C·E·P·S Journal

Center for Educational Policy Studies Journal

Revija Centra za študij edukacijskih strategij

Vol.3 | Nº4 | Year 2013

Revija Centra za študij edukacijskih strategij
Center for Educational Policy Studies Journal
ISSN 2232-2647 (online edition)
ISSN 1855-9719 (printed edition)
Publication frequency: 4 issues per year
Subject: Teacher Education, Educational Science
Publisher: Faculty of Education, University of Ljubljana, Slovenia

Managing editors: Mira Metljak / **English language editing:** Neville Hall / **Slovene language editing:** Tomaž Petek / **Cover and layout design:** Roman Ražman / **Typeset:** Igor Cerar / **Print:** Tiskarna Uradni list RS, d.o.o. Ljubljana

C·E·P·S *Journal*

Center for Educational Policy Studies Journal
Revija Centra za študij edukacijskih strategij

The CEPS Journal is an open-access, peer-reviewed journal devoted to publishing research papers in different fields of education, including scientific.

Aims & Scope

The CEPS Journal is an international peer-reviewed journal with an international board. It publishes original empirical and theoretical studies from a wide variety of academic disciplines related to the field of Teacher Education and Educational Sciences; in particular, it will support comparative studies in the field. Regional context is stressed but the journal remains open to researchers and contributors across all European countries and worldwide. There are four issues per year. Issues are focused on specific areas but there is also space for non-focused articles and book reviews.

About the Publisher

The University of Ljubljana is one of the largest universities in the region (see www.uni-lj.si) and its Faculty of Education (see www.pef.uni-lj.si), established in 1947, has the leading role in teacher education and education sciences in Slovenia. It is well positioned in regional and European cooperation programmes in teaching and research. A publishing unit oversees the dissemination of research results and informs the interested public about new trends in the broad area of teacher education and education sciences; to date, numerous monographs and publications have been published, not just in Slovenian but also in English.

In 2001, the Centre for Educational Policy Studies (CEPS; see http://ceps.pef.uni-lj.si) was established within the Faculty of Education to build upon experience acquired in the broad reform of the national educational system during the period of social transition in the 1990s, to upgrade expertise and to strengthen international cooperation. CEPS has established a number of fruitful contacts, both in the region – particularly with similar institutions in the countries of the Western Balkans – and with interested partners in EU member states and worldwide.

Revija Centra za študij edukacijskih strategij je mednarodno recenzirana revija, z mednarodnim uredniškim odborom in s prostim dostopom. Namenjena je objavljanju člankov s področja izobraževanja učiteljev in edukacijskih ved.

Cilji in namen

Revija je namenjena obravnavanju naslednjih področij: poučevanje, učenje, vzgoja in izobraževanje, socialna pedagogika, specialna in rehabilitacijska pedagogika, predšolska pedagogika, edukacijske politike, supervizija, poučevanje slovenskega jezika in književnosti, poučevanje matematike, računalništva, naravoslovja in tehnike, poučevanje družboslovja in humanistike, poučevanje na področju umetnosti, visokošolsko izobraževanje in izobraževanje odraslih. Poseben poudarek bo namenjen izobraževanju učiteljev in spodbujanju njihovega profesionalnega razvoja.

V reviji so objavljeni znanstveni prispevki, in sicer teoretični prispevki in prispevki, v katerih so predstavljeni rezultati kvantitativnih in kvalitativnih empiričnih raziskav. Še posebej poudarjen je pomen komparativnih raziskav.

Revija izide štirikrat letno. Številke so tematsko opredeljene, v njih pa je prostor tudi za netematske prispevke in predstavitve ter recenzije novih publikacij.

Contents

Varia

Reviews

Editorial

Dear Reader

Many researchers have dealt with, and continue to deal with, problem solving, definitions of the notion of a problem, the roles of problem solving in mathematics with regard to the development of procedural and conceptual knowledge, and differentiating between investigation and problem solving. In general, a problem in mathematics is defined as a situation in which the solver perceives the situation as a problem and accepts the challenge of solving it but does not have a previously known strategy to do so, or is unable to recall such a strategy. The best known strategies in mathematics are inductive reasoning and deductive reasoning. Inductive reasoning involves, on the basis of observations of individual examples, deriving a generalisation with a certain level of credibility (in mathematics, if the generalisation is not proven we must not take it as true, as always applicable). With deductive reasoning, on the other hand, we derive examples on the basis of a broadly accepted generalisation that serve to illustrate the generalisation. Both forms of reasoning are important in mathematical thinking. In addition to these two forms of reasoning – deductive and inductive – certain authors in the field of mathematics use other collocations, such as inductive inference, and reasoning and proof. In the majority of cases, researchers investigating problem solving associate the issues of problem solving with inductive reasoning. Research in the field of problem solving is focused on the cognitive processes associated with strategies used by enquirers (students of all levels) in solving selected problems, as well as on the significance of inductive reasoning for the development of the basic concepts of algebra. By analysing the process of solving problems, we gain insight into the strategies used by solvers, on the basis of which we can draw conclusions about the success of specific strategies in forming generalisations. An important finding in this regard is that not all strategies are equally efficient, and that the context of a problem can either support or hinder generalisation. However, selecting a good mathematical problem is not the only criteria for successful generalisation. Another important factor is the social interaction between the solvers of the problem, which means that when solving a problem, in addition to the dimension subject-object (solver-problem), it is also necessary to take into account the dimension subject-subject (solver-solver, solver-teacher).

The foundations of problem solving in mathematics instruction were established by Polya, who identified four levels of inductive reasoning within problem solving: observation of particular cases, conjecture formulation, based on previous particular cases, generalization and conjecture verification with new particular cases. Other researchers of problem solving have added further levels, such as: organization of particular cases, search and prediction of patterns, conjecture formulation, conjecture validation and general conjectures justification. It is rare that all of the levels are present when solving a particular example, as problem solving it is primarily dependent on the solver, on his/her knowledge and motivation, as well as on other factors.

The teacher plays an important role in problem solving in school, with knowledge, problem selection and the way the problem situation is conveyed, as well as by guiding students through the process of solving the problem. The greater the teacher's competence in problem solving, the greater the likelihood that s/he will include problem situations in mathematics instruction, and thus develop this competence in the students. A leading group in the field of researching problem-solving instruction and the inclusion of problem solving in mathematics instruction is ProMath, whose aim is to examine mathematical-didactical questions concerning problem solving in mathematics education. The group was formed on the initiative of Prof. Dr Günter Graumann (University of Bielefeld, Germany), Prof. Dr Erkki Pehkonen (University of Helsinki, Finland) and Prof. Dr Bernd Zimmermann (University of Jena, Germany). Originally founded in 1998 as a Finnish-German group, it has developed into an international collaboration with an European focus.

In the present edition of the CEPS Journal, members of ProMath have highlighted various issues regarding problem solving, from a reconsideration of the meaning and role of problem solving and a study of the factors determining successful problem solving, to the use of ICT in problem solving.

In the first article, On Teaching Problem Solving in School Mathematics, Erkki Pehkonen, Liisa Näveri and Anu Laine present an overview of the situation in the field of problem solving, as well as outlining the key activities and goals of the ProMath research group, whose aim is to improve mathematics instruction in school. They emphasise the importance of open problems in primary school education, as well as the role of the teacher in developing methods of instruction that include solving mathematical problems.

Benjamin Rott's contribution, Process Regulation in the Problem-Solving Processes of Fifth Graders, investigates how problem solving processes take place amongst fifth graders, as well as examining the influence of metacognition and self-regulation on these processes and whether transitions between the phases in the process of problem solving are linked with metacognitive activities. On the basis of an analysis of approximately one hundred fifth graders (aged 10–12 years) in German primary schools, the author concludes that there is a strong link between processes regulation and success (or lack of success) in problem solving.

The third article is by Ana Kuzle and is entitled Promoting Writing in Mathematics: Prospective Teachers' Experiences and Perspectives on the Process of Writing When Doing Mathematics as Problem Solving. It reports on research focused on gaps between writing and mathematical problem solving in instruction. At a problem-solving seminar, preservice teachers gained experience in writing in mathematics, and reported on how the experience influenced the process of problem solving and formed their attitude towards including writing in their own lessons. Those who perceived writing and mathematics instruction as one interwoven process expressed a positive attitude towards the use of writing in mathematics lessons, whereas those who viewed writing and mathematics instruction as two separate processes used writing purely as a method to create a formal document in order to satisfy the demands of teachers.

In an article entitled Applying Cooperative Techniques in Teaching Problem Solving, Krisztina Barczi presents cooperative learning as one way of overcoming the difficulty students face in making the transition from simple mathematical tasks to solving mathematical problems. The article describes the positive effects of the cooperative teaching techniques in a group of secondary school students aged from 16 to 17 years. These effects include a greater willingness amongst the students to share their opinions with other members of the group, and the development of independent thinking.

In the fifth article, Improving Problem-Solving Skills with the Help of Plane-Space Analogies, László Budai focuses on students' problems in dealing with the geometrical treatment of three-dimensional space. The author identifies the possibility of improving the situation in this field by creating teaching procedures that strengthen analogies between planar and spatial geometry. The article demonstrates the important role of the geometry programmes GeoGebra and DGS in developing spatial awareness and solving spatial geometry problems amongst secondary school students.

Zlatan Magajna's contribution, Overcoming the Obstacle of Poor Knowledge in Proving Geometry Tasks, presents one option for more successfully proving claims regarding planar geometry. Proving a claim in planar geometry involves several processes, the most salient being visual observation and deductive argumentation. These two processes are interwoven, but often poor observation hinders deductive argumentation. The article presents the results of two small-scale research projects involving secondary school students, both of which indicate that students are able to work out considerably more deductions if computer-aided observation is used. However, not all students use computer-aided observation effectively, as some are unable to choose the properties that are relevant to the task from the exhaustive list of properties observed by the computer programme.

In the Varia section we find one paper by Nada Turnšek, entitled Enjoying Cultural Differences Assists Teachers in Learning about Diversity and Equality. An Evaluation of Antidiscrimination and Diversity Training. In the paper a study based on a quasi-experimental research design in presented. The results of an evaluation of Antidiscrimination and Diversity Training that took place at the Faculty of Education in Ljubljana, rooted in the anti-bias approach to educating diversity and equality issues showed that ADT had a decisive impact on all of the measured variables. It was also found that self-assessed personality characteristics are predictors of the teachers' beliefs, especially the *enjoying awareness of cultural differences* variable, which correlates with all of the dependent variables.

In the last section a review by Darko Štrajn of a monograph The Globalisation Challenge for European Higher Education / Convergence and Diversity, Centres and Peripheries, edited by Zgaga, P., Teichler, U., and Brennan, J. (2012, Frankfurt/M.: Peter Lang. ISBN 978-3-631- 6398-5) is presented. As stated in the review the editors and authors of the book, which is an insightful product of a range of institutionally and informally based academic interactions, were obviously aware that the developments in European higher education systems expose a chain of meanings to different perceptions and to critical scrutiny.

Tatjana Hodnik Čadež and Vida Manfreda Kolar

On Teaching Problem Solving in School Mathematics

Erkki Pehkonen*[1], Liisa Näveri[2], and Anu Laine[3]

The article begins with a brief overview of the situation throughout the world regarding problem solving. The activities of the ProMath group are then described, as the purpose of this international research group is to improve mathematics teaching in school. One mathematics teaching method that seems to be functioning in school is the use of open problems (i.e., problem fields). Next we discuss the objectives of the Finnish curriculum that are connected with problem solving. Some examples and research results are taken from a Finnish–Chilean research project that monitors the development of problem-solving skills in third grade pupils. Finally, some ideas on "teacher change" are put forward. It is not possible to change teachers, but only to provide hints for possible change routes: the teachers themselves should work out the ideas and their implementation.

Keywords: Mathematics teaching; Open problems; Problem solving

1 *Corresponding Author. Department of Teacher Education, University of Helsinki, Finland
erkki.pehkonen@helsinki.fi
2 Department of Teacher Education, University of Helsinki, Finland
3 Department of Teacher Education, University of Helsinki, Finland

O poučevanju reševanja problemov v šolski matematiki

Erkki Pehkonen*, Liisa Näveri and Anu Laine

Začetek članka je posvečen pregledu razmer v svetu glede reševanja problemov. Nato so opisane dejavnosti skupine ProMath; namen te mednarodne raziskovalne skupine je izboljšanje pouka matematike v šoli. Metoda, ki bi lahko funkcionirala v šoli, je uporaba odprtih problemov (tj. problemska polja). Sledi razprava o ciljih finskega kurikuluma, ki so povezani z reševanjem problemov. Nekateri primeri so vzeti iz finsko-čilenskega raziskovalnega projekta, ki spremlja razvoj kompetenc reševanja problemov pri učencih tretjega razreda. Na koncu je podanih nekaj idej glede »spreminjanja učiteljev«. Ni namreč mogoče spremeniti učiteljev, mogoče je podati le namige za spremembo razmišljanja – učitelji bi morali sami poiskati ideje in način izvedbe.

Ključne besede: poučevanje matematike, odprti problemi, reševanje problemov

Problem solving has a long tradition in school mathematics, and has many facets and characterisations. In order to facilitate understanding, we therefore begin by providing a definition of problem solving (cf. Kantowski, 1980): a situation is said to be a *problem* when an individual must combine (for him/her) new information in a (for him/her) new way in order to solve the problem. If the individual can immediately recognise the procedures needed, the situation is a *standard task* (or a routine task or exercise). The term *non-standard task* is often used in reference to a task that one cannot usually find in mathematics books.

An Overview Of Problem-Solving Research

Since the United States is still the pioneer in the development of mathematics teaching, we will begin with the advances there. Schröder and Lester (1989), for instance, introduced three aims for using problem solving in mathematics teaching. They pointed out that problem solving should not be considered only as teaching content, but also as a teaching method. Later, in Standards for School Mathematics (NCTM, 2000), problem solving is mentioned as a teaching method with which one can improve the quality of mathematics teaching in school.

The key ideas of problem solving seem to have spread around the world, as we can see in the published overview papers. In the last ten years, a number of overview papers have been published in which the situation of problem solving has been described in several countries. The Proceedings of the ICME-9 Topic Study Group (Pehkonen, 2001), for example, is a compound of overview papers regarding problem solving from different continents. In this compound, the development of problem solving in all of the countries covered seems to be very similar. The collection of Törner, Schoenfeld and Reiss (2007) contains a description of problem solving in 15 countries, providing an even better account of the development.

One step further: open problem solving

When the constructivist view of learning was accepted in mathematics education about 30 years ago, there was a need to develop teaching methods that corresponded to the challenges set by constructivism. One such solution was the open approach (or the use of open problems) in Japan.

In Japan, the so-called open approach to mathematics teaching was developed in the 1970s. It was aimed at developing pupils' creativity and encouraging meaningful discussion in the classroom (Becker & Shimada, 1997; Pehkonen, 1995; Shimada, 1977, cf. Nohda, 1991). At the same time, so-called investigations were introduced and accepted as part of mathematics teaching in England, and soon became very popular (Wiliam, 1994). The notion of investigations was disseminated through the Cockcroft Report (1982) in particular. The idea of using open tasks in the classroom therefore spread throughout the world in the 1980s and 1990s, and research on the potential of open tasks in mathematics education was very lively in many countries (e.g., Clarke & Sullivan, 1992; Kwon, Park, & Park, 2006; Mason, 1991; Nohda, 1988; Pehkonen, 1989; Silver, 1995; Stacey, 1995; Williams, 1989; Zimmermann, 1991).

Almost 20 years ago, a number of articles critical of the use of open tasks were published. One American mathematician, for instance, wrote a very sceptical paper on learning mathematics with open problems (Wu, 1994), criticising the way open problems were used in Californian schools. At an international PME conference, Paul Blanc strongly criticised the implementation of investigations in British schools (Blanc & Sutherland, 1996), reproaching teachers for developing a new mechanical routine to solve investigations.

Using open tasks, we can respond to the challenges of developing mathematics teaching. Such teaching leads almost automatically to problem-centred teaching and clearly increases communication in class, thus approaching instruction that is more open and pupil-centred. Some ten years ago, Pehkonen (2004) wrote an overview on the situation of open problem solving. Later, Zimmermann (2010) described the development of open problem solving over the previous 20 years in Germany, while ProMath meetings have produced research results on the use of open problems for approximately 15 years (e.g., Bergqvist, 2012).

New approaches to teaching mathematics

Mathematics is not only calculation; the aim of teaching should also be the development of understanding and mathematical thinking. School teaching has been accused of viewing the act of teaching and the context in which it takes places entirely differently. However, psychological studies have shown that learning (even of mathematics) is strongly situation-bound (e.g., Bereiter, 1990; Brown, Collins, & Duguid, 1989; Collins, Brown, & Newman, 1989). Studies in learning psychology have, for instance, confirmed the hypotheses of Anderson (1980) that the learning of facts and procedures takes place with various mechanisms (e.g., Bereiter & Scardamalia, 1996). New elements should therefore be added to mathematics teaching in school.

Traditional teaching is well suited to the learning of facts, but new methods – emphasising, for example, pupils' self-regulated learning – are needed for learning procedures. Open learning environments offer such an opportunity, as within them real problems can be dealt with; pupils respond to the problems actively and learning takes place in natural situations. Learning arises through independent investigations and seeking solutions. It is believed that active learning of this kind leads to a better understanding of key principles and concepts. Active working sets a pupil in a real problem-solving environment and can thus combine the phenomena of real life and the classroom (cf. Blumenfeld, Soloway, Marx, Krajeik, Guzdial, & Palinscsar, 1991).

The development and formulation of ideas, pondering problem situations and balancing alternatives, require discussions between pupils and social interaction. This essential aspect of self-regulated active working is, in our culture, a natural way to rework ideas, conceptions and beliefs (cf. Brown et al., 1989). School culture has typically been characterised by a restriction of discussion; according to learning studies, however, free discussion amongst pupils should be encouraged and not inhibited. The only problem is how to guide the discussion in the right direction and keep it within reasonable bounds.

Open Problems In Focus (The Promath Group)

In this section, we will give a short description of the history of the ProMath group. The emphasis of the group is on open problems and the open approach in mathematics teaching, with the key question being how to use them.

A brief history of ProMath

More than ten years ago, the working group ProMath (Problem Solving in Mathematics) was established by a group of Finnish and German professors in mathematics education. A spontaneous meeting at the University of Bielefeld in 1999 can be considered to be the starting point for the series of ProMath workshops. At that meeting, the members decided to meet annually, and to establish the focus of the working group as follows:

> *the aim of the ProMath group is to study and examine those mathematical-didactical questions which arise through research on the implementation of open problem solving in school.*

The research group was designed to be open to everyone interested in mathematical problem solving. The group is based on voluntary organisation and strives to be as democratic as possible, e.g., there is no chair and each year the group votes where the next year's meeting will take place. Usually, the location hosting the annual meeting will publish proceedings in which the participants' papers are peer reviewed.

The ProMath group has now been active for more than ten years in Europe, holding annual meetings at various universities. As a rule, ProMath workshops take place at the beginning of autumn (i.e., August/September), and locations have circulated in the following countries (in alphabetical order): Finland, Germany, Hungary, Slovakia, Slovenia, Sweden. The list of all of the participants to date provides a boarder picture, as they originate from nine different countries (in alphabetical order): Australia (1), Denmark (1), Finland (15), Germany (9), Greece (1), Hungary (5), Slovakia (2), Slovenia (2), Sweden (2) and USA (1). Approximately ten presentations are given at each annual meeting.

When reading through the ProMath proceedings of the first ten years, one gains the impression that at the beginning of the 2000s there were more empirical studies focused on using examples of open problems. In recent years (since the end of the 2000s), the number of general theoretical papers has increased, i.e., there are some papers that could only marginally be regarded as problem solving. Another change concerns the number of examples: during the last few years, the number of examples of open problems has reduced significantly. Consequently, the proceedings of the workshops no longer function as a treasure trove for teaching open problems in school. For the development of school teaching, however, both aspects are needed: examples and theory.

On the use of the open approach

In line with the aim of the group (see above), the focus of ProMath workshops is open problems and their implementation in school. We therefore begin here with the concept of the 'open approach'.

One method, accepted all over the world, for a teacher to help pupils with optimal learning environments is the so-called open approach. In order to implement this method, which was developed in the 1970s in Japan (Becker & Shimada, 1997; Shimada, 1977, cf. also Nohda, 1991), one can use so-called open tasks. Such tasks have proved to be a promising solution for developing a proper learning environment, and appear to provide an opportunity for the meaningful teaching and learning of mathematics (cf. Boaler, 1998).

Amongst others, open problems include tasks from everyday life, problem posing, problem fields (or problem sequences), problems without a question, problem variations (the "what if" method), project work and investigations (cf. Pehkonen, 1995; Pehkonen, 1997; Schupp, 2002). For investigations, a starting situation is typically given within which the pupil first formulates a problem and then solves it. Such tasks are used extensively in England and Scotland, for example, as well as in Australia.

Problem fields

Investigations can be divided into two groups: structured and non-structured investigations. The latter are used in England: a pupil is given a starting situation and some starting problems, and then continues independently. Structured investigations are called *problem fields* (or problem domains). In this case, the teacher prepares a number of extension questions (problems) in advance and, depending on the solution activity of the class, decides which direction pupils will take and how far they will work with the given problem situation.

The purpose of using investigations is to promote pupils' creativity, and especially their divergent thinking (e.g., Kwon et al., 2006). In addition to problem solving, investigations also practise problem posing, as the pupil can, within the framework of the investigation, formulate and solve his/her own problem. When using open tasks in mathematics instruction, pupils have an opportunity to work like an active mathematician (Brown, 1997). It is also important for teachers to have experience with open problems during their education (cf. Zaslavsky, 1995).

Ideas for reforming mathematics teaching by Walsch

We will next consider the internal reform of mathematics education undertaken in East Germany commencing in the 1980s, which aimed at improving the quality of teaching within the existing curriculum (Walsch, 1984). The purpose was to move away from the method of model learning and towards the development of problem-solving thinking. According to Walsch (1984), didactic studies in East Germany showed that 85% of all tasks dealt with in mathematics lessons could be solved with a model known to the pupil. The reform was planned to be implemented "through working with tasks", i.e., the central idea was to deal with learning topics in the form of problems. Thus the central idea of the reform could be summarised as: *Ordinary mathematics tasks will be dealt with in an unordinary form!* (ibid) The following example demonstrates the idea.

Example 1. When the class is calculating the perimeter and area of a rectangle, the teacher can ask pupils to investigate whether the following statements are true or not (through experimenting, drawing, concluding logically, etc.):

- Two rectangles that have the same perimeter always have the same area.
- If the area of a rectangle is enlarged its perimeter will also always get longer.
- For each rectangle there is another rectangle that has the same area but a longer perimeter.

On teaching problem solving in Finland

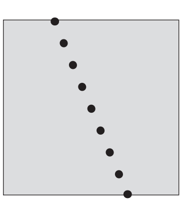
The purpose of mathematics learning for all age groups in Finland is the understanding of mathematical structures and the development of mathematical thinking, not merely mastering mechanical calculations (NBE, 2004). In order to develop correct learning habits, this should be the objective from the very beginning (Grade 1), as knowledge that is based on understanding can be more easily transferred to other contexts (cf. Sierpinska, 1994). According to the Finnish curriculum, it is not sufficient for pupils to be able to calculate mechanically, they should also be capable of providing reasoning and drawing conclusions, as well as being able to explain their activity verbally and in writing (NBE, 2004).

In the foundation of the kindergarten curriculum (NBE, 2010), it is already stated that in the development of mathematical thinking it is important that children learn to observe their own thinking. Children should be challenged to explain what they think and how they think, as well as to justify their thinking.

Some examples from the Finnish–Chilean comparison study

The three-year Finnish–Chilean comparison study has been financed in 2010–13 by the Academy of Finland (project #1135556) and CONICYT in Chile (project AKA 09). Its aim is to clarify the development of grade 3–5 pupils' mathematical understanding and problem-solving skills when using open problem tasks at least once a month. More details on the research project are available in, for example, the published paper (Laine, Näveri, Pehkonen, Ahtee, & Hannula, 2012).

Example 2. The task "Divide a Square", implemented in November 2010, was the second experimental task: "Divide a square into two identical pieces. In how many different ways can you make the division? Make a note of your solutions."



The results of the problem are published in, for example, a paper by Laine et al. (2012). The first research question in the study is: "How do pupils solve an open non-standard problem?" Pupils' solutions were categorised and classified as follows.

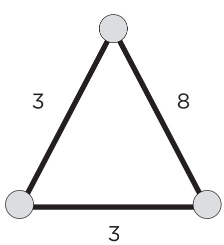
Pupils' performance in solving the problem can be divided into five hierarchical levels (cf. Table 1). The lowest level is *No solution* (Level 0), where the pupil has produced no solution during the lesson. Next is the *Basic level* (Level 1), where the pupil has found only the two obvious solutions (dividing with a diagonal into two triangles, and with a straight line parallel to the sides into two rectangles). The next level is labelled *Straight line* (Level 2), where the pupil has, in addition to the two obvious solutions, divided the square with a straight line that is neither a diagonal nor parallel to the side of the square. Finding such a solution requires a certain amount of creativity, i.e., the solver must be able to see outside of the frame of the basic solutions. There are, in fact, an infinite number of different solutions. The third level is *Curved line* (Level 3) where the dividing line can be arbitrarily curved, such as a fraction line or a curved line composed of arcs, whereby the solver breaks away from the barrier of the straight line. The number of such solutions is also infinite, but in this case the cardinality of potential solutions seems to be even greater than in the previous case. The highest level (Level 4) is represented by *Middle point thinking*, where the middle point of the square is seen as the essential element of the solutions, since all dividing lines – straight or curved – go through it and are symmetrical in relation to the middle point.

Table 1. *The distribution of pupils on the different achievement levels (N = 86).*

No solution	Basic level	Straight line	Curved line	Middle point thinking
Level 0	Level 1	Level 2	Level 3	Level 4
1 (1%)	33 (38%)	21 (25%)	18 (21%)	13 (15%)

Most of the pupils reached levels 1–3 in their solutions, but only 13 pupils (15%) reached the highest level (Level 4). The mode value in solutions was Level 1.

Example 3. The fourth experimental task, solved in February 2011, was "Arithmagon": "Arithmagons are specific number triangles. In arithmagons, there is a number in each corner of the triangle and their sum is between the corner numbers. Your task is to find the missing numbers in the corners. You should also explain your strategy to find the missing numbers in the case that two numbers on the sides of the triangle are the same. Additionally, construct a few arithmagons for your partner to solve."



Preliminary results in the Arithmagon task are given in the published paper (Näveri, Ahtee, Laine, Pehkonen, & Hannula, 2012). In this paper, the first research question also dealt with pupils' skill in solving such a non-standard problem.

The pupils' achievements were classified into three categories. Category A included achievements whereby pupils had come up with both a reasoning for the arithmagon solution and solvable additional arithmagons with sides with at least two of the same sums. Category B accounted for achievements in which no reasoning was provided for the solution, but pupils found additional arithmagons with sides with at least two of the same sums. In Category C were achievements in which pupils came up with neither reasoning nor additional arithmagons with sides with at least two of the same sums.

Furthermore, pupils' justifications for finding a solution method (i.e., Category A) when there are "two same sums" in the arithmagons were divided into three sub-classes according to the level of the justification. In the reasonings of sub-class A.1, it is clearly evident that the pupil took into account the fact that there are two same sums in the arithmagon. In the reasonings of sub-class A.2, it emerged that the pupil understood the fact that the solution was found using addition. In sub-class A.3 were those pupils who had at least tried to write something for a reasoning.

Table 2. *The distribution of pupils' achievements into different categories.*

	Category	Example of a justification used	Number of pupils
A.1	Two same sums in the arithmagon, and their meaning is understood	*"I pondered what +calculation is needed, and always calculated the same numbers, e.g., 1+1, 2+2 and 4+4."*	11
A.2	Two same sums in the arithmagon, and understood that it is a question of addition	*"I only calculated +calculations."*	12
A.3	Two same sums in the arithmagon, and an unclear explanation	*"I only calculated. Finally I just caught onto it."*	5
B	Two same sums in the arithmagon	e.g., 2 1 1, 4 9 9, 6 16 16, 200 500 500	35
C	Three different sums in the arithmagon	e.g., 8 4 6, 5 9 8, 11 17 14	39
Total			**102**

Summary of the results

In view of the results of the two examples, the first research question – "How do pupils solve an open non-standard problem?" – can be answered as follows (Table 1). On one hand, the mode value of pupils' solutions was Level

1 (38%), thus about two-fifths of the pupils only reached the basic level. On the other hand, 60% of the pupils' solutions showed some level of creativity, and as many as 15% of the pupils reached the midpoint thinking. However, we must remember that teachers guided the groups of pupils individually, and some of them may have helped their pupils more than the others.

In the second problem, about 30% of the pupils' answers included a description of a strategy (cf. Table 2), while in the rest of the answers it was only mentioned that addition is needed, and in some cases not even this was stated. Evens and Houssart (2004) claim that the skill of presenting general mathematical statements begins to develop by Grade 3 or 4 (cf. Pehkonen, 2000). It therefore seems to be important to give pupils from Grade 3 tasks in which they are compelled to explain how, based on the given information, they reach a particular conclusion, and to encourage them to explain their thinking to others (and to the teacher).

Observations based on the results presented in the present paper can be formulated as follows. Pupils (even those in Grade 3) have a great deal of potential that should be utilised and developed. The use of more creative tasks will develop pupils within the framework of the curriculum, as one objective of the Finnish curriculum is the development of creativity in all subjects (NBE, 2004). Pupils' reasoning skills, as demanded by the curriculum, will also be developed.

Conclusion

The Finnish curriculum demands that, in addition to calculation skills, problem solving and mathematical thinking should be taught in school (including primary school) (NBE, 2004). However, this does not seem to occur within ordinary mathematics teaching, where the teacher is too eager to use the textbook and its tasks. Therefore, new elements should be connected in instruction: open problem tasks with which the teacher can develop the pupils' problem solving and thinking skills.

In order for teachers to be able to implement such teaching, they should be interested in the development of teaching and committed to the new approach (cf. Shaw, Davis, & McCarty, 1991). The teachers in our experimental project were clearly ready to experiment with new solutions, and some of them were genuinely interested in the possibilities of open problem solving. Some of the teachers even made significant advances in this regard, creating the impression that they are ready to use the method in the future.

The rise of constructivism focused on teachers' mathematics-related beliefs. Here, the concept belief is understood as knowledge and feelings based on earlier experiences. Beliefs conduct and structure every teaching and learning process. In order to change teaching-learning processes, teachers' beliefs about good and successful instruction should be developed and changed. In the literature, one finds numerous research reports on the requirements for changing and developing teachers. However, none of the described intervention methods seem to be successful without problems. What is needed, therefore, is a new understanding of the problems of change and development in teachers' professional activities (for more on the problems of teacher change, see, for instance, Pehkonen (2007)).

Literature

Anderson, J. R. (1980). *Cognitive psychology and its implications*. San Francisco (CA): Freeman.

Becker, J. P., & Shimada, S. (1997). *The Open-Ended Approach.* Reston (VA): NCTM.

Bereiter, C. (1990). Aspects of an Educational Learning Theory. *Review of Educational Research*, *60*(4), 603–624.

Bereiter, C., & Scardamalia, M. (1996). Rethinking learning. In D. R. Olson & N. Torrance (Eds.), *The handbook of education and learning. New models of learning, teaching and schooling*. Cambridge (MA): Blackwell.

Bergqvist, T. (Ed.) (2012). *Learning Problem Solving And Learning Through Problem Solving.* University of Umeå.

Blanc, P., & Sutherland, R. (1996). Student teachers' approaches to investigative mathematics: iterative engagement or disjointed mechanisms? In L. Puig & A. Gutierrez (Eds.), *Proceedings of the PME-20 conference*, Vol. 2 (pp. 97–104). Valencia: University of Valencia.

Blumenfeld, P. C., Soloway, E., Marx, R. W., Krajcik, J. S., Guzdial, M., & Palincsar, A. (1991). Motivating project-based learning: Substaining the doing, supporting the learning. *Educational Psychologist*, *26*(3&4), 369–398.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for research in mathematics education, 29*(1), 41–62.

Brown, S. I. (1997). Thinking Like a Mathematician: A Problematic Perspective. *For the Learning of Mathematics, 17*(2), 36–38.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32–42.

Clarke, D. J., & Sullivan, P. A. (1992). Responses to open-ended tasks in mathematics: characteristics and implications. In W. Geeslin & K. Graham (Eds.), *Proceedings of the PME 16*, Vol I (pp. 137–144). Durham (NH): University of New Hampshire.

Cockcroft, W. (Chair) (1982). *Mathematics Counts*, Report of the Committee of Enquiry into the teaching of Mathematics in Schools. London: HMSO.

Collins, A., Brown, J. S., & Newman, S. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing and mathematics. In L. B. Resnick (Ed.), *Knowing, Learning and Instruction. Essays in Honor of Robert Glaser* (pp. 453–494). Hilldale, N. J.: Lawrence Erlbaum Associates.

Evens, H., & Houssart, J. (2004). Categorizing pupils' written answers to a mathematics test question: "I know but I can't explain". *Educational Research, 46*(3), 269–282.

Kantowski, M. G. (1980). Some Thoughts on Teaching for Problem Solving. In S. Krulik & R. E. Reys (Eds.), *Problem Solving in School Mathematics.* NCTM Yearbook 1980. (pp. 195–203). Reston (VA): Council.

Kwon, O. N., Park, J. H., & Park, J. S. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. *Asia Pacific Education Review*, *7*(1), 51–61.

