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Classification of skew morphisms of cyclic groups which are square roots of automorphisms*

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Abstract

The auto-index of a skew morphism φ of a finite group A is the smallest positive integer h such that φ^h is an automorphism of A. In this paper we develop a theory of auto-index of skew morphisms, and as an application we present a complete classification of skew morphisms of finite cyclic groups which are square roots of automorphisms.

Keywords: Skew morphism, auto-index, period, square root.

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1 Introduction

Throughout the paper, groups considered are all finite. A *skew morphism* of a group A is a permutation φ on A fixing the identity element of A and for which there is a function $\pi\colon A \to \mathbb{Z}_{|\varphi|}$ on A, called the *power function* of φ, such that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$ for all $a, b \in A$. It is apparent the notion of skew morphism is a generalization of that of group automorphism. A skew morphism of A is called *proper* if it is not an automorphism. Two skew morphisms φ and φ' of A are *conjugate* if there exists an automorphism θ of A such that $\varphi' = \theta\varphi\theta^{-1}$.

The concept of skew morphism was first introduced by Jajcay and Širáň in [13] as an algebraic tool to study regular Cayley maps, which are regular embeddings of graphs on orientable closed surfaces admitting a regular subgroup of automorphisms on the vertices of the embedded graph. In this direction, regular Cayley maps of cyclic groups and dihedral groups have been classified, see [8, 21] and [14, 15, 16, 19, 28, 27]. In contrast, classification of regular Cayley maps of non-cyclic abelian groups and other metacyclic groups is still in progress; see [4, 5, 7, 20, 22, 26] for details.

The connection between skew morphisms and regular Cayley maps reveals a deep relationship between skew morphisms and group factorizations with cyclic complements. Indeed, if a group G is expressible as a product $A\langle y\rangle$ of a subgroup A and a cyclic subgroup $\langle y\rangle$ with $A \cap \langle y\rangle = 1$, then left multiplication of elements of A by y gives rise to a skew morphism φ of A, determined by $ya = \varphi(a)y^{\pi(a)}$ for all $a \in A$. Conversely, if φ is a skew morphism of a group A, then for any $a, b \in A$, we have

$$\varphi L_a(b) = \varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b) = L_{\varphi(a)}\varphi^{\pi(a)}(b),$$

so $\langle\varphi\rangle L_A \subseteq L_A\langle\varphi\rangle$, where $L_A = \{L_a \mid a \in A\}$ is the left regular representation of A. Since $\langle\varphi\rangle \cap L_A = 1$, we have $|\langle\varphi\rangle L_A| = |L_A\langle\varphi\rangle|$, and hence $\langle\varphi\rangle L_A = L_A\langle\varphi\rangle$. Therefore, $G = L_A\langle\varphi\rangle$ is a factorization of a transitive permutation group with a cyclic complement, which is often referred to as the *skew-product group* of φ. The interested reader is referred to [6, 17] for more details.

A prominent problem in this field is the classification of skew morphisms of cyclic groups, which is closely related to regular Cayley maps [8] as well as edge-transitive embeddings of complete bipartite graphs [11]. Kovács and Nedela [17] showed that if $n = n_1n_2$ such that $\gcd(n_1, n_2) = 1$ and $\gcd(n_1, \phi(n_2)) = \gcd(\phi(n_1), n_2) = 1$, then every skew morphism φ of the cyclic additive group $\mathbb{Z}_n$ is a direct product $\varphi = \varphi_1 \times \varphi_2$ of skew morphisms φ_i of $\mathbb{Z}_{n_i}$, $i = 1, 2$. In a subsequent paper [18] the authors classified all skew morphisms of the cyclic groups $\mathbb{Z}_{p^e}$, where p is an odd prime. As for the case $p = 2$, the associated skew product groups are classified by Du and Hu in [9].

Recently, Bachratý and Jajcay introduced the notion of period of skew morphisms [1]. More precisely, the *period* of a skew morphism φ is the smallest positive integer d such that $\pi\big(\varphi^d(a)\big) = \pi(a)$ for all $a \in A$. In particular, if $d = 1$ then the skew morphism is said to be *smooth* (or *coset-preserving*). In [1, 23], it was shown that if φ is a skew morphism of period d, then φ^d is a smooth skew morphism. The smooth skew morphisms of cyclic groups and of dihedral groups were classified in [2] and [23] respectively. Let φ be a skew morphism of a group A with power function π. If for any $a \in A$ either $\pi(a) = \pi(\varphi(a)) = \cdots = \pi(\varphi^{|\varphi|-1}(a)) = 1$ or $\pi(a) = \pi(\varphi(a)) = \cdots = \pi(\varphi^{|\varphi|-1}(a)) = t$ where $|\varphi|$ is the order of φ and t is a fixed integer with $1 \le t < |\varphi|$, then φ is called *t-balanced*. Observe that every t-balanced skew morphism φ of a group A is necessarily smooth, and in particular φ^{t+1} is an automorphism of A (see [10] and Remark 3.2 in Section 3). Thus, any t-balanced skew morphism is a $(t+1)$-*th root* of a group automorphism.

Inspired by those results above, we propose the following two related problems:

Problem 1.1. Let A be a given group, and d a given positive integer.

(a) Classify all skew morphisms of A which are d-th roots of automorphisms of A.

(b) Classify all skew morphisms of A which have period d.

For $A = \mathbb{Z}_n$ and $d = 2$, the following main result of this paper is a solution to the first problem, and by Theorem 3.8 (a) in Section 4 it is also a partial solution to the second one (skew morphisms of period 2 of $\mathbb{Z}_n$ whose square is an automorphism are determined).

