & 2^{n-2} & 0 & \cdots & 0 \\ 0 & 0 & 2^{n-3} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2^{n-1} \end{bmatrix} \varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

so that $\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ is counter-diagonal. Since the last column of

$$\varphi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

is an eigenvector of $\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ at eigenvalue 1, we have

$$\varphi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so $\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right)$ is diagonal of the form

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & f_1(a) & 0 & \cdots & 0 \\ 0 & 0 & f_2(a) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_{n-1}(a) \end{bmatrix}$$

where $f_1, ..., f_{n-1} : \mathbb{F} \to \mathbb{F}$ are semigroup homomorphisms and $f_i(a)f_{n-i-1}(a) = f_{n-1}(a) = f(a)$ for every $i = 0, ..., n-1$ writing $f_0(a) = 1$. So

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} f_{n-1}(a) & f_{n-2}(a)f_1(b) & f_{n-3}(a)f_2(b) & \cdots & f_1(a)f_{n-2}(b) & f_{n-1}(b) \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ 0 & 0 \end{bmatrix},$$

we have

$$\sum_{k=1}^{i} \binom{i-1}{k-1} f_{n-k}(a) f_{k-1}(b) = f_{n-i}(a) f_{i-1}(a+b)$$

for every $i = 1, 2, ..., n$. Dividing by $f_{n-i}(a)$ and using

$$f_{n-i}(a) = f_{n-1}(a)/f_{i-1}(a),$$

it follows for $a \neq 0$ that

$$f_i(a+b) = \sum_{k=0}^{i} \binom{i}{k} f_i(a) f_k(b/a)$$

for $i = 0, 1, ..., n-1$. So f_1 is additive. Let us prove that $f_i(a) = f_1(a^i)$ by induction on i. Interchanging a and b we have

$$f_i(a+b) = \sum_{k=0}^{i} \binom{i}{k} f_i(b) f_k(a/b)$$

for all $a, b \neq 0$. Summing to zero the first and the last term we obtain

$$\sum_{k=1}^{i-1} \binom{i}{k} f_i(a) f_k(b/a) = \sum_{k=1}^{i-1} \binom{i}{k} f_i(b) f_k(a/b)$$

Dividing by $f_i(b)$ and writing $c = a/b$ we have

$$f_i(c) \sum_{k=1}^{i-1} \binom{i}{k} f_k(1/c) = f_i(c) \sum_{k=1}^{i-1} \binom{i}{k} f_1^k(1/c) = \sum_{k=1}^{i-1} \binom{i}{k} f_k(c) =$$

$$\sum_{k=1}^{i-1} \binom{i}{k} f_1^k(c) = f_1(c^i) \sum_{k=1}^{i-1} \binom{i}{k} f_1^{i-k}(1/c) = f_1(c^i) \sum_{k=1}^{i-1} \binom{i}{k} f_1^k(1/c)$$

so $f_i(c) = f_1(c^i)$ for all $c \in \mathbb{F}$ except possibly for those c for which $f_1(1/c)$ is zero of the polynomial

$$p(x) = \sum_{k=1}^{i-1} \binom{i}{k} x^k.$$

But the set of zeros of this polynomial is finite and f_i is multiplicative, so $f_i(c) = f_1(c^i)$ for all $c \in \mathbb{F}$.

Writing $f_1 = g$ we have

$$\varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & g(a) & 0 & \cdots & 0 \\ 0 & 0 & g(a^2) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g(a^{n-1}) \end{bmatrix}.$$

So we have

$$\varphi(A) = \mathrm{Sym}^{n-1} A$$

for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ where $0 \neq a \in \mathbb{F}$. But these matrices generate $\mathcal{M}_n(\mathbb{F})$ as a semigroup, so

$$\varphi(A) = \mathrm{Sym}^{n-1} A$$

holds for every $A \in \mathcal{M}_2(\mathbb{F})$. $\square$

Chapter 5

Homomorphisms from a dimension to one dimension higher

In this chapter we prove that every non-degenerate homomorphism from the multiplicative semigroup of all n-by-n matrices over an algebraically closed field of characteristic zero to the semigroup of $(n+1)$-by-$(n+1)$ matrices over the same field when $n \geq 3$ is reducible and that every non-degenerate homomorphism from the multiplicative semigroup of all 3-by-3 matrices over an algebraically closed field of characteristic zero to the semigroup of 5-by-5 matrices over the same field is reducible.

5.1 Singular matrices

We first look where an non-degenerate irreducible homomorphism sends singular matrices.

Proposition 5.1 *Let $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ a semigroup homomorphism, which sends* 0 *to* 0 *and identity to identity. Let*

$$k = \min\{\text{rank}A; \varphi(A) \neq 0\}.$$

Then

$$\binom{n}{k} \leq m.$$

If rank $A =$ rank B *then* rank $\varphi(A) =$ rank $\varphi(B)$.

Proof. A semigroup homomorphism which sends I to I, maps invertible matrices to invertible matrices. If rank $A =$ rank B, then there exist such invertible matrices P, Q that $A = PBQ$. So $\varphi(A) = \varphi(P)\varphi(B)\varphi(Q)$ and rank $\varphi(A) =$ rank $\varphi(B)$.
Let $E_1, E_2, ..., E_t$ be $t = \binom{n}{k}$ distinct diagonal idempotents of rank k. Then rank $\varphi(E_1) =$ rank $\varphi(E_2) = ... =$ rank $\varphi(E_t) \geq 1$. Since E_iE_j for $i \neq j$ has rank less than k, we have $\varphi(E_i)\varphi(E_j) = 0$, and $\varphi(E_1), \varphi(E_2), ..., \varphi(E_t)$ are disjoint idempotents. We conclude that $t($rank $\varphi(E_1)) \leq m$, implying $\binom{n}{k} \leq m$. □

Proposition 5.2 *Assume that $n \geq 3$ and $m < 2n$. Let $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ be a semigroup homomorphism, which is non-degenerate and sends* 0 *to* 0 *and identity to identity. Suppose that* rank $A = 1$ *implies* rank $\varphi(A) = 1$. *Then* rank $A = 2$ *implies* rank $\varphi(A) = 2$.