Laine, A., Näveri, L., Pehkonen, E., Ahtee, M., & Hannula, M. S. (2012). Third-graders' problem solving performance and teachers' actions. In T. Bergqvist (Ed.), *Learning Problem Solving And Learning Through Problem Solving* (pp. 69–81). University of Umeå.

Mason, J. (1991). Mathematical problem solving: open, closed and exploratory in the UK. *International Reviews on Mathematical Education* (= ZDM), *23*(1), 14–19.

Näveri, L., Ahtee, M., Laine, A., Pehkonen, E., & Hannula, M. S. (2012). Erilaisia tapoja johdatella ongelmanratkaisutehtävään - esimerkkinä aritmagonin ratkaiseminen alakoulun kolmannella luokalla [Different ways to introduce a problem task – as an example the solving of aritmagon in the third grade]. In H. Krzywacki, K. Juuti, & J. Lampiselkä (Eds.), *Matematiikan ja luonnontieteiden opetuksen ajankohtaista tutkimusta* (pp. 81–98). Helsinki: Suomen ainedidaktisen tutkimusseuran julkaisuja. Ainedidaktisia tutkimuksia 2.

NBE. (2004). *Perusopetuksen opetussuunnitelman perusteet 2004* [The basics of the curriculum for the basic instruction]. Helsinki: Opetushallitus.

NBE. (2010). Esiopetuksen opetussuunnitelman perusteet 2010 [Basics of the curriculum for pre-school instruction 2010]. Retrieved from www.oph.fi/download/131115_Esiopetuksen_opetussuunnitelman_perusteet_2010

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nohda, N. (1988). Problem solving using "open-ended problems" in mathematics teaching. In H. Burkhardt, S. Groves, A. Schoenfeld, & K. Stacey (Eds.), *Problem Solving – A World View. Proceedings of problem solving theme group at ICME-5* (Adelaide) (pp. 225–234). Nottingham: Shell Centre.

Nohda, N. (1991). Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematical problem solving. *International Reviews on Mathematical Education* (= ZDM), *23*(2), 32–37.

Pehkonen, E. (1989). Verwenden der geometrischen Problemfelder. In E. Pehkonen (Ed.), *Geometry Teaching – Geometrieunterricht.* Research Report 74 (pp. 221–230). University of Helsinki. Department of Teacher Education.

Pehkonen, E. (1995). Introduction: Use of Open-Ended Problems. *International Reviews on Mathematical Education* (= ZDM), *27*(2), 55–57.

Pehkonen, E. (Ed.) (1997). *Use of open-ended problems in mathematics classroom*. Research Report

176. University of Helsinki. Department of Teacher Education.

Pehkonen, E. (Ed.) (2001). *Problem Solving Around the World*. Report Series C:14. University of Turku. Faculty of Education.

Pehkonen, E. (2004). State-of-the-Art in Problem Solving: Focus on Open Problems. In H. Rehlich & B. Zimmermann (Eds.), *ProMath Jena 2003. Problem Solving in Mathematics Education* (pp. 93–111). Hildesheim: Verlag Franzbecker.

Pehkonen, E. (2007). Über "teacher change" (Lehrerwandel) in der Mathematik. In A. Peter-Koop & A. Bikner-Ahsbahs (Eds.), *Mathematische Bildung - mathematische Leistung: Festschrift für Michael Neubrand zum 60. Geburtstag* (pp. 349–360). Hildesheim: Franzbecker.

Pehkonen, L. (2000). Written arguments in a conflicting mathematical situation. *Nordic Studies in Mathematics Education, 8*(1), 23–33.

Schroeder, T. L., & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton (Ed.), *New Directions for Elementary School Mathematics.* NCTM 1989 Yearbook. (pp. 31–42). Reston, Va: NCTM.

Schupp, H. (2002). *Thema mit Variationen. Aufgabenvariation im Mathematikunterricht*. Hildesheim: Verlag Franzbecker.

Shaw, K. L., Davis, N. T., & McCarty, J. (1991). A cognitive framework for teacher change. In R. G. Underhill (Ed.), *Proceedings of PME-NA 13*, Vol 2 (pp. 161–167). Blacksburg (VA): Virginia Tech.

Shimada, S. (Ed.) (1977). *Open-end approach in arithmetic and mathematics – A new proposal toward teaching improvement*. Tokyo: Mizuumishobo. [in Japanese]

Sierpinska, A. (1994). *Understanding in mathematics*. Studies in mathematics education series: 2. London: Falmer.

Silver, E. (1995). The Nature and Use of Open Problems in Mathematics Education: Mathematical and Pedagogical Perspectives. *International Reviews on Mathematical Education* (= ZDM), *27*(2), 67–72.

Stacey, K. (1995). The Challenges of Keeping Open Problem-Solving Open in School Mathematics. *International Reviews on Mathematical Education* (= ZDM), *27*(2), 62–67.

Törner, G., Schoenfeld, A. H., & Reiss, K. M. (Eds.) (2007). Problem solving around the world: summing up the state of the art. *ZDM Mathematics Education, 39*(5/6), 353–551.

Wiliam, D. (1994). Assessing authentic tasks: alternatives to mark-schemes. *Nordic Studies in Mathematics Education, 2*(1), 48–68.

Williams, D. (1989). Assessment of open-ended work in the secondary school. In D. F. Robitaille (Ed.), *Evaluation and Assessment in Mathematics Education.* Science and Technology Education. Document Series 32. (pp. 135–140). Paris: Unesco.

Wu, H. (1994). The Role of Open-Ended Problems in Mathematics Education. *Journal of Mathematical Behavior, 13*(1), 115–128.

Zaslavsky, O. (1995). Open-ended tasks as a trigger for mathematics teachers' professional development. *For the Learning of Mathematics, 15*(3), 15–20.

Zimmermann, B. (1991). Offene Probleme für den Mathematikunterricht und ein Ausblick auf Forschungsfragen. *International Reviews on Mathematical Education* (= ZDM), *23*(2), 38–46.

Zimmermann, B. (2010). "Open ended problem solving in mathematics instruction and some perspectives on research question" revisited – new bricks from the wall? In A. Ambrus & E. Vasarhelyi (Eds.), *Problem Solving in Mathematics Education. Proceedings of the 11th ProMath conference in Budapest* (pp. 143–157). Eötvös Lorand University.

Biographical note

Erkki Pehkonen, Dr., is a full professor (retired) in the field of mathematics and informatics education in the Department of Teacher Education at the University of Helsinki in Finland. He is interested in problem solving with a focus on motivating middle grade pupils, as well as in understanding pupils' and teachers' beliefs and conceptions about mathematics teaching.

Anu Laine, Dr., is an adjunct professor in mathematics education, working as a university lecturer at the Department of Teacher Education at the University of Helsinki in Finland. Her research interests include affects, communication and problem solving in mathematics education.

Liisa Näveri, PhD., is a researcher in the research project on mathematics education (Academy of Finland and Chilean CONICYT) in the Department of Teacher Education at the University of Helsinki in Finland. Her research interests include problem solving and understanding in mathematics learning.

Process Regulation in the Problem-Solving Processes of Fifth Graders

Benjamin Rott[1]

It is well known that the regulation of processes is an important factor in problem solving from Grade 7 to university level (cf. Mevarech & Kramarski, 1997; Schoenfeld, 1985). We do not, however, know much about the problem-solving competencies of younger children (cf. Heinze, 2007, p. 15). Do the results of studies also hold true for students below Grade 7? The study presented here strongly suggests that metacognition and process regulation is important in Grade 5 as well.

The research questions are: *How do the (more or less successful) problem-solving processes of fifth graders occur? What is the impact of metacognition and self-regulation on these processes? Are the transitions between phases in the problem-solving process closely connected to metacognitive activities?*

An analysis of approximately 100 problem-solving processes of fifth graders (aged 10–12) from German secondary schools will be used to help answer these questions. The videotapes that supplied the raw data were parsed into phases called episodes using an adapted version of the "protocol analysis framework" by Schoenfeld (1985, ch. 9). The junctures between these episodes were additionally coded with the "system for categorizing metacognitive activities" by Cohors-Fresenborg and Kaune (2007a). There is a strong correlation between (missing) process regulation and success (or failure) in the problem-solving attempts.

Concluding suggestions are given for the implementation of the results in school teaching. These suggestions are currently being tested.

Keywords: Control; Mathematical problem solving; Metacognition; Process regulation

1 Leibniz University of Hannover, Germany; rott@idmp.uni-hannover.de

Usmerjanje procesov reševanja problemov petošolcev

Benjamin Rott

Znano je, da je usmerjanje procesov pomemben dejavnik pri reševanju problemov – od 7. razreda do univerzitetne ravni (Mevarech & Kramarski, 1997; Schoenfeld, 1985). Malo pa vemo o kompetencah reševanja problemov pri mlajših otrocih (Heinze, 2007, str. 15). Ali izsledki teh študij veljajo tudi za učence nižjih razredov? Predstavljena raziskava močno nakazuje pomen metakognicije in usmerjanja procesov tudi v 5. razredu. Raziskovalna vprašanja so: Kako (bolj ali manj uspešno) poteka postopek reševanja problemov petošolcev? Kakšen je vpliv metakognicije in samoregulacije na te postopke? Ali so prehodi med fazami procesa reševanja problemov tesno povezani z metakognitivnimi dejavnostmi? Analiza približno stotih postopkov reševanja problemov petošolcev (starih od 10 do 12 let) iz nemških osnovnih šol lahko pomaga pri odgovorih na ta vprašanja. Posnetki z videokaset, ki so podali neobdelane podatke, so bili z uporabo prilagojene različice Schönfeldovega »protokola analize podatkov« razčlenjeni v faze, imenovane epizode (1985, poglavje 9). Stičišča med temi epizodami so bila dodatno kodirana z »merili za razvrščanje metakognitivnih aktivnosti« avtorjev Cohors - Fresenborg in Kaune (2007a). Obstaja močna povezava med (manjkajočim) usmerjanjem procesov in uspehom (ali neuspehom) pri reševanju problemov. Podani so sklepni predlogi za uvajanje izsledkov naše raziskave v šolski pouk.

Ključne besede: matematično reševanje problemov, metakognicija, usmerjanje procesov

Background

Problem solving is important in everyday life, in situations where the solution path is not immediately obvious (cf. OECD, 2003), as well as in mathematics, because "what mathematics really consists of is problems and solutions" (Halmos, 1980, p. 519). It is widely accepted that solving problems is of importance for the learning of mathematics, and it is therefore part of many school curricula, e.g., in the United States and Germany (cf. KMK, 2003; NCTM, 2000).

In researching problem-solving, *metacognition* is an important factor to take into account (with resources, heuristics and beliefs being other factors, cf. Schoenfeld, 1985, p. 44 f.). According to Flavell (1976, p. 232), who was the first to describe this concept, the term metacognition "refers to one's knowledge concerning one's own cognitive processes and products or anything related to them, [...]. [It] refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes [...]".

The theoretical impact of metacognition: The latter part of Flavell's description, self-regulation or *control*, "deals with the question of resource management and allocation [...]" (Schoenfeld, 1985, p. 44 f.). The importance of self-regulation is highlighted in several models of the problem-solving process, some of which are presented in the following paragraphs.

Schoenfeld (1985, ch. 4), for example, describes design within problem solving as "something that pervades the entire solution process; its function is to ensure that you [as a problem solver] are engaged in activities most likely ([…]) to be profitable. Most generally, it means keeping a global perspective on what you are doing and proceeding hierarchically" (ibid., p. 108). Furthermore, Schoenfeld (1985, p. 300) claims that the *transitions* between phases in the problem-solving process are places "where managerial decisions (or their absence) will make or break a solution".

Mason, Burton, and Stacey (2010, ch. 7) suggest listening to an *internal monitor* that might guide a problem solver to go back to a planning step ("Attack") in his/her process, or to choose another strategy for the problem.

Finally, Wilson, Fernandez, and Hadaway (1993) claim that each movement from one (Pólya-like) stage to another in a problem-solving process represents a *managerial decision*.

The empirical impact of metacognition: There are also several studies that demonstrate the impact of self-regulation on the performance of problem-solving attempts. Schoenfeld (1992, p. 63), for instance, worked with university students and analysed their problem-solving processes. Approximately 60%

of the students showed a behaviour that Schoenfeld called "wild goose chase", whereby the students picked a solution direction and pursued it until they ran out of time, without reflecting on it. After a problem-solving course in which he regularly asked the students three questions – "What exactly are you doing?", "Why are you doing it?", and "How does it help you?" – the percentage of students not regulating their processes was reduced to 20%, accompanied by a proportional increase in the success of their problem-solving attempts.

Working with school students, Lester, Garofalo, and Kroll (1989, p. 115) obtained the following results in a study with two Grade 7 classes: "In general, it seems that the more successful problem solvers in our study were better able to monitor and regulate their problem-solving activity than the poorer problem solvers [...]. This observation is, of course, consistent with the preponderance of the research on expert-novice problem solving [...]."

Mevarech and Kramarski (1997) also worked with Grade 7 students, divided into two groups. The students in the experimental group (three classes), who had undergone metacognitive and self-regulatory training, showed significantly better results in mathematics tests than those in the control group (five classes).

A collection of several studies on metacognition is presented by Cohors-Fresenborg, Kramer, Pundsack, Sjuts, and Sommer (2010). All of these studies show a positive correlation between metacognitive behaviour and success in problem solving. Most notably, those high school students who demonstrated metacognitive activities[2] in problem-solving interviews scored significantly better results in a written mathematics test (cf. ibid., p. 234 ff.) than those who did not.

The research gap: As stated above, although we know the impact of metacognition and regulation on the problem-solving processes of students of Grade 7 onwards, we do not know much about the problem-solving abilities and processes of younger children, as there is a lack of research (cf. Heinze, 2007, p. 15). I therefore raise the following *research questions*:

- How do the (more or less successful) problem-solving processes of fifth graders occur?
- What is the impact of metacognition and self-regulation on these processes?

And, with the theoretical models (described above) in mind:

- Are the transitions between phases in the problem-solving process closely connected to metacognitive activities?

2 The activities were coded within the framework by Cohors-Fresenborg and Kaune (2007a), which is described in the Methodology section (see below).

Design of the study

Our support and research programme MALU[3] was an enrichment project for interested fifth graders (aged 10–12) from secondary schools in Hanover in Northern Germany. From November 2008 to June 2010, pupils came to our university once a week. A group of 10–16 children (45 altogether in four terms) was formed every new term. The sessions usually proceeded according to the following pattern. After some initial games and tasks, the pupils worked in pairs on 1–3 mathematical problems (about 30 different tasks in all) for about 40 minutes, during which time they were videotaped. They eventually presented their results to the whole group. The children's notes were also collected.

The pupils worked on the problems without interruptions or hints from the observers, because we wanted to study their uninfluenced problem-solving attempts. We decided not to use an interview or a think-aloud method, so as not to interrupt the students' mental processes. In order to gain an insight into their thoughts, we let the children work in pairs, thus providing an opportunity to interpret their communication as well as their actions.

In Tables 1 and 2, there are two examples of the problems we posed, four of which have been selected for analyses in the present paper (see Rott, 2012a for more examples).

Table 1. *The coasters task (idea: Schoenfeld, 1985, p. 77).*

Beverage Coasters

The two pictured squares depict coasters. They are placed so that the corner of one coaster lies in the centre of the other.

Examine the size of the area covered by **both** coasters.

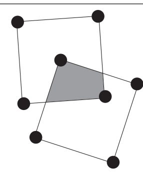
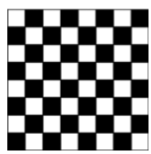


Table 2. *The chessboard task (idea: Mason, Burton, & Stacey, 2010, p. 17)*

Squares on a Chessboard

Peter loves playing chess. He likes playing chess so much that he keeps thinking about it even when he isn't playing. Recently he asked himself how many squares there are on a chessboard. Try to answer Peter's question!



3 **M**athematik **AG** an der **L**eibniz **U**niversität, which means Mathematics Working Group at Leibniz University.

Methodology

Product Coding: In order to determine the pupils' success in problem solving, their work results were graded into four categories: (1) *no access*, when the pupils did not work on the task meaningfully, (2) *basic access*, when they solved (parts of) the problem but the solution had notable flaws, (3) *advanced access*, when they solved the problem for the most part, and (4) *full access*, when the pupils solved the task properly and presented appropriate reasons.

This grading system was customised for each task with examples for each category (see Rott, 2012a for examples). All of the products were then rated independently by the author and a research assistant. We agreed in almost all cases (Cohen's kappa>0.9) and discussed the few differing ratings, reaching consensus every time. These discussions also led to better defined categories. It is important to note that the two members of a pair of problem solvers could, and sometimes did, achieve diverse ratings of their products when their written results differed.

Process Coding – Episodes: The pupils' behaviour was coded using the framework for the analysis of videotaped problem-solving sessions presented by Schoenfeld (1985, ch. 9). His intention was to "identify major turning points in a solution. This is done by parsing a protocol into macroscopic chunks called episodes […]" (ibid., p. 314). An episode is "a period of time during which an individual or a problem-solving group is engaged in one large task [...] or a closely related body of tasks in the service of the same goal [...]" (ibid., p. 292). Schoenfeld (1992, p. 189) continues: "We found […] that the episodes fell rather naturally into one of six categories:"

(1) *Reading or rereading the problem.*
(2) *Analyzing the problem (in a coherent and structured way).*
(3) *Exploring aspects of the problem (in a much less structured way than in Analysis).*
(4) *Planning all or part of a solution.*
(5) *Implementing a plan.*
(6) *Verifying a solution.*

We adopted this framework for our study with the following modifications. We initially experienced some difficulties in coding reliably (as predicted by Schoenfeld, 1992, p. 194). We therefore operationalised Schoenfeld's framework, which is constructed based on Pólya's famous list of questions and guidelines, by applying Pólya's suggestions to the episode descriptions (see Rott, 2012a for details).

Secondly, we added new categories of episodes, because our fifth graders – unlike university students – demonstrated plenty of non-task-related behaviour.

(7) *Digression*, when pupils show no task-related behaviour at all.

(8) *Organisation*, when working on the task is being prepared or followed up, e.g., by drawing lines to write on or by filing away worksheets.

(9) *Writing*, when pupils captured their results without gaining new insights.

(10) *Miscellaneous* – behaviour that is not covered by any other type of episode.

For the analyses presented in the present paper, only the task-related episodes are relevant, i.e., (2) – (6) of Schoenfeld's list.

This framework was used to code all of the MALU processes (see the Appendix for a sample coding). This coding was done independently by research assistants and the author. In order to compute the interrater-reliability, we applied the "percentage of agreement" approach as described in the TIMSS 1999 video study (cf. Jacobs et al., 2003, p. 99 ff.) for randomly chosen videos, gaining more than PA=0.7 for the parsing into episodes and more than PA=0.85 for the characterisation of the episode types. More importantly, however, each process was coded by at least two raters. Whenever these codes did not coincide (most of the time they did coincide), we attained agreement by recoding together (cf. Schoenfeld, 1992, p. 194).

Process Coding – Metacognitive Activities: The occurrence of metacognition should also have been coded in our pupils' processes. Schoenfeld (1985) included "local" and "global" assessments in his framework, local assessment being "an evaluation of the current state of the solution at a microscopic level" (ibid., p. 299). Unfortunately, in our team we were not able to use this description and Schoenfeld's examples to code assessments reliably.

Instead, we used another framework: Cohors-Fresenborg and Kaune (2007a; see 2007b for an English description) developed a "system for categorizing metacognitive activities during […] mathematics lessons" (2007b, p. 1182). There are three categories – *planning* (**P**), *monitoring* (**M**) and *reflection* (**R**) (as well as *discursivity*, which is not significant to our study) – to apply to passages in the transcript of a lesson, with subcategories such as "**M1**: Controlling of Calculation", "**M8**: Self-Monitoring" or "**R1**: Reflection on Concepts". Some of these subcategories also have specifications like "**P1**: Focus of attention" – "**P1a**: single-step" and "**P1b**: multi-step".

We adapted this system to identify metacognitive activities in our two-person problem-solving processes and used this framework to code some of our pupils' problem-solving processes (see the Appendix for a sample coding).

All of the processes were coded conjointly by three raters and discussed until all of the raters reached consensus, as described in the manual (cf. Cohors-Fresenborg & Kaune, 2007a).

Results

The results shown here combine the analyses of all of our pupils' processes working on four different tasks (for details, see Rott, 2012a). Please note that 10 of the 19 pupils who worked on the "Squares on a Chessboard" task misinterpreted the formulation of the task and answered "64 squares" within less than 3 minutes. This is a sure sign of missing metacognition or control that leads to bad results. These processes were excluded from the following analyses, as the children just followed routine patterns instead of showing problem-solving behaviour, whereas the focus of the present article is on control in problem-solving processes.

Wild Goose Chases: After dividing all of the processes into episodes to see how our pupils' processes occur, the codes had to be analysed. One of Schoenfeld's major findings, obtained with his video analysis framework, was the accentuation of the importance of metacognitive and self-regulatory activities in problem-solving processes. Problem solvers who missed out on such activities often engaged in a behaviour that Schoenfeld called "wild goose chase" (see above), whereas "successful solution attempts [...] consistently contained a significant amount of self-regulatory activity" (Schoenfeld, 1992, p. 195).

Most of the unsuccessful processes of our pupils did in fact demonstrate behaviour that fits the description of "wild goose chase" (most notably, these processes consisted almost exclusively of long *Exploration* episodes). In order to apply Schoenfeld's result to the MALU data, we had to operationalise the problem-solving type "wild goose chase", as he provided no real definition of it. In his book, Schoenfeld (1985, p. 307) denotes this type of behaviour as the "read/explore type". Thus, a process is considered to be a "wild goose chase" if it consists only of *Exploration* episodes.[4] It is possible, however, that pupils try to understand the task given to them for a short time before selecting a solution direction and pursuing it thoughtlessly. Accordingly, processes were also considered to be a "wild goose chase" if they consisted only of *Analysis & Exploration* episodes,[5] whereas processes that were not of this type mostly contained *planning* and/or *verifying* activities.

In order to check whether this kind of behaviour in the processes is interrelated with success or failure of the related products, a chi-square test was

4 I concentrate on the task-related episodes, disregarding *Reading* and the added types of episodes (see the Methodology section).

5 In our sample, there were no "wild goose chase" candidates in which the *Analysis* was nearly as long as the *Exploration*, thus we did not need to deal with the duration of the *Analysis* episodes.

used. The null hypothesis is "no correlation between the problem-solving type 'wild goose chase' and (no) success in the product". Due to the small size of the database, the product categories had to be subsumed by twos, to "no & basic approach" as well as "advanced and full access".

The entries in Table 3 consist of the observed numbers, while the expected numbers (calculated by the marginal totals) are added in brackets. The entries in the main diagonal are apparently above the expected values. The chi-square-test shows a significant correlation (p < 0.001) between the problem solvers' behaviour and their success.

Table 3. *Contingency table – process behaviour and product success (10 of the 19 processes belonging to the "Squares on a Chessboard" task have been excluded from these data).*

process / product categories	no & basic access	advanced & full access	sum
wild goose chase	27 (15.6)	5 (16.4)	32
miscellaneous	16 (27.4)	40 (28.6)	56
sum	43	45	88
χ^2=25.378	p<0.0001	Yates-p<0.0001	12

These results show the huge importance of self-regulation and process-regulation during problem-solving attempts. Wild goose chases imply missing changes between episodes and thus missing process regulation. Schoenfeld (1985, p. 300) emphasises that especially the "junctures between episodes [are parts], where managerial decisions (or their absence) will make or break a solution". This claim is in line with other models of the problem-solving process, e.g., Schoenfeld's (1985, ch. 4) "design", the "internal monitor" by Mason, Burton, and Stacey (2010, ch. 7), or the "managerial decisions" that are part of each transition between Pólya-like problem-solving phases by Wilson, Fernandez, and Hadaway (1993) (see Rott, 2012b for a comparison of these models).

Junctures between episodes: The result of the chi-square-test (see Table 3) and especially the theoretical assumptions of the models of the problem-solving process (see above), suggest the need for further investigation of the processes of our pupils, concentrating on the junctures between episodes. Are the transitions between phases in the problem-solving process closely connected to metacognitive activities? (research question 3)

Approximately 25% of the processes analysed with Schoenfeld's schema were additionally and independently coded with the system by Cohors-Fresenborg and Kaune. In almost all cases, the junctures between episodes also showed metacognitive activities – mostly "**P1**: Focus of Attention" and "**R6**: Reflective

Assessment / Evaluation" (see the Appendix for an example). This supports the theoretical assumptions, as well as highlighting the importance of metacognition and self-regulation during problem solving. The occurrence of mostly two codes (of about twenty different codes) should be explored further in subsequent studies.

Conclusions and implications

Self-regulation and process-regulation are very important factors in problem solving. In the present study, the junctures between (Pólya-like) episodes in problem-solving processes are closely related to metacognitive activities. Pupils who missed changing episode types (especially those who mostly conducted an *Exploration* episode, thus performing a "wild goose chase") regulated their processes badly. These pupils were significantly less successful than the pupils who did not show "wild goose chase" behaviour.

The sample of pupils used to obtain these results is not representative, as the children all came to our university voluntarily to participate in mathematical activities. Nonetheless, the results are in line with those of several studies that have consistently shown the importance of control and regulation (e.g., Cohors-Fresenborg et al., 2010; Lester, Garofalo, & Kroll, 1989; Mevarech & Kramarski, 1997; Schoenfeld, 1992), thus adding to the validity of the present study.

Fortunately for those pupils who performed badly, self-regulatory behaviour is learnable and can be taught, as has been demonstrated several times (e.g., by Schoenfeld, 1992 or Mevarech & Kramarski, 1997, see above). As an impetus for future studies, and following from our results in the classroom, I would like to present a training programme fostering students' self-regulation.

In our working group, we tried the following procedure. In addition to a two-column proof schema, Brockmann-Behnsen (2012a, b) used a set of questions similar to those of Schoenfeld (1992, see above) to help students to foster metacognitive activities. He let the students of his experimental classes regularly pose two questions whenever they tried to solve problems or to reason in a mathematical sense. However, unlike Schoenfeld, Brockmann-Behnsen used a model suitable for children: *Imagine, in a mathematical argumentation, you have to pass two gates, each one with a guardian that lets you pass only if you can answer his question: 1. Why are you allowed to do it? and 2. How does it help you?*[6]

The initial results of a small training study – admittedly, not with fifth graders – using these two questions seem to be very promising. A short pre-test (a

6 The German versions of these questions are "1. Warum darfst Du das?" and "2. Was bringt es Dir?"

geometric reasoning task) showed no significant differences between a set of four eighth grade classes (children aged 13–15). However, a post-test (using a comparable, slightly more difficult problem) after six weeks of training for two of those classes indicated significant differences in favour of the experimental groups. The control groups did not show any change in their level of success (mostly no success) or in the structure of their reasoning (mostly incoherent arguments). The experimental groups, on the other hand, displayed clear improvements in achieving correct solutions and in using mostly coherent, deductive reasoning.

Training programmes like the one presented by Brockmann-Behnsen should be extended to other age groups (such as fifth graders) and monitored scientifically. Additional video studies could analyse the students' behaviour with a focus on wild goose chases.

Related research intent would include a closer examination of possible correlations between the use of metacognitive activities (in general or special activities) and success in problem-solving attempts. Cohors-Fresenborg et al. (2010) present some studies that indicate such a correlation (see above). In particular, students who had shown a special kind of monitoring in a problem-solving interview were more successful in a written test: "**M8f**: Self-Monitoring of Monitoring", a meta-meta-category that supervised the use of monitoring in processes. In studies like the one presented in the present paper, it could be investigated whether there are similar special categories of metacognitive activities, or whether there is a general correlation to success.

On the theoretical side, the question as to whether junctures between phases (or episodes respectively) in the problem-solving process are (almost) always connected to metacognitive activities should be further explored. Personally, I have no knowledge of other studies that have independently coded and compared problem-solving phases and occurrences of metacognition.

References

Brockmann-Behnsen, D. (2012a). HeuRekAP – Erste Ergebnisse der Langzeitstudie zum Problemlösen und Beweisen am Gymnasium. In M. Ludwig & M. Kleine (Eds.), *Beiträge zum Mathematikunterricht 2012*. Münster: WTM.

Brockmann-Behnsen, D. (2012b). A long-term educational treatment using dynamic geometry software. In M. Joubert, A. Clarck-Wilson, & M. McCabe, *Proceedings of the 10th International Conference for Technology in Mathematics Teaching (ICTMT10)* (pp. 196–302).

Cohors-Fresenborg, E., & Kaune, C. (2007a). *Kategoriensystem für metakognitive Aktivitäten beim schrittweise kontrollierten Argumentieren im Mathematikunterricht*. Arbeitsbericht Nr. 44, Forschungsinstitut für Mathematikdidaktik, Universität Osnabrück.

Cohors-Fresenborg, E., & Kaune, C. (2007b). Modelling Classroom Discussions and Categorising Discursive and Metacognitive Activities. In *Proceedings of CERME 5* (pp. 1180 – 1189). Retrieved December 21 2012 from http://www.ikm.uni-osnabrueck.de/mitglieder/cohors/literatur/CERME5_discursivness_metacognition.pdf

Cohors-Fresenborg, E., Kramer, S., Pundsack, F., Sjuts, J., & Sommer, N. (2010). The role of metacognitive monitoring in explaining differences in mathematics achievement. *ZDM Mathematics Education, 42*, 231–244.

Jacobs, J., Garnier, H., Gallimore, R., Hollingsworth, H., Givvin, K. B., Rust, K., et al. (2003). *Third International Mathematics and Science Study 1999 Video Study Technical Report.* Volume 1: Mathematics. Washington: National Center for Education Statistics. Institute of Education Statistics, U. S. Department of Education.

Mason, J., Burton, L., & Stacey, K. (1982/2010). *Thinking Mathematically*. Dorchester: Pearson Education Limited. Second Edition.

Mevarech, Z. R., & Kramarski, B. (1997). IMPROVE: A Multidimensional Method for Teaching Mathematics in Heterogeneous Classrooms. *American Educational Research Journal, 92*(4), 365–394.

Pólya, G. (1945). *How to Solve It*. Princeton, NJ: University Press.

Rott, B. (2012a). Problem Solving Processes of Fifth Graders – an Analysis of Problem Solving Types. In *Proceedings of the 12th ICME Conference. Seoul, Korea*. Retrieved November 25 2012 from http://www.icme12.org/upload/UpFile2/TSG/0291.pdf

Rott, B. (2012b). Models of the Problem Solving Process – a Discussion Referring to the Processes of Fifth Graders. In T. Bergqvist (Ed.), *Proceedings from the 13th ProMath conference, Sep. 2011* (pp. 95–109).

Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. Orlando, Florida: Academic Press, Inc.

Schoenfeld, A. H. (1992). On Paradigms and Methods: What do you do when the ones you know don't do what you want them to? Issues in the Analysis of data in the form of videotapes. *The Journal of the learning of sciences, 2*(2), 179–214.

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics*. Chapter. 4. (pp. 57–77).

Biographical note

Benjamin Rott attended the University of Oldenburg to become a secondary teacher for mathematics and physics; his thesis was on the topic of problem solving and dynamic geometry software. His studies were followed by a two-year teacher training at a school near Braunschweig to gain a full teaching license. Starting at the end of 2008, he wrote his PhD thesis on the topic of mathematical problem solving at the University of Hanover, which he defended in 2012. Since then, he works at the University of Education Freiburg as a post-doctoral researcher on the topic of epistemic beliefs.

Appendix

L and E are two girls working on the “Squares on a Chessboard” task. The codes in the last column refer to the coding of metacognitive activities by Cohors-Fresenborg and Kaune (2007a, b).

Table 4. *L & E – Squares on a Chessboard – excerpt from the transcript, part 1 of 2.*

time	L	E	commentary, codes
00:27	*Turns her sheet around, reads for 24 seconds. Turns to the observer.* “Do squares just mean…” *No reaction from the observer.*	*Turns her sheet around, reads for 22 seconds.*	**M2**: control of terminology
00:54	“One colour of the area?”	*Looks to L.* “no.”	*Refers to black/white*
00:57		“This can, look, this can be a square” *Points to a 3x3-square that is build of 9 little squares.*	
01:01	“But then, this is …” “oh.”	“a square. Have a look. First, we have to count them.”	
01:07	“We have to detect all of the squares that exist.”	“that exist.” *Looks to L.*	
01:09	*Draws lines on her chessboard for 16 seconds.* “But, look, this is a square. But then, this is not.”		**M5a**: control of the consistency of the argumentation
01:14		*Looks to the observer.* “Shall we write the answer on this sheet?” *Draws on her sheet for 4 seconds.* <*quiet* “Now, we just have to …”>	
01:24	*Looks at E.* “At first, all of the white ones?” *Draws lines on the white squares.* “The black ones are missing sides, when they belong to the white ones.”		**bP1a**: justified single-step planning
01:32	*Draws on her sheet for 13 seconds.*	*Draws on her sheet for 13 seconds.*	
01:44		“Just colour the white ones blue.” *Laughs, begins to shade the white squares.*	
01:48	“I wouldn’t do that. Because there are more squares in it.” *Looks at E’s sheet.*		**M4a**: control of methods. **bR3c**: reflection of the markings
01:55		“Pha!” *Laughs. Looks to L.* “Now you said it. Thanks”	
01:59	*Laughs.* “Doesn’t matter.”	“I don’t want to…” <*unclear* “I don’t want to position it.”>	

Table 5. *L & E – Squares on a Chessboard – excerpt from the transcript, part 2 of 2.*

time	L	E	commentary, codes
02:05	*Draws borders around the white squares.*	*Shades squares for 8 seconds.*	(see Figure 1)
02:11		"Who is faster? Hah."	
02:13	*Draws borders around squares.*	*Shades squares for 14 seconds.*	
02:26		"Done it!" *Laughs, looks at L.*	
02:30	"Then, this one is a square." *Draws a border around the whole board.*		
02:37		*Looks at L's sheet.* "The whole."	
02:40		*Draws a border around the whole chessboard for 8 seconds.*	
02:45	"Hmm, no."	"Did you take that one?"	
02:48	*Draws a thicker line around the whole chessboard for 11 seconds.*	*Draws a thicker line around the whole chessboard for 11 seconds.*	(see Figure 1)
02:58		*Looks at L's sheet.* "That is a square. And now we have to count the little black ones. That could be squares as well." *Draws something for 4 sec.*	**P1a**: single-step planning
03:08		*Offers her pen.* "Let's exchange pens, so that we don't confuse the colours, okay?"	**fP1c**: requests the use of tools
03:14	"Just take this one." *Gives E a black pen.*	"Can I?"	
03:18		*Takes the pen.* "So we don't confuse the colours." *Draws for 6 seconds.*	
03:28	*Draws on her chessboard.* "But, but then, this is (..) I think (.) all of these are squares." *Looks to E.*	"There could be (..) such a"	**R6a**: reflection / evaluation of an important situation
03:40		"So, we just have to. Count how many there are, so" *Starts to count*: <*quiet* "one, two, three, four, five>	**P1a**: single-step planning: count the squares
03:47	*Counts* "one, two, three, seven, eight"	"six, seven, eight."	