Theorem 1.2. *Every proper skew morphism of the cyclic additive group $\mathbb{Z}_n$ which is a square root of an automorphism is conjugate to a skew morphism of the form*

$$\varphi(x) \equiv sx - \frac{x(x-1)n}{2k} \pmod{n},$$

where the pair (k, s) of positive integers satisfy the following conditions:

(a) *k^2 divides n and $s \in \mathbb{Z}_n^*$ if k is odd, and $2k^2$ divides n and $s \in \mathbb{Z}_{n/2}^*$ if k is even,*

(b) *$s \equiv -1 \pmod{k}$, s has multiplicative order 2ℓ in $\mathbb{Z}_{n/k}$ and $\gcd(w, k) = 1$ where*

$$w = \frac{k}{n}(s^{2\ell} - 1) - \frac{s(s-1)}{2}\ell.$$

The power function of φ is given by $\pi(x) \equiv 1 + 2xw'\ell \pmod{m}$, where $w'w = 1 \pmod{k}$ and $m = 2k\ell$ is the order of φ. Moreover, two such skew morphisms corresponding to distinct integer pairs are not conjugate.

The paper is organized as follows. After a summary of preliminary results in Section 2, we develop a more comprehensive theory of powers of skew morphisms by defining a new notion called auto-index in Section 3. In Section 4 we show that if φ is a proper skew morphism of a group A which is a square root of an automorphism, then its power function has the property $\pi(xy) \equiv \pi(x) + \pi(y) - 1 \pmod{|\varphi|}$ for all $x, y \in A$; in particular, if $A = \mathbb{Z}_n$, then $\pi(x) \equiv (\pi(1) - 1)x + 1 \pmod{|\varphi|}$ for all $x \in \mathbb{Z}_n$. As an application of the theory, we present a proof of Theorem 1.2 in Section 5. Finally, for the special case when $n = p^e$ is a prime power, we enumerate proper skew morphisms of $\mathbb{Z}_n$ which are square roots of automorphisms in Section 6.

2 Preliminaries

In this section we summarize some preliminary results on skew morphisms for future reference.

Proposition 2.1 ([1, 13])**.** *Let φ be a skew morphism of a group A, and let $\pi\colon A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. Then for any positive integer k,*

$$\varphi^k(ab) = \varphi^k(a)\varphi^{\sigma(a,k)}(b), \qquad \text{for all} \quad a, b \in A,$$

where $\sigma(a,k) = \sum_{i=1}^{k} \pi(\varphi^{i-1}(a))$; moreover, φ^k is a skew morphism if and only if the congruence $kx \equiv \sigma(a,k) \pmod{m}$ is solvable for every $a \in A$.

Proposition 2.2 ([13]). *Let φ be a skew morphism of a group A, and let $\pi\colon A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. Then for any $a, b \in A$,*

$$\pi(ab) \equiv \sum_{i=1}^{\pi(a)} \pi(\varphi^{i-1}(b)) \pmod{m}.$$

Proposition 2.3 ([23]). *Let φ be a skew morphism of a group A, and let $\pi\colon A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. Then for any automorphism θ of A, $\varphi' = \theta\varphi\theta^{-1}$ is a skew morphism of A with power function $\pi' = \pi\theta^{-1}$.*

It follows that the automorphism group $\mathrm{Aut}(A)$ of A acts by conjugation on the set $\mathrm{Skew}(A)$ of all skew morphisms of A. Two skew morphisms of A are conjugate if they belong to the same orbit under such action.

An important subgroup related to skew morphisms is the *kernel* of φ defined by

$$\mathrm{Ker}\,\varphi = \{a \in A \mid \pi(a) \equiv 1 \pmod{m}\}.$$

It is well known that, for any $a, b \in A$, $\pi(a) \equiv \pi(b) \pmod{m}$ if and only if $ab^{-1} \in \mathrm{Ker}\,\varphi$, so π takes exactly $|A : \mathrm{Ker}\,\varphi|$ distinct values in $\mathbb{Z}_m$. The index $|A : \mathrm{Ker}\,\varphi|$ is called the *skew-type* of φ. It is obvious that φ is an automorphism if and only if it has skew-type 1. A skew morphism which is not an automorphism will be called *proper*.

The subset

$$\mathrm{Fix}\,\varphi = \{a \in A \mid \varphi(a) = a\}$$

of fixed-points of φ forms a subgroup of A. A subgroup N of A is *φ-invariant* if $\varphi(N) = N$. Clearly, $\mathrm{Fix}\,\varphi$ is φ-invariant, but $\mathrm{Ker}\,\varphi$ may not be. However, the subset

$$\mathrm{Core}\,\varphi = \bigcap_{i=1}^{m} \varphi^i(\mathrm{Ker}\,\varphi)$$

forms the largest φ-invariant subgroup of A contained in $\mathrm{Ker}\,\varphi$, and in particular, it is normal in A [28]. Thus $\mathrm{Ker}\,\varphi$ is φ-invariant if and only if $\mathrm{Ker}\,\varphi = \mathrm{Core}\,\varphi$, in which case the skew morphism is called *kernel-preserving*. It is apparent that if φ is kernel-preserving, then the restriction of φ to $\mathrm{Ker}\,\varphi$ is an automorphism of $\mathrm{Ker}\,\varphi$. The following result is well known.

Proposition 2.4 ([5]). *Every skew morphism of an abelian group is kernel-preserving.*

The importance of φ-invariant normal subgroups is reflected by the following result.