Proof. Denote by E_{ij} the matrix which has 1 in the i-th row and the j-th column, and 0 elsewhere. Matrices $\varphi(E_{11}), \varphi(E_{22}), ...\varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F})$ are disjoint commuting idempotents of rank 1. Let

$$P_2 = E_{11} + E_{22}, P_3 = E_{11} + E_{33}, ..., P_n = E_{11} + E_{nn}.$$

Rank $\varphi(P_2)$ cannot be 1. Suppose rank $\varphi(P_2) \geq 3$. Then $\varphi(P_2), \varphi(P_3), ...,$ $\varphi(P_n)$ are commuting idempotents of equal rank with products $\varphi(P_i)\varphi(P_j) = \varphi(E_{11})$. So

$$\text{rank } (\varphi(P_2) + \varphi(P_3) + ... + \varphi(P_n)) \geq 2(n-1) + 1 \geq m.$$

Now $\varphi(E_{22} + E_{33})$ has products of rank 1 with $\varphi(P_2)$ and $\varphi(P_3)$, and it is disjoint from $\varphi(P_4), ..., \varphi(P_n)$, so

$$\text{rank } (\varphi(P_2) + \varphi(P_3) + ... + \varphi(P_n) + \varphi(E_{22} + E_{33})) =$$

$$= \text{rank } (\varphi(P_2) + \varphi(P_3) + ... + \varphi(P_n)) + 1,$$

which is a contradiction. So rank $\varphi(P_2) = 2$ and, finally, rank $A = 2$ implies rank $\varphi(A) = 2$. $\square$

The next proposition is trivially true for $n = 3$ and $m < 6$. We prove it also for larger n.

Proposition 5.3 *Assume that $n > 4$ and $m < 2n$ or that $n = 4$ and $m \leq 5$. Let $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ be a semigroup homomorphism, which is non-degenerate and sends 0 to 0 and identity to identity. Then we have two possibilities:*

(a) if rank $A = 1$ *then* rank $\varphi(A) = 1$, *and if* rank $A = 2$ *then* rank $\varphi(A) = 2$, *or*

(b) if rank $A < n - 1$ *then* $\varphi(A) = 0$, *and if* rank $A = n - 1$ *then* rank $\varphi(A) = 1$.

Proof. Let

$$k = \min\{\text{rank} A; \varphi(A) \neq 0\}.$$

Since φ is non-degenerate, $1 \leq k \leq n - 1$. If $n > 4$, then $m < 2n \leq \binom{n}{2}$. If $n = 4$, then $m \leq 5 < \binom{4}{2}$. So by Proposition 5.1 $k = 1$ or $k = n - 1$.

Case (a): $k = 1$. The matrices $E_{11}, E_{22}, ...E_{nn} \in \mathcal{M}_n(\mathbb{F})$ are idempotents of rank 1, so $\varphi(E_{11}), \varphi(E_{22}), ...\varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F})$ are disjoint commuting idempotents of the same rank, say l. Since they are disjoint, $nl \leq m$, so $l = 1$.

Thus rank $A = 1$ implies rank $\varphi(A) = 1$. Proposition 5.2 now gives us the asserted result.

Case (b): $k = n-1$. We have that rank $A < n-1$ implies $\varphi(A) = 0$. Let $P_1, P_2, ...P_n \in \mathcal{M}_n(\mathbb{F})$ be distinct diagonal idempotents of rank $n-1$. Then $\varphi(P_1), \varphi(P_2), ...\varphi(P_n) \in \mathcal{M}_m(\mathbb{F})$ are disjoint commuting idempotents with the same rank, say l. Since they are disjoint, $nl \leq m$, so $l = 1$. Thus rank $A = n-1$ implies rank $\varphi(A) = 1$. □

5.2 Two possibilities

We will now explore the two possibilities which appear in Proposition 5.3. The first one is that only 0 maps to 0.

Proposition 5.4 *Assume that* $n \geq 2$ *and* $m \geq n$. *Let* $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ *be a semigroup homomorphism, which is non-degenerate and sends* 0 *to* 0 *and identity to identity. Suppose that* rank $A = 1$ *implies* rank $\varphi(A) = 1$ *and that* rank $A = 2$ *implies* rank $\varphi(A) = 2$. *Then*

$$\varphi(A) = S \begin{bmatrix} \hat{f}(A) & * \\ * & * \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ *is a field homomorphism and* $S \in \mathcal{M}_m(\mathbb{F})$ *is an invertible matrix.*

Proof. Denote by E_{ij} the matrix which has 1 in the i-th row and the j-th column, and 0 elsewhere.