They start counting and recounting until they decide to write an answer at 07:48:

> "64 little, one very big (the whole board), 16 four-part, 4 sixteen-part. Altogether: 85 squares."

This was coded as "(2) Basic access" as both girls discovered that there are more than 64 squares, but they only identified squares that fill the chessboard completely without overlapping.

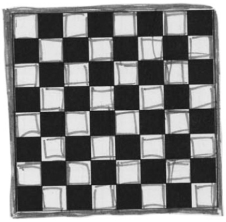
a)  b) 

Figure 1. Drawings of L (a) and E (b) – squares on a chessboard.

All of the junctions between the episodes in this process are connected with a metacognitive activity. The following table 6 summarises the activities at those junctions.

Table 6. *L & E – Squares on a Chessboard – episode junctions.*

time	L	E	commentary, codes
(00:30 – 00:50)	*Reading*	Pupils L & E read the task formulation.	The *Reading* ends with a question by L that was coded as *monitoring* (**M2**).
(00:50 – 01:30)	*Analysis*	They struggle with the task formulation and try to make sense of it.	The last statement of the *Analysis* that leads to the following *Exploration* is justified *planning* (**bP1a**).
(01:30 – 03:30)	*Exploration*	They try some unstructured ideas like shading the squares or drawing boarders around the white ones.	The *Exploration* ends with the finding that there are more than the little (1x1) squares, which is a *reflection* (**R6a**). The new episode starts with *planning* (**P1a**).
(03:30 – 07:50)	*Planning-Implementation*	In a structured way, they try to count the squares (without realising all of them).	This combined episode of *Planning-Implementation* ends with drawing an interim balance (**R6a**).
(07:50 – 08:10)	*Writing*	They write down the results without new ideas.	- - -

Promoting Writing in Mathematics: Prospective Teachers' Experiences and Perspectives on the Process of Writing When Doing Mathematics as Problem Solving

Ana Kuzle[1]

Despite a great deal of research on the benefits of writing in mathematics, writing plays a minimal role, if any, in secondary and tertiary mathematics education. In order for teachers to use writing in their classrooms, they themselves have to experience writing mathematics within the teacher education programme. The present paper reports on a study aimed at addressing this gap. In a problem-solving seminar, preservice teachers had an opportunity to experience writing in mathematics and report how this affected their problem-solving processes and shaped their attitudes towards incorporating writing in their classrooms. In order to provide a more detailed description of the phenomenon, four participants were chosen based on their beliefs about mathematics. All of the participants struggled with writing their explanations. Those who used writing as a method to support metacognitive processes while exploring mathematics tended to respond positively to the writing process. The others used writing merely as a method to produce a formal document to be evaluated by the instructor. Consequently, those who viewed writing and doing mathematics as an intertwined process expressed a positive attitude towards using writing in their mathematics classroom. This was, unfortunately, not the case when writing and doing mathematics were seen as two separate processes. Implications for teacher education programmes are presented at the end of the report.

Keywords: Attitudes; Beliefs; Metacognition; Problem solving; Prospective mathematics teachers; Writing in mathematics

1 University of Paderborn, Germany; akuzle@math.upb.de

Spodbujanje pisanja pri matematiki – izkušnje in pogledi bodočih učiteljev na proces pisanja pri reševanju problemov pri matematiki

Ana Kuzle

Kljub številnim raziskavam o koristi pisanja pri matematiki ima ta dejavnost – če že – minimalno vlogo v sekundarnem in terciarnem izobraževanju. Da bi se učitelji posluževali pisanja pri učnih urah, morajo tudi sami dobiti izkušnjo pisanja pri matematiki, in sicer med svojim pedagoškim izobraževanjem. Članek poroča o raziskavi, ki je bila namenjena obravnavi te vrzeli. Na seminarju iz reševanja problemov so prihodnji učitelji dobili izkušnjo pisanja pri matematiki; poročali so, kako je to vplivalo na njihov proces reševanja problemov in oblikovalo njihov odnos do vključevanja pisanja v njihove učne ure. Z namenom podrobnejšega opisa pojava so bili glede na prepričanje o matematiki izbrani štirje udeleženci. Vsi so se spopadali s pisanjem svojih razlag. Tisti, ki so uporabili pisanje kot metodo za podporo metakognitivnih procesov pri raziskovanju matematike, so se nagibali k pozitivnemu odzivu do procesa pisanja. Preostali so uporabili pisanje samo kot metodo za oblikovanje pisnega dokumenta, ki služi za pregled profesorja. Posledično so tisti, ki so dojeli pisanje in pouk matematike kot en sam prepleten proces, izrazili pozitiven odnos do uporabe pisanja pri matematičnih učnih urah. Tako pa ni bilo pri tistih, ki so pisanje in pouk matematike videli kot dva ločena procesa. Predlogi za program izobraževanja učiteljev so podani na koncu članka.

Ključne besede: stališča, prepričanja, metakognicija, reševanje problemov, prihodnji učitelji matematike, pisanje pri matematiki

Introduction

Students often ask: What does writing have to do with mathematics? They are not open to the idea of writing in mathematics and very often view writing only as a part of language and social studies classes. On the other hand, educational organisations and researchers advocate using writing in mathematics. In its report *An Agenda for Action*, the National Council of Teachers of Mathematics ([NCTM], 1980) strongly recommended that the process of writing in mathematics become an integral part of mathematics lessons. In their view, "writing as a process [that] emphasizes brainstorming, clarifying, and revising … can readily be applied to solving a mathematical problem" (p. 142). Later, they added that "writing in mathematics can also help students consolidate their thinking because it requires them to reflect on their work and clarify their thoughts about the ideas" (2000, p. 61). Hence, they considered writing to be a method or tool to both learn and communicate mathematics. Since 1989, psychologists and researchers (e.g., Brown, 1987; Cross, 2009; Pugalee, 2001; Sfard, 2001; Vygotksy, 1987) have studied the use of writing in the mathematics classroom, reporting its positive benefits for the problem solver: it promotes the development of metacognitive behaviours, it helps to construct meaning and to organise one's ideas into a new structure of ideas, and so on. Nevertheless, writing has not found a place in the mathematics classroom, especially at secondary level. In order to create a more positive writing climate in school, dissemination at the tertiary level, and/or directly in schools, is of great importance.

The present article focuses on work on writing in mathematics with prospective secondary mathematics teachers in the middle of their studies, which is the optimal time for innovation. The prospective teachers are confronted with didactical ideas that may be quite different from what they have experienced thus far, and through such conflict can examine the benefits that writing brings, as well as examining their beliefs about teaching mathematics.

Theoretical considerations about writing with respect to problem solving, metacognition and beliefs, and research questions

Writing in mathematics

Underachievement in mathematics has led to several education reforms that place the focus on instruction methods fostering higher-order thinking, such as flexible and critical thinking, and mathematical argumentation. In this regard, numerous studies (e.g., Komorek, 2009; Kramarski, Mevarech,

& Arami, 2002; Kuzle, 2011, 2013; Lester, 1994; Mayer, 1998; Schoenfeld, 1987, 1992; Silver, 1987) have reported that improvement in problem-solving abilities is dependent on mathematical knowledge as well as cognitive and metacognitive abilities. Hence, mathematics instruction calls for methods that support students' acquisition and development of these processes. Writing has been recognised as one possible method to do so.

One of the earliest reports on writing in mathematics came from Geeslin (1977), who stressed that students of all ages need to write about mathematics for two reasons: "as a diagnostic tool for the teacher and as a learning device for the student" (p. 113). This would then help students develop a more precise idea of mathematical concepts, as well as helping prospective teachers learn how to explain mathematics. In a report of the College Entrance Examination Board (1983) published a few years later, Kilpatrick also addressed the usefulness of writing in gaining a better understanding of mathematics and constructing individual knowledge, which influenced subsequent NCTM's reports (1989, 2000). For Countryman (1992), mathematics learning occurs when "students construct it for themselves. They can only do that by exploring, justifying, representing, discussing, using, describing, investigating, predicting, in short by being active in the world. Writing is an ideal activity for such processes" (p. 2). Writing in the mathematics classroom ranges from informal, unstructured journal writing (concept development) to formal assessments of mathematical reasoning (portfolios, homework) (e.g., Bruder & Collet, 2011; Komorek, 2009). However, writing is not an easy process, but takes time and deep consideration from the writer: "a writer in the act of discovery is hard at work searching memory, forming concepts, and forging a new structure of ideas" (Flower & Hayes, 2009, p. 467).

Writing and metacognition

Metacognition in problem solving is considered to be a "driving force" that influences cognitive behaviour at all stages of problem solving (Lester, 1994). Mathematical instruction that focuses on the metacognitive aspects of mathematical thinking is therefore important. Various methods aid and support the development of this higher-order thinking, one of them being writing (e.g., Brown, 1987; Bruder & Collet, 2011; Bereiter & Scardamalia, 1987; Cross, 2009; Kuzle, 2011, 2013; Pugalee, 2001; Sfard, 2001). From a psychological perspective, writing is planned and conscious, and is therefore a valuable method of reflecting on, consolidating and strengthening what one knows. Pugalee (2001), for instance, concluded that writing serves as a monitoring tool that allows students to record what they know, orchestrate the mathematical resources and decide on a problem-solving path. Moreover, it sustains the development of mental reasoning, the ability to make connections and communication skills, ultimately contributing to the enhancement of metacognitive processes. Cross (2009) confirmed that writing activities help to develop a deeper conceptual understanding of students' current knowledge, while at the same time serving as heuristics. In other words, writing is a communication tool that allows students to transmit their mathematical ideas, while enabling teachers to model their students' mathematics. However, Cross also concluded that writing is a challenging cognitive process that requires a careful examination of the thinking one wants to articulate.

Beliefs about the nature of mathematics and teaching mathematics

In his work, Ernest (1989, 1991) defined three types of beliefs about the nature of mathematics and described how these provide a basis for teachers' conceptions of mathematics teaching and learning: (1) the instrumentalist view, (2) the Platonist view, and (3) the problem-solving view. According to the instrumentalist view, mathematics is "an accumulation of acts, rules and skills to be used in the pursuance of some external end" (p. 250). Hence, mathematics is viewed as a finished product, whereby the teacher takes the role of an instructor and learning is viewed as "skills mastery with correct performance" (p. 250). The student, on the other hand, is a passive receiver and consumer of knowledge and skills that must be mastered by practising on routine problems. The Platonist view is also a product-oriented perspective. Mathematics is viewed as "a static but unified body of certain knowledge" (p. 250). Moreover, mathematics is discovered, not created. In other words, mathematics is perceived as a consistent, connected and objective structure. A teacher with Platonist views takes the role of an explainer, whereby learning is conceived as the reception of knowledge. However, a Platonist teacher emphasises the conceptual understanding of unified knowledge. The problem-solving view of mathematics has a more process-oriented perspective. Mathematics is seen as a "dynamically organized structure located in a social and cultural context" (p. 250). Thus, mathematics is not a finished product, but rather includes activities such as generating ideas and solving problems, as well as communicating ideas and solutions. Through these activities, mathematics is a result of human inquiry and creation. The teacher takes the role of a facilitator; in his/her classroom, learning is an active process of construction of understanding, also by the means of problem posing and problem solving. The student is an active participant in the learning process and a creator of mathematical understanding and knowledge, communicating and sharing his/her mathematics results and discussing them with peers. This taxonomy of beliefs was used as a theoretical framework for the present study, both as a measurement instrument and to analyse prospective mathematics teachers' views on mathematics, teaching and problem solving.

Teachers' personal beliefs and theories about mathematics, learners and learning, teaching, subjects or curriculum, learning to teach, and about the self are widely considered to play a significant role in teaching practices (Pajares, 1992; Thompson, 1992; Wilson & Cooney, 2002). Confronting and changing prospective teachers' beliefs about mathematics and teaching has been promoted by some researchers (Cooney, 1999; Llinares, 2002) as one of the many goals of teacher preparation. Liljedahl, Rolka, and Rösken (2007), Conney (1999) and others have asserted that teacher education programmes are capable of helping to remedy the preconceived beliefs of preservice teachers. Emphasising the culture of a continuous process of personal reflection in education courses, teachers can become aware of their beliefs, theories or philosophies, so that they come to understand their own implicit theories and the ways these theories influence their professional practice (deFreitas, 2008). Teachers can then re-evaluate their beliefs and gradually replace existing beliefs with more relevant beliefs (Nespor, in Thompson, 1992). However, in order for this re-evaluation of beliefs to occur, teachers have to experience innovation for themselves, otherwise innovation lacks resonance. Given that writing is a reflective activity, it may help preservice teachers to become aware of their beliefs about it and to reflect on its effect with respect to learning, ultimately shaping their teaching practices.

Porter and Masingila (2001) gave an overview of the vast research conducted on writing in mathematics as a valuable tool for student learning in the mathematics classroom. Despite the calls of numerous research organisations and researchers, however, many mathematics teachers remain reluctant to use writing in their lessons, thus creating a gap between research and the realities of practice. As summarised in the previous section, this disconnection may lie in the fundamental belief that the process of writing is removed from the process of mathematical problem solving. If writing is to become standard in the mathematics classroom, as has been advocated for the last three decades, it is our role as mathematics educators to move teachers towards a view of mathematics and writing as a deeply related and intertwined process, rather than as two disjointed products. In order to achieve this, however, it is crucial to first understand how teachers respond to writing when doing mathematics. Miller and Hunt (1994) suggested that engagement in writing had the power to initiate change, as the actors reflect on its process.

The research on writing in mathematics lacks structured research focusing on prospective teachers' experiences with writing in mathematics and how such experiences might shape their attitudes towards incorporating writing in their lessons, from the standpoint of both a learner and a teacher. With these considerations in mind, a cohort of prospective secondary mathematics teachers had an opportunity to experience writing in a problem-solving seminar through exploring various mathematical problems, preparing written reports on the problem-solving process, and reflecting on it from the perspective of both a problem solver and a practitioner. Three questions were of interest for the current study:

- What metacognitive behaviours, if any, are supported by prospective secondary mathematics teachers writing reports?
- How do prospective secondary mathematics teachers react to writing mathematics during problem solving and to reflective writing after problem solving?
- What are prospective secondary mathematics teachers' perspectives on using writing with respect to problem solving in their mathematics classrooms?

Methodology

Context and participants

This was an exploratory qualitative study conducted in the problem-solving seminar *Problem Solving in Mathematics*, held at a large state university in Germany. The seminar took place once per week for 90 minutes, and was organised by both the author of this paper and the students. The seminar concentrated on learning about problem solving (e.g., problem-solving models, heuristics, self-regulated problem solving, teaching problem solving, problem solving with technology) facilitated through student presentations, while at the same time focusing on solving mathematics problems and how problem-solving activities can be implemented in mathematics instruction led by the instructor. The aim of the seminar was to provide the participants with a deeper understanding of problem solving through self-study, inquiry, investigation and exploration.

A cohort of 24 students in their third to sixth semester participated in the study, 13 of whom were elementary preservice teachers (Grades 1–4) and 11 of whom were lower secondary preservice teachers (Grades 5–10). Their own school memories of the mathematics classroom portrayed a traditional classroom in which learning materials and algorithms were presented by the teacher, followed by drill and practice. Very few had experience with problem solving, and those who did associated problem solving with solving puzzles, modelling and word problems. Thus, the participants had limited practical experience with both the theoretical and practical aspects of problem solving.

Data Collection

Data collection methods included a survey and various written materials. At the beginning of the semester, the participants completed a VAMS survey (adapted from Carlson, 1997) based on a contrasting alternative design developed by Halloun and Hestens (1996), which was used to examine participants' beliefs about mathematics as well as problem solving and its teaching. During the semester, various written instruments were administered. Every 3–4 weeks, the author of the paper administered the students' homework with 1–3 mathematical problems. The students were asked to keep a booklet comprising a problem-solving protocol and a post-reflection protocol about problem solving to allow for active engagement in knowledge construction. The problem-solving protocol served as an instrument to help students structure and guide their own problem-solving process. It was divided into four sections: the goal of the problem, the plan(s) to solve the problem, the implementation of the devised plan(s), and the conclusion(s) with respect to the problem. The students were encouraged to write down all of the ideas and questions that arose during the problem-solving process. After solving the problem, they had to reflect on the experience of writing guided by several questions in the post-reflection protocol. At the end of the semester, the students submitted nine booklets in the form of a portfolio. In addition, they had to write 1-2 page reflection papers designed to encourage them to relate what they were learning in class to their own practice or experience. In particular, they had to reflect on the semester-long experience of writing mathematics, both through the eyes of a student and a future practitioner, and report on aspirations for implementing writing when undertaking problem solving in their own mathematics classrooms.

Data analysis

Data analysis of both the quantitative and qualitative data went through several stages, as suggested by Yoo (2008) and Patton (2002), respectively. The data analysis began by administering the VAMS survey to a community of mathematicians and mathematics educators. The survey consisted of 50 items pertaining to two dimensions: epistemological (the nature of mathematics, connections and problem solving) and pedagogical (the learnability of mathematics and problem solving, and the personal relevance of mathematics and problem solving). Each item consisted of a statement followed by two contrasting alternative views. The participants were asked to identify their level of agreement with the two alternatives on a scale from 1 to 8. This contrasting alternative design increases the validity and reliability of the belief measurement (Halloun & Hestenes, 1996; Philipp, 2007), allowing the researcher to distinguish between the product view (instrumentalist), the mixed view (Platonist view), and the problem-solving view (process) of mathematics and its teaching, as suggested by Carlson (1997) and Yoo (2008). Based on the experts' answers, student answers were scored on a scale from 0 to 2. Hence, the maximum total score for the 50 survey items was 100 points. The student's response to the VAMS item was considered ideal if it fell into the first category, which contained a high frequency of mathematicians' responses. It was considered mixed if it fell into the second category, which contained a low frequency of mathematicians' responses, and it was considered non-ideal if it fell into the third category, which contained zero or a very low number of mathematicians' responses. Students who achieved less than 55 points were designated as having a product or instrumentalist view, those scoring between 55 and 79 points were considered to have a mixed or Platonist view, and those scoring from 80 to 100 points were deemed to have a process or problem-solving view. Out of the 24 students, 5 were assigned a product view of mathematics, 16 a mixed view and 3 a process view.

After completing the quantitative data analysis, the qualitative data analysis commenced. Given that the examination of beliefs is rather complex, field notes containing data from the seminar actions and conversations were balanced against the survey results. This enabled the confirmation or repudiation of the data, as well as the refinement of the characterisation of participants' beliefs, as suggested by Philipp (2007). Hence, both the quantitative and qualitative data allowed the identification of the three types of participants based on their views. In the second step, each of the participants' booklets was read and their responses were analysed based on the three research questions. The analytical inductive method (Patton, 2002) was used for the data for convergence, whereby analysis of the data is first deductive and then inductive. The deductive analysis was coded and analysed based on the theoretical framework, which was then refined using inductive analysis through emerging themes and additional codes. The categories of codes were used to interpret and understand data for a more in-depth discussion according to the theme. After the analysis of the booklets was complete, the final reflection paper described above was analysed using textual analysis (Patton, 2002).

Results

This section presents the results of the study with respect to the research questions. The first section focuses on metacognitive behaviours supported by writing, the next section contains a report on the participants' experiences with respect to writing, and the third and final section focuses on the participants' attitudes and beliefs about using writing in their future classrooms. In order to allow a richer description of the phenomenon, four cases are examined. The four participants – Chloe, Hannah, James and Leonard – were chosen randomly within their belief category.

Participants' backgrounds and belief structure

Leonard achieved 80 points in the survey and was therefore labelled as a process-oriented type. He was in his sixth semester of a teacher education programme for *Hauptschule, Realschule and Gesamtschule*.[2] For him, mathematics was mainly a dynamic and continuously growing field in which humans create their own knowledge. The role of a teacher was more that of a facilitator guiding students to construct mathematical knowledge and understanding it on their own, rather than that of a transmitter. He viewed learning as an active process in which students participate in the learning activity in order to work out and discuss the solution with others. For Leonard, solving problems was mostly an enjoyable experience that allowed the development of his reasoning skills. He believed that a good problem solver primarily needs to think flexibly, but is facilitated to a large extent by resources, skills and strategies rather than persistence.

Hannah achieved 69 points and was therefore labelled as a mixed type. She was in the third semester of a teacher education programme for primary school. For her, mathematics was a static but unified body of knowledge, perceived more as a formal than a creative representation of the real world. Hence, doing mathematics was more like following a recipe than an individual's creative way of explaining the world around him/her. For Hannah, the teacher had the role of a mediator, while emphasising conceptual understanding. Solving problems was both an enjoyable and a frustrating experience for her, but she noted that it helped to develop her reasoning skills. She believed that a good problem solver primarily needs to think flexibly, but is aided more by resources, skills and strategies than persistence.

James achieved 67 points and was therefore labelled as a mixed type. He was in his fifth semester of a teacher education programme for *Hauptschule, Realschule and Gesamtschule*. For him, mathematics was a static but unified

2 Hauptschule, Realschule and Gesamtschule are types of secondary school in Germany.

body of knowledge, perceived as a more creative representation of the real world rather than formalisation. Doing mathematics was much like following a recipe, and the result was a piece of artwork. The goal of instruction, for him, was to transmit knowledge, while at the same time guiding students to understand the transmitted material. Solving problems was more an enjoyable experience than a frustrating one, and was more dependent on his resources than on perseverance. James believed that a good problem solver needed only to think flexibly and know how and when to apply various types of reasoning and skills.

Chloe achieved 45 points and was therefore labelled as an absolutist type. She was in her fifth semester of a teacher education programme for primary school. For her, mathematics was primarily a formal way of representing the real world. She held mathematics to be a static body of facts independent of human invention. The role of the teacher was that of a transmitter of knowledge, with students absorbing mathematical concepts and practising routine problems for accurate performance rather than actively participating in the learning process. For Chloe, solving mathematical problems was both an enjoyable and frustrating experience, but she recognised that it helped to develop her reasoning skills. For her, an organised memory (formulas, procedures), flexible thinking and perseverance were the characteristics of a good problem solver.

Metacognitive activities supported by the writing process

Throughout the booklets, the participants demonstrated reasoning that included not only cognitive behaviours, but also metacognitive behaviours. The use of a problem-solving protocol as an instrument to analyse metacognitive behaviours was somewhat limited as the participants did not write a narrative of their problem-solving processes. However, in combination with the post-reflection protocol, it allowed an examination of which metacognitive behaviours were prompted through the writing process. Writing supported various metacognitive processes. For instance, Leonard most often reported that through writing he was able to organise his thinking: he drew a sketch of the problem and noted possible problem-solving approaches before he decided on the final problem-solving approach. Hence, writing enabled him to manage the various resources he possessed (knowledge, strategies) and to regulate his problem-solving processes in a productive way. In addition, by writing down his ideas, he was able to control the reasonableness of his arguments and thus the correctness of his problem solution. Through writing, he was therefore able to explore the problem-solving space before arriving at a solution.

Hannah added that writing helped "to consciously think about problem-solving processes from a metacognitive perspective". That is, in having to state the problem goal, she focused on understanding the problem before choosing a perspective to solve the problem; before choosing a perspective, she wrote down possible solution paths based on her knowledge, and then evaluated the plausibility of each approach before deciding on one. Lastly, after she had carried out the plan, she read through her arguments again before writing down the solution. Hence, she monitored, regulated and evaluated her work continuously. However, such behaviour was only present when she knew how she might solve the problem.

James initially found it difficult to provide clear goals and adequate explanations of his work, and to use proper mathematical language to communicate his problem-solving process clearly. Nevertheless, as the semester progressed, this picture changed and growth in metacognitive activities occurred. Instead of merely writing the solution steps, he started using writing for exploration: he gathered strategies and accessed mathematical content that might be useful for the problem before choosing a problem-solving path. When the plan did not work out, he was able to look back and decide on another perspective. Thus, writing helped him to systematically gather relevant information, to organise his thoughts, to regulate the available resources, and to refine them when evaluation was lacking. Redirection and reorganising thinking in productive directions were supported by the writing process. For Leonard, Hannah and James, writing thus generated an awareness of their thinking and helped them to develop a deeper conceptual understanding of their current knowledge, to analyse the current problem-solving state, and to move towards identifying a successful solution plan. Sfard (2001) described this as a dialogical endeavour, whereby we inform ourselves, we argue, we ask questions and we wait for our responses (pp. 4–5). As a result of such dialogic endeavour, the students were able to construct new knowledge through the interaction between their problem-solving space and their writing space, in order to meet specific goals.

Chloe's booklet, on the other hand, did not exhibit any evidence of metacognitive behaviour; she completely neglected exploration and arrived directly at the problem solution, adding in the post-reflection guide that the writing protocol only helped her to structure her work in four sections. Such behaviour was consistent with her absolutist view of mathematics, in which mathematics is detached from exploration and individual creation.

Writing to reflect on problem solving

During the semester, the participants experienced writing in a problem-solving seminar through exploring different mathematical problems, preparing written reports of the problem-solving process, and reflecting on it through the perspective both as a problem solver and a practitioner. However, the participants responded differently to completing the protocols. Some found it helpful immediately, some after some time, and others not at all. Some found it helpful only with respect to a specific problem in which the method helped them to organise their thoughts. Leonard completed each protocol, offering rich descriptions of his reflection. Through writing, he realised how difficult it is to note down what one is thinking in a comprehendible manner, but added that describing his processes "helped [him] go back, follow his train of thought and check the reasonableness of his solution", as well as to check whether his arguments were correct. He also added that it helped him check whether "another person reading the problem-solving path could arrive at the solution as well". The process of writing helped him to "intensively engage in problem solving", "put down his ideas immediately", "revise work" and "structure his approach". The problem-solving protocol prompted metacognitive behaviours – such as planning, monitoring, regulation and evaluation – which were beneficial for his work.

Hannah added that having a protocol helped her to structure her work by preventing it from becoming chaotic. In addition, she "put the solution in the background and focused on the process". However, when she was unable to solve a problem, she left the problem-solving protocol empty, writing in the post-reflection protocol that the problem-solving protocol was not always helpful. In their final paper, Leonard and Hannah added that the post-reflection guide prompted them to go over their problem-solving protocol and re-examine the quality of their work and of the problem-solving process. In addition, they believed that protocols would allow them to assess their students' thinking and possible knowledge deficits.

James's attitude towards writing changed positively as the semester progressed. James, like Leonard and Hannah, stated that preparing the problem-solving protocol helped him to consciously organise his thoughts. All three felt that writing could help them better understand their thinking processes and remember key ideas of the problem-solving process, which they could then use in future problem solving. Writing allowed participants "to look back at their thoughts and reflect on their growth", as noted previously in the literature (Flores & Brittain, 2003, p. 114) and as observed in Leonard, Hannah and James. It was, however, clear that as the semester progressed their post-reflection became repetitive. Leonard wrote in his final paper "at the beginning it was very helpful to write everything down, to reflect on the experience … but afterwards it become boring to explain the same things over and over again". As time passed, Chloe found little use for either the problem-solving or the post-reflection protocol, stating that they were time consuming and that she did not see the point in describing her problem-solving processes and reflecting on the writing down of her problem-solving process. For her, writing was extremely removed from exploring mathematics. In contrast to the other participants, she said that preparing booklets did not help her to organise ideas.

Beliefs about writing from a practitioner's perspective

Participants established the connection between problem solving and mathematics from a practitioner's perspective with varying intensity. Both Leonard and Hannah indicated that they found preparing booklets very helpful and articulated a rather strong belief that writing should be incorporated in regular mathematics lessons. For both of them, writing was an instrumental part of problem-solving activity, and therefore an important part of the mathematics classroom, enabling students to consolidate of their knowledge and supporting the development of conceptual understanding. Leonard noted that, from a learner's perspective, he was able to discern benefits of writing for his students, "I was able to systematically structure my work and intensively engage in thinking what the next step should be. My students could benefit from writing as well". He added that it could help students to "learn how to justify their thinking and support their individual problem-solving process", while at the same time enabling him to "understand [his] students' thinking and gaps in knowledge" they might have.

From the start, Hannah was open to writing, although she was not initially sure how mathematics and writing related to each other. However, as the semester progressed, her problem-solving and post-reflection protocols changed in their nature and quality. She was willing to explore connections between writing and problem solving. By doing so, her mixed views tended to give way to new conceptual views, enabling positive beliefs and attitudes towards writing and doing mathematics. Hence, this experience and her reflection on it allowed her to connect the writing process to building conceptual understanding and constructing new knowledge. In her final paper, she wrote, "by writing, students can actively experience problem solving as a mathematician does and construct new knowledge. They can explain thinking in their own words". She stated she would use writing in her classroom, adding that we should "add prompts within problem-solving protocols to help the student overcome barriers", as she found that the protocol was not helpful when she was stuck. Last but not least, she reported that she would use booklets as resources in her classroom.

James, on the other hand, much like Hannah, held that teachers should promote conceptual understanding, but was somewhat confused about the role of writing in the mathematics class. Many questions remained open for him:

"Writing helped me organise my thoughts and helped go on the right path. But it was time consuming. It will be overwhelming to teach what needs to be taught, prepare students for exams, and on top of it use writing." He added, "I am not sure how often I should use booklets. I do not feel ready to use writing in my classroom." Hence, the participants who were aware of their writing and learning through writing seemed to benefit most from the overall process, and were thus more likely to use writing in their own classrooms. It seems that direct instruction in writing is needed for teachers to feel competent to use writing in their lessons.

Unlike the other participants, Chloe held to a product view of mathematics and tended towards a traditional classroom rather than a student-oriented classroom. She indicated that preparing booklets was extremely time consuming and after the second booklet the quality of her work dropped. She viewed writing as a process disconnected from problem solving and merely as a means to produce a formal document to be evaluated by the instructor. She summarised her thoughts: "Writing is useful, but it does not belong in a mathematics class. When students write, they focus on writing and not on the mathematics. For me, it is important that students can follow procedures I give them, and to do so they do not need writing protocols." Hence, it seems that not only beliefs about mathematics influence whether teachers use writing in their classroom, but also their beliefs about writing in mathematics.

Conclusions and final thoughts

Mathematics is more than just numbers. Writing is a challenging cognitive process that requires a careful examination of the thinking one wants to articulate (Bereiter & Scardamalia, 1987; Flower & Hayes, 1980). In this project, the students were asked to solve various mathematical problems based on their knowledge and report on this experience. In order to do so effectively, the participants needed to engage in various metacognitive processes: to orient themselves with respect to the problem, to decide on a strategy to solve the problem based on the vast resources they possessed, to monitor and regulate their processes, and to evaluate the reasonableness of their planned processes and/or of the solution (Kuzle, 2011, 2013; Pugalee, 2001). This was, however, not an easy task; issues of providing clear goals, of adequate explanations of their thinking, and of the integration of mathematics and words sometimes interfered with their ability to effectively communicate the mathematics.

The quality of writing differed. For instance, while Chloe just reported what she already knew, Hannah, James, and Leonard were able to construct new knowledge through the interaction between their problem-solving space and their writing space in order to meet specific goals, as suggested by Vygotsky (1978). Hence, much like verbal communication (Cross, 2009), the act of producing convincing arguments through writing created an additional cognitive demand on the participant. This ability to "efficiently generate adequate content so that one has the flexibility to select from what is available and discard what is deemed unnecessary or irrelevant (a skill of more expert writers) appears to be one's knowledge of the subject being written about and the ability to readily access this knowledge" (Cross, 2009, p. 925). The participants who were aware of their writing and learning through the process of writing seemed to benefit most from the overall process, making them most likely to use writing in their own classroom. However, it seems that beliefs about writing also play a significant role, as James and especially Chloe were not convinced to use writing in their classrooms.

The communication principle is one of the standards outlined in the mathematics curriculum (NCTM, 2000). As one of the communication methods, writing is implemented in mathematics classrooms with varying intensity, despite its benefits (e.g., Bereiter & Scardamalia, 1987; Cross, 2009; Pugalee, 2001; Sfard, 2001). This may be a result of a misconception that the process of writing and that of doing mathematics are unrelated. With respect to Hannah and Leonard, the results of the present study showed that when writing helped support the metacognitive processes essential for productive problem solving, the distinction between the two disappeared. Thus, although beliefs are extremely difficult to change (Pajares, 1992), rich and meaningful experiences may help promote awareness of the benefits of writing in mathematics, and encourage the development of positive beliefs with regard to the process of writing and mathematics, as suggested by Miller and Hunt (1994).