Proposition 2.5 ([29]). *Let φ be a skew morphism of a group A, and let $\pi\colon A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. If N a φ-invariant normal subgroup of A, then $\overline{\varphi}$ defined by $\overline{\varphi}(\overline{x}) = \overline{\varphi(x)}$ is a skew morphism of the quotient group $\overline{A} := A/N$. In particular, the order m_1 of $\overline{\varphi}$ is a divisor of m, and the power function $\overline{\pi}$ of $\overline{\varphi}$ is determined by $\overline{\pi}(\overline{a}) \equiv \pi(a) \pmod{m_1}$ for all $a \in A$.*

Since $\mathrm{Core}\,\varphi$ is a normal subgroup of A, φ induces a skew morphism $\overline{\varphi}$ of the quotient group $\overline{A} = A/\mathrm{Core}\,\varphi$. Define

$$\mathrm{Smooth}\,\varphi = \{a \in A \mid \overline{\varphi}(\overline{a}) = \overline{a}\},$$

which is the preimage of the fixed-point subgroup $\mathrm{Fix}\,\overline{\varphi}$ of $\overline{\varphi}$ under the natural epimorphism of A onto $A/\mathrm{Core}\,\varphi$. Since $\mathrm{Fix}\,\overline{\varphi}$ is a $\overline{\varphi}$-invariant subgroup of $\overline{A}$, $\mathrm{Smooth}\,\varphi$ is a φ-invariant subgroup of A.

In the extremal case that $\mathrm{Smooth}\,\varphi = A$, the skew morphism φ is called *smooth*. In [23] it is shown that a skew morphism φ of A is smooth if and only if $\pi(a) \equiv \pi(\varphi(a)) \pmod{m}$ for all $a \in A$. More generally, the *period* of φ is the smallest positive integer d such that $\pi(\varphi^d(a)) \equiv \pi(a) \pmod{m}$ for all $a \in A$. Thus, φ is smooth if and only if it has period 1. The following properties on the periodicity of skew morphisms are fundamental, see [23] for details.

Proposition 2.6 ([23]). *Let φ be a skew morphism of a group A, and let $\pi\colon A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. If φ has period d, then the following hold:*

(a) *d is equal to the order of the induced skew morphism $\overline{\varphi}$ of $\overline{A} = A/\mathrm{Core}\,\varphi$;*

(b) *d is the smallest positive integer such that φ^d is a smooth skew morphism of A;*

(c) *for any $a \in A$, $\sum_{i=1}^{d} \pi(\varphi^{i-1}(a)) \equiv 0 \pmod{d}$;*

(d) *conjugate skew morphisms have identical periods.*

Note that for any positive integer k, by Proposition 2.6 (a), if φ^k is a smooth skew morphism, then the period d of φ divides k.

3 Skew morphisms and automorphisms

Lemma 3.1. *Let φ be a skew morphism of a group A, and let $\pi\colon A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. Then for any positive integer k, φ^k is a group automorphism if and only if*

$$\sum_{i=1}^{k} \pi\big(\varphi^{i-1}(a)\big) \equiv k \pmod{m}$$

for all $a \in A$. In particular, if φ is smooth, then φ^k is an automorphism if and only if $k\pi(a) \equiv k \pmod{m}$ for all $a \in A$.

Proof. By Proposition 2.1, φ^k is a skew morphism of A if and only if the congruences

$$kx \equiv \sigma(a,k) \pmod{m} \tag{3.1}$$

are solvable for all $a \in A$, where

$$\sigma(a,k) = \sum_{i=1}^{k} \pi\big(\varphi^{i-1}(a)\big).$$

Note that if π_μ is the power function of $\mu := \varphi^k$, then $\pi_\mu(a)$ is the solution of (3.1), and therefore μ is an automorphism if and only if $\sigma(a,k) \equiv k \pmod{m}$ for all $a \in A$. In addition, if φ is smooth, then $\sigma(a,k) = k\pi(a)$, so μ is an automorphism if and only if $k\pi(a) \equiv k \pmod{m}$ for all $a \in A$. □

Remark 3.2. If φ is a t-balanced skew morphism of a group A, then φ is smooth and for all $a \in A \backslash \operatorname{Ker} \varphi$, $\pi(a) \equiv t \pmod{m}$ where $t^2 \equiv 1 \pmod{m}$ [5]. Therefore $(t+1)t \equiv t+1 \pmod{m}$. By Lemma 3.1, φ^{t+1} is a group automorphism. This is a generalization of [10, Lemma 3.4].

Definition 3.3. For a skew morphism φ of a group A, the *auto-index* of φ is defined to be the smallest positive integer h such that φ^h is a group automorphism of A.

Clearly, φ is an automorphism if and only if it has auto-index 1. Lower and upper bounds of the auto-index of a skew morphism are given as follows.

Lemma 3.4. *Let φ be a skew morphism of a group A. Suppose that φ has order m, period d and auto-index h, then d divides h and h divides m.*

Proof. Note that d is the smallest positive integer such that φ^d is a smooth skew morphism. Since φ^h is an automorphism which is necessarily smooth, the minimality of d implies that $d \mid h$. Since $\varphi^m = 1$ is the identity automorphism, the minimality of h implies that $h \mid m$, as required. □

Corollary 3.5. *If φ is a proper skew morphism of prime order, then it is smooth with auto-index equal to its order.*

Proof. Let d and h denote the period and auto-index of φ, respectively. As φ is proper, $d \leq |A : \operatorname{Ker} \varphi| < |\varphi|$ and $h > 1$. By Lemma 3.4, d divides h and h divides $|\varphi|$. Since $|\varphi| = p$ is prime, we obtain $d = 1$ and $h = p$, as required. □

As an example of Corollary 3.5, $\varphi = (0)(153)(2)(4)$ is a proper skew morphism of the cyclic group $\mathbb{Z}_6$. It is smooth, and both its order and auto-index are equal to 3.