Matrices $E_{11}, E_{22}, ..., E_{nn} \in \mathcal{M}_n(\mathbb{F})$ are disjoint commuting idempotents of rank 1, so $\varphi(E_{11}), \varphi(E_{22}), ...\varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F})$ are disjoint commuting idempotents of rank 1. It follows that they are simultaneously similar to

$$E_{11}, E_{22}, ..., E_{nn} \in \mathcal{M}_m(\mathbb{F}).$$

Thus we may assume without loss of generality that

$$\varphi(E_{ii}) = E_{ii}.$$

Let δ_{ij} be the Kronecker symbol, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. We have

$$\delta_{ki}\delta_{jl}\varphi(E_{ij}) = \varphi(\delta_{ki}\delta_{jl}E_{ij}) = \varphi(E_{kk}E_{ij}E_{ll}) = E_{kk}\varphi(E_{ij})E_{ll},$$

so

$$\varphi(E_{ij}) = \begin{bmatrix} t_{ij}E_{ij} & 0 \\ 0 & * \end{bmatrix}.$$

Since $E_{ij}E_{ji} = E_{ij}$, we obtain $t_{ij} \neq 0$, and since $\varphi(E_{ij})$ has rank 1, we have $* = 0$. Thus

$$\varphi(E_{ij}) = t_{ij}E_{ij}.$$

We may now apply a simultaneous similarity with a diagonal matrix

$$\mathrm{diag}(1, t_{12}, ..., t_{1n}, 1, ..., 1)$$

to obtain $\varphi(E_{1j}) = E_{1j}$. Now

$$E_{1j} = \varphi(E_{1j}) = \varphi(E_{1i}E_{ij}) = E_{1i}t_{ij}E_{ij} = t_{ij}E_{1j},$$

so t_{ij} equals 1 for all i, j and therefore

$$\varphi(E_{ij}) = E_{ij}.$$

Let a be an element in $\mathbb{F}$.

$$\varphi(aE_{11}) = \varphi(E_{11}aE_{11}E_{11}) = E_{11}\varphi(aE_{11})E_{11},$$

so the only non-zero entry of $\varphi(aE_{11})$ is at the $(1,1)$ position. So there exists such mapping $f : \mathbb{F} \to \mathbb{F}$ that

$$\varphi(aE_{11}) = f(a)E_{11}.$$

Mapping f is obviously multiplicative. Furthermore

$$\varphi(aE_{ij}) = \varphi(aE_{i1}E_{11}E_{1j}) = E_{i1}\varphi(aE_{11})E_{1j} = E_{i1}f(a)E_{11}E_{1j} = f(a)E_{ij}$$

Now let $A = [a_{ij}]_{i,j=1}^{n}$ be a matrix in $\mathcal{M}_n(\mathbb{F})$. We have

$$E_{ii}\varphi(A)E_{jj} = \varphi(E_{ii}AE_{jj}) = \varphi(a_{ij}E_{ij}) = f(a_{ij})E_{ij},$$

so the ij-th entry of $\varphi(A)$ is $f(a_{ij})$ and

$$\varphi(A) = \begin{bmatrix} \hat{f}(A) & * \\ * & * \end{bmatrix}.$$

Further, the matrix

$$\varphi(E_{11} + E_{22}) = \begin{bmatrix} E_{11} + E_{22} & * \\ * & * \end{bmatrix}$$

has rank 2; thus we may assume

$$\varphi(E_{11} + E_{22}) = \begin{bmatrix} E_{11} + E_{22} & * \\ 0 & 0 \end{bmatrix}.$$

Let $A = [a_{ij}]_{i,j=1}^{n}$ be a matrix in $\mathcal{M}_n(\mathbb{F})$, such that $a_{ij} = 0$ if $i \geq 3$. Then

$$\varphi(A) = \varphi((E_{11} + E_{22})A) = \begin{bmatrix} E_{11} + E_{22} & * \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{f}(A) & * \\ * & * \end{bmatrix} = \begin{bmatrix} \hat{f}(A) & * \\ 0 & 0 \end{bmatrix}.$$

Let us now prove that f is additive. For $a, b \in \mathbb{F}$ we have

$$f(a+b)E_{11} = \varphi((a+b)E_{11}) = \varphi((aE_{11} + bE_{12})(E_{11} + E_{21})) =$$

$$= \begin{bmatrix} f(a)E_{11} + f(b)E_{12} & * \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_{11} + E_{21} & * \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (f(a) + f(b))E_{11} & * \\ 0 & 0 \end{bmatrix},$$

so $f(a+b) = f(a) + f(b)$, and thus $\hat{f}$ is multiplicative. $\square$

The second possibility is that only almost full rank matrices map to non-zero matrices.

Proposition 5.5 *Assume that* $n \geq 3$ *and* $m \geq n$. *Let* $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ *be a semigroup homomorphism, which is non-degenerate and sends* 0 *to* 0 *and identity to identity. Suppose that* rank $A < n-1$ *implies* $\varphi(A) = 0$ *and that* rank $A = n-1$ *implies* rank $\varphi(A) = 1$. *Then*

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & * \\ * & * \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ *is a homomorphism of the multiplicative semigroup* $(\mathbb{F}, \cdot)$ *and* $S \in \mathcal{M}_m(\mathbb{F})$ *is an invertible matrix.*

Proof. Denote by E_{ij} the matrix which has 1 in the i-th row and the j-th column, and 0 elsewhere. Introduce $P_{ii} = I - E_{ii} \in \mathcal{M}_n(\mathbb{F})$, and let I_i be the identity matrix in $\mathcal{M}_i(\mathbb{F})$. Further, let N_i be the matrix in $\mathcal{M}_i(\mathbb{F})$, defined by $N_i = E_{12} + ... + E_{i-1,i}$. Denote $P_{ij} = I_{i-1} \oplus N^T_{j-i+1} \oplus I_{n-j}$ if $i < j$, and $P_{ij} = I_{j-1} \oplus N_{i-j+1} \oplus I_{n-i}$ if $i > j$.