Mathematics educators cannot assume that student teachers come with experience and knowledge of how to write effective mathematical explanations. They need experience in writing in order to build awareness of the merits of writing with respect to promoting mathematical understanding. Moreover, they need direct instruction in what it means to target an audience, to state the goal in a well-defined introduction, to link and explain representations, and to properly integrate mathematical notation and figures with words. If writing is to become an accepted method for both teaching and learning mathematics, teachers need to experience high quality writing for themselves, to raise awareness of its benefits, and to be trained in how to use writing in their classroom, as demonstrated by both Hannah and Leonard. Moreover, both processes need to transform into a single process. Only then will teachers use writing as a method of critical thinking that can help students learn how to think mathematically.

References

Bereiter, C., & Scardamalia, M. (1987). *The psychology of written composition*. Hillsdale, NJ: Erlbaum.

Brown, A. (1987). Metacognition, executive control, self-regulation and other more mysterious mechanisms. In F. Weinert & R. Klume (Eds.), *Metacognition, motivation and understanding* (pp. 65–116). Mahwah, NJ: Erlbaum.

Bruder, R., & Collet, C. (2011). *Problemlösen lernen im Mathematikunterricht.* Berlin: Cornelsen.

Carlson, M. P. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success. *Educational Studies in Mathematics, 40*(3), 237–258.

College Entrance Examination Board. (1983). *Academic preparation for college: What students need to know and be able to do*. New York: Author.

Cooney, T. J. (1999). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics, 38*, 163–187.

Countryman, J. (1992). *Writing to learn mathematics*. Portsmouth, NH: Heinemann.

Cross, D. I. (2009). Creating optimal mathematics learning environments: Combining argumentation and writing to enhance achievement. *International Journal of Science and Mathematics Education, 7*(5), 905–930.

deFreitas, E. (2008). Troubling teacher identity: Preparing mathematics teachers to teach for diversity. *Teaching Education, 19*(1), 43–55.

Ernest, P. (1989). Philosophy, mathematics and education: The state of the art. *International Journal of Mathematics Education in Science and Technology, 20*, 555–559.

Ernest, P. (1991). *The philosophy of mathematics education*. Abingdon, Oxon, UK: Routledge Farmer.

Flores, A., & Brittain, C. (2003). Writing to reflect in a mathematics methods course. T*eaching Children Mathematics, 10*, 112–118.

Flower, L., & Hayes, J. R. (2009). The cognition of discovery: Defining a rhetorical problem. In S. Miller (Ed.), *The Norton book of composition studies* (pp. 467–478). New York: W.W. Norton.

Geeslin, W. E. (1977). Using writing about mathematics as a teaching technique. *Mathematics Teacher, 70*, 112–115.

Halloun, I., & Hestenes, D. (1996). Views About Sciences Survey: VASS. *Paper presented at the annual meeting of the National Association of Research in Science Teaching*, St. Louis, MO. (ERIC Document Reproduction Service No. ED394840).

Komorek, E. (2009). *Mit Hausaufgaben Problemlösen und eigenverantwortliches Lernen in der Sekundarstufe I fördern. Entwicklung und Evaluation eines Ausbildungsprogramms für Mathematiklehrkräfte*. Berlin: Logos Verlag.

Kramarski, B., Mevarech, Z. R., & Arami, M. (2002). The effects of metacognitive instruction on solving mathematical authentic tasks. *Educational Studies in Mathematics, 48*, 225–250.

Kuzle, A. (2011). *Preservice teachers' patters of metacognitive behavior during mathematics problem solving in a dynamic geometry environment.* Doctoral dissertation. The University of Georgia–Athens.

Kuzle, A. (2013). Patterns of metacognitive behavior during mathematics problem-solving in a dynamic geometry environment. *International Electronic Journal of Mathematics Education, 8*(1), 20–40.

Lester, F. K. (1994). Musing about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25*(6), 660–675.

Liljedahl, P., Rolka, K., & Rösken, B. (2007b). Affecting affect: The reeducation of preservice teachers' beliefs about mathematics and mathematics teaching and learning. In W. G. Martin, M. E. Strutchens, & P. C. Elliott (Eds.), *The learning of mathematics* (pp. 319–330). Reston, VA: National Council of Teachers of Mathematics.

Llinares, S. (2002). Participation and reification in learning to teach: The role of knowledge and beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 195–209). Dordrecht, The Netherlands: Kluwer.

Mayer, R. E. (1998). Cognitive, metacognitive, and motivational aspects of problem solving. *Instructional Science, 26*(1–2), 49–63.

Miller, L. D., & Hunt, N. P. (1994). Professional development through action research. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics* (pp. 296–303). Reston, Va: The National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Pajares, F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Patton, M. Q. (2002). *Qualitative research and evaluation methods*. Thousand Oaks, CA: Sage.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, Vol. 2 (pp. 257–315). Charlotte, NC: Information Age.

Porter, M., & Masingila, J. (2001). Examining the effects of writing on conceptual and procedural knowledge in calculus. *Educational Studies in Mathematics, 42*(2), 165–177.

Pugalee, D. K. (2001). Writing, mathematics, and metacognition: Looking for connections through students' work in mathematical problem solving. *School Science and Mathematics, 101*(5), 236–245.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Sfard, A. (2001). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of the Twenty-first Conference of PME-NA* (pp. 23–44). Columbus, OH: ERIC Clearing House for Science, Mathematics, and Environmental Education.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem-solving instruction. In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 33–60.) Hillsdale, NJ: Erlbaum.

Thompson, A. (1992). Teacher's beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wilson, M., & Cooney, T. J. (2002). Mathematics teacher change and development. In G. C. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 127–147). Dordrecht, The Netherlands: Kluwer.

Yoo, S. (2008). *Effects of traditional and problem-based instruction on conceptions of proof and pedagogy in undergraduates and prospective mathematics teachers*. Doctoral dissertation. The University of Texas, Austin.

Biographical note

Ana Kuzle, Dr., is a research assistant in the Institute for Mathematics at the University of Paderborn. After finishing her dissertation on preservice teachers' metacognitive processes while solving nonroutine geometry problems in a dynamic geometry environment at the University of Georgia, she moved to Paderborn and continued working in the area of problem solving with both preservice and inservice mathematics teachers. She founded DUPLO project (Durch Problemlosen Mathematik lernen), in which the main goal is that students develop understanding of mathematical concepts and methods by teaching these through problem solving. Within the project she focuses on problem solving processes and its promotion as well as beliefs towards problem solving and possibility of their change through new innovative learning and teaching methods and environments.

Applying Cooperative Techniques in Teaching Problem Solving

Krisztina Barczi[1]

Teaching how to solve problems – from solving simple equations to solving difficult competition tasks – has been one of the greatest challenges for mathematics education for many years. Trying to find an effective method is an important educational task. Among others, the question arises as to whether a method in which students help each other might be useful. The present article describes part of an experiment that was designed to determine the effects of cooperative teaching techniques on the development of problem-solving skills.

Keywords: Cooperative techniques; Problem solving; Investigation; Open problem

1 PhD student, University of Debrecen and Teacher of Mathematics, Neumann János Secondary School, Hungary; bkrixta@gmail.com

Uporaba sodelovalnih tehnik pri poučevanju reševanja problemov

Krisztina Barczi

Poučevanje reševanja problemov – od reševanja preprostih enačb pa vse do reševanja zapletenih tekmovalnih nalog – je že več let eden izmed največjih izzivov pri pouku matematike. Iskanje učinkovite metode je pomembna izobraževalna naloga. Med drugim se postavlja vprašanje, ali je tehnika sodelovalnega učenja, pri kateri si učenci medsebojno pomagajo, lahko uporabna. Predstavljeni članek opisuje del raziskave, ki je bila zasnovana z namenom, da pokaže vpliv tehnik sodelovalnega poučevanja na razvoj spretnosti reševanja problemov.

Ključne besede: sodelovalne tehnike, reševanje problemov, raziskovanje, odprt problem

Introduction

Teaching problem solving is one of the most important aims of mathematics education. The reason for this is that problem solving develops cognitive abilities, emphasises the application of mathematical knowledge, improves creativity, etc. (Schoenfeld, 1992), while also being a basic skill that is needed in all areas of life (Kirkley, 2003). In Hungary, mathematical problem solving has a strong tradition, but mainly for gifted students; for students of average ability, there are currently some issues regarding the teaching of problem solving. The first and most widespread method is front teaching. Due to the fact that class sizes in many schools are quite large (approximately 30 students), this method is not the most effective in some situations. There is no doubt that, when teaching problem solving, a kind of dialogue needs to be present between the teacher and the students. This enables the teacher to simultaneously guide the children towards both the right solution and the right way of thinking. If the teacher asks the whole class questions, there are usually only a small number of students who take part in the dialogue actively, students who understand what is going on and would probably be able to solve the tasks on their own. What about the others? Some students will listen in silence and try to follow the discussion, while others will copy everything from the board but not necessary understand it. It is precisely this situation that gave rise to the idea of using cooperative teaching. Moreover, the problem collections used in class contain hardly any open problems and lack investigations. Many tasks can be solved simply by following "recipes", algorithms. The problems that are closest to the open problem type are those involving geometric constructions or parametric equations, as well as those that provide an opportunity for discussing multiple solutions. These rarely appear in lessons but are frequently present in mathematics study sessions and among mathematics competition problems.

In order to try to improve this situation, an experiment was planned and carried out that was designed to determine the effects of the regular use of cooperative teaching techniques on the development of students' problem solving skills, and to provide improved insight into how students solve problems. The article presents a part of this experiment.

Theoretical Background

Problem solving

Problems – A problem is a task in which we attempt to find a solution on the basis of certain known data and conditions. First of all, we need to distinguish between routine and non-routine problems. Routine problems are those in which the problem solver immediately knows and recognises the correct process for the solution, whereas in non-routine problems the solver does not identify how to solve the problem immediately (Mayer & Hegarty, 1996). Of course, the final goal is to find the solution, but this is often impeded by various obstacles that depend on many factors, such as the problem solver's age, ability, etc. How to recognise and overcome these obstacles is not always obvious. According to Fisher (2005), the following elements are required to define what a problem is: (1) what is given, (2) obstacles, (3) aims and (4) effort. Another description of a problem is provided by Lénárd (1987), who says that a problem is a situation where we want to reach a specific goal, but the way of reaching this goal is hidden. Pólya (1962) describes problems similarly, stating that if we have a problem it means that we are trying to find some means with the help of which we can reach a clearly stated but not necessary easily achieved aim. All of these definitions regard problem as a rather broad concept, and they agree that in problem solving we know where to start and in most cases we know where we want to end up, but the "HOW" is as yet unknown to the problem solver.

Open problems – Open problems are problems (1) that are unsolved, or (2) whose solution depends on the interpretation of the problem solver, or (3) where multiple ways of solving the problem are possible, or (4) that suggest further questions and possible generalisations (Ambrus, 2000). Investigations, real-life situations, projects or problem fields are considered to be open problems according to Pehkonen (1999), who says that open-ended tasks can be created by problem posing, problem variations or by working with problem fields. The problem presented in the present article belongs to (3) and (4). The main benefits of using open mathematical problems are that they provide an opportunity for children with different mathematical abilities to experience success, they allow students to progress at their own pace, and with multiple solutions they provide an excellent basis for mathematical discussions (Way, 2013). Solving open problems can increase students' need to prove and justify ideas that arise during problem solving, enabling them to return to the original problem and investigate it from a different perspective, thus creating kind of a problem solving cycle (Hähkiöniemi, Leppäaho, & Francisco, 2012).

When teaching problem solving, it is good to let students try to have their own experiences with the task at hand, encouraging them not only to solve the problem but also to formulate and find new problems (Zimmermann, 1986), since formulating problems plays a vital role in problem solving. Clearly, the easiest way to start is by using already existing problems. One option for making new tasks is to look back at a previously solved problem. Here, the solution may suggest new problems, or we can verify whether changing the conditions of the original problem results in something new. Another way of posing new problems from old ones arises when we have not found the solution. In this case, the problem solver usually breaks the original task up into smaller parts and attempts to solve these new, probably easier problems (Kilpatrick, 1987; Pólya, 1973).

Problem solving phases

In his renowned book How to Solve It (1973), Pólya distinguishes between the following four phases of problem solving: (1) *Understand the problem*: we have to understand the given information and we need to see where we want to get; (2) *Make a plan*: we need to determine how the given information can be connected to what we are looking for, and we need to decide which tools to use in order to obtain a solution; (3) *Carry out the plan*: we need to do what we have planned; and (4) *Look back*: this phase is more than just checking whether the answer is correct, we need to review how we solved the problem and discuss the difficulties we had to face, the ideas that helped us carry on, etc.

Schoenfeld suggests similar steps in problem solving, which are extended versions of Pólya's phases (Ambrus, 2004).

Cooperative teaching and learning

Cooperative learning is a teaching arrangement whereby people work together in order to achieve a common goal, which often means solving a problem. During this work, group members depend on each other, and the success of the team depends on their ability to cooperate. They must support each other, trust each other and respect each other if they want to overcome the difficulties that might hinder them (Kagan, 2004).

In Hungarian education, cooperative learning appeared as a result of the work of József Benda. He believed that cooperative learning could bring about a positive change in the way Hungarian schools worked, resulting in improved achievement, integration and development of students (Józsa & Székely, 2004).

It is important to note that cooperative learning is not simple group work. According to Johnson and Johnson (1994) and Kagan (2004), in cooperative learning the following four principles should always be present: positive interdependence, individual accountability, equal participation and simultaneous interactions (PIES). Furthermore, for effective work, team members should possess social skills that they can apply appropriately. If necessary, these skills should be taught to students in advance (Johnson & Johnson, 2009).

As for the ideal group size, Crabill (1990) recommends groups of four. The reason for this is that, in a group of four, two simultaneous dialogues can coexist and an ideal seating arrangement can easily be achieved, whereas groups with more members are more likely to become passive.

Examples of cooperative structures

In this section, we present some examples of cooperative structures that have been found useful and are efficiently applicable in mathematics classes. The structures were given catchy, easy-to-remember names so that students and teachers can remember them more easily, thus creating an opportunity to use the same structure in many different contexts (Kagan, 2003).

Pairs Check: In this structure, two students work together. One is the "coach" and only checks the work of the other student, or, if necessary, offers advice on how to carry on. The second student has to write everything down while explaining aloud what s/he is doing. (Kagan & Kagan, 1998).

Jigsaw Expert Groups: The main idea of this structure is that every group is an expert in a topic or a task. They are given some time to prepare – either to collect ideas or solve a task – then new groups are formed such that each new group contains one person from each of the original groups. The new groups consequently contain students who are experts in each task. In the new groups, students share their topics with each other, notes are taken and comments are discussed (Kagan & Kagan, 1998).

Gallery Walk: Students have to collect information or solve a task in groups of four. The information gathered is written on a poster that is displayed in the classroom. Everybody then walks around so that they can check the work of the different groups. The mingling students are allowed to write comments or ideas on each other's posters (Kagan, 2004).

The teacher's role in cooperative learning

Obviously, when using cooperative teaching techniques it is not only the classroom setting and the students' role that change, but also the teacher's role. The teacher is transformed from an instructor to a tutor, someone who guides students in the teaching/learning situation. While the students work cooperatively, the teacher's task in class is to monitor and observe their work, ensuring that they make progress, helping them if they are stuck on a problem and cannot continue alone, and providing extension exercises for groups that finish sooner (Burns, 1990). The teacher should still be the leader and, in addition to explaining the guidelines of cooperative work, it is his/her responsibility to maintain a suitable working environment. In cooperative class work, students are supposed to talk to each other, so the classroom becomes noisy, but the teacher should prevent the classroom turning into a chaotic environment (Dees, 1990).

Research question

How does the regular use of cooperative teaching techniques contribute to effective teaching and learning of problem-solving skills?

Research Methodology

Background information

The experiment was action research, which means that the researcher was also the teacher of the class. This type of experiment has become popular amongst practitioners who would like to carry out research related to professional development. Koshy (2005) defines action research as a kind of enquiry whose aim is to constantly refine practice and ultimately contribute to the teacher's professional development. The writer states that action research means researching one's own practice, and is therefore participatory and situation-based, as well as being emergent and focused primarily on improvement. Action research is a useful tool for narrowing the gap between the goals of mathematics educators and teachers (Zimmermann, 2009).

Since the researcher was interested in the effects of cooperative techniques on the development of students' problem-solving skills, the question of using control groups arises. According to Slavin (1996), however, when comparing the outcomes of cooperative teaching and learning to other programmes, there are many factors that differ between the two alternative programmes, such as the subjects, the duration, etc., and these factors can account for the differences in the outcomes.

The *school* where the experiment took place is a mixed comprehensive secondary school whose strength lies in scientific subjects and computer science. The *students* taking part in the action research were 16–17 years old. All of the 16 participants were attending a class that prepares students for tertiary education in technology and specialises in foreign languages. Following a preparatory year, these students have four years to complete their secondary school studies. As mentioned above, the writer of the present article was the teacher of this group. The 2012/2013 academic year was the students' third year at our school. In their first year, the students attended three mathematics lessons per week, increasing to four in the subsequent two years. In the year in question, they followed the year 10 scheme of work for secondary school students. Since they had more mathematics lessons than a "normal" class, we often had an opportunity to discuss a topic in more detail or to solve problems from mathematics competitions. These students are not necessary gifted in mathematics, but the majority of them certainly have a great interest in mathematics and other scientific subjects. Their mathematics grades were good (4) or excellent (5), with only one student having a grade of satisfactory (3).[2] These students are mainly *motivated*, although it is not always easy to activate them in class. Some of them still regularly take part in mathematics competitions and they attend group study sessions weekly.

Methods of data collection

The students participating in the experiment started by completing various psychological tests focusing on their communication skills, their attitude towards learning, their attitude towards working in groups and their attitude towards mathematics. They also took a mathematics pre-test, post-test and delayed test (Ambrus, 2004; Tóth, 2007), all of which were constructed so that the selected problems test the mathematical knowledge and thinking methods that are needed during the experiment.

Half of the lessons were recorded on video and the discussions of the different groups were also recorded with a voice recorder. During the lessons, the teacher observed the groups' work and the reaction of the individual students, and each student had a so-called "reflection book". As we have seen, one of the most important steps of problem solving is reflecting on both the solution and the strategy used. In order to record all of the ideas, thoughts and comments that might occur during the experiment, each student had a so-called "reflection book", a small exercise book in which they were instructed to take notes. These notes not only included mathematical ideas and ways of problem solving, but also personal feelings, reactions, etc. The students first described their working method in their own words, but in the discussions some heuristic strategies were named as well.

Forming groups

In order to form efficient groups, the following aspects had to be taken into consideration: a) friends in a group?; b) some students are difficult to work with; c) some students are less talkative; d) some students are tolerant, so can work with anyone, etc.

As stated above (Crabill, 1990), the optimal group contains four members. There were 16 students taking part in the research, so four groups of four were created. Each group was constituted so that it included a weaker student, a more able student, a quiet student and a more talkative student; moreover, "difficult" students were grouped with patient ones. The group settings were

2 In Hungary, we use a five-grade marking scale, in which 1 means fail and 5 means excellent achievement.

changed once during the first part of the experiment.[3] The reason for this was to give the students an opportunity to work with as many fellow students as possible. In order to achieve efficient, cooperative work, however, the group members needed time to get used to each other, so more frequent changes in the group settings were avoided.

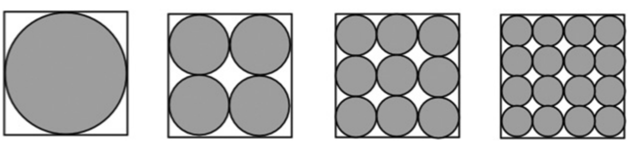
The problems

Five curriculum-based problems were selected, derived from different fields of mathematics (algebra, geometry, number theory, combinatorics).[4] As Hungarian textbooks contain hardly any open problems or investigations, these problems were either open problems already or they were chosen so that they could be "opened". Possible ways of extension were suggested either by the teacher or the students. Twelve 45-minute lessons were devoted to discussing these tasks. The main aim was to explore and discuss each problem in great detail, and if possible extend it as well, which is why two or three lessons were planned to be used for each problem. The present article describes the discussion of one of these problems in detail.

Since our plan was to examine the effect of the regular use of cooperative teaching techniques on the development of problem-solving skills, each of the aforementioned 12 lessons was planned with this method. Although the tasks might suggest otherwise, providing fun mathematics for the students was not the main aim of the experiment, which is why all of the problems are curriculum-based and develop mathematical competencies.

The problem presented

The problem presented in this article is an area investigation task. Students received the following figure:



Problem: From a square measuring 60 cm by 60 cm we cut circles as you can see on the figure above. What percentage of the square is wasted? Do you recognise a pattern? Can you generalise your idea? Can you prove your conjecture for n circles?

3 The first part contained 12 lessons. See *The problems*.
4 List of problems: Appendix

Extension: How could you modify the problem? Can you think of any other regular shapes that could be used?

Lesson plan and experiences

Firstly, it should be pointed out that by the time the students received this problem they had already had some experience in working in groups, so forming the groups went smoothly. Secondly, the detailed discussion of the whole problem took three consecutive 45-minute lessons. Solving and reflecting on the original problem was followed by the students' modification of the problem as well as the solution and discussion of part of this subsequent problem.

Starter activity

As a starter activity, the groups were given a worksheet on which they listed all of the mathematical knowledge – this could be a mathematical word or a formula – that they thought would be useful in solving the problem. One interesting point should mentioned in this regard: all of the groups noted down that they would use percentages during the solution, as this was mentioned in the statement of the problem, but in the end only one group actually worked out the percentage of wasted material, while the rest worked with fractions. Instead of using posters, the mathematical information that the groups had collected was written on the white board, which was divided into four parts. Each group had a section to write their formulae in.[5] After the groups had finished collecting useful mathematical ideas, they sent one of their members to the white board to share the group's notes with the others. Each group had time to modify their notes on the basis of what was written on the board, and they then started solving the problem.

Solving the original problem[6]

To start with, the easiest part for the groups was to determine that the area of the wasted material is the same for all four arrangements. Admittedly, calculating the area of a square and some circles and doing some basic arithmetic should not be a challenge for a 16- or 17-year-old student, so it is no wonder that they completed this part of the task quite quickly and none of the groups needed extra help.

However, when I asked the groups: "How about n circles?", all of them said that the waste must be the same in that case too. I then asked them to prove their statement, in order to generalise what they had found. The reason for this is that students do not usually feel the need for proof, and have little

5 Method: Gallery Walk
6 Method: Pair Check

experience with generalising let alone proving statements. Consequently, it was at this point that the students became puzzled. Nobody had any idea how to start, and therefore *I had to provide some guidelines, some helping questions and comments* (CBS, 2011), as follows:

- What can you say about the number of circles?
- What is the relationship between the number of circles and the length of the side of the square? Can you write an expression for the side length in terms of the number of circles?
- Can you write an expression for the area of the circles?
- Can you express the wasted area?

Since the working pace of the groups was different, this discussion took place four times.

Why did the students struggle with generalising their ideas? It could be that in everyday mathematics lessons, problem solving usually ends here: you do some calculations on the basis of the given data, you attain some results that seem correct, job done, you can go to the next exercise. It is therefore no wonder that when I encouraged the students to read the whole question again, and they realised that they had to find an answer in case of the nth figure, they looked a bit puzzled. After the groups received the aforementioned "helping questions", they were asked to continue working in pairs using the "Pairs Check" method.

The pairs were given some time to brainstorm ideas but were asked to continue the solution in the original groups. After some productive thinking and discussion, the groups came up with some seemingly different formulae for the amount of wasted material in the nth square. Although it was a lesson based on cooperative teaching methods, to ensure that everybody was on the right track an occasional whole-class discussion was unavoidable. In order to do this in an effective way, the white board was again divided into four sections, and after each group had worked out a formula for the nth square, the formulae were written in the different sections. Each formula was then interpreted with the help of the groups. In this discussion, the teacher led the students through their explanations with the help of questions.

The groups came up with four different formulae that were meant to express the same thing, so first of all we had to clarify how they had created these formulae and what the different parts meant. A speaker from each groups was therefore selected to explain their formula. It was not surprising that all of the formulae turned out to be equivalent and correct. After generalising the problem, the question arose as to whether the task could be modified to create a problem field (Pehkonen, 1997), and if so, how.

Modifying the problem

Although the students came up with various ideas, due to a lack of time we settled on one that suggested replacing the square with an equilateral triangle and trying to arrange congruent circles inside it. At first, the students only had to work out the possible arrangements, as, in my experience, it often helps if they receive a task in easily understandable chunks.

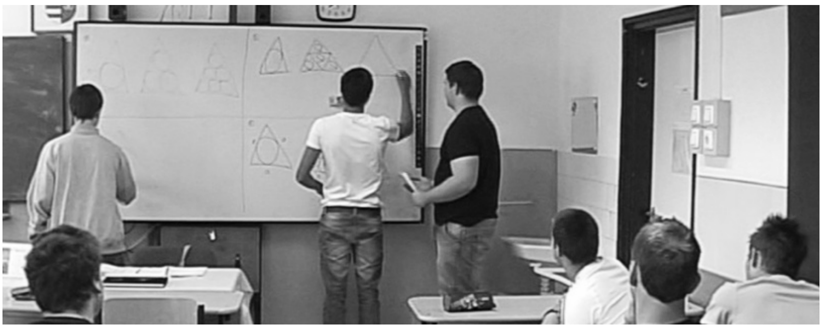


Figure. Students drawing their arrangements on the board.

The new problem

Now that we had some pictures to work with, the groups received the following questions regarding both arrangements (see picture above):

- How many congruent circles fit into each triangle? Do you notice a pattern?
- How many circles would be in the n^{th} triangle?
- If we cut out the circles, how much of the triangle would be wasted?

The students soon noticed that in the first arrangement the total number of circles can be calculated from the following pattern:

1   1, 2   1, 2, 3   1, 2, 3, 4

After each group had discovered this sequence, we paused for another whole-class discussion, as it was a perfect opportunity to introduce the concept of triangle numbers; furthermore, the students had noticed that discovering this pattern leads to yet another problem, i.e., how to find the sum of the first n natural numbers.[7]

Once the pattern of the circles was determined and the formula for the n^{th} triangle number[8] explained, the groups found a new obstacle. Clearly, this

7 At this stage, the students were neither familiar with the explicit definition of number sequences nor did they know the formula for the sum of the first n terms in a sequence.

8 The same as the sum of the first n natural numbers.

time a new method was needed for deriving the radius of the circle that was inscribed the equilateral triangle.[9] The groups therefore had to work together again using "Pairs Check". The same procedure as before was used. Following the pair work, the groups came together and compared their results, then the different ways of calculating the length of the radius were presented on the board divided into four sections.

Unfortunately, due to a lack of time, we were unable to proceed any further with the detailed analysis of the problem in class. The rest of the problem was therefore left to be discussed in group study sessions.

Evaluation

In general, the groups were able to work together quite effectively, as can be observed on the video recordings. The voice recordings provide further evidence that the communication in the different groups was efficient, although the students sometimes struggled to express their ideas with the correct mathematical terms. Even students who usually just sit and listen in class participated rather actively, as shy students were "forced" to communicate and express their ideas. There is one student in the class who is quite capable in mathematics but tends to display disruptive behaviour; however, in cooperative work he managed to control himself and contributed to the work of his team very well. It was also interesting to see that there was a so-called "silent group", in which the members restricted communication to the minimum necessary. Despite the fact that they did not talk as much, this group managed to proceed well with the solution of the task.

The voice recordings also demonstrate that students opened up and were braver than in previous lessons in terms of sharing their ideas with each other and asking questions. They did not mind being wrong or suggesting something that might sound unreasonable at first. Let us now review what the students said about cooperative learning.

Students' comments

"Working in groups is sometimes good, but sometimes I can't contribute to the discussions." K. A.

"It was helpful when K. D. had some ideas and I could carry on from his, but the behaviour of B. P. was often annoying." H. M.

"I learned a lot from my group mates." M. Sz.

9 Since the lessons were only 45 minutes long, we did not have an opportunity to check how to calculate the radii of the circles in the second, third, etc., arrangements. This was left for discussion in group study sessions.

"I enjoyed working in groups but we shouldn't do it this often. Not every lesson." P. R.

"It was easier to work together than alone, because we had people thinking differently and this helped a lot." K-M. M.

"Cooperative work brings the members of the groups closer, but I think frontal teaching is more effective." N. B.

"It could have been better if we had been allowed to choose who to work with." B. P.

From the teacher's perspective

Using cooperative teaching techniques changed my role as teacher. The bulk of the work was done before the lesson; planning cooperative lessons requires creativity from the teacher as well, which was sometimes challenging. In class, my task was simply to monitor the students and guide them towards the right solution. While my students were working on the given task, I was checking whether they were proceeding in the right direction, making sure that they did not misunderstand anything. If a group got stuck and the members could not help each other, it was my task to help them continue their work with useful comments or questions. This way of working gave me an opportunity to gain a better insight into how my students think, how they solve problems. I could help students who were slower while the others were busy working on their tasks.

Since the last phase of the experiment – completing the psychological questionnaires again and administering the delayed mathematics test – took place in June 2013, the complete analysis of the quantitative data is still in progress. In answering the research question, we must for now rely on the students' comments in their "reflection books", as well as the teacher's notes made during the lessons. Based on these two sources, we can say that cooperative learning:

- can be considered to be an effective tool for developing problem solving skills, as it (1) contributed to the development of students' individual thinking: as they became accustomed to cooperative work they required less and less assistance in solving problems; and (2) provided more opportunity for students to think creatively than frontal teaching: their "reflection books" show that they often came up with multiple solutions for a problem;
- should be used alongside and mixed with other methods: students' comments show that using cooperative techniques exclusively for an extended period of time is not the best approach.

Future work

This experiment is far from completed. There is, of course, a massive amount of data to be analysed: the recordings of the lessons, the reflection books and the pre- and post-tests. In addition, this particular class continued working with cooperative structures once every two weeks and they then completed further psychological and mathematical questionnaires at the end of the school year, which provided further data to work with. Based on the experience of these 12 lessons, a similar experiment will also be planned for another class.

Acknowledgements

This research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund, within the framework of TAMOP 4.2.4. A/2-11-1-2012-0001 "National Excellence Program".

References

Ambrus, A. (2004). *Bevezetés a matematika didaktikába.* (Introduction to Mathematical Didactics). Budapest: ELTE Eötvös kiadó.

Ambrus, G. (2000). "Nyitott" és "nyitható" feladatok a tanárképzésben és a matematikaoktatásban. ["Open" and "Openable" Problems in Teacher Training and Mathematics Education]. *Matematika tanítása, VIII*(1), 7–15.

Burns, M. (1990). Using Groups of Four. In N. Davidson (Ed.), *Cooperative-learning in Mathematics* (pp. 21–46). Boston: Addison-Wesley Publishing Company.

Capacity Building Series. (2011). Asking Effective Questions. Retrieved 13 January 2013 from http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_AskingEffectiveQuestions.pdf

Crabill, C. D. (1990). Small-group Learning in the Secondary Mathematics Classroom. In N. Davidson (Ed.), *Cooperative-learning in Mathematics* (pp. 201–227). Boston: Addison-Wesley Publishing Company.

Dees, R. L. (1990). Cooperation in the Mathematics Classroom: A User's Manual. In N. Davidson (Ed.), *Cooperative-learning in Mathematics* (pp. 160–200). Boston: Addison-Wesley Publishing Company.

Fisher, R. (2005). *Teaching Children to Think.* Cheltenham: Nelson Thornes.

Gardner, M. (1988). *Riddles of the Sphinx.* Washington D. C.: The Mathematical Association of America.

Hähkiöniemi, M., Leppäaho, H., & Francisco, J. (2012). Model for teacher assisted technology enriched open problem solving. In T. Bergqvist (Ed.), *Learning Problem Solving and Learning Through Problem Solving, proceedings from the 13th ProMath conference, September 2011* (pp. 30–43). Umeå: UMERC.

Johnson, R. T., & Johnson, D. W. (1994). An Overview of Cooperative Learning. In J. Thousand., A. Villa, & A. Nevin (Eds.), *Creativity and Collaborative Learning*. Baltimore: Brookes Press.

Johnson, D. W., & Johnson, R. T. (2009). An Educational Psychology Success Story: Social Interdependence Theory and Cooperative Learning. *Educational Research, 38*(5), 365–379.

Józsa, K., & Székely, Gy. (2004). Kísérlet a kooperatív tanulás alkalmazására a matematika tanítása során. [Experiment for Using Cooperative Learning in Teaching Mathematics]. *Magyar pedagógia, 104*(3), 339–362.

Kagan, S. (2003). A Brief History of Kagan Structures. *Kagan Online Magazine.* San Clemente, CA: Kagan Publishing.

Kagan, S. (2004). *Kooperatív tanulás.* [Cooperative Learning]. Budapest: Önkonet.

Kagan, S., & Kagan, M. (1998). *Multiple Intelligences: the Complete MI Book*. San Cemente, CA: Kagan Publishing.

Kilpatrick, J. (1987). Problem Formulating: Where do Good Problems Come From? In A. H. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education.* New Jersey: Lawrence Erlbaum Associates Inc.

Kirkley, J. (2003). Principles for Teaching Problem Solving. Retrieved February 20 2012 from: http://citeseerx.ist.psu.edu/viewdoc/downloaddoi=10.1.1.117.8503&rep=rep1&type=pdf

Koshy, V. (2005). *Action Research for Improving Practice*. London: Paul Chapman Publishing.