Lemma 3.6. *Let φ be a skew morphism of the cyclic group $\mathbb{Z}_n$ and let $\pi : \mathbb{Z}_n \to \mathbb{Z}_m$ be the associated power function, where m is the order of φ. If φ has period 2 and auto-index h, then h is an even positive divisor of m and there exists some $u \in \mathbb{Z}_h$ such that*

$$\pi(x) \equiv \big(\pi(1) - 1\big) \sum_{i=1}^{x} \Big(1 + \frac{um}{h}\Big)^{i-1} + 1 \pmod{m}, \quad \textit{for all } x \in \mathbb{Z}_n. \tag{3.2}$$

Proof. Since φ has period 2, by Proposition 2.6 (c), $\pi(x) + \pi(\varphi(x)) \equiv 0 \pmod{2}$ for all $x \in \mathbb{Z}_n$. By Lemma 3.4, h is an even positive divisor of m. By Lemma 3.1, we have

$$h \equiv \sum_{i=1}^{h} \pi(\varphi^{i-1}(1)) \equiv \frac{1}{2}\Big(\pi(1) + \pi\big(\varphi(1)\big)\Big)h \pmod{m},$$

and then

$$\frac{1}{2}\Big(\pi(1) + \pi\big(\varphi(1)\big)\Big) = 1 + um/h,$$

for some $u \in \mathbb{Z}_h$.

Moreover, since φ has period 2, by Proposition 2.6 (a), $\overline{\varphi}$ is an automorphism of order 2. Thus, $\pi(1) \equiv \overline{\pi}(\overline{1}) \equiv 1 \pmod 2$. Consequently, by Proposition 2.1, we have

$$\begin{aligned}\pi(2) \equiv& \sum_{i=1}^{\pi(1)} \pi\big(\varphi^{i-1}(1)\big)\\ \equiv& \pi(1) + \frac{\pi(1)-1}{2}\Big(\pi(1) + \pi\big(\varphi(1)\big)\Big)\\ \equiv& \pi(1) + \big(\pi(1)-1\big)(1+um/h)\\ \equiv& \big(\pi(1)-1\big)\big(1+(1+um/h)\big)+1 \pmod m.\end{aligned}$$

By induction, we obtain (3.2), as required. □

In what follows we study skew morphisms of auto-index 2. These skew morphisms are all square roots of automorphisms. Clearly, every permutation of order 2 on A is a square root of the identity automorphism of A. Generally, a square root of an automorphism of A maybe not a skew morphism of A. It seems too difficult to determine all square roots of automorphisms for a family of groups. In the following example, all square roots of nonidentity automorphisms of $\mathbb{Z}_8$ are determined.

Example 3.7. The cyclic group $\mathbb{Z}_8$ has three nonidentity automorphisms as follows:

$$\sigma_1 = (0)(2)(4)(6)(1,5)(3,7),\ \sigma_2 = (0)(4)(2,6)(1,3)(5,7),\ \sigma_3 = (0)(4)(2,6)(1,7)(5,3).$$

Since the square of every permutation of order 4 on $\mathbb{Z}_8$ either fixes no element or fixes 4 elements, σ_2 and σ_3 have no square roots. Set $\mu = (0)(2)(4)(6)(1,3,5,7)$ and use C_μ to denote the set of all square roots of the identity automorphism of $\mathbb{Z}_8$ which commute with μ. Then every square root of σ_1 can be represented as a product $\tau\mu$ where $\tau \in C_\mu$. It is straightforward to check that μ and μ^3 are the only two square roots of σ_1 which are skew morphisms. Since $\mu^3 = \sigma_3^{-1}\mu\sigma_3$, $\mathbb{Z}_8$ has a unique conjugate class of skew morphism of auto-index 2.

We are only concerned with square roots of automorphisms which are also skew morphisms. For convenience, skew morphisms of auto-index 2 are called *proper square roots of automorphisms* throughout this paper.

Theorem 3.8. *Let φ be a skew morphism of a group A, and let $\pi : A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. If φ is a proper square root of an automorphism, then*

(a) *φ is kernel-preserving of period at most 2;*

(b) *$\pi(x)$ is odd for all $x \in A$;*

(c) *$\pi(xy) \equiv \pi(x) + \pi(y) - 1 \pmod m$ for all $x, y \in A$;*

Proof. Take an arbitrary element $x \in A$. Since φ^2 is an automorphism and φ is not an automorphism, by Lemma 3.1, we have

$$\pi(x) + \pi(\varphi(x)) \equiv 2 \pmod m \quad \text{and} \quad \pi\big(\varphi(x)\big) + \pi\big(\varphi^2(x)\big) \equiv 2 \pmod m. \quad (3.3)$$

(a) From (3.3) we deduce $\pi(x) \equiv \pi\big(\varphi^2(x)\big) \pmod{m}$, so the period of φ is at most 2. In particular, we see that $\pi\big(\varphi(x)\big) = 1$ whenever $\pi(x) = 1$. It follows that φ is kernel-preserving.

(b) If φ has period 1, then $\pi(x) \equiv \pi\big(\varphi(x)\big) \pmod{m}$, and hence $2\pi(x) \equiv \pi(x) + \pi\big(\varphi(x)\big) \equiv 2 \pmod{m}$. Since φ is not an automorphism, m must be even. Since π is a group homomorphism from A to $\mathbb{Z}_m^*$ [23, Theorem 4.9], $\pi(x)$ is an odd integer. Now assume φ has period 2. Since φ is kernel-preserving, $\operatorname{Ker}\varphi = \operatorname{Core}\varphi$ is normal in A. By Proposition 2.6 (a), the induced skew morphism $\overline{\varphi}$ of $A/\operatorname{Ker}\varphi$ is an automorphism of order 2. Thus, $\pi(x) \equiv \overline{\pi}(\overline{x}) \equiv 1 \pmod{2}$, and $\pi(x)$ is also odd.