The matrices $P_{11}, P_{22}, ..., P_{nn} \in \mathcal{M}_n(\mathbb{F})$ are disjoint commuting idempotents of rank $n-1$, so $\varphi(E_{11}), \varphi(E_{22}), ..., \varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F})$ are disjoint commuting idempotents of rank 1. So they are simultaneously similar to $E_{11}, E_{22}, ..., E_{nn} \in \mathcal{M}_m(\mathbb{F})$. Without loss of generality we may thus assume that

$$\varphi(P_{ii}) = E_{ii}.$$

Observe that $P_{ij} = P_{ik}P_{kj}$ and $P_{ik}P_{lj}$ has rank less than $n-1$ if $k \neq l$. We now have

$$\delta_{ki}\delta_{jl}\varphi(P_{ij}) = \varphi(\delta_{ki}\delta_{jl}P_{ij}) = \varphi(P_{kk}P_{ij}P_{ll}) = E_{kk}\varphi(P_{ij})E_{ll},$$

so

$$\varphi(P_{ij}) = \begin{bmatrix} t_{ij}E_{ij} & 0 \\ 0 & * \end{bmatrix}.$$

The matrix $\varphi(P_{ij})$ has rank 1, so $t_{ij} \neq 0$ and $* = 0$. This implies

$$\varphi(P_{ij}) = t_{ij} E_{ij}.$$

We may now apply a simultaneous similarity with a diagonal matrix

$$\mathrm{diag}(1, t_{12}, ..., t_{1n}, 1, ..., 1)$$

to obtain $\varphi(P_{1j}) = E_{1j}$. Now

$$E_{1j} = \varphi(P_{1j}) = \varphi(P_{1i}P_{ij}) = E_{1i} t_{ij} E_{ij} = t_{ij} E_{1j},$$

so $t_{ij} = 1$ for all i, j and

$$\varphi(P_{ij}) = E_{ij}.$$

Let $A \in \mathcal{M}_{n-1}(\mathbb{F})$ be arbitrary matrix and $A' = 0_1 \oplus A \in \mathcal{M}_n(\mathbb{F})$. Then

$$\varphi(A') = \varphi(P_{11} A' P_{11}) = E_{11} \varphi(A') E_{11},$$

so the only non-zero entry of $\varphi(A')$ is at the $(1,1)$ position. Thus we have a multiplicative mapping $\varphi' : \mathcal{M}_{n-1}(\mathbb{F}) \to \mathbb{F}$. By Proposition 2.1 there exists a multiplicative mapping $f : \mathbb{F} \to \mathbb{F}$ such that

$$\varphi'(A) = f(\det A)$$

and

$$\varphi(A') = f(\det A) E_{11} = f(\det A'_{11}) E_{11}.$$

Now let $B \in \mathcal{M}_n(\mathbb{F})$. We have

$$E_{ii} \varphi(B) E_{jj} = \varphi(P_{ii} B P_{jj}) = \varphi(P_{i1} P_{1i} B P_{j1} P_{1j}) = E_{i1} \varphi(P_{1i} B P_{j1}) E_{1j}.$$

The matrix $P_{1i} B P_{j1}$ has the form of A', so $\varphi(P_{1i} B P_{j1}) = f(\det B_{ij}) E_{11}$ and

$$E_{ii} \varphi(B) E_{jj} = E_{i1} f(\det B_{ij}) E_{11} E_{1j} = E_{ii} f(\det B_{ij}) E_{jj}.$$

Thus the ij-th entry of $\varphi(A)$ is $f(\det A_{ij})$ and

$$\varphi(A) = \begin{bmatrix} \hat{f}(\text{Cof(A)}) & * \\ * & * \end{bmatrix}.$$

This ends the proof. $\square$

We will now give the proof of the Theorem 2.2.

Proof. (of the Theorem 2.2) Let $n = 2$. Since φ is non-degenerate, also $m = 2$ and φ maps matrices of rank 1 to matrices of rank 1. By Proposition 5.4 we obtain the asserted form. Assume now that $n \geq 3$ and let

$$k = \min\{\text{rank}A; \varphi(A) \neq 0\}.$$

Then $\binom{n}{k} \leq m$ by Proposition 5.1. Suppose that $m < n$. Then $k = 0$, which is impossible or $k = n$, which gives us a degenerate homomorphism. Thus m is equal to n. By Proposition 5.3 we have two possibilities:

(a) rank $A = 1$ implies rank $\varphi(A) = 1$ and rank $A = 2$ implies rank $\varphi(A) = 2$ or

(b) rank $A < n-1$ implies $\varphi(A) = 0$ and rank $A = n-1$ implies rank $\varphi(A) = 1$.

In case (a) we obtain by Proposition 5.4 a form

$$\varphi(A) = S\hat{f}(A)S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_m(\mathbb{F})$ is an invertible matrix. In case (b) we obtain by Proposition 5.5 a form

$$\varphi(A) = S\hat{f}(\text{Cof}(A))S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism and $S \in \mathcal{M}_m(\mathbb{F})$ is an invertible matrix. It remains to prove that f is additive. To show this, we observe that

$$\begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix} \oplus I_{n-2} = \left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \oplus I_{n-2} \right) \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus I_{n-2} \right)$$

so that

$$f(a+b)E_{22} = \varphi\left(\begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix} \oplus I_{n-2}\right) =$$

$$= \varphi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \oplus I_{n-2}\right) \varphi\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus I_{n-2}\right) =$$

$$= (f(b)E_{21} + f(a)E_{22})(E_{12} + E_{22}) = (f(a) + f(b))E_{22}.$$

Thus we have $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathbb{F}$ and this ends the proof. □

5.3 Case $m = n + 1$

We will now prove the main theorem of this chapter. We will assume that $m = n + 1$ and show that in this case either of the two possibilities of the previous section gives us reducibility.

Theorem 5.6 *Assume that $n \geq 3$. Every non-degenerate semigroup homomorphism $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{n+1}(\mathbb{F})$ is reducible.*

Proof. Suppose $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{n+1}(\mathbb{F})$ is an irreducible non-degenerate semigroup homomorphism. An irreducible semigroup homomorphism maps 0 to 0, I to I and invertible matrices to invertible matrices. By Proposition 5.3 we have two possibilities:

(a) rank $A = 1$ implies rank $\varphi(A) = 1$ and rank $A = 2$ implies rank $\varphi(A) = 2$ or

(b) rank $A < n-1$ implies $\varphi(A) = 0$ and rank $A = n-1$ implies rank $\varphi(A) = 1$.