Lénárd, F. (1987). *A problémamegoldó gondolkodás.* [Problem Solving Thinking]. Budapest: Akadémia kiadó.

Mayer, R. E., & Hegarty, M. (1996). The Process of Understanding Mathematical Problems. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The Nature of Mathematical Thinking.* New Jersey: Lawrence Erlbaum Associates Inc. More Beads. Retrieved August 15 2012 from *http://nrich.maths.org/2048*

Pehkonen, E. (1997). Use of problem fields as a method for educational change. In E. Pehkonen (Ed.), *Use of open-ended problems in mathematics classroom* (pp. 73–84). Helsinki: University of Helsinki.

Pehkonen, E. (1999). Open-ended Problems: A Method for an Educational Change. In *International Symposium on Elementary Maths Teaching (SEMT 99).* Prague: Charles University.

Pólya, Gy. (1962). *Mathematical Discovery.* New York: John Wiley & Sons Inc.

Pólya, Gy. (1973). *How to Solve it.* New Jersey: Princeton University Press.

Slavin, R. E. (1996). Research for the Future. Research on Cooperative Learning and Achievement: What We Know, What We Need to Know. *Contemporary Educational Psychology, 21*, 43–69.

Schoenfeld, A. H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense-making in Mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334–370). New York: MacMillan.

Tóth, L. (2007). *Pszichológiai vizsgálati módszerek a tanulók megismeréséhez.* [Psychological Methods for Getting to Know the Students]. Debrecen: Pedellus.

Way, J. (s.d.). Problem Solving: Opening up Problems. Retrieved October 21 2013 from http://nrich.maths.org/2471

Zimmermann, B. (1986). From Problem Solving to Problem Finding in Mathematics Education. In

P. Kupari (Ed.), *Mathematics Education Research in Finland, Yearbook 1985* (pp. 81–103). Jyväskylä: Institute for Educational Research.

Zimmermann, B. (2009). "Open ended Problem Solving in Mathematics Instruction and some Perspectives on Research Questions" revisited – New Bricks from the Wall. In A. Ambrus & É. Vásárhelyi (Eds.), *Problem Solving in Mathematics Education,* proceedings from the 11th ProMath conference, September 2009 (pp. 143–157). Budapest: ELTE.

Biographical note

Krisztina Barczi is a PhD student at the University of Debrecen in Hungary and a secondary school Mathematics teacher at Janos Neumann Secondary School where she has been teaching for 4 years. She started taking part in education related research as a university student (Krygowska Project of Professional Development of Teacher-Researchers) and continued her work in different British secondary schools as a Mathematics teacher where she carried out further research through the University of Cambridge (HertsCam Network) on motivating low achievers in Mathematics. Presently her field of interest is applying cooperative teaching techniques for mathematical problem solving. The article *Applying cooperative techniques in teaching problem solving* summarizes part of an action research related to this topic.

Appendix

1. *Matchstick game*: Two players and 27 matchsticks are needed. The two players take turns and remove 1, 2 or 3 matchsticks. The winner is the one who removes the last matchstick. Task for the groups: to find a winning strategy for both players (Ambrus, 2004).

2. *Number magic*: In this problem field, the individual problems are related to simple number tricks that can be explained using number theory. For example: type the number 15,873 into your calculator. Select a number from 1 to 9 and multiply 15,873 by that number. Now multiply the product by 7. What do you notice? Try with different digits. Can you explain what is going on? (Gardner, 1988).

3. *Area investigation*: From a square measuring 60 cm x 60 cm we cut out circles as you can see on the figure. What percentage of the square is wasted in each case? Do you notice a pattern? Can you generalise your idea? Can you prove your conjecture for n circles?

4. *More beads*: Three beads are threaded on a circular wire and are coloured either red or blue. You repeat the following actions over and over again: between any two beads of the same colour place a red bead, and between any two beads of different colours put a blue bead, then remove the original beads. Discuss all of the possible outcomes. What happens when you do the same thing with 4, 5 or 6 beads? (nrich)

5. *Primes and factors*: This problem field contains algebraic problems that can be solved using factorisation, special products and other algebraic modifications. For example: Think of a two-digit number. Reverse its digits to obtain a new number and subtract the smaller number from the bigger one. Can you get a prime number as the result? Why/Why not? Can you prove that it is impossible to get a prime number? What if you use three digit numbers? Four digit numbers? N digit numbers? (nrich)

Improving Problem-Solving Skills with the Help of Plane-Space Analogies

László Budai[1]

We live our lives in three-dimensional space and encounter geometrical problems (equipment instructions, maps, etc.) every day. Yet there are not sufficient opportunities for high school students to learn geometry. New teaching methods can help remedy this. Specifically our experience indicates that there is great promise for use of geometry programs, GeoGebra and DGS, combined with plane space analogies for the development of spatial thinking and problem-solving skills in the three dimensions of solid geometry.

Keywords: Problem solving; Plane-space analogies; GeoGebra; Teaching; Secondary school

1 Budapest Business School, University of Applied Sciences, College of International Management and Business, Institute of Business Teacher Training and Pedagogy, Hungary; budai0912@gmail.com

Izboljšanje sposobnosti reševanja problemov s pomočjo prostorsko-ravninske analogije

László Budai

Živimo v tridimenzionalnem prostoru in se dnevno srečujemo z geometrijskimi problemi (navodila za uporabo različne opreme, zemljevidi idr.). Po drugi strani pa učenci nimajo veliko priložnosti za učenje tovrstnih vsebin v šoli. Z oblikovanjem novih učnih pristopov pa situacijo lahko bistveno izboljšamo. V prispevku prikazujemo svoje izkušnje, ki potrjujejo pomembno vlogo geometrijskih programov GeoGebra in DGS skupaj z razvijanjem prostorsko-ravninske analogije pri razvijanju prostorske predstavljivosti in reševanju problemov iz prostorske geometrije pri učencih.

Ključne besede: reševanje problemov, prostorsko-ravninska analogija, GeoGebra, DGS, poučevanje, srednješolsko izobraževanje

Introduction

Spatial abilities are so important in modern life that several research projects addressed evaluation of these abilities in students (Gorska & Cižmešija, 2007; Hoffmann & Németh, 2007; Milin-Šipuš, 2012; Nagy-Kondor, 2007; Nagy-Kondor, 2012).

Cultural, social, technological and mathematical-didactical changes over the past ten years have modified the benchmarks for development of spatial perception. New teaching methods use analogies/analogues to solve several of these problems together and understand the relationships between different sets of problems (Lénárd, 1978; Pólya, 1988; Pólya, 1989).

Analogues refer to the similarity and parity between things. In geometry, we may talk about analogies between objects, theorems and problems, as well as between problem solving, proof methods and processes.

Here we deal with analogies involving the spatial (three-dimensional) generalisation of plane geometrical theorems (for other interpretations see McGee, 1979 or Nagy, 2000). Using analogy we can generalise plane geometric theorems in the plane itself and also spatial geometric theorems in three dimensions.

The examples below consider properties that are the similar for a plane object and its spatial generalisation and so establish an analogy.

The circle and the sphere are both a set of points at a given distance (the radius) from a central point, the circle in a plane and the sphere in 3 dimensions. The tetrahedron in three dimensions is a generalization of the triangle in two dimensions and the polyhedron in three dimensions is analogous to the polygon in two dimensions. Because the sides of a square are equal length sections any of its two adjacent sheets give a right triangle. The sheets of its spatial equivalent, the cube, are congruent squares, and any of its two adjacent sheets also create a right triangle. The perpendicular bisector of the segment in the plane and the perpendicular bisector of the segment plane in space are a set of points that are equidistant from the two end points of the section.

A plane geometrical object or two dimensional theorem can have two or more analogues in 3D-space: the tetrahedron can be considered to be the analogue of the triangle as well as of the three-sided tetrahedron and the spherical triangle. Many theorems can be applied to them that are also true for the triangle. Furthermore, we can interpret the triangle not only on the spherical surface, but also on other surfaces (e.g. non-Euclidean geometries).

From these few examples, we can see how to create analogies in different ways in geometry, and many analogies can be found in secondary school material. However more time in the secondary school curriculum is needed for mathematics including concepts such as analogues. Table 1 contains the average secondary school class frame numbers in a year broken down into grades.

Table 1. *Number of annual mathematics classes in particular secondary school grades.*

The degree of education	High school education			
	9.	10.	11.	12.
Total number of mathematics class hours per school year (average)	108	108	108	108
Total number of geometry class hours per school year	39	59	45	35
Number of three dimensional type class hours per school year	3	6	4	21
Percentage rate of three dimensional geometry type class hours compared to the total class hours	2,7%	5,4%	3,7%	19,5%

Many students first encounter the pyramid in mathematics classes at the age of 18 and yet despite this slow pace of learning, the new basic educational curriculum in Hungary starting September 2013 unfortunately further reduces the time for mathematics classes.

Plane-space analogies in public secondary education

The following discussions of analogies would be valuable within the framework of the mathematics curriculum:

- definition of the concept of planar and spatial objects, their mutual position and their distance (for example, the distance of two straight lines and the distance of two planes or the distance of two lines not in the same plane),
- geometric transformations in the plane and space,
- loci (e.g., the perpendicular bisector of a segment in the plane and in space, circle, sphere),
- application of angle functions in two dimensions and three dimensions (triangle and spherical triangle).

Let us examine some specific examples that might be considered in mathematics classes.

Geometric basic insertion concepts

The basic insertion concepts of point, straight line, plane and space are the basis developing geometrical concepts, theorems and definitions.

The straight line, for example, acts analogously in the plane to how the plane acts in space (it is worth identifying and discussing the features that make up the analogy together with the students). Thus, we can conclude that the analogue of the straight line in space is the plane. We can also provide students with further formulation of theorems and definitions. For example, two straight lines intersect if they have exactly one mutual point, or two planes are intersecting if they have exactly one mutual straight line. The analogy concept here is the idea of overlapping points.

A further example of analogising a basic geometrical concept is the interpretation of distance. By the distance between two parallel lines, we mean the distance from one arbitrary point of the straight line to the other straight line. In the case of the distance between two parallel planes we mean the distance from one arbitrary point of the plane to the other plane. Here we formulate the analogy by extending the concept of distance.

The axis-mirroring concept is a similar example of geometric transformation. A geometric transformation is called axis mirroring when every point of a given straight line t is self-mapped and it assigns the point P' of the plane to every other point P in such a way that the perpendicular bisector PP' of the segment is precisely the t axis. The geometric transformation is called mirroring to the plane when every point of a given plane S is self-mapped and point P' is assigned to every other spatial point P so that the section PP' would be perpendicularly bisected by the plane S.

The determination of geometrical locations in the plane and in space is the same as we have seen previously. Here again it is worth having the students formulate the question of the spatial analogue. Let us begin with a simple example:

Teacher: What is the geometrical location of those points in the plane that are equidistant from a given point?

Student: A circle.

Teacher: Now, define the spatial analogue of the question!

Student: What is the geometrical location of those points in space that are equidistant from a given point?

Teacher: That is right. What can that object be?

Student: A sphere.

The student needs to think logically to answer these questions and even more carefully in the case of defining the analogues of more difficult geometrical locations: What is the geometrical location of those points in the plane that are equidistant from a given straight line? (A parallel straight-line pair). What is the geometrical location of those points in space that are equidistant from a given straight line? (An infinite right circular cylinder).

Subdivision of 2-D and 3-D space

One of the most difficult typical problems that can occur in the classroom is related to the subdivision of two and three dimensional space. These problems can be discussed together. The question is: A maximum of how many sections are created in the plane by *n* number of straight lines? The spatial analogue formulation in this case is: A maximum how many sections are created in space by *n* number of planes? The formulation of the problem itself is not difficult, but the students rarely succeed in finding a solution, especially in the case of using an analogue. An outline version of the deduction of the problem is shown in Figure 1.

The division of planes with straight lines	The division of three dimensions with planes
s(1)=s(0)+1 s(2)=s(1)+2 s(3)=s(2)+3 ... s(n)=s(n-1)+n s(n+1)=s(n)+n+1 Adding together the appropriate sides of the above equation than sorted we will get: $s(n)=\frac{1}{2}(n^2+n+2)$ A more easy to remember form: $s(n)=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}$	t(1)=t(0)+s(0) t(2)=t(1)+s(1) ... t(n)=t(n-1)+s(n-1) So t(n)=t(0)+s(0)+s(1)+...+s(n-1) Therefore $t(n)=\frac{1}{6}(n^3+5n+6)$ A more easy to remember form: $t(n)=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$

Figure 1. Subdivision of the plane and space.

Out of 47 students, there was only one who solved the spatial analogue problem successfully.

Textbooks (with some exemplary exceptions) do not even refer to spatial analogies, thus denying students an opportunity to understand analogues or read about them in mathematics classes. More talented students are forced into the background and their thought and abilities are suppressed. A spatial skill development study group session was introduced in the 2012–2013 school year. It took place once a week, resulting in total 36 study hours. Out of these 36 hours, plane-spatial analogies were taught for 16 hours, with an average of 10–12 students participating in each session. A qualitative-type test was performed with reference to these 16 hours. This test evaluated the attitude of the students about the topic and included an examination of affective psychosomatic factors. It also included the evolution of the rate of motivation, continuous observation of the students, student interviews and questionnaires, the attitudes of the participants to the problems, and the opinions of the students at the end of the session (regarding the change of approach and its development).

Let us examine the analogues that occurred in the study group sessions.

Problems occurring at the study group sessions

I believe that the following two dimensional analogies can be considered in study group sessions based on secondary school knowledge:

- triangle-tetrahedron,
- circle-sphere,
- parallelogram-parallelepiped (square-cube, rectangle-cuboid),
- trapeze, triangular-based truncated pyramid,
- coordinate geometric analogies,
- analogue entry and paraphrasing problems,
- analogue extreme value problems.

The scope of the present paper does not permit a full presentation of the study group material; instead, some analogies are illustrated in detail with student reactions and didactical comments. The analogies selected are those that would be the most interesting for students and that have a stronger link to standard mathematical material.

A possible analogue of the cosine theorem

The triangle-tetrahedron analogue is a huge and extensive topic, and is alone sufficient to occupy an entire study group session. Table 2 shows the related theorems and correlations in detail.

Table 2. *Triangle and tetrahedron analogies.*

General triangles, tetrahedrons	Right triangle, tetrahedron
3 bi-sectoral-6 page bi-sectoral	First theorem of Euclide
Inscribed circle inscribed sphere	Height theorem
3 side perpendicular bisector-6-edge and section midpoint perpendicular bisector, two dimensional object	Pythagorean theorem
Added circle - added sphere	Euler-straight line
Radius formulas of circles-radius formulas of sphere	Feurbach circle-Feurbach-sphere
Sine theorem	
Cosine theorem	

The average secondary school students were especially interested in the Pythagorean Theorem, the height theorem, the leg theorem, or the cosine and sine theorem. Let us examine the analogue of the cosine theorem in detail followed by some of the students' opinions regarding the theorem.

The cosine theorem related to the triangle is:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$,

where a, b and c are the length of the sides, and α is the angle opposite the side.

Let us consider two different types of proof, one algebraic and the other geometrical. It is important to show the students that we can approach the problem in a number of ways, all of which can lead to the correct solution, irrespective of the method used.

According to the algebraic proof, the following statements are true for any kind of triangle (acute angle, obtuse angle, right angle):

$a = b\cos\gamma + c\cos\beta$,

$b = a\cos\gamma + c\cos\alpha$,

$c = b\cos\alpha + a\cos\beta$.

Multiplying the first equation with a, the second with b and the third with c, and adding the resulting equations together, we arrive at the solution.

Based on the vector-type proof, let us direct a, b and c side vectors such that we rotate around the triangle counter clockwise. In this case:

$a + b + c = 0$,

and from this we get:

$a = -(b + c)$.

If we then square both sides, i.e., self-multiply it in *a* scalar way, we get:
$a^2 = b^2 + 2bc + c^2$.

It should be known that the square of *a* vector is equal to its absolute squared value. This follows from the scalar product definition. If we draw *a*, *b* and *c* vectors in line with their direction but from their mutual starting point, we can see that their angle of inclination is not α, but 180° α, thus the *a*, *b* and *c* vector's scalar product is:
$bc = bc\ cos(180° - \alpha) = -bc\ cos\alpha$.

In the study group session, we developed the formulation and proof of the spatial analogue of the problem just mentioned. We had sufficient prior knowledge to be aware that in the special case of α = 90°, i.e., in case of right triangle, the cosine theorem is provided by the Pythagorean Theorem; superficially, the cosine theorem can also be considered as a generalisation of the Pythagorean Theorem. We started from this point with the students, and they therefore had certain ideas about what the three-dimensional Pythagorean Theorem would look like.

Some of the ideas put forward by the students for the three-dimensional shape of the Pythagorean Theorem were:
$a^3 + b^3 = c^3$,
$a^3 + b^3 + c^3 = d^3$.

The first student could to try this problem not explain what *a*, *b* and *c* might indicate; and thought they could be the length of the edges. The second student got somewhat closer to the truth by stating that the letters could mean the areas of the 1-1 sheet of the tetrahedron. All 12 students agreed that in the three-dimensional Pythagorean Theorem the side variables *a*, *b* and *c* are raised to cube power. Let us look at the deduction:

Consider the generalisation of the cosine theorem related to the tetrahedron. If we mark the sheet areas of the tetrahedron with *ti* (i = 1, 2, 3, 4), and if we mark the plane angle created by *ti* and *tj* sheet areas with α_{ij} (i,j = 1, 2, 3, 4; $i \neq j$), we get:
$t_1^2 = t_2^2 + t_3^2 + t_4^2 - 2\ t_2 t_3\ cos\alpha_{23} - 2\ t_2 t_4\ cos\alpha_{24} - 2\ t_3 t_4\ cos\alpha_{34}$.
If the occurring angles are 90°, we get the following correlation:
$t_1^2 = t_2^2 + t_3^2 + t_4^2$.

This is the Pythagorean Theorem related to the right angle tetrahedron.

It was surprising to the students that the degree number of the formula remains 2 despite the spatial extension. They understood this better when considering the proof below:

Let us project perpendicularly the further three sheets to the plane of each tetrahedron sheet and determine the correlation between 1-1 sheet area and the area of the other sheets' incidental projection. This way, we get the following correlation:

$$t_1 = t_2\, cos\alpha_{12} + t_3\, cos\alpha_{13} + t_4\, cos\alpha_{14},$$
$$t_2 = t_1\, cos\alpha_{12} + t_3\, cos\alpha_{23} + t_4\, cos\alpha_{24},$$
$$t_3 = t_1\, cos\alpha_{13} + t_2\, cos\alpha_{23} + t_4\, cos\alpha_{34},$$
$$t_4 = t_1\, cos\alpha_{14} + t_2\, cos\alpha_{24} + t_3\, cos\alpha_{34}.$$

We can easily check that these correlations are true in each case, even if there are obtuse and right angles among the plane angles of the tetrahedron. If the above equation is multiplied by t_1-, $(-t_2)$-, $(-t_3)$- and $(-t_4)$ respectively, adding the given equations together we arrive at the correct formula.

In accordance with the plane theorem, the proof with vectors works here as well. We have seen that the sum of the side length vectors of the triangle is a null vector. The spatial equivalent of this is the following statement: The sum of the sheet area vectors of the tetrahedron is a null vector. A sheet area vector belongs to every sheet of the tetrahedron. This is a vector whose size is equal to the area of the sheet and whose direction is perpendicular to the sheet pointing outwards. From this principle, we will deduce the cosine theorem of the tetrahedron.

After the deduction, the students judged the formula to be logical and also understood why we do not have to increase the number of degrees in the exponents. The next step was the collection and solution of specific analogue problem pairs. The students were more successful in this than in the theoretical background deduction.

Circle-sphere analogies

From the circle-sphere analogues, the following can come into the picture at the study group session:

- similarity points,
- Apollonius-type problems,
- inversions,
- the power of a point concerning circles and spheres,
- lines of circles, lines of spheres,
- circle crowds with one parameter and their spatial analogue.

The most familiar of these is the inversion. The fact that there are rather "strange" geometric transformations could also be interesting to students. The interpretation of the inversion is: The inversion maps regarding a circle with centre *O* and radius *r* so that in the plane it transfers point *P* differing from any point *O* of the plane to point P' on the half-straight line of *OP*, and for which the following is true $OP \cdot OP' = r^2$.

Using a couple of examples, let us analyse the concrete construction method, and then let the students formulate the definition of the spatial analogue based on the concept of an inverted plane Here is an example of the correct wording of a student: The inversion related to a sphere with centre *O* and radius *r* maps so as to transfer point P differing from any point O in space to point *P'* on the *OP'* half line. For this statement, the equation $OP \cdot OP' = r^2$ is true.

The formulation of the spatial analogue is very clear and occurred naturally to the students. The representation of the exact problem was performed with the help of GeoGebra (discussed in more detail in the next section).

After the constructions and discussions for the different objects and locations are made in GeoGebra, the students can easily compose a few theorems related to the inversion:

T1: the straight line going through the pole (plane) is the inverse of itself,

T2: the straight line not going through the pole (plane)-its inverse is the circle going through the pole (sphere),

T3: the circle going through the pole (sphere)-its inverse is the straight line not going through the pole (plane),

T4: the circle not going through the pole (sphere)-its inverse is the circle not going through the pole (sphere),

T5: there are many such circles (spheres) for which the inverses are themselves.

After this, we discussed the inversion from the perspective of coordinate geometry. The plane theorem is as follows: If the equation of the plane inversion of the base circle is $x^2 + y^2 = 1$, then the inverse of the arbitrary point $P(x,y)$ differing from the plane pole, is:

$$P'\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

The three-dimensional analogue formulation can be considered as a problem of medium difficulty. 7 of the 12 students were able to formulate correct spatial analogue theorems. A perfect formulation by a student is as follows: If the equation of a base circle's spatial inversion is $x^2 + y^2 + z^2 = 1$, then the inverse of the arbitrary point $P(x, y, z)$ differing from the pole is:

$$P'\left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right).$$

In our experience, the degree of the exponents does not necessarily change when setting up the plane-spatial (two dimensional) analogue.

The students participating in the study group sessions really liked the inversion; they found it extraordinary compared to the familiar geometric transformations. They prepared and collected several theorem pairs to be studied at home. One student said the following about the inversion: "It is fantastic that we can close the whole world into a circle/sphere."

Parallelepiped and parallelogram analogues

The parallelepiped and parallelogram analogues can be very colourful; we can even consider special cases originating from the definitions, such as square-cube or rectangular-cuboid. In the case of the formulation of the theorems, we aimed for the more ordinary features, (*p* marks the parallelogram and *P* the parallelepiped):

T1: *p* and *P* are centrally symmetric,

T2: any two opposite sides of *p* are equal length, any two opposite sheets of *P* are congruent,

T3: *p*'s and *P*'s centre of gravity and their centre of symmetry coincide,

T4: the opposite angles of *p* are equal, *P*'s opposite plane angles are equal,

T5: any straight line going through the symmetry centre (plane) splits the area of *p* (the volume of *P*) in half.

The students were able to understand the above theorems. However, if problems are encountered it is worth using the aforementioned special cases, thus gradually introducing the use of analogue theorem pairs. Let us consider the following problem in the plane. The ratio of the sides of a rectangle is 2:1, and the ratio of numbers for its circumference and area are equal. How large are the sides of the rectangle?

Teacher: What could the problem pair of the spatial analogue be? Try to formulate it.

Student's train of thought:

rectangle→ cuboid

the ratio of the sheets is 1 : 2 and the → ratio of the edges is 1 : 2 : 3,

circumference → surface,

area → volume.

This is an example of the independent creation of an analogue problem pair by a student. Quite rightly, he placed every concept in the problem into the next higher dimension, thus reaching a spatial analogue. This type of problem aims to develop an important competence for finding similar problems.

Coordinate geometry

Let us examine a typical classroom example from the topic of coordinate geometry: an equation of a straight line. Below are the analogues of the direction vectored equation for the planar and spatial straight line:

- The plane equation of the straight line that goes through point $P_0(x_0,y_0)$, with a direction vector $v(v_1,v_2)$ is:
  $v2(x\text{-}x_0) = v_1(y\text{-}y_0)$
- The equation system for the spatial straight line with a direction vector $v(v_1,v_2,v_3)$, going through point $P_0(x_0,y_0,z_0)$ is:
  $V_2(x\text{-}x_0) = v_1(y\text{-}y_0)$,
  $V_3(y\text{-}y_0) = v_2(z\text{-}z_0)$.

After a discussion of the theory, the solution of a specific problem was considered. What is the equation of the straight line that goes through point (3,-2) and is perpendicular to the straight line with the equation $5x + 6y + 1$?

This problem is a typical classroom problem required for graduation. Let us place this problem into space, and then the task of the students is to find a spatial analogue problem (What is the equation of the straight line that goes through point (3, -2, 1) and is perpendicular to the straight line with the equation?) and to work out its solution based on the experience already gathered.

Extreme value

Finally, let us examine a problem related to extreme value calculations. This is a typical example of how analogical thinking can play a role in the solution of a problem, specifically:

- setting up and solving a simpler analogue problem,
- showing how the more difficult problem can be solved if the sample problem is reshaped to a certain degree.

Problem: Let us fix the base of a regular three-sided pyramid and change its height m. Let us select the value of m such that the radius of a sphere drawn around the pyramid would be the smallest possible!

This problem was solved in one of the later study sessions when the students had already acquired insight into analogue based problem solving. At first, they looked for a simpler solution, i.e., the analogue of the original problem in two dimensions. After having solved this and discussed the solution, they returned to the original problem and applied the related (spatial) theorems, knowledge based on the problem solved in the plane (two dimensions).

A possible formulation of a simple plane problem by a student is: Let us fix the base of an isosceles triangle and change its height *m*! What size of m will produce the smallest possible radius of the circle drawn around the triangle?

In summary, we discuss a plane problem with the students and then present a spatial analogue of that problem with a variety of proof methods. After the similar properties are found in the plane and space analogues, the students themselves search independently for further similar properties. Then there is a discussion of the concrete problem related to the plane and the spatial analogies, followed by the collection and elaboration of the analogue problem pairs.

Development using GeoGebra

Several recent research studies indicate that we can achieve better results in mathematics classes with the use of GDS than with traditional mathematics tools. Version 4.2 of GeoGebra does not include the display of spatial objects, but after defining one's own base system it is possible to display such objects. Version 5.0 Beta is still undergoing significant development, and under certain conditions its operation may therefore be unstable. Taking all of this into consideration, the best solution is to become familiar with both versions.

Let us briefly examine some examples, without attempting to be comprehensive.

The simplest approach is to demonstrate the conventional plane geometric transformation's spatial analogue (Figure 2).

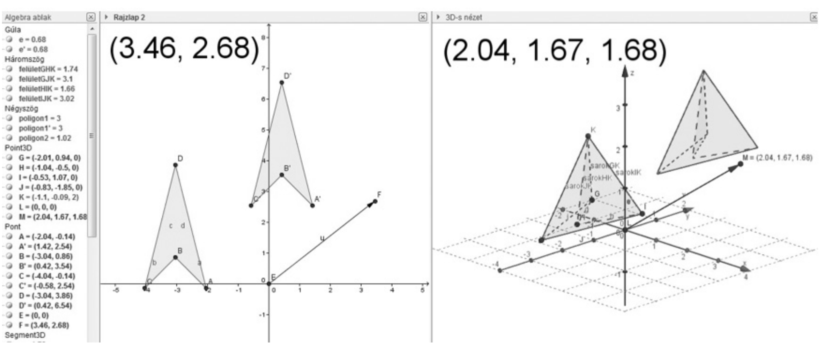


Figure 2. Analogue of plane and space translation.

GeoGebra offers many possibilities, allowing the user to work in more than one window at the same time. The algebra window contains the equations of the selected objects and both a plane worksheet and a spatial worksheet are displayed. We can work in these three windows simultaneously, and if we change a feature in one window it dynamically changes in the other windows, although it is also possible to disable this feature. Another very important feature is that we can choose from a variety of projections; for example, we can select how we would like to display the spatial object in a plane (parallel projection, axonometries), thus broadening the approach related to the spatial skills of students.

We can modify any features interactively and dynamically, even in the case of translation (Figure 2). In addition to working in the classroom, it can be very useful to develop a self-learning environment in which the student him/herself can discover correlations. The coordinates may be written out, which is useful for those who find it easier to understand or associate in this way.

The presentation of proofs is very valuable for classroom use. For example:

$(a + b)^2 = a^2 + 2\ ab + b^2$ and

$(a + b)^2 = a^2 + 3\ a^2b + 3\ ab^2 + b^2$.

Students always have problems with the sameness of algebraic equations; for example, the double product is often left out. Instead of mechanically memorising the formula, they see graphic evidence dynamically, enabling them to understand the origin of the double product. For spatial cases, this exists exponentially (Figure 3).

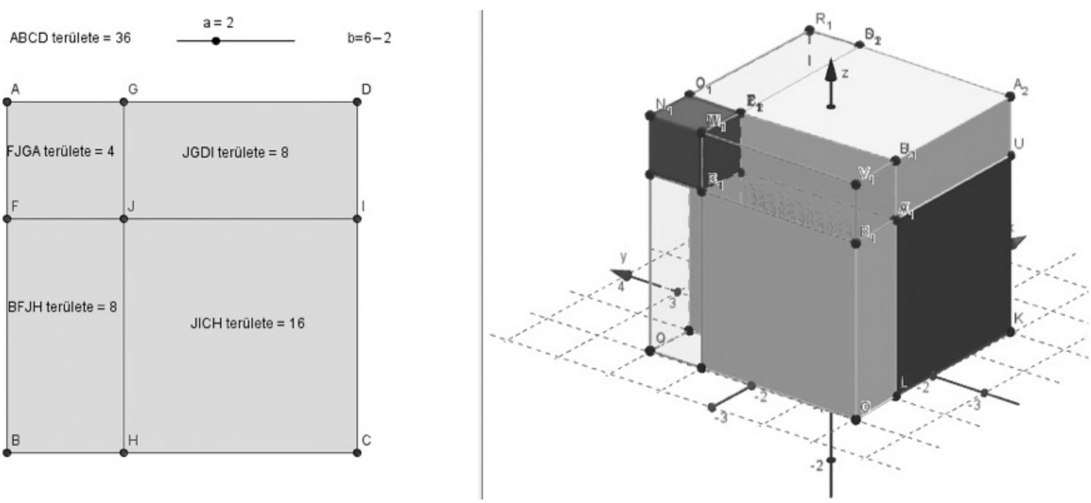


Figure 3. Algebraic proof of sameness in plane and space.

Students can set up the subdivisions dynamically, i.e., the value of *a* and *b*. By adjusting these parameters to the levels of volume and area extent, they can then also read them dynamically. After the subdivisions have been made, we can see what kinds of shapes were created and how it generates the whole square/cube. Remembering the formula was much easier for students this way.

The first major challenge of the study group session was to discover the cosine theorem's analogue. As we saw in the previous section, it is worth considering a special case, namely the Pythagorean Theorem (Figure 4).

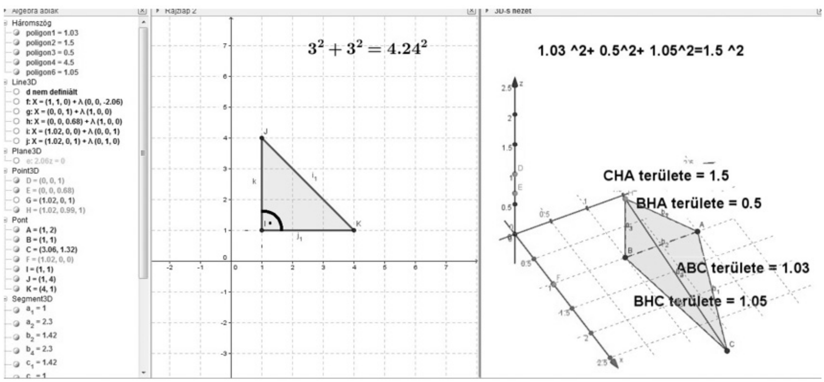


Figure 4. Plane and space analogues of the Pythagoras theorem.

We can see in Figure 4 that the correlation is given by the squared sum of the appropriate areas. The students can attempt the correlation even in the case of more values by moving the vertexes with the help of the dynamic figure. They can even determine the equation of the spatial analogue independently. By moving the respective vertex point, a general tetrahedron can be obtained instead of a right triangle tetrahedron. This allows the cosine theorem correlations to be derived, a problem more complicated for students.

The students participating in the study group sessions particularly liked the inversion, since they had not met such transformations directly in their everyday lives. Figure 5 shows a possible GeoGebric adaptation of this transformation.

The inversion in the plane exists as a built-in option in GeoGebra 4.2. If we extend this, we have an opportunity to present the spatial inversion to students, a task that would be very cumbersome using traditional spatial tools. The analogue features are clearly visible in the figure and the students can independently read the major theorems. While using the worksheet, the plane and spatial geometric windows were connected; therefore, if we move the straight line of the plane figure, the plane of the spatial figure moves at the same rate. Of course, it is also possible to handle the worksheets separately.