(c) By Proposition 2.2, we have

$$\begin{aligned}\pi(xy) \equiv &\sum_{i=1}^{\pi(x)} \pi(\varphi^{i-1}(y))\\ \equiv&\pi(y) + \frac{\pi(x)-1}{2}\big(\pi(y) + \pi(\varphi(y))\big)\\ \equiv&\pi(x) + \pi(y) - 1 \pmod{m}\end{aligned}$$

for all $x, y \in A$. □

Corollary 3.9. *Let φ be a proper square root of an automorphism of a group A, and let $\pi : A \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. Then*

(a) *if φ is smooth, then it has skew-type two, 4 divides m, and $\pi(x) = 1 + m/2$ for all $x \in A \setminus \operatorname{Ker}\varphi$;*

(b) *if φ is not smooth, then it has skew-type at least* 3.

Proof. If φ is smooth, then from the proof of Theorem 3.8, we see that m is even and $2\pi(x) \equiv 2 \pmod{m}$ for any $x \in A$. Hence $\pi(x) = 1$ or $1 + m/2$. Since φ is proper and $\pi(x)$ is odd, 4 divides m. If φ is not smooth, then the skew-type of φ is at least 3 since φ is kernel-preserving of period 2. □

Example 3.10 ([25])**.** The cyclic group $\mathbb{Z}_9$ has four skew morphisms of period 2:

$$\begin{aligned}\varphi_1 &= (0)(1,2,7,5,4,8)(3,6), & \pi_1 &= [1][3,5,3,5,3,5][1,1];\\ \varphi_2 &= (0)(1,5,4,2,7,8)(3,6), & \pi_2 &= [1][3,5,3,5,3,5][1,1];\\ \varphi_3 &= (0)(1,8,4,5,7,2)(3,6), & \pi_3 &= [1][5,3,5,3,5,3][1,1];\\ \varphi_4 &= (0)(1,8,7,2,4,5)(3,6), & \pi_4 &= [1][5,3,5,3,5,3][1,1].\end{aligned}$$

It can be directly verified that φ_i^2 $(i = 1,2,3,4)$ are automorphisms of $\mathbb{Z}_9$, so that all of these skew morphisms are proper square roots of automorphisms. Note that up to conjugation by automorphisms they are divided into two classes $\{\varphi_1, \varphi_4\}$ and $\{\varphi_2, \varphi_3\}$.

Example 3.11. Define two functions φ and π on the cyclic group $\mathbb{Z}_{8n}$ where n is a positive integer as follows:

$$\varphi(x) \equiv \begin{cases} 2i \pmod{8n}, & \text{if } x = 2i;\\ 2(n+i)+1 \pmod{8n}, & \text{if } x = 2i+1\end{cases}$$

and

$$\pi(x) = \begin{cases} 1, & \text{if } x = 2i; \\ 3, & \text{if } x = 2i + 1. \end{cases}$$

It is straightforward to check that φ is a skew morphism of $\mathbb{Z}_{8n}$ with power function π whose square is an involutory automorphism.

4 Technical lemmas

In what follows we restrict our discussion to proper square roots of automorphisms of the cyclic groups.

Lemma 4.1. *Let φ be a skew morphism of the cyclic group $\mathbb{Z}_n$, and let $\pi\colon \mathbb{Z}_n \to \mathbb{Z}_m$ be the power function of φ, where m is the order of φ. If φ is a proper square root of an automorphism and it has skew-type k, then the following hold:*

(a) *there is some integer $\ell \geq 1$ such that $m = 2k\ell$;*

(b) *there is some integer $u \in \mathbb{Z}_k^*$ such that $\pi(x) \equiv 1 + 2xu\ell \pmod{m}$ for all $x \in \mathbb{Z}_n$;*

(c) *the number $r = \varphi^2(1)$ is coprime to n and there exists some integer $v \in \mathbb{Z}_k^*$ such that $r^\ell \equiv 1 + vn/k \pmod{n}$;*

(d) *k^2 is a divisor of n;*

(e) *the multiplicative order of r in $\mathbb{Z}_{n/k}$ is equal to ℓ.*

Proof. By Theorem 3.8, φ has period 1 or 2 and

$$\pi(x + y) \equiv \pi(x) + \pi(y) - 1 \pmod{m}$$

for all $x, y \in \mathbb{Z}_n$. Thus $\pi(2) \equiv 2\pi(1) - 1 \equiv 2\big(\pi(1) - 1\big) + 1 \pmod{m}$ and by induction

$$\pi(x) \equiv x\big(\pi(1) - 1\big) + 1 \pmod{m}, \quad \forall x \in \mathbb{Z}_n.$$

In particular, $\pi(m) \equiv m\big(\pi(1) - 1\big) + 1 \equiv 1 \pmod{m}$, and therefore $m \in \operatorname{Ker}\varphi$. Since φ is of skew-type k, $\operatorname{Ker}\varphi = \langle k \rangle$, and hence $k \mid m$. Noting that

$$1 \equiv \pi(k) \equiv k\big(\pi(1) - 1\big) + 1 \pmod{m},$$

we get $\pi(1) = 1 + um/k$ for some $u \in \mathbb{Z}_k$. Consequently, $\pi(x) \equiv 1 + xum/k \pmod{m}$. Since π takes k distinct values of the form $1 + im/k$ ($i = 0, 1, \ldots, k-1$) in $\mathbb{Z}_m$, we have $u \in \mathbb{Z}_k^*$. By Theorem 3.8, $1 + m/k$ is odd, that is, m/k is even. Thus we can write $m = 2k\ell$, where ℓ is a positive integer. Then $\pi(x) \equiv 1 + 2xu\ell \pmod{m}$.