In case (a)

$$\varphi(A) = S \begin{bmatrix} \hat{f}(A) & * \\ * & * \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $S \in \mathcal{M}_{n+1}(\mathbb{F})$ is an invertible matrix. So for arbitrary $A \in \mathcal{M}_n(\mathbb{F})$ we now have

$$\varphi(A) = \begin{bmatrix} \hat{f}(A) & \varphi_{12}(A) \\ \varphi_{21}(A) & \varphi_{22}(A) \end{bmatrix}.$$

If also $B \in \mathcal{M}_n(\mathbb{F})$, then

$$\varphi(AB) = \begin{bmatrix} \hat{f}(AB) & \varphi_{12}(AB) \\ \varphi_{21}(AB) & \varphi_{22}(AB) \end{bmatrix} = \begin{bmatrix} \hat{f}(A) & \varphi_{12}(A) \\ \varphi_{21}(A) & \varphi_{22}(A) \end{bmatrix} \begin{bmatrix} \hat{f}(B) & \varphi_{12}(B) \\ \varphi_{21}(B) & \varphi_{22}(B) \end{bmatrix} =$$

$$= \begin{bmatrix} \hat{f}(A)\hat{f}(B) + \varphi_{12}(A)\varphi_{21}(B) & * \\ * & * \end{bmatrix}.$$

So $\varphi_{12}(A)\varphi_{21}(B) = 0$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$. If $\varphi_{12}(A) \neq 0$ for some $A \in \mathcal{M}_n(\mathbb{F})$, we have a nonzero linear functional, which is zero on the image of φ. So φ is reducible Proposition 4.1. If $\varphi_{12}(A) = 0$ for every $A \in \mathcal{M}_n(\mathbb{F})$, φ is reducible by the same argument.

In case (b)

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & * \\ * & * \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism and $S \in \mathcal{M}_{n+1}(\mathbb{F})$ is an invertible matrix.

We consider the images under φ of the permutation matrices. Denote by R_i the transposition matrix $I_{i-1} \oplus (E_{12} + E_{21}) \oplus I_{n-i-1}$ for $i = 1, 2, ..., n-1$. If $j < i$ or $j > i+1$, we have $P_{jj}R_i = P_{jj}R_iP_{jj}$, so $E_{jj}\varphi(R_i) = E_{jj}\varphi(R_i)E_{jj}$, thus the only non-zero element in the j-th row of $\varphi(R_i)$ is in the j-th position. The same holds for the j-th column. On the other hand, $P_{ii}R_i = P_{i(i+1)}$, so $E_{ii}\varphi(R_i) = E_{i(i+1)}$, thus the only non-zero element in the i-th row of $\varphi(R_i)$ is in the $(i+1)$-st position and vice versa. The same holds for the i-th and the $(i+1)$-st column. We have thus seen that

$$\varphi(R_i) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(R_i)) & 0 \\ 0 & * \end{bmatrix} S^{-1}.$$

The entry in the last row and column must be ± 1, since R_i is an involution. Since the matrices R_i generate the whole group of permutation matrices, we have for every permutation matrix P

$$\varphi(P) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(P)) & 0 \\ 0 & \pm 1 \end{bmatrix} S^{-1}.$$

Now let $A = A' \oplus I_{n-2}$, where

$$A' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_2(\mathbb{F}).$$

So

$$\varphi(A) = S \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) \oplus f(ad-bc)I_{n-2} & * \\ * & * \end{bmatrix} S^{-1},$$

Multiplying A by $P_{33}, ...P_{nn}$ on the left or on the right side we obtain

$$\varphi(A)_{n+1,i} = 0 \quad \text{and} \quad \varphi(A)_{i,n+1} = 0$$

for $i = 3, ..., n$ Thus

$$\varphi(A) = S \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & 0 & * \\ 0 & f(ad-bc)I_{n-2} & 0 \\ * & 0 & * \end{bmatrix} S^{-1}.$$

Let $C_{1,2,(n+1)}$ be a compression to the first, second and last rows and columns of a matrix. Define

$$\psi(A') = C_{1,2,(n+1)}\left(S^{-1}\varphi(A' \oplus I_{n-2})S\right).$$

It is obvious that ψ is multiplicative and we have just seen that

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & * \\ * & * \end{bmatrix}.$$

By Theorem 3.4 we have two possibilities:

(i)

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & 0 \\ 0 & * \end{bmatrix}$$

and f is additive. In this case we have

$$\varphi(A) = S \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & 0 & 0 \\ 0 & f(ad-bc)I_{n-2} & 0 \\ 0 & 0 & * \end{bmatrix} S^{-1} =$$

$$= S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & 0 \\ 0 & * \end{bmatrix} S^{-1},$$

for $A = A' \oplus I_{n-2}$. The same holds for permutation matrices. Since matrices of the form $A = A' \oplus I_{n-2}$ and permutation matrices generate the complete $\mathcal{M}_n(\mathbb{F})$, we obtain

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & 0 \\ 0 & * \end{bmatrix} S^{-1},$$

for all $A \in \mathcal{M}_n(\mathbb{F})$, and consequently φ is reducible.