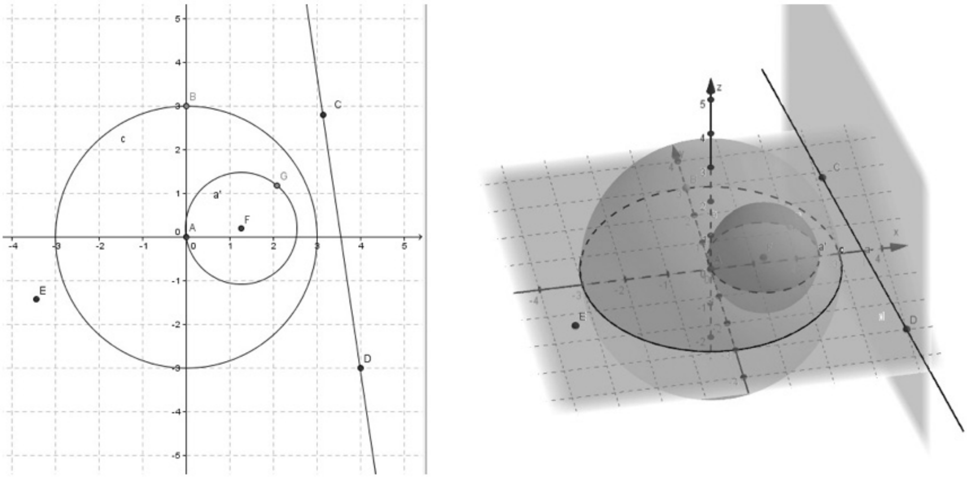


Figure 5. Plane and spatial analogues of inversions.

The plane and spatial analogues of coordinate geometry can be presented richly with the help of GeoGebra, since part of the concept of GeoGebra is to show the algebraic and geometric view of the objects in parallel. Several possibilities may arise here: the extension of the coordinates of points to spatial cases, the equations of straight lines in the plane and space, the equations of circles in the plane and space, the equation of a sphere, equations of surface areas and curves and the determination of intersections (just one click in GeoGebra!).

Last but not the least, let us examine the calculation of the extreme value. Here the students may apply GeoGebra very effectively, as they have an opportunity to quickly prepare the figures that suit the different conditions and then formulate the conjecture (Figure 6).

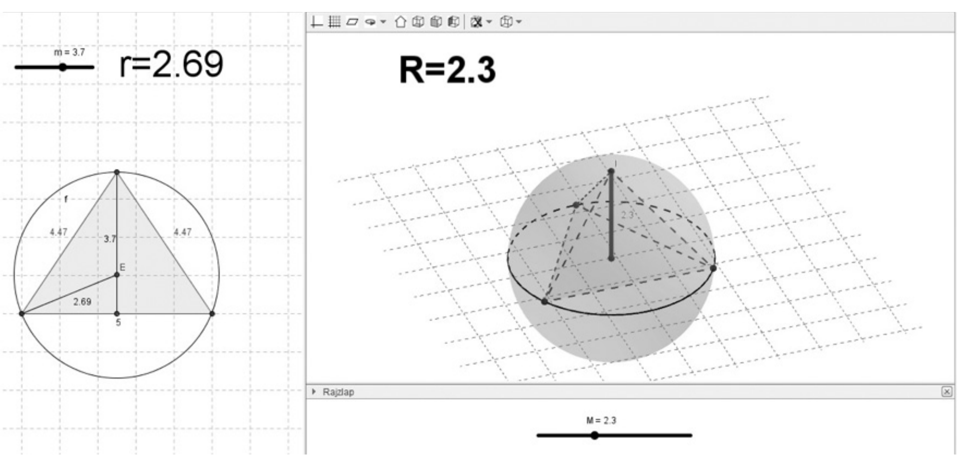


Figure 6. Extreme value calculation, example of searching for and solving simple analogue problems.

In this case, it is worth separating the worksheets from each other, i.e., the planar and spatial analogue problems should be moved separately.

Using GeoGebra to solve the 2-D analogue of the previous problem he students realized that the radius of a circle around the triangle is smallest when its diameter is equal to the base of the triangle, i.e., $r=a/2$ and then $m=a/2$.

The students could then return to solving the more difficult analogue problem, and again with the help of GeoGebra formulate their conjectures. The sphere drawn around each and every joint based pyramid goes through the vertexes of the joint base sheet, i.e., it fits on the circle drawn around the joint base sheet.

Of these spheres, the smallest is the one whose main circle is the circle drawn around the base sheet, i.e., $r=(a\sqrt{3})/2 \cdot 2/3 = (a\sqrt{3})/3$, and the height is also this length.

By their own admission, GeoGebra helped the students a great deal in solving and analysing the problems. After finding the basic ideas necessary to solve the problems, the search for the analogue problem pairs was simpler. The students successfully used dynamic sketch drawings in GeoGebra and were able to analyse and discuss solutions to the problems.

Conclusion

In my experience based on the responses of the students who participated in the study group sessions the discussion of analogues provides an excellent opportunity to develop problem-solving skills. The analogue mindset helps students handle the problem with a different approach. Presenting the proofs may also enable students to absorb different proof strategies. Studying analogue problem pairs, developing new pairs, and solving them promotes understanding and use of analogues. In the case of more difficult problems, the approach can be further facilitated by solving a simpler but similar type of problem first and then returning to the original more difficult problem.

In summary, the following observations were made at the study group sessions.

The students who participated in the study group sessions could do the following well:

- formulating spatial analogue problems and theorems,
- stating attributes of plane objects that are related to their spatial analogue,
- solving problems using the knowledge related to spatial analogues,
- choosing the suitable supplementary tools for solving the 1-1 analogue problem.

The students who participated in the study group sessions could not do the following very well:

- formulating notable triangle theorem analogues independently,
- proving spatial theorem analogues based on plane theorems (even if they knew the basic ideas).

Based on the qualitative-type surveys, the study group sessions were useful for the participants. The wide range of possible uses of analogues may help students to develop their cognitive operation and creativity, encouraging:

- independent ideas for possible analogues,
- increased student motivation for the correct justification of ideas
- the use of IT tools (GeoGebra 5.0).

Analogies help the students with problem solving, developing their creative response and contributing to the retention of new information. These two functions are fulfilled by helping users to think in new ways and facilitate the acquisition of abstract concepts.

Acknowledgements

I would like to express my particular gratitude for the help of Miklós Hoffmann, Ján Gunčaga and András Ambrus.

References

Cižmešija, A., & Milin-Šipuš, Ž. (2012). Spatial ability of students of mathematics education in Croatia evaluated by the Mental Cutting Test. *Annales Math. et Inf., 40*, 203-216.

Gorska, R., & Jušcakova, Z. (2007). TPS Test Development and Application into Research on Spatial Abilities. *Jour. Geom. Graph., 11*, 223–236.

Hoffmann, M., Németh, B., & Sörös, Cs. (2007). Typical Mistakes in Mental Cutting Test and Their Consequences in Gender Differences. *Teaching Mathematics and Computer Science, 5*(2), 385–392.

Hoffmann, M., & Németh, B. (2006). Gender differences in spatial visualization among engineering students. *Annales Mathematicae et Informaticae, 33,* 169–174

Leclére, P., & Raymond, C. (2010). *Use of the GeoGebra software at upper secondary school, FICTUP Public case report*, INPL, France.

Lénárd, F. (1978). *A problémamegoldó gondolkodás.* Budapest: Akadémiai Kiadó.

Lord, T. R. (1985). Enhancing the Visuo-spatial Aptitude of Students. *Journal of Research in Science Teaching, 22*, 395–405.

McGee, M. G. (1979). Human Spatial Abilities: Psychometric studies and environmental, genetic, hormonal and neurological influences. *Psychological Bulletin, 86*, 899–918.

Nagy-Kondor, R. (2007). Spatial ability of engineering students. *Annales Mathematicae et Informaticae, 34,* 113–122.

Nagy-Kondor, S. (2012). Engineering students' spatial abilities in Budapest and Debrecen. *Annales Math. et Inf., 40*, 187-201.

Nagy-Kondor (2010). Spatial Ability, Descriptive Geometry and Dynamic Geometry Systems. *Annales Math. et Inf., 37*, 199-210

Nagy, L. (2000). Analógiák és az analógiás gondolkodás a kognitív tudományok eredményeinek tükrében. Magyar Pedagógia.

Pólya, Gy. (1985). *A problémamegoldás iskolája I.* Budapest: Typotex Kft.

Pólya, Gy. (1988). *A gondolkodás iskolája*. Budapest: Typotex Kft.

Pólya, Gy. (1988). *A problémamegoldás iskolája II*. Budapest: Typotex Kft.

Pólya, Gy. (1989). *A plauzibilis következtetés*. Budapest: Gondolat Kiadó.

Pólya, Gy. (1989). *Indukció és analógia.* Budapest: Gondolat Kiadó.

Reiman, I. (1986). *A geometria és határterületei.* Budapest: Gondolat Kiadó.

Schröcker, H.-P., Stachel, H., Tsutsumi, E., & Weiss, G. (2005). Evaluation of Students' Spatial Abilities in Austria and Germany. *Jour. Geom. Graph., 9*, 107–117.

Biographical note

László Budai is a PhD students of Mathematics and Computer Sciences at the University of Debrecen (Hungary). He was a mathematic-IT teacher, till 2013 a research assistant, instructor at Budapest Business School, University of Applied Sciences, College of International Management and Business, Institute of Business Teacher Training and Pedagogy. His research interests are in Spatial geometry and GeoGebra methodology applications. His current projects is GeoMatech.

Overcoming the Obstacle of Poor Knowledge in Proving Geometry Tasks

Zlatan Magajna[1]

Proving in school geometry is not just about validating the truth of a claim. In the school setting, the main function of the proof is to convince someone that a claim is true by providing an explanation. Students consider proving to be difficult; in fact, they find the very concept of proof demanding. Proving a claim in planar geometry involves several processes, the most salient being visual observation and deductive argumentation. These two processes are interwoven, but often poor observation hinders deductive argumentation. In the present article, we consider the possibility of overcoming the obstacle of a student's poor observation by making use of computer-aided observation with appropriate software. We present the results of two small-scale research projects, both of which indicate that students are able to work out considerably more deductions if computer-aided observation is used. Not all students use computer-aided observation effectively in proving tasks: some find an exhaustive computer-provided list of properties confusing and are not able to choose the properties that are relevant to the task.

Keywords: Computer-aided observation; Dynamic geometry; OK Geometry; Proof

1 Faculty of Education, University of Ljubljana, Slovenia; zlatan.magajna@pef.uni-lj.si

Premagovanje ovire šibkega znanja pri geometrijskih dokazovalnih nalogah

Zlatan Magajna

Pri dokazovanju v šolski geometriji ne gre le za dokazovanje resničnosti trditve. Pri pouku matematike je bistvo dokazovanja prepričljiva razlaga, zakaj je neka trditev resnična. Učenci doživljajo dokazovanje kot zahtevno; zahteven se jim zdi že pojem dokaza. Dokazovanje trditev o ravninski geometriji vključuje več procesov, najizrazitejša pa sta vizualno opazovanje in deduktivno argumentiranje. Ta procesa sta prepletena, pri čemer pa šibko opazovanje pogosto ovira deduktivno argumentacijo. V članku preučujemo možnost premagovanja ovire učenčevega šibkega opazovanja z uporabo računalniško podprtega opazovanja z ustrezno programsko opremo. Predstavljamo izsledke dveh manjših raziskav. Obe pokažeta, da učenci ob uporabi računalniško podprtega opazovanja oblikujejo bistveno več deduktivnih sklepov kot sicer. Vendar pa niso vsi učenci ob uporabi računalniško podprtega opazovanja učinkoviti: nekatere zmede izčrpen nabor lastnosti, ki jih opazi računalniški program, in niso zmožni med lastnostmi izbrati tistih, ki so pomembne za nalogo.

Ključne besede: računalniško podprto opazovanje, dinamična geometrija, OK Geometry, dokaz

Introduction

Proving and problem solving

As a mathematical discipline, geometry is a formalisation of reasoning about shapes that occurs in everyday situations. However, everyday geometry and geometry as a formal discipline are two different systems of practices. The basic difference between them is well known: everyday geometry is essentially empirical, while formal geometry is an axiomatic system. In everyday geometry, the truth of a statement is commonly validated by experience, while in formal geometry the truth of propositions is validated using deductive arguments, usually organised in a proof. The two aspects of geometry – everyday and formal – are also constituent parts of school mathematics. Learning the basic concepts of geometry is, obviously, based on experience; however, virtually all mathematics curricula at some stage include deductive proofs as a means to ascertain the validity of geometric propositions. Herbst (2002) pointed out that the role of proof in school mathematics has changed considerably in the last two centuries. At first, proofs were presented only for the sake of establishing the truth of the considered theorems. Students were not supposed to produce their own proofs, but just to reproduce the proofs presented to them. With time, proofs acquired an additional role: they became a means for developing mathematical reasoning, especially deduction. A novel type of proof-like exercises developed: proof was associated with exercise. Textbooks gradually incorporated didactically elaborated exercises about geometric facts to be proved, and students had to invent their own proofs based on deductive arguments. In current terminology, such exercises can be considered as closed problem situations (Orton & Frobisher, 1996).

Despite the essential role of proofs in mathematics, students barely accept proofs and proving as a part of 'their mathematics'. Hadas, Hershkowitz and Schwarz (2000) and Raman (2003), among others, reported on several studies of students' perception of proofs in mathematics. These studies indicate that students have difficulty not only in producing proofs, but even in recognising what a proof is. The fact that geometric objects and properties are easily visualised makes proving in the field of geometry both more difficult and easier. The visual nature of geometry facilitates the representation of the studied objects and the presentation of arguments. For this reason, planar geometry has traditionally been considered, and is still considered, an appropriate context for introducing the concept of proof and for developing deductive reasoning (Lingefjard, 2011). On the other hand, since geometric propositions can be easily visualised by simple sketches on a piece of paper, students barely find it reasonable to provide deductive arguments for facts that are empirically evident via visualisation. The introduction of dynamic geometry software, an exceptional didactic instrument for the visualisation of geometric objects and properties, makes the visual evidence even more convincing, encouraging students to adhere to empirical argumentation. Thus, it is necessary to clarify the necessity for proofs and the nature of deductive argumentation (Hadas, Hershkowitz, & Schwarz, 2000).

Demonstrating the truth of a claim is not the only reason, and often not the main reason, for proving in school mathematics. Hanna (2000) compiled the following list of functions of proofs and proving: *verification* that something is true, *explanation* why something is true, *systematisation* of concepts, theorems and various results, *discovery* of new results, *communication* of mathematical knowledge, *construction* of an empirical theory, *exploration* of the meaning of definitions, and *incorporation* of known facts into new frameworks. Although proofs are usually presented as a justification (to show the truth of a claim), their real value in the school context is to clarify why something is true (idem). This holds for exemplary proofs (e.g., proofs of relevant theorems) as well as for proofs produced by students when solving proof-like exercises.

There is no general agreement on what a proof is in school mathematics. Stylianides and Stylianides (2009) consider a proof to be an argument for the truth of a statement: the argument should be general, valid and accessible to members of the community involved. The validity of a proof is commonly related to the concept of derivation. In this sense, a proof consists of "a sequence of steps leading from premises to conclusion by way of valid reasoning" (Hanna & Sidoli, 2007). According to Hanna (2000), a proof should be legitimate and should "lead to real mathematical understanding", as only such a proof is convincing. Pedemonte (2007) lists four characteristics of argumentations and proofs in mathematics: 1) proofs are rational justifications; 2) proofs should convince; 3) proofs are addressed to a universal audience; and 4) proofs need to be considered in the context of specific fields (e.g., school geometry). From the perspective of situated cognition, the proof is an artefact that mediates between the individual and social practice (Hemmi, 2010). Due to the different conceptions and various functions of proof, it is not surprising that mathematics teachers develop different subjective theories, so that the way they treat proofs in classrooms ranges from almost ignoring them to including them systematically, from presenting only the key idea of the proof to emphasising the formal derivations (Furinghetti & Morselli, 2011; Hemmi, 2010; Knuth, 2002).

Proving and previous knowledge

Proving a property of a geometric configuration is a mathematical problem. A good paradigm for researching proving as solving problems is the information theory (Kahney, 1993). This approach has been used extensively in researching problem solving in many fields, including mathematics (Schoenfeld, 1985). From this perspective, a problem consists of a set of states called the problem space, a system of rules that define the possible transformation of states, as well as two special states, called the starting point and the goal. Solving a problem means transforming the starting point to the goal with a series of permitted transformations within the problem space. Consider, for example, the problem of proving a fact about a given geometric configuration. The starting point consists of the premises of the configuration, and the goal is, obviously, the claim to be proved. The problem space (of a solver) consists of all configuration-related statements that come to the solver's mind. The transformation rules in geometry are clear: only known facts (known theorems or previously ascertained facts), assumptions and deductive argumentation are allowed.

The problem space is subjective and depends on the solver's knowledge base. High achievers have a rich prior geometric knowledge that is effectively organised into schemas (Chinnappan, 1998). On the other hand, a poor problem space, resulting from poor, confused or disorganised prior knowledge, hinders the solving process. Being capable of deductive argumentation does not help much in proving geometric facts if one is not able to generate an appropriate problem space.

Observation in solving geometry problems

By an observation, we mean a conscious interpretation of a (visual) perception. Thus, observation refers to concrete properties of visualised geometric configurations. In observing a geometric property, the observer relates the visual perception to his/her understanding of the involved concepts and properties. Thus, observation is associated with the observer's knowledge.

An observation of a property may occur by chance, but usually, when solving geometry tasks, observing is an active process. Observation occurs in an interpretative context and can be more or less focused. We do not discuss here various observation strategies in solving geometry problems, as we take it for granted that a good solver is aware of the importance of non-focused observation and, on the other hand, is able to identify which properties are relevant to specific problems and focus on them during observation.

Observing is an essential process in learning geometry. Jahnke (2007) suggested that the introduction of the concept of proof to pupils should be based on experiential observation, for this is the pupils' natural way of establishing the truth of geometric facts. The underlying idea of this approach is: in a context where events are highly or completely predictable, observations are also predictable. Since geometry is a predictable context *par excellence*, proofs can be thought of as an effective substitute for observation. For example, one can prove that in a triangle the congruence of two of the triangle's sides implies the congruence of triangle's angles opposite to the congruent sides. Thus, there is no need to observe and check the congruence of the base angles each time we encounter an isosceles triangle. The proof is a way for establishing the truth of a claim once and for all.

The role of observation in proving geometric facts is quite complex. In theory, the proof of a geometric fact should not depend on the observation of a visual representation (Hanna & Sidoli, 2007). In practice, however, especially in school geometry, visual representations are essential: sometimes their purpose is to illustrate a concept or a claim, sometimes they serve as informal justifications, and there are situations where a proof can be reduced to a 'visual argument' (not just observation) (Hanna & Sidoli, 2007). When solving geometry problems, visualisation is a thinking aid for representing geometric facts (which can be true or false). An observer may or may not be aware of a property related to a represented configuration. Furthermore, a property that is observed may or may not be true. The awareness of an observed property is thus associated with various degrees of certainty of its truth. The degree of certainty ranges from very hypothetical to absolute certainty. Observing thus allows the student to become aware of properties, while also offering some degree of certainty. Let us recall the case of the isosceles triangle. One observer may not be aware that its base angles are congruent, whereas someone else may perceive the angles as being congruent and consider this as a hypothesis. If measuring the angles shows the same angle size, this raises the degree of certainty that the base angles are congruent. Finally, someone may consider, on the basis of previous knowledge, the congruence of base angles in isosceles triangles as unquestionably true.

The role of observation in solving geometric problems can also be explained in terms of the information paradigm. Given a geometric problem, the solver first constructs a problem space, i.e., a set of properties related to the geometric problem. To solve a problem means to connect a subset of properties in a proper way. Toulmin's model of argumentation (Pedemonte, 2007; Fujita, Jones, & Kunimune, 2010) provides further insight into this process. The solver needs (besides a strategy) some guidance when moving in the problem space. In order to take a property into consideration in constructing a proof, one needs some guarantee that, to be possibly considered in the proof, the property holds. In the case of geometric problems, the solver uses observation to obtain hypothetical properties that could be eventually be used in the solution (or proof). If the solver has some doubts whether a property (claim) is true, a more careful or elaborated observation may be used in order to reject the claim or to provide an additional guarantee for the claim. In the case that the hypothesis appears to be true and is of use in the solution to the problem, the solver should, at some stage, provide a backing, i.e., convincing (unquestionable, deductive) arguments for the hypothesised claim. A similar approach was used by Nunokawa (2010), who stresses the dynamic nature of the problem space. According to him, the construction of the problem space is, to some extent, parallel to the justification of the observed properties: in fact, it is the justification of an observation that often leads to new focuses and to observation of properties that otherwise would not be noticed.

Computer-Aided Observation – OK Geometry

Solving a geometric problem requires the construction of an appropriate problem space. It is essential that, at some stage of proving, the solver is aware of the properties – established or hypothesised – that could possibly be used in building up a proof. Note that the solver's problem space is, in general, not static: during the proving process new insights may lead to new focuses, while previously considered ones are ignored. However, not being aware of the relevant properties at any stage of the proof is an obstacle to building up proofs and, consequently, to learning to prove. The reason for not being aware of a property (not observing it) may be simply 'not paying attention' or it may be poor knowledge that prevents an appropriate interpretation of a perception. In any case, this lack of awareness prevents the solver from connecting facts, proving facts and upgrading proving skills.

In order to research the nature of this obstacle, a research tool called OK Geometry was developed by the author.[2] In simple terms, OK Geometry is a tool for the computer-aided observation of dynamic geometric constructions. As opposed to static geometric constructions made by paper and pencil, dynamic geometry constructions are computer representations that allow the dragging of non-constructed objects and the dynamic display of the constructed objects. Dynamic geometry systems are widely used in school mathematics. Given a dynamic construction, obtained by some of the widely used dynamic geometry systems, OK Geometry provides a list of properties related to the studied construction (together with their visual representations). OK Geometry does not

2 OK Geometry is available at http://z-maga.si/index?action=article&id=40

prove facts, it only lists the properties that are detected by the software. Due to the method of observation used, the possibility of an observational error is rather remote. In the school setting, the list of observed properties can be used for various purposes, e.g., the exploration or connecting of facts. We shall focus here on just one purpose: proving facts.

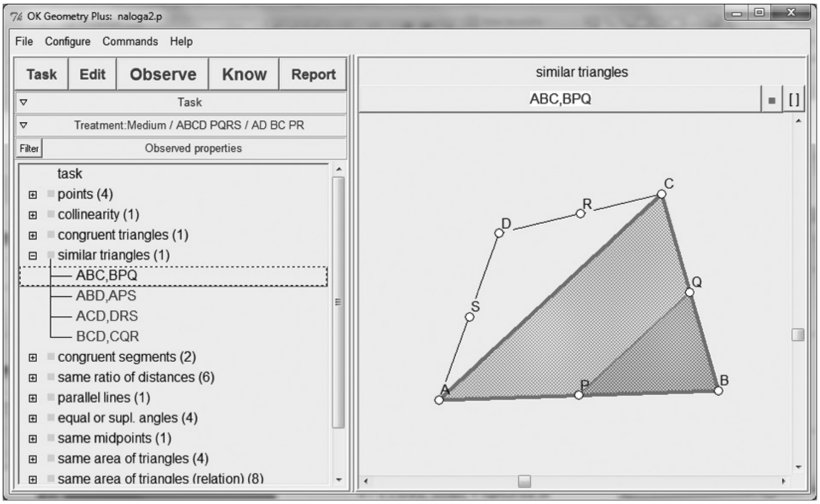


Figure 1. Using computer-aided observation in a proving task.

Computer-aided observation assures that solvers notice a variety of properties of the studied geometric configuration, including those that are relevant to the solution of the geometric problem. The list is presented in a structured way so that the various properties are easily retrieved and visualised (Figure 1). However, even in simple geometric configurations, the computer-provided list of properties can be rather extensive, for it includes many trivial properties and properties that are irrelevant to the solution of the problem. Evaluation of the list of properties should be directed by the aim or by the clear idea of what the list of properties was created for: in this case, the need to solve the given problem. The solver needs to decide which properties from the computer-provided list are trivial or probably irrelevant to the solution, and which are possibly related to the solution. The latter are then organised into a solution/proof.

OK Geometry allows a further simplification of the above described proving process (Figure 2). In the simplified form, the proving task consists of a claim to be proved together with a list of 'observed' properties to be used in the proof. The student needs only organise the given properties into a proof and provide arguments for the claims in the proof.

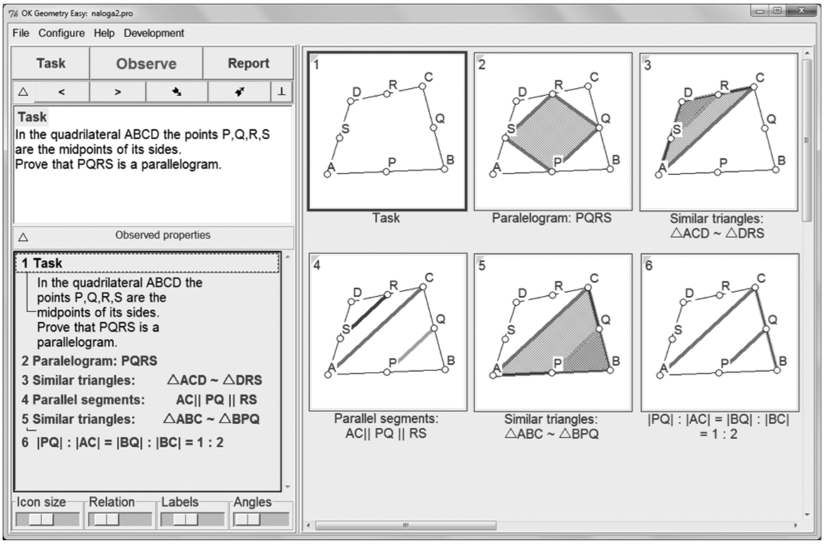


Figure 2. A proving task with given selected properties.

Research Question

Observing is unquestionably an important cognitive process that should be developed during mathematics education. Proving in geometry involves observing, especially when the solver constructs the problem space related to the task. Obviously, a poor problem space is an obstacle to working out deductive arguments related to the solved task.

We conjecture that computer-aided observation can extend students' problem space related to geometry proving tasks and, consequently, can facilitate the expression of deductive argumentations.

Put simply, we conjecture that in solving geometry proving tasks, computer-aided observation can help students (novices) to overcome the obstacle of poor observation ability. In such tasks, computer-aided observation can be a facilitator for expressing deductive reasoning.

In the sections that follow, we present the results of two small-scale research projects.

The First Research Project

The participants were six above-average students aged 15 years, all of whom were attending the first year of gymnasium (general upper secondary school). They had been studying planar geometry in the months directly preceding the research. The students worked on two problems:

1. *The quadrilateral task* (Figure 1 left). In the quadrilateral ABCD, the points P, Q, R, S are the midpoints of its sides.

1a. Write down a list of all of the properties (not explicitly mentioned in the task) that you observe. For each property, if possible, explain why it is true.

1b. Prove that PQRS is a parallelogram.

2. *The trapezium task* (Figure 1 right). In the trapezium ABCD, let E be the intersection of diagonals AC and BD.

2a. Write down a list of all of the properties (not explicitly mentioned in the task) that you observe. For each property, if possible, explain why it is true.

2b. Prove that the triangles AED and BCE have the same area.

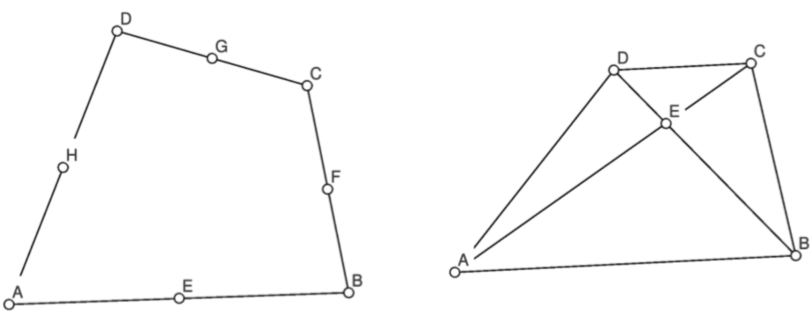


Figure 3. The illustrations for the quadrilateral and the trapezium tasks.

The students first worked individually with paper and pencil (i.e., in the way they were used to solving geometry problems). They solved the tasks 1a, 1b, 2a, and 2b one by one, spending approximately 5-10 minutes on each task.

In the second phase, the same students worked on the same four tasks (1a, 1b, 2a, 2b), except that:

1. In tasks 1a and 2a, they used OK Geometry to obtain a computer-provided list of properties. They only had to select the properties they considered interesting and non-trivial. They were also asked to try to prove the selected properties. Tasks 1b and 2b were worked out with paper and pencil or using OK Geometry after tasks 1a and 2a, respectively.
2. The students worked in pairs (they were supposed to work alone, but as OK Geometry revealed the properties of the studied configuration the students could not resist discussing them with the student next to them).

The students' solutions were analysed as follows. First, all non-trivial geometric properties of the two configurations (Figure 3) within the reach of the students were identified: these were considered the 'canonical list of properties' of each configuration. The list comprised claims about congruent angles, parallel lines (in the quadrilateral task), congruent triangles, similar triangles, etc. Then, the students' answers to each task were considered. For each property in the 'canonical list of properties' of a configuration, it was established whether the student noted the property and whether s/he gave reasonable arguments for its validity. The trivial claims in the answers (e.g. 'the quadrilateral has four sides', 'the bases of trapezium are parallel') were ignored. Table 1 displays the percentage of properties (relative to the 'canonical list of properties') that were identified and the percentage of those for which the students gave reasonable arguments in the first (paper and pencil) phase and in the second (OK Geometry) phase.

Table 1. *Observed and proved properties for various observational methods. The percentages refer to the canonical list of properties related to each task.*

Task	Paper and pencil		OK Geometry	
	Observed facts	Proved facts	Observed facts	Proved facts
The trapezium task	0%	0%	73%	40%
The quadrilateral task	6 %	4 %	48%	24%

In the first phase, the students were virtually unable to identify any non-trivial properties, and, obviously, did not prove these properties. The high percentage of observed facts in the second phase is not surprising, as the students only had to select among the properties provided by the computer. In the trapezium case, for example, they 'missed' 27% of relevant properties (they either considered them to be trivial/irrelevant or they did not understand them). What is striking is that they were able to prove approximately half of the facts they did not even notice before. Obviously, being aware of some facts helped them to prove other facts.

The Second Research Project

The participants in this research were 38 prospective mathematics teachers at the beginning of their fourth year of study. During their university study, they attended some courses in advanced geometry.

Each student was asked to solve four tasks, which shall be referred as Task 1, 2, 3 and 4. Note that the order in which the tasks were presented varied from student to student. Each task contained a situation and a property to be proved.

The four problem tasks (not listed here) to be solved by the participants were comparable in form and difficulty to tasks 1b and 2b in the first research project. The tasks were solved in three phases:

Phase 1. The students were asked to work individually using paper and pencil on the first two tasks, using approximately five minutes for each task.

Phase 2. In the second phase, the students solved the first three problems individually using computer-based observation. Using OK Geometry, their task was to import a ready-made dynamic construction, to generate a list of properties, and to select the properties to eventually be used in the proof. They then tried to work out the strategy of the proof by organising the selected properties into an appropriate order. Finally, they tried to provide arguments for each step in the proof. OK Geometry served as an observational tool and as a tool for organising and documenting their work. The students had a limited time (15 minutes) to complete all three tasks. If a student claimed that s/he had already solved a problem in Phase 1, his/her solution of Phase 1 was also accepted for Phase 2, and s/he could skip the task.

Phase 3. In the third phase, each student solved all four tasks individually in the same order as in the previous phases. In this phase, they solved the problems using OK Geometry, but instead of an extensive list of properties related to each task, they used a short list of selected properties to be used in a proof. The students only had to work out the strategy of a proof by organising the selected properties into an appropriate order, while also providing arguments for each step of the proof. As in the previous step, they worked individually on computers, also using OK Geometry to document their work. They had a limited time (20 minutes) to complete all four tasks; however, none of the students worked out more than three tasks. The students were allowed to claim they had already solved a problem in a previous phase and skip to the next problem; in this case, the solution of the previous phase was also accepted in Phase 3.

Table 2. *The plan of task presentation in the second study.*

Phase Time	Method	Order of the presented tasks		
		Group 1	Group 2	Group 3
Phase 1 10 min.	Paper and pencil	1, 2	1, 3	1, 4
Phase 2 15 min.	Complete list of properties (computer-aided observation)	1, 2, 3	1, 3, 4	1, 4, 2
Phase 3 20 min.	Selected list of properties (computer-aided observation)	1, 2, 3, 4	1, 3, 4, 2	1, 4, 2, 3

Using this arrangement, all of the problems (except Problem 1) were solved by some students first by paper and pencil, then (if not solved) using the computer-generated list of observations, and then (if still not solved) by making use of selected properties. The same problem was solved by other students initially by computer-generated observations, and then (if not solved) by making use of the list of selected properties to be used in the proof.

For each task, the various possible strategies of solutions were divided into the same number of steps (claims). For each proposed solution, the following points were considered:

- whether the overall strategy (the basic idea) of the solution was correct,
- the number of relevant properties (solution steps) for which a student gave correct arguments,
- the number of incorrect claims (i.e., observations that were false),
- the number of claims (proved or unproved) that were not relevant to the solution of the problem.

The results are summarised in Figure 4 and Figure 5.