Set $r = \varphi^2(1)$. Since $\varphi^2 \in \operatorname{Aut}(\mathbb{Z}_n)$, r is coprime to n and $\varphi^2(x) \equiv rx \pmod{n}$ for all $x \in \mathbb{Z}_n$. In particular, $\varphi^{2\ell}(k) \equiv r^\ell k \pmod{n}$. On the other hand, there exists $u' \in \mathbb{Z}_n$ such that $\pi(u') \equiv 1 + 2\ell \pmod{m}$. Therefore

$$\varphi(k) + \varphi(u') \equiv \varphi(k + u') \equiv \varphi(u' + k) \equiv \varphi(u') + \varphi^{1+2\ell}(k) \pmod{n}$$

and then $\varphi^{2\ell}(k) = k$. Thus, $r^\ell \equiv 1 \pmod{n/k}$. Write $r^\ell = 1 + vn/k$. Recalling that φ has period at most 2, we have $\pi\big(\varphi^{2\ell}(1)\big) \equiv \pi(1) \pmod{m}$ and hence $\varphi^{2\ell}(1) \equiv 1$

$\pmod{k}$. It follows that $1 + vn/k \equiv r^{\ell} \equiv \varphi^{2\ell}(1) \equiv 1 \pmod{k}$, and hence k is a divisor of vn/k. Note that

$$\varphi^{2\ell j}(1) \equiv r^{\ell j} \equiv \Big(1 + \frac{vn}{k}\Big)^j \equiv 1 + \frac{jvn}{k} + \sum_{i=2}^{j} \binom{j}{i} \Big(\frac{vn}{k}\Big)^i \equiv 1 + \frac{jvn}{k} \pmod{n}$$

for any positive integer j. By [29, Lemma 3.1], the length of the orbit of 1 under φ is equal to the order $m = 2k\ell$ of φ. If $0 < j < k$, then $1 \not\equiv \varphi^{2j\ell}(1) \equiv 1 + jvn/k \pmod{n}$. Consequently, $v \in \mathbb{Z}_k^*$ and k^2 divides n.

If the multiplicative order of r in $\mathbb{Z}_{n/k}$ is i, then $r^i = 1 + tn/k$ for some positive integer t. Since $r^{\ell} \equiv 1 \pmod{n/k}$, we have $i \mid \ell$. On the other hand, since $k^2 \mid n$ for all $x \in \mathbb{Z}_n$, we have

$$\varphi^{2ik}(x) \equiv r^{ik}x \equiv (1 + tn/k)^k x \equiv x \pmod{n}.$$

Since the order of φ is $2k\ell$, we get $\ell \mid i$, and therefore $\ell = i$. □

Corollary 4.2. *Let φ be a skew morphism of the cyclic group $\mathbb{Z}_n$. If φ is a proper square root of an automorphism, then the induced skew morphism $\overline{\varphi}$ of $\mathbb{Z}_n/\operatorname{Ker}\varphi$ maps each $\overline{x}$ to $-\overline{x}$.*

Proof. Let m and k be the order and the skew-type of φ, respectively. By Lemma 4.1, $m = 2k\ell$ for some positive integer ℓ, and

$$2 \equiv \pi(x) + \pi\big(\varphi(x)\big) \equiv 2 + 2\big(x + \varphi(x)\big)u\ell \pmod{2k\ell}$$

for all $x \in \mathbb{Z}_n$, where $u \in \mathbb{Z}_k^*$. Thus $2\big(x + \varphi(x)\big)u\ell \equiv 0 \pmod{2k\ell}$ and then $\varphi(x) \equiv -x \pmod{k}$, as required. □

The converse of Corollary 4.2 is generally not true, see [6, Theorem 6.5] for a counterexample. However, we have the following result.

Lemma 4.3. *Let φ be a proper skew morphism of the cyclic group $\mathbb{Z}_n$. If the induced skew morphism $\overline{\varphi}$ of $\mathbb{Z}_n/\operatorname{Ker}\varphi$ maps each $\overline{x}$ to $-\overline{x}$, then φ^2 is a skew morphism of skew-type at most 2. In particular, if the skew-type of φ is odd, then φ^2 is an automorphism of $\mathbb{Z}_n$.*

Proof. Throughout the proof, we denote the order and the skew-type of φ by m and k, and the power functions of φ and $\overline{\varphi}$ by π and $\overline{\pi}$, respectively.

If $k = 2$, then the result is obviously true. In what follows we assume $k > 2$. Since $\overline{\varphi}$ maps each $\overline{x}$ to $-\overline{x}$, $\overline{\varphi}$ is an automorphism of order 2. By Proposition 2.6 (a), φ has period 2. It follows that m is even, $\pi\big(\varphi^2(x)\big) \equiv \pi(x) \pmod{m}$ and $\pi\big(\varphi(x)\big) \equiv \pi(-x) \pmod{m}$ for all $x \in \mathbb{Z}_n$. Since $\pi(x) \equiv \overline{\pi}(\overline{x}) \equiv 1 \pmod{2}$, $\pi(x)$ is odd.