(ii)

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \hat{g}\left(\begin{bmatrix} d^2 & c^2 & dc \\ b^2 & a^2 & ba \\ 2db & 2ca & da+cb \end{bmatrix}\right),$$

where $f(x) = g(x^2)$ and g is additive. In this case we have

$$\varphi(A) = S\hat{g}\left(\begin{bmatrix} d^2 & c^2 & 0 & dc \\ b^2 & a^2 & 0 & ba \\ 0 & 0 & (ad-bc)^2 I_{n-2} & 0 \\ 2db & 2ca & 0 & da+cb \end{bmatrix}\right) S^{-1}$$

for $A = A' \oplus I_{n-2}$. Now let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3}$$

and $B = R_2 A R_2$, so

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3},$$

We have

$$AB = BA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3},$$

but on the other hand

$$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1},$$

$$\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix} S^{-1} = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \pm 1 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ \pm 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1},$$

so

$$\varphi(A)\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 \pm 2 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \pm 1 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 2 \pm 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}$$

and

$$\varphi(B)\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 \pm 2 & 0 & 1 & 0 & \pm 1 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 2 \pm 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}.$$

This is a contradiction, so that the possibility (ii) cannot occur. □

Remark: Case (a) in the proof is general: If $n \geq 3$, $m > n$ and $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ ia a non-degenerate semigroup homomorphism such that rank $A = 1$ implies rank $\varphi(A) = 1$ and rank $A = 2$ implies rank $\varphi(A) = 2$, then φ is reducible.

5.4 Case $n = 3$ and $m = 4, 5$

We will now explore the case $n = 3$ a little further.

Theorem 5.7 *Assume that $m = 4$ or $m = 5$. Every non-degenerate semigroup homomorphism $\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is reducible.*

Proof. If $m = 4$, this is a special case of Theorem 5.6, so let $m = 5$. Suppose $\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_5(\mathbb{F})$ is an irreducible non-degenerate semigroup homomorphism. Again we have two possibilities:
(a) rank $A = 1$ implies rank $\varphi(A) = 1$ and rank $A = 2$ implies rank $\varphi(A) = 2$ or
(b) rank $A = 1$ implies $\varphi(A) = 0$ and rank $A = 2$ implies rank $\varphi(A) = 1$.

In case (a) the same proof as in Theorem 5.6 works.

In case (b)

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & * \\ * & * \end{bmatrix} S^{-1},$$

where $f : \mathbb{F} \to \mathbb{F}$ is a semigroup homomorphism and $S \in \mathcal{M}_5(\mathbb{F})$ is an invertible matrix. Similarly as in Theorem 5.6 we prove, that if P is a permutation matrix, then

$$\varphi(P) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(P)) & 0 \\ 0 & * \end{bmatrix} S^{-1}, \tag{5.1}$$

and if

$$A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$\varphi(A) = S \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & 0 & * \\ 0 & f(ad-bc) & 0 \\ * & 0 & * \end{bmatrix} S^{-1}.$$

Let $C_{1,2,4,5}$ be a compression to the first, second and fourth and fifth rows and columns of a matrix. Define $\psi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_4(\mathbb{F})$

$$\psi(A') = C_{1,2,4,5}\left(S^{-1}\varphi(A' \oplus I_1)S\right).$$

It is obvious that ψ is multiplicative and we have just seen that

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & * \\ * & * \end{bmatrix}.$$

The map ψ may be irreducible or reducible. If it is irreducible it has one of the forms (a) or (b) of Theorem 4.4. If it is reducible, its image has an irreducible invariant subspace of dimension at least two, so this irreducible subspace is of dimension two or three. Thus we have four possibilities to explore:

(i)

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \hat{g}\left(\begin{bmatrix} d^3 & c^3 & c^2d & cd^2 \\ b^3 & a^3 & a^2b & ab^2 \\ 3b^2d & 3a^2c & a^2d+2abc & 2abd+b^2c \\ 3bd^2 & 3ac^2 & 2acd+bc^2 & ad^2+2bcd \end{bmatrix}\right),$$

where $f(x) = g(x^2)$ and g is additive. In this case we have

$$\varphi(A) = S\hat{g}\left(\begin{bmatrix} d^3 & c^3 & 0 & c^2d & cd^2 \\ b^3 & a^3 & 0 & a^2b & ab^2 \\ 0 & 0 & (ad-bc)^3 & 0 & 0 \\ 3b^2d & 3a^2c & 0 & a^2d+2abc & 2abd+b^2c \\ 3bd^2 & 3ac^2 & 0 & 2acd+bc^2 & ad^2+2bcd \end{bmatrix}\right)S^{-1}$$

for $A = A' \oplus I_1$. Furthermore, we have

$$\varphi(R_1) = S \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} S^{-1},$$

and, using (5.1), it follows that

$$\varphi(R_2) = S \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & 0 & e_3 & e_4 \end{bmatrix} S^{-1},$$

where the lower-right corner $E = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$ is an involution similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the product $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} E$ is of order three or one. In particular, $e_1 + e_4 = 0$ and $e_2 + e_3 \neq 0$, so that $e_1 + e_2 + e_3 + e_4 \neq 0$. Now let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = R_2 A R_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices A and B commute, but

$$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1},$$

$$\varphi(B) = S \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} E \\ E \begin{bmatrix} -3 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} & E \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} E \end{bmatrix},$$

so the upper-left 3-by-3 corner of $S^{-1}\varphi(A)\varphi(B)S$ is equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} E \begin{bmatrix} -3 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 - 3(e_1 + e_2 + e_3 + e_4) & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

and the upper-left 3-by-3 corner of $S^{-1}\varphi(B)\varphi(A)S$ is equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} E \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1+3(e_1+e_2+e_3+e_4) & 0 & 1 \end{bmatrix}$$

This is a contradiction, possibility (i) cannot occur.