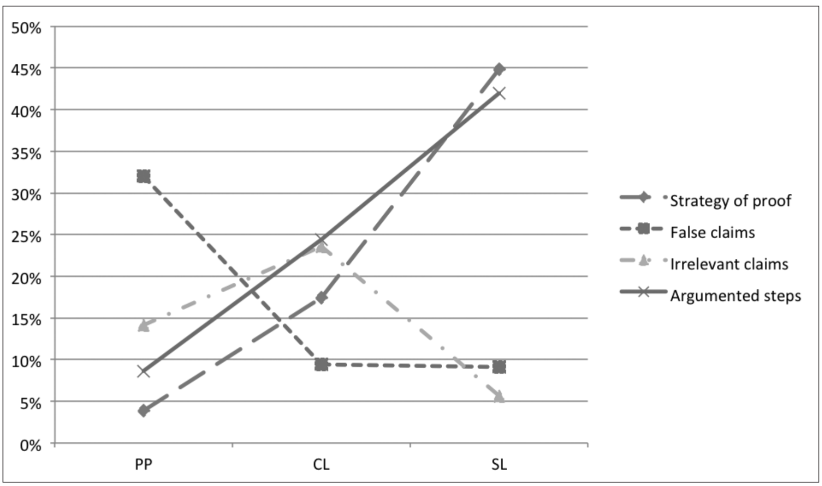


Figure 4. The structure of claims in the solutions of the proving tasks for various observational methods (PP – paper and pencil, CL – complete list of properties obtained by computer-aided observation, SL – selected list of properties obtained by computer-aided observation).

Let us first consider the results shown in Figure 4. The students solved the geometric problems in three modalities: 1) without any help, just using paper and pencil (PP); 2) provided with an extensive computer-generated list of properties (CL); and 3) provided with a reduced list of properties to be used in the proof (SL). When working with paper and pencil (PP), the students found the solution strategies for a total of only approximately 4% of the tasks, and provided an argument for approximately 9% of the solution steps. The respective results rose to 17% and 24% when an extensive list of properties was provided (CL). Providing the students with a list containing only essential properties (SL) produced even better results: 45% and 42%, respectively.

It is not surprising that more tasks are solved if the students are provided with an extensive list of properties, and that even more tasks are solved if they are provided with a list of essential properties to be considered in the proof. This is why authors of textbooks often add some hints to proving exercises in order to make them easier. However, we considered this phenomenon from another perspective: poor observation is an obstacle in proving facts in geometry, and, by extension, hinders the developing and demonstrating of deductive argumentation. Observing is unquestionably an essential process in proving, one that should be promoted and emphasised. However, there is no reason for poor observation to prevent students developing argumentation abilities, and it appears that computer observation may help students in this respect.

Figure 4 also indicates that poor observation manifests in two ways: 1) not seeing (not being aware) of relevant properties, and 2) observing 'false' properties, i.e., properties that do not hold. On average, when working with paper and pencil (PP), the students considered and gave arguments for approximately 9% of essential properties and 14% of irrelevant properties. Approximately 33% of the properties the students claimed or hypothesised to be true (whether they provided some arguments for them or not) were false. Obviously, there is nothing wrong with considering false or irrelevant properties (although sometimes they may be a symptom of poor expertise): considering false properties may, in fact, be a good source of new conceptual knowledge. On the other hand, false and irrelevant claims hinder the proving process. Figure 4 indicates, as is reasonable to expect, that if the students are provided with an extensive list of properties (CL), the number of irrelevant claims increases and the false claims, though still present, decrease in number. The reason for the presence of false statements will be explained shortly.

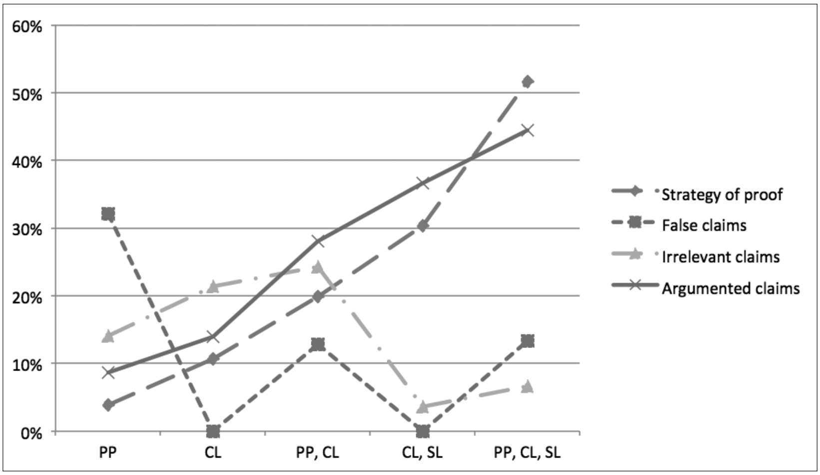


Figure 5. The structure of the claims in the solutions of the proving tasks for various combinations of observational methods (PP – paper and pencil, CL – complete list of properties obtained by computer-aided observation, SL – selected list of properties obtained by computer-aided observation).

Figure 5 presents some aspects of solving geometric tasks for selected combinations of observation methods. As already explained, paper and pencil (PP), a complete computed-provided list of properties (CL) and a selection of essential properties (SL) were associated with various degrees of help in solving proving problems. Obviously, combining two or more of these methods (i.e., one method after another in succession) improved success, as more time was available for finding a solution. One interesting phenomenon is the persistence of false claims when the paper and pencil method was followed by computer-aided observation: in some cases, the student tried to prove a claim even after computer-aided observation did not confirm its correctness. Perhaps this can be explained as fixation or confirmation bias, but we prefer to interpret it as the student's need to explore the configuration by themselves and achieve a personal conviction. The question as to whether it is profitable to combine various observation methods, and how to combine them, requires further investigation.

Conclusions

We have presented the results of two small-scale studies on the role of observation in solving geometric problems that require deductive argumentation. Although there are certain validity issues in these studies (e.g., the relatively short time for paper and pencil work), the results indicate that computer-aided observation can help students to build up an appropriate problem space related to geometry tasks. Consequently, this facilitates the expressing of deductive argumentations in geometry proving tasks. Since a poor problem space may also result from poor observation ability, computer-aided observation can, to some extent, overcome the obstacle of poor observation in solving such tasks.

Most of the participants in the studies used computer-aided observation effectively, but not all and not always. Some found the large number of properties identified by the computer software confusing, even though the properties were presented in a structured way. Some focused their attention rigidly on a particular property that they were convinced would lead to the solution even though the property was not on the computer-provided list (and was false). Evidently, solving problems using computer-aided observation requires the adoption of appropriate strategies, especially if the solver's knowledge is poor. An expert in the field knows which type of properties to look for in specific problems, while a novice has to develop a technique or strategy for selecting the potentially relevant properties. The novices' strategies for solving geometry proving tasks using computer-aided observation are certainly worth researching in the future, as they may find suitable other strategies besides "working forwards" or "working backwards".

Observation is an essential process in building up proofs, as it provides the necessary hypotheses that need deductive backing. In this sense, observation is a prerequisite for deductive argumentation. Current school-oriented software tools for learning planar geometry (dynamic geometry systems) are powerful tools for visualising and checking properties. In working out proving tasks, such software can help students to check observed properties that serve as hypothesised steps in the proof (Mariotti, 2000). However, if the solver is not able to identify the relevant properties to be used in a proof, dynamic geometry software will not be of any help, as the solver does not know which properties to check and, eventually, use in deductive argumentation. Poor observation ability is thus an obstacle to developing deductive reasoning. The two pilot studies indicate that computer-aided observation may be used to overcome the obstacle of poor observation and enable students to make deductions.

References

Chinnappan, M. (1998). Schemas and Mental Models in Geometry Problem Solving. *Educational Studies in Mathematics*, *36*(3), 201–217.

Fujita, T., Jones, K., & Kunimune, S. (2010). Students' geometrical constructions and proving activities: a case of cognitive unity? In M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Annual Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3 (pp. 9–16). Belo Horizonte, Brazil.

Furinghetti, F., & Morselli, F. (2011). Beliefs and beyond: hows and whys in the teaching of proof. *Zentralblatt für Didaktik der Mathematik*, *43*, 587–599.

Hadas, N., Hershkowitz, R., & Schwarz, B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, *44*, 127–150.

Hanna, G., & Sidoli, N. (2007). Visualization and proof: a brief survey of philosophical perspectives, *Zentralblatt für Didaktik der Mathematik*, *39*, 73–78.

Hanna, G. (2000). Proof, Explanation and Exploration: an Overview. *Educational Studies in Mathematics*, *44*, 5–23.

Hemmi, K. (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics*, *75*, 271–291.

Herbst, P. G. (2002). Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics, 49*, 283–312.

Jahnke, H. N. (2007). Proofs and Hypotheses. *Zentralblatt für Didaktik der Mathematik*, *39*(1-2), 79–86.

Kahney, H. (1993). *Problem Solving: Current Issues*. Buckingham: Open University Press.

Knuth, E. J. (2002). Teachers' Conceptions of Proof in the Context of Secondary School Mathematics. *Journal of Mathematics Teacher Education*, *5*, 61–88.

Lingefjärd, T. (2011). Rebirth of Euclidean geometry? In L. Bu & R. Schoen (Eds.), *Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra* (pp. 205–215). Rotterdam: Sense Publishers.

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics*, *44*, 25–53.

Nunokawa, K. (2010). Proof, mathematical problem-solving, and explanation in mathematics teaching. In G. Hanna (Ed.), *Explanation and proof in mathematics. Philosophical and educational perspectives* (pp. 223–236). Berlin: Springer.

Orton, A., & Frobisher, L. J. (1996). *Insights into Teaching Mathematics*. London: Cassel.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analyzed? *Educational Studies in Mathematics*, *66*, 23–41.

Raman, M. (2003). Key ideas: what are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, *52*, 319–325.

Schoenfeld, A. H. (1985). *Mathematical Problem Solving.* San Diego: Academic Press.

Stylianides, A., & Stylianides, G. J. (2009). Proof Constructions and Evaluations. *Educational Studies in Mathematics*, *72*(2), 237–253.

Biographical note

Zlatan Magajna is an assistant professor for didactics of mathematics at the Faculty of Education University of Ljubljana. After studying theoretical mathematics at the University of Ljubljana he worked for many years as a development engineer in the field of computer aided design. He received his PhD in the field of mathematics education at the University in Leeds. His main fields of research are: mathematics in out of school and working environment, using computer technology at teaching mathematics. He also works on mathematics curriculum at primary, technical and vocational level; and also analyzing international comparative mathematics acchievement studies.

Enjoying Cultural Differences Assists Teachers in Learning about Diversity and Equality. An Evaluation of Antidiscrimination and Diversity Training

Nada Turnšek[1]

The present study is based on a quasi-experimental research design and presents the results of an evaluation of Antidiscrimination and Diversity Training that took place at the Faculty of Education in Ljubljana, rooted in the anti-bias approach to educating diversity and equality issues (Murray & Urban, 2012). The experimental group included 52 in-service early childhood teachers attending the training, which consisted of a total of 120 hours. There was also a control group comprising 130 teachers. The ADT had a decisive impact on all of the measured variables: on an improvement in the participants' knowledge of discrimination, and on increased support for positive measures and for the preservation of the cultural traditions and language of immigrant children. It was found that self-assessed personality characteristics are predictors of the teachers' beliefs, especially the *enjoying awareness of cultural differences* variable, which correlates with all of the dependent variables.

Keywords: Early childhood; Diversity; Equality; Discrimination

1 Faculty of Education, University of Ljubljana, Slovenia; turnsek.nada@gmail.com

Uživanje v kulturni raznolikosti je v pomoč vzgojiteljem pri izobraževanju za različnost in enakost – evalvacija izobraževanja za nediskriminacijo in raznolikost

Nada Turnšek

Študija temelji na kvazieksperimentalnem raziskovalnem načrtu; predstavlja izsledke evalvacije izobraževanja za nediskriminacijo in raznolikost, ki je potekalo na Pedagoški fakulteti v Ljubljani; utemeljeno je na antipristranskem pristopu k izobraževanju za vprašanja raznolikosti in enakosti (Murray in Urban, 2012). V eksperimentalni skupini je bilo 52 vzgojiteljev, zaposlenih v vrtcu, ki so bili deležni 120-urnega usposabljanja; v kontrolni skupini je bilo 130 vzgojiteljev. Izobraževanje je imelo odločilen vpliv na vse merjene spremenljivke: izboljšalo se je znanje udeležencev o diskriminaciji, povečala se je podpora pozitivnim ukrepom ter ukrepom ohranjanja kulturnih tradicij in jezika otrok priseljencev. Samoocene osebnostnih lastnosti so napovedovalci prepričanj vzgojiteljev, še posebej spremenljivka uživanje v zavedanju obstoja kulturnih razlik, ki je povezana z vsemi odvisnimi spremenljivkami.

Ključne besede: zgodnje otroštvi, raznolikost, enakost, diskriminacija

Introduction

Early childhood education is seen as playing a key role in combating educational disadvantages stemming from socioeconomic, cultural and/or language factors (Eurydice, 2009). Studies demonstrate the impact of quality preschools on all children's educational achievement, showing that disadvantaged children benefit the most; even long-term individual and societal benefits have been demonstrated in terms of reducing educational inequality (Heckman, 2011). However, there is growing doubt about the "formula" according to which early childhood education represents a good investment in the social state (Ruhm & Waldfogel, 2011). Moss (2012) points out the incredulity of the 'story of high-returns', claiming that even in countries that have for decades implemented national compensatory early interventions, social inequality is still increasing. Experts problematise the implications of such an 'instrumentalisation' of preschool education – manifested in imposing the function of eliminating inequalities on preschool programmes – especially if the state does not provide effective mechanisms for reducing income inequality and social disparities in societies. As many studies show that inequality persists, Gaber and Marjanovič (2009) draw attention to the illusion that educational institutions (as well as educators) regard themselves as socially neutral and just, often combined with a belief that, solely by increasing accessibility, education institutions are doing enough to reduce inequality. It is therefore essential for those responsible for university study programmes to ask themselves what kind of conceptions, beliefs and practices they promote with regard to teachers within the existing conceptualisations of teacher education.

We argue that early childhood teacher preparation programmes need to undergo structural reforms (Florian, Young, & Rouse, 2010) in order to adequately address issues of diversity and equality. Firstly, university programmes have to incorporate inclusive competencies as the "core competencies" of prospective early childhood teachers, representing a *central role* in the conceptualisation of their profession. Secondly, a significant shift away from the multicultural approach towards the *anti-bias approach* (Murray & Urban, 2012) is essential, as it fosters critical reflection (Mezzirow, 1990) and an awareness of the social construction and reproduction of inequality, along with an understanding of the social and psychological mechanisms of discrimination and its prevention. The processual aspect of the reform concerns social learning occurring as a collective exercise (Ainscow & Sandill, 2010) involving interactive methods that consider participants as dynamic actors (Boal, 2000; Clements & Jones, 2006; Kaikkonen, 2010). Such an approach represents a collective emancipatory movement that undermines the foundations of the micro-ideologies of everyday life (Ule, 2004, pp. 191-193). In this paper, we present the results of an evaluation of Antidiscrimination and Diversity Training (ADT) based on the described characteristics.

Tackling inequality through early childhood education

Transforming preschools into inclusive settings relies on a broad definition of inclusion understood in terms of responding to the diversity of *all* children, particularly those who are vulnerable to exclusionary pressures within society (Ainsclow et al., 2006). In Slovenia, apart from children with special educational needs due to disabilities and/or illness, children from low-income families, immigrant children (mainly from former Yugoslavia) and Roma children are also considered 'at risk' (Turnšek & Bastič-Zorec, 2009).

Slovenian early childhood legislation and national strategic documents combine social justice policies at societal and institutional levels with recommendations on inclusive pedagogy. Priority in admission to preschools is given to children with special needs and to socially disadvantaged children. The White Paper of the Republic of Slovenia (Krek & Metljak, 2011) envisages the increased preschool inclusion of children from socially and culturally less stimulating environments, as it promotes school readiness as well as compensating for deficiencies in learning and development (Gaber & Marjanovič Umek, 2009). Nevertheless, as Bennett (2008) points out, family poverty and background remain significantly linked to poor educational outcomes; this is also reflected in social disparities in Slovenia's education system (Flere, 2005).

As Slovenian society shows a high level of intolerance within the European context, especially towards the Roma (Kirbiš, Flere, & Tavčar Krajnc, 2012), Roma children are often exposed to discrimination and exclusion (Klopčič & Polzer, 2003). They are not enrolled in preschools in adequate proportions (Strategy..., 2004), which contributes to their failure at school (Macura-Milovanović, 2006). Despite clear policy recommendations (Strategy for Roma/Gypsy Education..., 2011), national measures such as a more favourable child/adult ratio and staff reinforcement (Turnšek & Batistič-Zorec, 2009), along with recommendations on teachers' practices (Supplement to the Curriculum..., 2002), the social inclusion of Roma children is not progressing as envisaged. Possible causes include a lack of perceived responsibility for the social inclusion of Roma children (Peček & Macura-Milovanović, 2012; Peček Čuk, & Lesar, 2008).

Policies fostering the inclusion of immigrant children from former Yugoslavia were introduced relatively late (National Strategy…, 2007), which, together with a high level of intolerance towards immigrants (Kirbiš, Flere, &

Tavčar Krajnc, 2012), contributed to the prevailing belief that immigrant status does not represent a barrier to a child's learning and development. The policies that were adopted later (Guidelines..., 2011) now provide recommendations on adapting teaching practices in line with individual learning programmes. The White Paper obliges preschools to organise lessons for learning Slovenian, as well as support in the child's mother tongue. Still, immigrant children often achieve worse learning results than other children due to linguistic and cultural differences combined with their low socioeconomic status (Dekleva & Razpotnik, 2002; Macura-Milovanović, 2006; Peček & Lesar, 2006).

From multiculturalism to an anti-bias approach

Until recently, two approaches to educating diversity and equality prevailed in the conceptualisation of teacher education: *the multicultural approach* and *the intercultural approach*. The multicultural approach acknowledges the need for the recognition and celebration of different cultures in early childhood settings. However, its sole focus on cultural diversity, particularly on those aspects of culture that appear interesting, exotic and different, poses the risk of confirming or strengthening stereotypes, and thus perpetuating the perception of minority cultures being different to what is normal. Even though the intercultural approach goes one step further by fostering understanding and respect between majority and minority cultures, both fail to address the broader societal context in which diversity and inequality exists, including prejudice, discrimination, racism and power (Murray & Urban, 2012, pp. 116–117).

The fundamental characteristics of the anti-bias approach to educating diversity and equality issues described by Murray & Urban (2012, pp. 118–127) can be summarised as its *proactive*, *value-based* and *activist orientation* that recognises the influence of the societal context on generating equality or inequalities along with power issues in societies and preschools. It supports teachers in the process of becoming critically reflective practitioners who are able to reflect on belief systems as a significant element of the institutional 'hidden curricula' (see Jackson, 1968; Apple, 1982), and who are "conscious of the influence of their assumptions and belief systems on behaviour" towards others (Mezzirow, 1990, as cited in Murray & Urban, 2012, pp. 92-98). As Mac Naughton (2003, p. 3) points out, early childhood professionals also need to examine "the social and political factors that produce knowledge and practices, together with the use of this knowledge to strategically transform education in socially progressive directions". By providing a transformative teacher education experience, the anti-bias approach therefore promotes a consciousness of teaching as a political act (Kozleski & Waitoller, 2010).

Antidiscrimination and Diversity Training: Goals and content

The first stage of the ADT focuses on participants' self-perception and identity, followed by an exploration of the influences that affect our perceptions of others. Participants explore the images and stories concerning other groups that we create on the basis of our own societal or cultural background; the aim is to foster an understanding that a perception of one's identity can only be constructed through its opposition, through the ultimately different Other (Hall, 2007).

The next stage involves identifying dominant negative stereotypes and prejudices (including those relevant to preschools) in order to recognise their evaluative, affective and conative components (Augoustinos & Walker, 1995, p. 230). The training supports the recognition of stereotypes and prejudices as those attitudes that foster *maximisation of the perceived differences between groups* (Tajfel, 1978).

Further activities promote an understanding that the emergence and reinforcement of stereotypes and prejudices is not an individual or cognitive process, but rather a collective and ideological one. Participants are made aware of the written and spoken messages that surround us in everyday life, such as fairy tales, songs and sayings, which influence our thinking about minority groups. Exploring the so-called "common knowledge" of the society can lead to the realisation of the limiting and stereotyping effects of such messages. Here, the focus is also on the role of the political sphere, the media and education in maintaining inequality. Participants become aware that stereotypes function as a motivation and legitimisation of discriminatory practices (Van Dijk, 2005), e.g., through exclusion from, or the unequal distribution of, social resources or human rights, which in turn produce, determine and objectify cognitive and affective structures concerning minority groups (Augoustinos & Walker, 1995, p. 222).

Further steps involve the promotion of tolerance and empathy; participants are encouraged to identify with various marginalised positions. Activities lead to the acknowledgement that our own cultural background also equips us with distinctive 'cultural glasses' that influence our perceptions of others. Analysis of discriminatory situations represents the basis for embracing the advocacy aspects of the professional role. By addressing the concept of equal opportunities, participants are made aware that children start their lives from different positions; at this stage, the key goal is to raise awareness of the importance of positive measures as a key means of achieving equality.

The problem

Although undergraduate early childhood study prior to the Bologna reform did contribute to Slovene teachers' pro-democratic orientations and to respecting diversity on the whole, it did not challenge deeply rooted ethnocentric positions (Turnšek, 2006; Turnšek & Pekkarinen, 2009). Teachers' attitudes concerning immigrant families and children represented significant exceptions in the general democratic orientation (Turnšek & Pekkarinen, 2009), as (only) about half of teachers advocated the preservation of immigrants' customs, traditions and languages; among the rest, a tendency towards the denial of cultural rights or assimilation was identified. Since the data for teachers did not deviate from general Slovenian public opinion, it was concluded the prevailing cultural values had been accepted.

In teachers' interpretations, equality of opportunities was often confined to the notion of preschools' (*social*) *accessibility*; the strong presence of a common-sense conception has also been identified, characterised by the "demand" to *treat all children in the same way*. Less than one third of teachers supported the interpretation requiring *individualisation and differentiation*.

At the time, the research results indicated that early childhood teachers mainly associate justice with formal equality, but not with differential treatment. In response to the research findings, the Counter-Discriminatory Practice course was introduced within the framework of Bologna-reformed postgraduate early childhood study, within which the Antidiscrimination and Diversity Training was implemented. This course focused special attention on the teachers' understanding of the differential treatment – in terms of pedagogical practices and policies – with which teachers and preschools can promote justice. This is also in line with the White Paper (Krek & Metljak, 2011, pp. 14-15), which defines equity in education as a "key element of social justice", obliging the state to "adopt various measures and policies including positive discrimination for children from socially and culturally disadvantaged backgrounds", stressing the demand for non-biased or non-discriminatory treatment.

The study presented here answers the following research questions: Does ADT have a significant impact on the participants' knowledge of discrimination, on their attitudes towards the differential treatment of at-risk children and towards maintaining the cultural identity of immigrant children? To which factors are these positions related?

Method

The experimental group consisted of 52 early childhood student teachers (*participants*) who had completed the three-year undergraduate Early

Childhood Education Study Programme[2] and had continued studying at the two-year postgraduate level; within this programme, they attended the ADT, consisting of 60 hours of workshops and 60 hours of independent study (e.g., writing reflections, etc.). The control group included 130 teachers with the same level and type of education extracted from a random sample of Slovenian teachers (*non-participants*).[3] A comparison of the characteristics of both groups shows no significant differences, apart from the average older age of the participants due to the fact that the postgraduate study programme is also attended by teachers and headmasters with many years of work experience.

In order to assess the participants' initial knowledge and beliefs, they were asked to complete an evaluation pre-questionnaire prior to the ADT in September 2010. The questionnaire was designed to evaluate the achievement of the following key ADT objectives: *improving knowledge of discrimination, understanding the importance of differential treatment in achieving equal opportunities, and changing the participants' attitudes in favour of preserving immigrant children's mother tongue and their culture, customs and habits in preschool.*

Shortly thereafter, the questionnaire was sent to the non-participants by post; 98% of the completed questionnaires were returned in the following three weeks. At the end of the last ADT workshop in December 2010, the participants completed a post-questionnaire containing the same set of questions. The daily ADT schedule consisted of three to four workshops, lasting about five hours in total.

An analysis of variance was used to determine the significance of the differences in the participants' knowledge and beliefs between the pre-testing and post-testing compared to those of the control group. Using Tukey's HSD post-hoc test, we established which group means differ significantly from each other. A multiple regression analysis was performed to establish which independent variables are predictors of the respondents' knowledge and beliefs (Table 1). These predictors were assumed to be: personal (age), professional (length of service in preschools), family-related (parents' education), personality characteristics (preferences indicating "openness/closeness" towards people of a different ethnicity and/or cultural background, preference to live in a diverse environment), environmental (population diversity in the place of residence), and lifestyle factors (having immigrant friends, travelling abroad) (Table 2).

2 At any of the universities in Slovenia: in Ljubljana, Koper or Maribor.

3 First, the teachers working in the same preschool institutions as the participants were included in the whole Slovenian sample. In the second phase, we randomly selected preschools in the remaining 12 statistical regions in Slovenia and added these teachers to the sample. In the third phase, we formed the control group by selecting only those teachers who had completed the same formal education: the 3-year undergraduate Early Childhood Education Study Programme.

Table 1. *The dependent variables.*

Variable name	Variable description	Range
DISCRIMINATION	The total number of correctly identified situations of discrimination or non-discrimination. Correct answer = value 1; Incorrect answer = value 0.	0 – 8
POSITIVE MEASURES	Assessment of positive measures (a list of 12 statements) as being *just* (value = 10) or *unjust* (value = 0) on a semantic differential scale.	0 – 120
ATTITUDES-IMMIGRANTS	Agreement with attitudes to immigrants (a list of 12 attitudes) on a Likert attitude scale from *strongly disagree* = value 1 to *fully agree* = value 5.	12 – 60

Table 2. *The independent variables.*

group	Experiment – T (pre-testing), Experiment – T2 (post-testing), Control
cautious	I am usually cautious during contact with people from another culture (from Not at all true = 1 to Very true = 5).
enjoy	I enjoy my awareness that there are cultural differences between myself and other people (from Not at all true = 1 to Very true = 5).
try to learn	When I make contact with people from different cultures, I try to learn as much as possible about their life (from Not at all true = 1 to Very true = 5).
avoid	If possible, I prefer to avoid situations that involve contacting people from a different cultural environment (from Not at all true = 1 to Very true = 5).
age	Number of years.
neighbourhood	How would you describe the neighbourhood in which you live? As an area where almost nobody (some people, many people) has a different ethnical origin than the majority of residents of Slovenia.
education: mother	Please indicate the last level of education your mother completed (incomplete primary school, primary school, 2- or 3-year vocational school, 4- or 5-year secondary school, 2-year post-secondary school, 3-year higher vocational programme, university education (faculty, academy, specialisation, master's degree, doctorate)).
education: father	Please indicate the last level of education your father completed (same modalities as for the education: mother variable).
work experience	Number of years.
travel	How often do you travel abroad? (A few times a year, 2 –3 times a year, once a year, every few years, never).
friends	Do you have any friends who have moved to Slovenia from another country? (No, I have none; yes; I have some; yes, I have many).
like to live	Suppose you could choose where you would like to live. In which of the three areas listed below would you like to live if you had the choice? (In an area where almost nobody (some people, many people) has a different ethnic origin than the majority of residents of Slovenia).

Results

Understanding discrimination

The ADT aimed at improving the participants' ability to recognise various manifestations of *direct* and *indirect* discrimination, as well as to distinguish them from cases of positive discrimination (or *positive measures*). They were asked to read descriptions of various situations and decide whether the acts mentioned constitute discrimination or not.

Table 3. *The cases of discrimination – the shares of correct answers.*

	Group Experiment–T1 N = 52	Group Experiment–T2 N = 52	Group Control N = 130
Musical ear	29.4%	55.8%	19.4%
Roma/employment	73.1%	76.9%	67.7%
Age/marital status	55.8%	76.9%	53.5%
Women/quota	55.8%	80.8%	27.7%
Gay	96.2%	98.1%	88.5%
Asthma	80.8%	98.1%	80.8%
Roma/restaurant	88.5%	100.0%	75.0%
Enrolment criteria	40.4%	76.9%	20.5%

After completing the training, the participants' ability to identify *direct discrimination* (see Appendix, statements No. 1, No. 2 and No. 3) improved. There was an increase in the share of participants who understood that a Roma person is discriminated against if they are forbidden to enter a restaurant, that a child is discriminated against if they are not allowed to attend preschool because of asthma, and that a gay teacher is discriminated against if they lose their job because of their sexual orientation (Table 3). In the process of learning through group experiences and discussions, the participants increased their ability to use the *comparable situations* concept (Manual for Trainers, 2006): they learned that discrimination occurs if one person is treated less favourably than another in a similar situation on any grounds, including racial or ethnic origin as well as other characteristics defined by law.

The share of participants identifying *indirect discrimination* (see Appendix, statements No. 4, No. 5 and No. 6) increased in all cases (Table 3). The participants learned that enrolment criteria that give priority to children whose parents are employed and have a permanent residence are only seemingly justified. Although the argument that these children are most "in need" of full-day care is plausible, the teachers learned that such criteria also place the children of unemployed parents, who often have a low income, at a disadvantage, as well as those with a temporary residence or without citizenship (potentially immigrants, foreigners and Roma children). Furthermore, an improvement was shown in the participants' understanding that the restriction on enrolment in an early childhood study programme based on the candidates' musical ear is only apparently neutral, and is therefore discriminatory for two reasons: firstly, because a musical ear is not an essential competence for the early childhood profession and, secondly, because access to the study programme is limited on the basis of a person's inborn characteristic. A similar substantive analysis was required in the case of job reductions; after completing the training, more participants recognised discrimination in criteria whereby a teacher loses their job on the grounds of their personal attributes, such as age and marital status, rather than on those related to their job description.

There was also an increase in the share of participants who fully understood *positive measures*, such as a quota for women in political parties or a job quota (see Appendix, statements No. 7 and No. 8), as a means of ensuring equal opportunities (Table 3).

Table 4. *Tukey HSD test – variable DISCRIMINATION.*

	{1}	{2}	{3}
Control {1}		0.2237	**0.0000**
Experiment – T1 {2}	0.2237		**0.0000**
Experiment – T2 {3}	**0.0000**	**0.0000**	

Marked differences are significant at p < .05000

Initially, there were no significant differences between the participants and the control group, while the differences between the participants' positions prior to and after the ADT were clearly significant (Table 4).

Supporting positive measures

The ADT aimed at raising awareness of the role of positive measures in ensuring equal opportunities of the following "at-risk" groups: children with special needs, immigrant children, Roma, and socially disadvantaged children. The participants assessed the measures listed on a ten-point scale ranging from *unjust* to *just*.

Table 5. *Positive measures – Means, Std. Dev.*

The preschool provides and finances …	Experim–T1 N = 52		Experim–T2 N = 52		Control N = 130	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
… lessons in learning the Slovenian language for immigrant children.	7.2	1.4	9.4	2.1	6.1	2.7
… children's picture books in Braille and stories on CDs for blind children.	7.5	0.9	9.6	1.2	7.2	1.9
… the admission of Roma children to preschools free of charge.	4.8	2.7	7.4	2.8	3.5	3.6
… an afternoon programme providing reading and socialising for immigrant children and parents in their mother tongue.	6.9	1.6	9.2	1.6	5.4	3.1
… an additional expert worker offering support to autistic children.	7.6	0.8	9.9	1.1	7.4	1.6
… counselling support for Roma families in their homes.	6.5	1.8	8.7	1.9	4.6	3.3
… the admission of chronically ill children to preschools free of charge.	6.5	2.2	8.8	2.7	5.7	2.8
… playtime hours carried out by teachers inside a Roma neighbourhood for the purpose of gaining the greater trust of Roma families in the preschool.	6.6	2.0	8.7	2.1	5.4	2.9
… holidays for socially disadvantaged children free of charge.	7.4	1.2	9.5	1.6	6.8	2.3
… an afternoon preschool programme for children of families where both parents are unemployed.	5.5	3.0	7.7	2.7	4.1	3.8
… the admission of children from families that receive social assistance benefits to preschools free of charge.	6.5	2.0	8.8	2.0	5.6	3.0
… a temporary translator for the adjustment period of a child who has immigrated to Slovenia.	6.5	1.9	8.5	2.2	5.4	3.0

Prior to the ADT, the participants' scores ranged from 6.5 to 7.6, revealing a slight tendency towards assessing the measures as *just* (with the exception of a lower score for *admission for Roma children to preschools free of payment*); on completing the ADT, all of the scores increased. In all cases, the scores of the non-participants were lower (Table 5). A similar pattern emerged when observing the scores of all respondent groups: the set of measures aimed at children with special needs on average gained the highest scores, followed by those aimed at immigrants and socially disadvantaged children, with the lowest scores concerning Roma children. The lowest overall support for measures targeting Roma children is particularly apparent when comparing the same measure for the different at-risk children, i.e., admission to preschool free of payment; again there was greater support for poor and chronically ill children.

Table 6. *Tukey HSD test – variable POSITIVE MEASURES.*

	{1}	{2}	{3}
Control {1}		0.0035	**0.0000**
Experiment – T1 {2}	**0.0035**		0.0451
Experiment – T2 {3}	0.0000	**0.0451**	
Marked differences are significant at p < .05000			

The Tukey's HSD shows that the participants' overall agreement with the measures was significantly higher than that of the non-participants; however, the ADT still significantly increased the participants' assessment of the measures as being closer to just (Table 6).

Preserving immigrant children's cultural traditions and language

Previous research showed that nearly half of teachers were not in favour of preserving immigrant children's mother tongue and their culture, customs and habits in preschool (Turnšek & Pekkarinen, 2009). Even those whose rhetoric strongly advocated equality often expressed "reservations". The counter-arguments were based on the belief that using the mother tongue and exposing cultural habits does not benefit immigrant children's socialisation and their subsequent school performance; we label these arguments as *pragmatic*, as they express "what is best for children". The second type of argumentation is grounded on an ethnocentric position claiming that immigrant children should adjust to Slovenian culture. We label these beliefs as *ideological*, because they reflect "what is right or wrong in principle". The respondents in our research indicated their agreement with the statements on a Likert scale.