Take two arbitrary elements $x, y \in \mathbb{Z}_n$. By Proposition 2.2, we have

$$\pi(x + y) \equiv \sum_{i=1}^{\pi(x)} \pi(\varphi^{i-1}(y)) \equiv \pi(y) + \frac{\pi(x) - 1}{2}\big(\pi(y) + \pi(-y)\big) \pmod{m}.$$

In particular,

$$1 = \pi(x-x) \equiv \pi(-x) + \frac{\pi(x)-1}{2}\big(\pi(x)+\pi(-x)\big) \pmod{m}, \tag{4.1}$$

$$1 = \pi(-x+x) \equiv \pi(x) + \frac{\pi(-x)-1}{2}\big(\pi(x)+\pi(-x)\big) \pmod{m}, \tag{4.2}$$

$$\pi(2x) \equiv \pi(x) + \frac{\pi(x)-1}{2}\big(\pi(x)+\pi(-x)\big) \pmod{m}, \tag{4.3}$$

$$\pi(-2x) \equiv \pi(-x) + \frac{\pi(-x)-1}{2}\big(\pi(x)+\pi(-x)\big) \pmod{m}, \tag{4.4}$$

$$\pi(2x+1) \equiv \pi(2x) + \frac{\pi(1)-1}{2}\big(\pi(2x)+\pi(-2x)\big) \pmod{m}, \tag{4.5}$$

$$\pi(-2x-1) \equiv \pi(-2x) + \frac{\pi(-1)-1}{2}\big(\pi(2x)+\pi(-2x)\big) \pmod{m}. \tag{4.6}$$

Adding (4.1) to (4.2) and (4.3) to (4.4), we get

$$\frac{1}{2}\big(\pi(x)+\pi(-x)\big)^2 \equiv 2 \pmod{m}$$

and

$$\frac{1}{2}\big(\pi(x)+\pi(-x)\big)^2 \equiv \pi(2x)+\pi(-2x) \pmod{m}.$$

Thus,

$$\pi(2x)+\pi(-2x) \equiv 2 \pmod{m}. \tag{4.7}$$

Substituting 2 for $\pi(2x)+\pi(-2x)$ in (4.5) and (4.6) we obtain

$$\pi(2x+1) \equiv \pi(2x)+\pi(1)-1 \pmod{m}$$

and

$$\pi(-2x-1) \equiv \pi(-2x)+\pi(-1)-1 \pmod{m}.$$

It follows that

$$\pi(2x+1)+\pi(-2x-1) \equiv \pi(1)+\pi(-1) \pmod{m}. \tag{4.8}$$

From (4.7) and (4.8) we deduce that

$$\varphi^2(x+y) = \varphi^2(x)+\varphi^2(y)$$

if x is even, and

$$\varphi^2(x+y) = \varphi^2(x)+\varphi^{\pi(1)+\pi(-1)}(y)$$

if x is odd. Thus, φ^2 is a skew morphism of skew-type at most 2. In particular, if the skew-type k of φ is an odd number, then

$$\pi(1)+\pi(-1) \equiv \pi(k+1)+\pi(k-1) \equiv 2 \pmod{m}$$

and therefore φ^2 is an automorphism, as claimed. □

5 Classification

In this section, we classify proper square roots of automorphisms of $\mathbb{Z}_n$.

Theorem 5.1. *Define a quadratic polynomial over the ring* $(\mathbb{Z}_n, +, \times)$ *by*

$$\varphi(x) \equiv sx - \frac{x(x-1)n}{2k} \pmod{n}, \ \ x \in \mathbb{Z}_n, \tag{5.1}$$

where k *and* s *are positive integers satisfying the following conditions:*

(a) k^2 *divides* n *and* $s \in \mathbb{Z}_n^*$ *if* k *is odd, and* $2k^2$ *divides* n *and* $s \in \mathbb{Z}_{n/2}^*$ *if* k *is even,*

(b) $s \equiv -1 \pmod{k}$, s *has multiplicative order* 2ℓ *in* $\mathbb{Z}_{n/k}$ *and* $\gcd(w, k) = 1$ *where*

$$w = \frac{k}{n}(s^{2\ell} - 1) - \frac{s(s-1)}{2}\ell.$$

Then φ *is a proper square root of an automorphism of the cyclic additive group* $\mathbb{Z}_n$ *whose skew-type is* k *and power function is given by*

$$\pi(x) \equiv 1 + 2xw'\ell \pmod{m},$$

where $w'w \equiv 1 \pmod{k}$ *and* $m = 2k\ell$ *is the order of* φ. *Moreover, up to conjugation* φ *is uniquely determined by the parameters* k *and* s.

Proof. First, we show that φ is a permutation on $\mathbb{Z}_n$. Assume $\varphi(x) \equiv \varphi(y) \pmod{n}$ where $x, y \in \mathbb{Z}_n$. Then it suffices to prove that $x \equiv y \pmod{n}$. Since

$$sx - \frac{x(x-1)n}{2k} \equiv sy - \frac{y(y-1)n}{2k} \pmod{n},$$

we get

$$s(x-y) \equiv \frac{(x-y)(x+y-1)n}{2k} \pmod{n}.$$

By (a) and (b) we have $s \in \mathbb{Z}_n^*$. Thus, from the above equation we deduce that $x - y \equiv 0 \pmod{n/k}$. By (a) again we obtain

$$\frac{(x-y)(x+y-1)n}{2k} \equiv 0 \pmod{n},$$

and hence $x \equiv y \pmod{n}$.