(ii)

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} g(d)h(d) & g(c)h(c) & g(c)h(d) & g(d)h(c) \\ g(b)h(b) & g(a)h(a) & g(a)h(b) & g(b)h(a) \\ g(b)h(d) & g(a)h(c) & g(a)h(d) & g(b)h(c) \\ g(d)h(b) & g(c)h(a) & g(c)h(b) & g(d)h(a) \end{bmatrix} S^{-1},$$

where $f(x) = g(x)h(x)$ and g, h are additive. In this case we have

$$\varphi(A) =$$

$$S \begin{bmatrix} g(d)h(d) & g(c)h(c) & 0 & g(c)h(d) & g(d)h(c) \\ g(b)h(b) & g(a)h(a) & 0 & g(a)h(b) & g(b)h(a) \\ 0 & 0 & g(ad-bc)h(ad-bc) & 0 & 0 \\ g(b)h(d) & g(a)h(c) & 0 & g(a)h(d) & g(b)h(c) \\ g(d)h(b) & g(c)h(a) & 0 & g(c)h(b) & g(d)h(a) \end{bmatrix} S^{-1}$$

for $A = A' \oplus I_1$. Again

$$\varphi(R_2) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & 0 & e_3 & e_4 \end{bmatrix} S^{-1},$$

where lower-right corner $E = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$ is involution similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $e_1 + e_2 + e_3 + e_4 \neq 0$. For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = R_2AR_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}$$

and

$$\varphi(B) = S \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} E \\ E \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & E \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E \end{bmatrix}.$$

so the upper-left 3-by-3 corner of $S^{-1}\varphi(A)\varphi(B)S$ is equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 + e_1 + e_2 + e_3 + e_4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and the upper-left 3-by-3 corner of $S^{-1}\varphi(B)\varphi(A)S$ is equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 + e_1 + e_2 + e_3 + e_4 & 0 & 1 \end{bmatrix}.$$

Since A and B commute, this is a contradiction, possibility (ii) cannot occur.

(iii)

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \hat{f}\left(\begin{bmatrix} d & c \\ b & a \end{bmatrix}\right) & * \\ 0 & * \end{bmatrix}$$

and f is additive. In this case we have

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & * \\ 0 & * \end{bmatrix} S^{-1}$$

for $A = A' \oplus I_1$. The same holds for permutation matrices. Since matrices of the form $A = A' \oplus I_1$ and permutation matrices generate complete $\mathcal{M}_3(\mathbb{F})$, we obtain

$$\varphi(A) = S \begin{bmatrix} \hat{f}(\mathrm{Cof}(A)) & * \\ 0 & * \end{bmatrix} S^{-1},$$

for all $A \in \mathcal{M}_3(\mathbb{F})$, and consequently φ is reducible.

(iv)

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \hat{g}\left(\begin{bmatrix} d^2 & c^2 & dc & * \\ b^2 & a^2 & ba & * \\ 2db & 2ca & da+cb & * \\ 0 & 0 & 0 & * \end{bmatrix}\right),$$

where $f(x) = g(x^2)$ and g is additive. In this case we have

$$\varphi(A) = S\hat{g}\left(\begin{bmatrix} d^2 & c^2 & 0 & dc & * \\ b^2 & a^2 & 0 & ba & * \\ 0 & 0 & (ad-bc)^2 & 0 & 0 \\ 2db & 2ca & 0 & da+cb & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}\right) S^{-1}$$

for $A = A' \oplus I_1$. Now

$$\varphi(R_1) = S \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix} S^{-1}.$$

If the last entry in the last row is equal to 1, then $a = 0$ and the lower-right 2-by-2 corner of every permutation matrix is equal to I_2, and consequently φ is reducible. So the last entry in the last row is equal to -1. We may now apply a simultaneous similarity with a matrix of the form $I + \alpha E_{45}$ to obtain $a = 0$ and without disturbing the first four columns. Further,

$$\varphi(R_2) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & 0 & e_3 & e_4 \end{bmatrix} S^{-1},$$

where the lower-right corner

$$E = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$$

is an involution similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the product $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E$ is of order three or one. So either $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or it has the form

$$E = \begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{4b} & \frac{1}{2} \end{bmatrix}$$

where $b \neq 0$. In the first case again φ is reducible, in the second case we may apply a simultaneous similarity with a diagonal matrix of the form $I_4 \oplus [\beta]$ to obtain $b = \frac{1}{2}$. So

$$\varphi(R_2) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} \end{bmatrix} S^{-1}.$$

For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = R_2 A R_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we now have

$$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & x \\ 1 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}$$

and

$$\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 & \frac{3x}{2} & \frac{x}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -\frac{1}{2} + \frac{3y}{2} & \frac{1}{2} + \frac{y}{2} \\ -1 & 0 & 0 & 1 - \frac{3z}{4} & -\frac{z}{4} \\ 3 & 0 & 0 & \frac{9z}{4} & 1 + \frac{3z}{4} \end{bmatrix} S^{-1}.$$

Since A and B commute, $\varphi(A)$ and $\varphi(B)$ must also commute. Thus we obtain $x = 0$, $y = \frac{1}{3}$ and $z = 0$. Now let

$$C = R_1R_2R_1AR_1R_2R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The matrices A and C commute, but

$$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}$$

and

$$\varphi(C) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 0 & 1 \end{bmatrix} S^{-1}$$

do not commute. Again we get a contradiction and this ends the proof. □

5.5 Case $m = 6$

In this concluding section we will give three examples of irreducible non-degenerate homomorphisms, which go to the dimension 6. We have seen in previous section that every non-degenerate homomorphism from dimension 3 to dimension 5 is reducible. But there exist an irreducible non-degenerate homomorphism from dimension 4 to dimension 6, and two different irreducible non-degenerate homomorphisms from dimension 3 to dimension 6.