Table 7. *Attitudes to immigrants – Mean, Std. Dev.*

	Experim–T1 N = 52		Experim–T2 N = 52		Control N = 130	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Immigrant parents should speak Slovenian with their children since in this way they show their respect for the country to which they have moved.	1.83	0.90	1.40	0.53	2.45	1.16
Immigrant parents provide the best assistance for their children's social integration if they also speak Slovenian with them in their home environment.	2.62	1.21	1.92	1.01	3.00	1.21
Children's perception of the Slovenian language as the national language is one of the most important objectives of the early childhood curriculum.	3.50	0.94	3.25	1.15	3.73	1.08
It is most beneficial for immigrant children to only speak the Slovenian language because this ensures they will have good academic performance later in school.	1.87	0.82	1.71	0.78	2.13	0.91
It would set a bad example for others if a preschool teacher were to speak with immigrant children in their own language.	1.60	0.66	1.60	0.80	2.12	0.86
At preschool, it is best for immigrant children to only speak Slovenian, as this ensures that they will be accepted by their peers when playing together.	2.37	1.01	2.02	1.06	2.58	1.10
The teacher would act unprofessionally if she/he were to allow Serbian parents to present their Orthodox holiday to children in a Slovenian preschool institution.	1.46	0.85	1.21	0.46	1.68	0.77
It is best for immigrant children to not visit their native country often to avoid experiencing distress.	1.19	0.44	1.17	0.38	1.47	0.61
I find it disrespectful of immigrant parents if they talk with their children in their language in the preschool centre's cloakroom.	1.33	0.51	1.19	0.40	1.65	0.75
In order to avoid circumstances in which immigrant children would feel vulnerable, it is better that at preschool the teacher does not carry out any activities related to their culture.	1.44	0.57	1.23	0.47	1.62	0.74

The participants' agreement with the statements, indicated at a level between 1 and 2, demonstrate their (*strong*) *disagreement* with most statements even prior to the ADT, except for two statements concerning immigrant children learning the Slovenian language (Table 7).

Table 8. *Tukey HSD test – variable ATTITUDES-IMMIGRANTS.*

	{1}	{2}	{3}
Control {1}		0.0033	**0.0000**
Experiment – T1 {2}	0.0033		0.0334
Experiment – T2 {3}	**0.0000**	**0.0334**	
Marked differences are significant at p < .05000			

The participants' overall agreement with the statements relating to immigrant children/parents was significantly lower than that of the non-participants; on completion of the ADT, the participants' agreement decreased significantly (Table 8).

Factors related to the teachers' positions

Multiple regression analysis shows that among all of the independent variables only the variable *ENJOY* is a significant predictor of (all of) the respondents' knowledge of discrimination (*DISCRIMINATION*); R^2 explains 18% of the variance. The positive relationship indicates that those respondents who described themselves as people who *enjoy their awareness that there are cultural differences between themselves and other people* performed better in identifying the discriminatory situations (Beta = 0.16; B = 0.282; p = 0.044). The variables *ENJOY* and *TRY TO LEARN* are predictors of the respondents' assessments of the positive measures; R^2 explains a total of 17% of the variance. Those respondents who described themselves as people who *enjoy cultural differences* ... and/or t*ry to learn as much as possible about the life of people from different cultures* consider the positive measures to be just to a greater extent (*ENJOY*, Beta = 0.19; p = 0.014; *TRY TO LEARN*, Beta = 0.18; p = 0.032). The variables *ENJOY*, *TRY TO LEARN*, *FRIENDS*, *CAUTIOUS* and *AVOID* are predictors of the respondents' attitudes towards immigrant children; R^2 explains a total of 33% of the variance. Respondents who *enjoy cultural differences; try to learn more about other people's lives*; are not *cautious during contact with people from another culture*; and *do not avoid them, as well as those who have many friends with an immigrant background* express significantly less agreement with the arguments opposing the preservation of immigrant children's culture and language (*ENJOY*, Beta = 0.24; p = 0.001; *TRY TO LEARN*, Beta = 0.23; p = 0.001; *FRIENDS*, Beta = 0.16; p = 0.021; *CAUTIOUS*, Beta = – 0.16; p = 0.028; *AVOID*, Beta = – 0.15; p = 0.041).

Conclusions and discussion

The Antidiscrimination and Diversity Training clearly helped reshape the student teachers' orientations regarding diversity and equality issues, as after the training significant differences were found in their beliefs and knowledge, or the initial differences between those not involved in the ADT and the participants increased "in favour" of the latter. The overall positive impact might also be attributed to the participants' high satisfaction with the ADT, especially with the methods of learning; according to the participants, sharing and exchanging their experiences and views within the group helped to raise their awareness of the key concepts related to diversity and equality.

The ADT had a decisive impact on the participants' ability to identify those key circumstances that define discrimination through the use of the *comparable situations concept*, as well as the *apparently neutral concept*, which is crucial for defining hidden discrimination. Equipping teachers with knowledge of discrimination represents a good starting point for asserting the advocacy aspects of their professional role. Moreover, *differential* and/or *preferential treatment* became strongly incorporated into the teachers' conceptualisation of equality, partly due to the ADT, which strengthened their perception of positive measures as important *instruments* in reaching the political aim of equal opportunities. The ADT thus contributed to eliminating the main barrier to the implementation of inclusion, i.e., the prevailing belief that any kind of special treatment represents injustice, as was recognised in earlier studies.

However, the teachers do not perceive all vulnerable life circumstances as involving the same level of risk: not surprisingly, children with special needs are seen as those who are the most "entitled to" additional support. We assume that the *attribution of helplessness* and *responsibility* plays an important role in determining who is entitled to special treatment. In other words, teachers tend not to doubt that children with physical impairments are indeed in the most disadvantaged position, as they cannot adequately help themselves without the support of others, and cannot assume responsibility for improving their own situation. On the other hand, it does not seem equally self-evident that a child who does not speak the Slovenian language, or who comes from another cultural background, also needs support in order to learn and socialise. Recognition of the need for additional support is the weakest in the case of Roma children, which is surprising given the fact that the accumulation of risk factors is particularly associated with a Romany background. In this respect, the study indicates the teachers' "*differentiated approach to differential treatment*".

The present study suggests that teachers' *intrinsic motivation* for improving their knowledge and professional growth might be a more reliable predictor of their positions than the education they have acquired; even though the teachers from the representative Slovenian sample had the same level and type of education as the teachers involved in the ADT, they initially demonstrated less support for positive measures and agreed more with attitudes opposing immigrants' right to preserve their own culture. We argue that the determining feature of the participants was that they have *continued* their education at postgraduate level. In this regard, the results correspond to previous research results (Turnšek & Pekkarinen, 2009).

Finally, the study draws attention to teachers' *self-perception* as an important factor related to their orientation towards diversity and equality (whereas personal and professional circumstances, such as work experience, living in a culturally diverse environment, parental education, etc., show no such relation). *Enjoying* (not simply *accepting*) *cultural differences* between people is the key determining factor, especially when combined with a genuine interest in understanding others or their way of life, and having others within one's friendship network. This set of interrelated self-assessed personality characteristics reflecting an "openness towards otherness" is unambiguously related to greater support for immigrants' cultural rights, as well as for positive measures supporting at-risk children, even with a better knowledge of discrimination. Further research is needed in order to explore the role of other potential (subjective) factors in the professional development of teachers.

References

Ainscow, M., Booth, T., Dyson, A., Farrell, P., Frankham, J., Gallannaugh, F., Howes, A., & Smith, R. (2006). *Improving schools, developing inclusion*. London: Routledge. Retrieved July 3 2013 from http://www.oecd.org/development/anewparadigmfordevelopment.htm

Ainscow, M., & Sandill, A. (2010). Developing inclusive education systems: The role of organisational cultures and leadership. *International Journal of Inclusive Education, 14*(4), 401–416.

Apple, M. W. (1982). *Education and power*. Boston: Routledge and Kegan Paul.

Augoustinos, M., & Walker, I. (1995). *Social cognition*. London: Sage.

Bennett, J. (2008). *Early childhood services in the OECD countries: A review of the literature and of current policy in the early childhood field*, Innocenti Working Paper No. 2008-01. Florence: UNICEF, Innocenti Research Centre.

Berger, L. M., Paxson, C., & Waldfogel, J. (2009). Income and child development. *Children and Youth Services Review, 31*(9), 978–989.

Boal, A. (2000). *Theater of the oppressed*. London: Pluto Press.

Clements, P., & Jones, J. (2006). *The diversity training handbook*. London: Kogan Page.

Dekleva, B., & Razpotnik, Š. (2002). *Čefurji so bili rojeni tu: življenje mladih priseljencev druge generacije v Ljubljani.* [Chefurs were Born Here: The Life of Second-Generation Young Migrants in Ljubljana]. Ljubljana: Faculty of Education, Institute for Criminology at the Faculty of Law.

Dodatek h Kurikulu za vrtce za otroke Romov. [Supplement to the curriculum for educational work with Roma children]. (2002). Ljubljana: Ministry of Education and Sport. Retrieved July 3 2013 from http://www.mss.gov.si/ fileadmin/mss.gov.si/pageuploads/podrocje/vrtci/pdf/vrtci_Dodatek_-_ROMI.pdf

Eurydice 2009. *Tackling social and cultural inequalities through early childhood education and care in Europe*. Brussels: EACEA

Flere, S. (2005). Social inequity and educational expansion in Slovenia. *Educational Studies, 31*(4), 449–464.

Florian, L., & Kershner, R. (2009). Inclusive pedagogy. In H. Daniels, J. Porter, & H. Lauder, (Eds.), *Knowledge, values and educational policy: A critical perspective* (pp. 173–183). London: Routledge.

Gaber, S., & Marjanovic Umek, L. (2009). *Študije (primerjalne) neenakosti.* Znanstvena poročila Pedagoškega inštituta 21/09. [Studies of (comparative) inequality. Scientific reports of the Educational Research Institute of 21/09]. Ljubljana: Educational Research Institute.

Hall, S. (2007). Who needs "identity"? In P. Du Gay, J. Evans, & P. Redman (Eds.), *Identity: A reader* (pp. 15–30). London: SAGE.

Heckman, J. J. (2011). The economics of inequality: The value of early childhood education. *American Educator, 35*(1) 31–35.

Kaikkonen, L. (2010). Promoting teacher development for diversity. In R. Rose (Ed.), *Confronting obstacles to inclusion: International responses to developing inclusive education* (pp. 1-6). Abingdon: Routledge.

Kirbiš, A., Flere, S., & Tavčar Krajnc, M. (2012). Netolerantnost v Sloveniji in Evropi: primerjalna longitudinalna analiza. [Intolerance in Slovenia and Europe: A comparative longitudinal analysis]. *Družboslovne razprave, XXVIII*(70), 27–50.

Klopčič, V., & Polzer, M. (2003). *Evropa, Slovenija in Romi.* [Europe, Slovenia and Roma]. Retrieved July 3 2013 from http://www.inv.si/DocDir/PublikacijePDF/2003/evropa,%20slovenija%20in%20 romi_optimized.pdf

Kozleski, E. B., & Waitoller, F. (2010). Teacher learning for inclusive education: Understanding teaching as a cultural and political practice. *International Journal of Inclusive Education, 14*(7), 655–666.

Krek, J., & Metljak, M. (Eds.) (2011). *Bela knjiga o vzgoji in izobraževanju v Republiki Sloveniji 2011.* [White Paper on Education in the Republic of Slovenia 2011]. Ljubljana: Educational Research Institute.

Mac Naughton, G. (2003). *Shaping early childhood: Learners, curriculum and contexts.* Berkshire: Open University Press.

Macura-Milovanović, S. (2006). Socijalni aspekt inkluzije romske dece iz naselja Deponija u obrazovni sistem. [Social aspect of the inclusion of Roma children from the Deponija settlement into the education system]. *Pedagogija, LXI*(3), 304–320.

Manual for Trainers – Workshops to Counteract Discrimination. (2006). Ljubljana: ZARA.

Mezzirow, J. (1990). *Fostering critical reflection in adulthood: A guide to transformative and emancipator learning.* San Francisco: Jossey-Bass Publishers.

Murray, C., & Urban, M. (2012). *Diversity and equality in early childhood: An Irish perspective*. Dublin: Gill & Macmillan.

Peček, M., & Lesar, I. (2006). *Pravičnost slovenske šole: Mit ali realnost* [Justice of the Slovenian school: Myth or reality?]. Ljubljana: Sophia.

Peček Čuk, M., & Lesar, I. (2008). Teachers' perceptions of the inclusion of marginalised groups. *Educational Studies, 34*(3), 225–239.

Peček, M., & Macura-Milovanović, S. (2012). Who is responsible for vulnerable pupils? The attitudes of teacher candidates in Serbia and Slovenia. *European Journal of Teacher Education, 35*(3), 327–346.

Ruhm, C. J., & Waldfogel, J. (2011). Long-term effects of early childhood care and education. Institute for the Study of Labor. IZA Discussion Paper No. 6149. Retrieved July 3 2013 from http://ssrn.com/abstract=1968100

Smernice za vključevanje otrok priseljencev v vrtce in šole. [Guidelines for the integration of immigrant children in preschools and schools]. (2011). Ljubljana: Institute of RS for Education and Sport. Retrieved July 3 2013 from http:// www.zrss.si/pdf/250811092039_smernice_-dopolnitev.pdf

Strategija vzgoje in izobraževanja Romov v Republiki Sloveniji. [Strategy of Education and Training of Roma Children in the Republic of Slovenia]. (2004). Retrieved July 3 2013 from www.mizks.gov.s/fileadmin/mizks.../0721_- strategija Romi.doc

Strategija vzgoje in izobraževanja Romov v Republiki Sloveniji. [*Strategy of Education and Training of Roma Children in the Republic of Slovenia*]. (2011). Retrieved July 3 2013 from http:// www.mss.gov.si/fileadmin/mss.gov.si/pageuploads/podrocje/razvoj_solstva/projekti/Strategija_Romi_dopolnitev_2011.pdf

Strategija vključevanja otrok, učencev in dijakov migrantov v sistem vzgoje in izobraževanja v Republiki Sloveniji. [National strategy for the social inclusion of migrant children and youth into the education system in the Republic of Slovenia]. (2007). Ljubljana: Ministry of Education and Sport.

Tajfel, H. (1978). *Introducing social psychology*. London: Penguin Books.

Thomas, K. M., & Chrobot-Mason, D. (2005). Group-level explanations of workplace discrimination. In R. L. Dipboye & A. Colella (Eds.), *Discrimination at work: The psychological and organisational bases* (pp. 63–88). London: LEA.

Turnšek, N., & Batistič-Zorec, M. (2009). *Early childhood education and care in Europe: Tackling social and cultural inequality: Slovenia*. Brussels: EACEA. Retrieved July 3 2013 from http://eacea.ec.europa.eu/about/eurydice/documents/098_en_v2.pdf

Turnšek, N., & Pekkarinen, A. (2009). Democratisation of early childhood education in the attitudes of Slovene and Finnish teachers. *European Early Childhood Education Research Journal, 17*(1), 23–42.

Ule, M. (2004). *Socialna psihologija* [Social psychology]. Ljubljana: Faculty of Social Sciences.

Van Dijk, T. A. (2005). Elite discourse and institutional racism. Retrieved July 3 2013 from http://www.discourses.org/UnpublishedArticles/Elite%20discourse%20and%20institutional%20racism.html

Biographical note

Nada Turnšek is an assistant professor of sociology of education at the Faculty of Education of the University of Ljubljana. She is lecturing Social Studies at the Undergraduate Early Childhood Education Study Program as well as subjects such as Democratization of preschool education, Comparative studies of education and Non-discriminatory practice at the Postgraduate study-level. Her main research work concentrates on exploring the teachers' subjective theories and their professional development, on comparative/culture studies of early childhood and educational values, as well as on exploring and promoting diversity and equality in pre-school settings.

Appendix:

1. *The security guard of the restaurant-bar 'Four Roses' is told to prevent Roma people from entering the premises. The owner argues that during the last few months there have been numerous thefts in the restaurant, and that he wants his clients to feel safe. (**Roma/restaurant**)*

2. *A preschool centre is attended by a girl who is often absent because she suffers from asthma. The teacher claims that her pedagogical work is affected due to the girl's frequent absences and late arrivals in the morning. Therefore, the principal proposes to the parents that they take the girl out of the preschool and find more appropriate day care. In response to the parents' complaint, the principal explains that their preschool centre has an internal policy according to which children who do not attend the preschool at least three times a week do not need day care. (**Asthma**)*

3. *Mr Favili teaches geography in a Catholic boys' boarding school. He has never spoken about his private life, but one day at the school's annual Christmas party he presents his male partner to his colleagues and the principal. During the Christmas holidays, he is given notice with an explanation that his lifestyle is not in line with the schools' ethical values and he is therefore no longer suitable as a teacher. (**Gay**)*

4. *A preschool centre has a problem because it cannot enrol all of the children who need day care. Therefore, the committee at the centre decides that enrolment priority will be given to children whose parents are employed, because they need day care the most, and to children whose permanent residence is in the area of the preschool centre. (**Enrolment criteria**)*

5. *In the process of selecting candidates for the Early Childhood Education course at the Faculty of Education, the candidates' musical abilities are tested. A candidate who did not demonstrate any musical ear was rejected. (**Musical ear**)*

6. *Due to downsizing the number of units in a preschool centre, the principal is forced to dismiss staff. Among the first on the list who will lose their jobs are single women under 30 years of age. The principal explains to the collective that young and single people have more opportunities to find a new job. (**Age/marital status**)*

7. *A political party decided to offer 50% of the leading functions to women. The "affected" men complain because they think they are as capable of performing the leading functions as women. (**Women/quota**)*

8. *A well-known Slovenian company wishes to help young people of Roma origin in their entrance to the labour market, so the management agrees to establish a temporary quota of four jobs for apprentices of Roma origin. Frank, a young boy of Slovenian origin, applies for an apprenticeship in the company. In reply, he receives a letter saying: "Unfortunately, we cannot offer you an apprenticeship due to giving priority to candidates of Roma origin in accordance with our new programme. We will keep your application in our register for any potential future needs. We wish you every success." (**Roma/employment**)*

Zgaga, P., Teichler, U., & Brennan, J. (Eds.) (2012). The Globalisation Challenge for European Higher Education / Convergence and Diversity, Centres and Peripheries. Frankfurt/M.: Peter Lang. 389 pp., ISBN 978-3-631-6398-5.

Reviewed by Darko Štrajn[1]

The present book suggests that the notion of "globalisation" is not only a content- empty term, but an unavoidable overarching concept. It covers a long chain of events, realities, ideas, views and standpoints, perhaps simply including many meanings of words that designate the complexities we have to deal with. Higher education is an extremely complex "organism" within the wider complexities of social spaces, cultural diversities, economic relationships and representations in a variety of relevant and irrelevant discourses. The editors and authors of this book, which is an insightful product of a range of institutionally and informally based academic interactions, were obviously aware that the developments in European higher education systems expose the aforementioned chain of meanings to different perceptions and to critical scrutiny. Hence, terms such as Europeanisation, internationalisation, diversification, etc., became linked to "Bolognisation" as an underlying, ongoing process present both before and after the introduction of the crucial declaration in Bologna at the end of the previous millennium. In their introduction to the book, the editors point out that: "The *Zeitgeist* called for the creation of more 'unity' in the European 'diversities'; it was in this context that the political momentum was accumulated to establish the European Higher Education Area (EHEA)" (p. 13). However, the intention to create more unity in diversity rearranged the pattern in which particular nations and regions stand against each other as parts of "centres" and "peripheries". As the editors hint, trends are also taking opposite directions to those prevalent at the start of the process. Of course, the editors do not try to simplify the complex outcomes of the "process" of the last two decades; they stress the importance of research and critical analysis, which are actually performed in fifteen chapters in the three parts of this interesting and engaging book. The contributors provide well-grounded observations and numerous research-based considerations of different aspects, contexts and spaces in which "Bologna" has instigated many changes in accordance and/or in conflict with social changes. However, the book as a whole suggests that the

1 Educational Research Institute (Pedagoški inštitut), Ljubljana, Slovenia

way forward starts by taking into account the different phenomena, realities, frameworks and perspectives of European higher education. Although all of the contributions share a common spirit, they also make the particular chapters diverse and specific, which in the end makes us see the big picture of the EHEA after years of transformations.

Part 1 of the book examines the *Front Issues*. In the first text of this part, Janja Komljenovič and Klemen Miklavič concentrate on the discourses generated by EU institutions. Under the title *Imagining Higher Education in the European Knowledge Economy*, the authors point to the fact that nation-state boundaries have been crossed, creating "new arenas of policy making" (p. 339). The chapter then goes on to show how the documents of EU institutions above all reflect the economic imaginary, i.e., the economic instrumentalisation of higher education based on the paradigm of the so-called knowledge society. The history of interacting concepts and actions in the area of the EHEA points towards a supranational "new constitutionalism." The next paper by Ulrich Teichler, *The Event of International Mobility in the Course of Study*, deals with the cross-border mobility of students as "a key policy objective in Europe" (p. 55). Unfortunately, this is not evident in statistics, which Teichler very precisely reveals as being extremely untrustworthy by citing many aspects of imprecise meaning, definitions, etc. Still, the author gives some interesting estimations, stating that, in his critical judgment, international mobility, as one of the main objectives of the Bologna reform, appears to be only vaguely attained. I must add that this is one of the best recent papers I have came across that distinguishes between policy declarations and hard facts; indeed, the paper is as enjoyable to read as a crime story. Ellen Hazelkorn and Martin Ryan examine *The Impact of University Rankings on Higher Education Policy in Europe*. This chapter does not question the methodology and purpose of University rankings, but in its conclusion argues "that the emergence of global university rankings was not only a challenge to the perceived wisdom about the status and reputation of European higher education, but has stimulated significant changes in European higher education policy" (p. 94). There should be more research in this area, as this chapter remains focused primarily on three central countries: France, Germany and the UK. The reader learns the difference between *Bildung* (education) and *Ausbildung* (vocational education and training) through the chapter written by Elsa Hackl: *Diversification in Austrian Higher Education*. The author somewhat reluctantly acknowledges the impact of the EU on the national level, and especially on the process of the diversification of higher education. Researchers in other countries should take Hackl's contribution as a good example and do their own thinking about national and supranational influences in what, over last two decades, has been labelled as the modernisation of higher education. That which Manja Klemenčič had in mind when writing about "diversification" is slightly different from Hackl's definition. In her article *The Effects of Europeanisation on Institutional Diversification in the Western Balkans*, Klemenčič presents a case study on four Balkan countries: Slovenia, Serbia, Croatia and Albania. The paper undoubtedly demonstrates how tricky introducing European policies – in order to achieve the goals of accessibility and excellence at the same time – can become when it comes to legislation involving new mechanisms to stimulate diversification of aims and quality within a system. The author determines that Slovenia and Croatia have come closer to European goals than the other countries studied. Still, it seems that Klemenčič sees the benefit of the whole process in strengthening research excellence as "the single most powerful element of institutional diversification" (p. 135).

The next five papers in Part 2 of the book are grouped around the topics of *Massified and Internationalised Higher Education*. The first chapter, *The Monolithic Un-intentionality of Higher Education Policies*, written by Voldemar Tomusk, invokes "Karl Marx, Joseph Stalin and the minor classics less known". Tomusk critically challenges a number of established beliefs and notions concerning higher education, building on some broad arguments by such authors as Trow and Clark. In such a brief overview of the whole book, it is impossible to present the scope of this paper. It makes a witty, critical and even somewhat provocative point about the contradictions of the reform process, which unintentionally "fights vigorously" thinking and knowledge. I cannot resist the temptation to observe that this paper reminds one of the spirit of the 1960s and the push for the democratisation of higher education in those times. Close to this spirit is the next chapter by Susan L. Robertson "*Hullabaloo in the Groves of Academe*". The author deals with recent British changes – of course, also taking into account the history – aimed at developing a "competitive higher education market open to for-profit providers". After a rather detailed examination of leading notions and policies, Robertson arrives at a conclusion that should be taken as an axiom: "The differentiation of higher education institutions cannot be separated from the differentiation of the societies of which they are part" (p. 198). Leon Cremonini's assertion that "the preoccupation with league tables and excellence may lead to a state of *bellum omnium contra omnes* that is detrimental rather than beneficial to higher education systems and societies" (p. 201) should be taken very seriously if we are ever going to contemplate going back to the emphasis on social equality as a structuring principle of higher education. Otherwise, Cremonini, in his chapter *The Recognition of Prior Learning and Dutch Higher Education*, gives a detailed overview of what is going on in the

Dutch system and in the policies surrounding it, arriving at the conclusion that recognition of acquired competences "has the potential to address inequality in higher education" (p. 226). The following chapter, *From System Expansion to System Contraction* by Marek Kwiek, provides rich information on developments in Poland, where – similarly to in other post-communist transition countries – a wave of expansion of higher education has been followed by contraction, above all due to demographic causes. Apart from this, Kwiek examines public-private relations in light of the consequences for access to higher education.

Part 3 of this overall inspiring book is focused on *Higher Education in Eastern and South-East Europe*. The chapter written by Martina Vukasović and Mari Elken, entitled *Higher Education Policy Dynamics in a Multi-level Governance Context*, is a good example of a comparative study, involving four countries (Croatia, Estonia, Slovenia and Serbia). The authors analyse meanings of the label 'Europeanisation' within a context that they uncover by presenting the results of their inquiries into some recent developments, which, due to the Bologna process, have to an extent shifted the focus of changes to international concerns. Jana Baćević provides a very informative, instructive and condensed overview of higher education in the post-Yugoslav space in her chapter *What Kind of University for What Kind of Society?* At the same time, Baćević takes care to critically repudiate any doctrinal approaches to the notion of society and its higher education. On the basis of her study of the changes in higher education in the Western Balkans, she clearly corroborates changed social realities, since "the 'society' that universities are supposed to 'belong' to has changed" (p. 305). *The Bosnian Puzzle of Higher Education in the Perspective of the Bologna Process* by Tatjana Sekulić is another chapter that examines a very specific country of the 'periphery'. The article provides quite ample information on the main historical traits of Bosnian higher education and describes the political and social framework of the implementation of "Bologna" from 2007 onwards, when the legal basis was introduced. Bosnia could be one of the rare countries where academics – if we are to believe what some of Sekulić's interviewees report – see more positive than negative impacts of the reform. Higher education reform otherwise came in a package of complex solutions for this war-torn country, now slowly making its way towards the EU. Drawing on some critical educational and social theorists such as Giroux, Apple, Saunders and others, in the chapter *Reclaiming the Role of Higher Education in Croatia*, Danijela Dolenec and Karin Doolan present a case study focusing on the recent Croatian student movement. With the extended critical and theoretical opening section of the paper, the authors clearly join the ranks of the rapidly growing numbers of thinkers in the field of education studies who are developing analyses of the neoliberal economy and politics and their disastrous impact on education, particularly in the domain of higher education. The student movement in Croatia is resisting the privatisation, commodification and marketisation of universities in favour of the accessibility of education and a redefinition of its social role. In the final chapter, *Reconsidering Higher Education Reforms in the Western Balkans*, Pavel Zgaga reflects on higher education reforms in the Western Balkans (comprising all former Yugoslav republics plus Albania). Zgaga focuses particular attention on notions of "centre and periphery", and in this light ponders shifts, developments, positions and perceptions concerning the aforementioned peripheral countries. He provides well-founded reasons to doubt the simple idea that these countries just happen to be areas of colonisation within the currents of globalisation and the implementation of "Bologna." At the same time, some tendencies in the direction of autarchy cannot be generalised as prevailing trends. However, an important feature of this chapter is its conceptually clear narrative, which presents the most important historical facts and relevant developments in the area of higher education before, during and after the Bologna reform.

As seen by the contributors to the present volume, the Bologna reform (as an agency of globalisation) is entering a period of critical evaluation, theoretical reflection and possibly thorough revision. Still, as most of the contributors point out, "Bologna" makes up part of the globalisation process in its best and worst features. In its form and content, this book is one of the pioneering steps forward in the context of recent efforts to reconsider the role of higher education. The relationship between centres and peripheries – by virtue of their existence and irrespective of the different views on them – seems to be the crucial element by which "success" or failure of the EHEA could be judged in the not so distant future – provided that there is a future beyond the neoliberal world and its power frameworks.

List of Referees in Year 2013

The members of the editorial board would like to thank the reviewers for their professional review of the contributions.

Igor Radeka
Jože Rugelj
Vesna Ferk Savec
Gregor Torkar
Nika Golob
Saša A. Glažar
Davorin Kralj
Dušan Krnel
Milan Kubiatko
Mehmet Erdogan
Silvia Markić
Mariana Moynova
Marjan Šimenc
Miha Dešman
Grozdanka Gojkov
Darko Likar
Mojca Peček Čuk
Vlasta Vizek Vidović
Ljubica Marjanovič Umek
Urška Sešek
Beatriz G. Tomšič Čerkez
Vlatka Domović
Tomaž Vec
Valentina Kranželić
Bojan Dekleva
Neven Hrvatić
Robi Kroflič
Ninoslava Pećnik
Olga Poljšak Škraban
Daniela Bratković
Pavel Zgaga
António Magalhães
Lucilla Lopriore
Milena Valenčič Zuljan
Vida Medved Udovič
Simona Kranjc
Stojan Kostanjevec
Pavel Prokop
Alenka Polak
Jana Kalin
Robert Repnik
Dagmara Sokolowska
Leopold Mathelitsch
Katarina Susman
Andras Ambrus
Erkki Pehkonen
Bernd Zimmermann
Barbro Grevholm
Benjamin Rott
Fulvia Furinghetti
Lynn Hart
Avikam Gazit
Zlatan Magajna
Michal Tabach
Torsten Fritzlar
Jarmila Novotna
Peter Liljedahl
Eva Swoboda
Janez Krek
Marija Kavkler
Marcela Batistič Zorec
Alenka Kobolt
Tuula Keinonen

Instructions for Authors for publishing in CEPS Journal (www.cepsj.si – instructions)

Submissions

Manuscript should be from 5,000 to 7,000 words long, including abstract and reference list. Manuscript should be not more than 20 pages in length, and should be original and unpublished work not currently under review by another journal or publisher.

Review Process

Manuscripts are reviewed initially by the Editors and only those meeting the aims and scope of the journal will be sent for blind review. Each manuscript is reviewed by at least two referees. All manuscripts are reviewed as rapidly as possible, but the review process usually takes at least 3 months. The CEPS Journal has a fully e-mail based review system. All submissions should be made by e-mail to: editors@cepsj.si.

For more information visit our web page www.cepsj.si.

Abstracting and indexation

Cooperative Online Bibliographic System and Services (COBISS) | Digital Library of Slovenia - dLib | DOAJ - Directory for Open Access Journals | Academic Journals Database | Elektronische Zeitschriftenbibliothek EZB (Electronic Journals Library) | Base-Search | DRJI - The Directory of Research Journal Indexing | GSU - Georgia State University Library | MLibrary - University of Michigan | NewJour | NYU Libraries | OhioLINK | Open Access Journals Search Engine (OAJSE) | peDOCS: open access to educational science literature | ResearchBib | Scirus | Ulrich's International Periodicals Directory; New Providence, USA

Annual Subscription (4 issues). Individuals 45 €; Institutions 90 €. Order by e-mail: **info@cepsj.si**; postal address: CEPS Journal, Faculty of Education, University of Ljubljana, Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia.

Online edition at **www.cepsj.si**.

Navodila za avtorje prispevkov v reviji (www.cepsj.si – navodila)

Prispevek

Prispevek lahko obsega od 5.000 do 7.000 besed, vključno s povzetkom in viri. Ne sme biti daljši od 20 strani, mora biti izvirno, še ne objavljeno delo, ki ni v recenzijskem postopku pri drugi reviji ali založniku.

Recenzijski postopek

Prispevki, ki na podlagi presoje urednikov ustrezajo ciljem in namenu revije, gredo v postopek anonimnega recenziranja. Vsak prispevek recenzirata najmanj dva recenzenta. Recenzije so pridobljene, kolikor hitro je mogoče, a postopek lahko traja do 3 mesece. Revija vodi recenzijski postopek preko elektronske pošte. Prispevek pošljite po elektronski pošti na naslov: editors@cepsj.si.

Več informacij lahko preberete na spletni strani www.cepsj.si.

Povzetki in indeksiranje

Cooperative Online Bibliographic System and Services (COBISS) | Digital Library of Slovenia - dLib | DOAJ - Directory for Open Access Journals | Academic Journals Database | Elektronische Zeitschriftenbibliothek EZB (Electronic Journals Library) | Base-Search | DRJI - The Directory of Research Journal Indexing | GSU - Georgia State University Library | MLibrary - University of Michigan | NewJour | NYU Libraries | OhioLINK | Open Access Journals Search Engine (OAJSE) | peDOCS: open access to educational science literature | ResearchBib | Scirus | Ulrich's International Periodicals Directory; New Providence, USA

Letna naročnina (4 številke). Posamezniki 45 €; pravne osebe 90 €. Naročila po e-pošti: **info@cepsj.si**; pošti: Revija CEPS, Pedagoška fakulteta, Univerza v Ljubljani, Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia.

Spletna izdaja na **www.cepsj.si**.

University *of Ljubljana*
Faculty of Education

CONTENTS

C·E·P·S *Journal*

Center for Educational Policy Studies Journal

Revija Centra za študij edukacijskih strategij

Vol.3 | Nº4 | Year 2013

www.cepsj.si

ISSN 1855-9719