Example: There exist two essentially different irreducible non-degenerate semigroup homomorphisms $\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F})$:

(a) Symmetric square:

$$\varphi(A) = \mathrm{Sym}^2 A;$$

explicitly

$$\varphi\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) =$$

$$\begin{bmatrix} a^2 & b^2 & c^2 & ab & ac & bc \\ d^2 & e^2 & f^2 & de & df & ef \\ g^2 & h^2 & i^2 & gh & gi & hi \\ 2ad & 2be & 2cf & ae+bd & af+cd & bf+ce \\ 2ag & 2bh & 2ci & ah+bg & ai+cg & bi+ch \\ 2dg & 2eh & 2fi & dh+eg & di+fg & ei+fh \end{bmatrix}.$$

(b) Symmetric square of exterior power:

$$\varphi(A) = \mathrm{Sym}^2(A \wedge A);$$

explicitly, we give it column by column:

$$\varphi\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

where

$$b_1 = \begin{bmatrix} b^2d^2 - 2abde + a^2e^2 \\ b^2g^2 - 2abgh + a^2h^2 \\ e^2g^2 - 2degh + d^2h^2 \\ 2b^2dg - 2abeg - 2abdh + 2a^2eh \\ 2bdeg - 2ae^2g - 2bd^2h + 2adeh \\ 2beg^2 - 2bdgh - 2aegh + 2adh^2 \end{bmatrix},$$

$$b_2 = \begin{bmatrix} c^2d^2 - 2acdf + a^2f^2 \\ c^2g^2 - 2acgi + a^2i^2 \\ f^2g^2 - 2dfgi + d^2i^2 \\ 2c^2dg - 2acfg - 2acdi + 2a^2fi \\ 2cdfg - 2af^2g - 2cd^2i + 2adfi \\ 2cfg^2 - 2cdgi - 2afgi + 2adi^2 \end{bmatrix},$$

$$b_3 = \begin{bmatrix} c^2e^2 - 2bcef + b^2f^2 \\ c^2h^2 - 2bchi + b^2i^2 \\ f^2h^2 - 2efhi + e^2i^2 \\ 2c^2eh - 2bcfh - 2bcei + 2b^2fi \\ 2cefh - 2bf^2h - 2ce^2i + 2befi \\ 2cfh^2 - 2cehi - 2bfhi + 2bei^2 \end{bmatrix},$$

$$b_4 = \begin{bmatrix} bcd^2 - acde - abdf + a^2ef \\ bcg^2 - acgh - abgi + a^2hi \\ efg^2 - dfgh - degi + d^2hi \\ 2bcdg - aceg - abfg - acdh + a^2fh - abdi + a^2ei \\ cdeg + bdfg - 2aefg - cd^2h + adfh - bd^2i + adei \\ ceg^2 + bfg^2 - cdgh - afgh - bdgi - aegi + 2adhi \end{bmatrix},$$

$$b_5 = \begin{bmatrix} bcde - ace^2 - b^2df + abef \\ bcgh - ach^2 - b^2gi + abhi \\ efgh - dfh^2 - e^2gi + dehi \\ bceg - b^2fg + bcdh - 2aceh + abfh - b^2di + abei \\ ce^2g - befg - cdeh + 2bdfh - aefh - bdei + ae^2i \\ cegh + bfgh - cdh^2 - afh^2 - 2begi + bdhi + aehi \end{bmatrix},$$

$$b_6 = \begin{bmatrix} c^2de - bcdf - acef + abf^2 \\ c^2gh - bcgi - achi + abi^2 \\ f^2gh - efgi - dfhi + dei^2 \\ c^2eg - bcfg + c^2dh - acfh - bcdi - acei + 2abfi \\ cefg - bf^2g + cdfh - af^2h - 2cdei + bdfi + aefi \\ 2cfgh - cegi - bfgi - cdhi - afhi + bdi^2 + aei^2 \end{bmatrix}.$$

Example: There exists an irreducible non-degenerate semigroup homomorphism $\varphi : \mathcal{M}_4(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F})$, an exterior power:

$$\varphi(A) = A \wedge A;$$

explicitly, we give it column by column:

$$\varphi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}$$

where

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \\ a_{11}a_{42} - a_{12}a_{41} & a_{11}a_{43} - a_{13}a_{41} \\ a_{21}a_{32} - a_{22}a_{31} & a_{21}a_{33} - a_{23}a_{31} \\ a_{21}a_{42} - a_{22}a_{41} & a_{21}a_{43} - a_{23}a_{41} \\ a_{31}a_{42} - a_{32}a_{41} & a_{31}a_{43} - a_{33}a_{41} \end{bmatrix},$$

$$\begin{bmatrix} c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a_{11}a_{24} - a_{14}a_{21} & a_{12}a_{23} - a_{13}a_{22} \\ a_{11}a_{34} - a_{14}a_{31} & a_{12}a_{33} - a_{13}a_{32} \\ a_{11}a_{44} - a_{14}a_{41} & a_{12}a_{43} - a_{13}a_{42} \\ a_{21}a_{34} - a_{24}a_{31} & a_{22}a_{33} - a_{23}a_{32} \\ a_{21}a_{44} - a_{24}a_{41} & a_{22}a_{43} - a_{23}a_{42} \\ a_{31}a_{44} - a_{34}a_{41} & a_{32}a_{43} - a_{33}a_{42} \end{bmatrix},$$

$$\begin{bmatrix} c_5 & c_6 \end{bmatrix} = \begin{bmatrix} a_{12}a_{24} - a_{14}a_{22} & a_{13}a_{24} - a_{14}a_{23} \\ a_{12}a_{34} - a_{14}a_{32} & a_{13}a_{34} - a_{14}a_{33} \\ a_{12}a_{44} - a_{14}a_{42} & a_{13}a_{44} - a_{14}a_{43} \\ a_{22}a_{34} - a_{24}a_{32} & a_{23}a_{34} - a_{24}a_{33} \\ a_{22}a_{44} - a_{24}a_{42} & a_{23}a_{44} - a_{24}a_{43} \\ a_{32}a_{44} - a_{34}a_{42} & a_{33}a_{44} - a_{34}a_{43} \end{bmatrix}.$$

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Izjava

Izjavljam, da je to delo rezultat lastnega raziskovalnega dela.

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