Second, we show that φ^2 is an automorphism of $\mathbb{Z}_n$. By (a) and (b), we derive from formula (5.1) that

$$\varphi\Big(\frac{jn}{k}\Big) \equiv \frac{sjn}{k} - \frac{jn(jn-k)n}{2k^3} \equiv -\frac{jn}{k} \pmod{n} \tag{5.2}$$

for all positive integers j. Now for any $x, y \in \mathbb{Z}_n$,

$$\begin{aligned}\varphi(x+y) &\equiv s(x+y) - \frac{(x+y)(x+y-1)n}{2k}\\ &\equiv sx - \frac{x(x-1)n}{2k} + sy - \frac{y(y-1)n}{2k} - \frac{xyn}{k}\\ &\equiv \varphi(x) + \varphi(y) - \frac{xyn}{k} \pmod{n}.\end{aligned}$$

It follows that

$$\begin{aligned}\varphi^2(x) &\equiv \varphi\Big(sx - \frac{x(x-1)n}{2k}\Big)\\ &\equiv \varphi(sx) + \varphi\Big(-\frac{x(x-1)n}{2k}\Big) + \frac{n}{k}\frac{sx^2(x-1)n}{2k}\\ &\equiv \varphi(sx) + \varphi\Big(-\frac{x(x-1)n}{2k}\Big)\\ &\overset{(5.2)}{\equiv} s^2x - \frac{sx(sx-1)n}{2k} + \frac{x(x-1)n}{2k}\\ &\equiv \Big(s^2 - \frac{s(s-1)n}{2k}\Big)x - \frac{(s^2-1)x(x-1)n}{2k}\\ &\overset{(b)}{\equiv} \Big(s^2 - \frac{s(s-1)n}{2k}\Big)x \pmod n.\end{aligned}$$

Since $s \in \mathbb{Z}_n^*$ and $k^2 \mid n$, we have $\gcd\big(s^2 - \frac{s(s-1)n}{2k}, n\big) = 1$. Thus, φ^2 is an automorphism of $\mathbb{Z}_n$.

Next we show that φ is a skew morphism of $\mathbb{Z}_n$ with associated power function π defined by $\pi(x) \equiv 1 + 2w'\ell \pmod m$ for any $x \in \mathbb{Z}_n$, where $w'w \equiv 1 \pmod k$. Take arbitrary $x, y \in \mathbb{Z}_n$. By the conditions (a) and (b), we have

$$\begin{aligned}\varphi(x) + \varphi^{\pi(x)}(y) &\equiv \varphi(x) + \varphi^{1+2xw'\ell}(y) \equiv \varphi(x) + \varphi^{2xw'\ell}(\varphi(y))\\ &\equiv \varphi(x) + \varphi(y)\Big(s^2 - \frac{s(s-1)n}{2k}\Big)^{\ell w' x}\\ &\equiv \varphi(x) + \varphi(y)\Big(s^{2\ell} - \frac{s(s-1)\ell n}{2k}\Big)^{w'x}\\ &\equiv \varphi(x) + \varphi(y)\Big(1 + \frac{wn}{k}\Big)^{w'x}\\ &\equiv \varphi(x) + \varphi(y)\Big(1 + \frac{nx}{k}\Big) \pmod n\end{aligned}$$

and

$$\begin{aligned}\varphi(x+y) &\equiv \varphi(x) + \varphi(y) - \frac{nxy}{k} \equiv \varphi(x) + \Big(sy - \frac{y(y-1)n}{2k}\Big) - \frac{nxy}{k}\\ &\equiv \varphi(x) + \Big(sy - \frac{y(y-1)n}{2k}\Big) + \frac{snxy}{k}\\ &\equiv \varphi(x) + \Big(sy - \frac{y(y-1)n}{2k}\Big)\Big(1 + \frac{nx}{k}\Big)\\ &\equiv \varphi(x) + \varphi(y)\Big(1 + \frac{nx}{k}\Big) \pmod n.\end{aligned}$$

Therefore, $\varphi(x+y) \equiv \varphi(x) + \varphi^{\pi(x)}(y)$ and thus φ is a skew morphism of $\mathbb{Z}_n$.

Finally, we prove that up to conjugation φ is uniquely determined by the parameters k and s. It is evident that if two such skew morphism are conjugate, then they must have the same skew-type k. Suppose now that φ_i $(i = 1, 2)$ are two conjugate skew morphisms of $\mathbb{Z}_n$ defined by

$$\varphi_i(x) \equiv s_i x - \frac{x(x-1)n}{2k} \pmod n,$$

where n, k and s_i satisfy the stated conditions. Then there exists an automorphism θ of $\mathbb{Z}_n$ such that $\varphi_1\theta = \theta\varphi_2$. Set $r = \theta(1)$. Then

$$s_1 rx - \frac{rx(rx-1)n}{2k} \equiv \varphi_1\theta(x) \equiv \theta\varphi_2(x) \equiv s_2 rx - \frac{rx(x-1)n}{2k} \pmod{n}.$$

Since $\gcd(r, n) = 1$, this is reduced to

$$s_1 x - \frac{x(rx-1)n}{2k} \equiv s_2 x - \frac{x(x-1)n}{2k} \pmod{n},$$

or equivalently,

$$(s_1 - s_2)x \equiv \frac{x(rx-1)n}{2k} - \frac{x(x-1)n}{2k} \equiv \frac{x^2(r-1)n}{2k} \pmod{n}.$$

If we choose $x = \pm 1$, then $\pm(s_1 - s_2) \equiv (r-1)n/2k \pmod{n}$. Therefore $2(s_1 - s_2) \equiv 0 \pmod{n}$ and $r \equiv 1 \pmod{k}$. If k is even, so is n, and hence $s_1 \equiv s_2 \pmod{n/2}$. If both k and n are odd, then $s_1 \equiv s_2 \pmod{n}$. If k is odd but n is even, then r is odd. Since $r \equiv 1 \pmod{k}$, we obtain $r - 1 \equiv 0 \pmod{2k}$. Thus, we also get $s_1 \equiv s_2 \pmod{n}$, as required. □

Now we are ready to prove the main result of the paper.

Proof of Theorem 1.2. By Theorem 5.1, the quadratic polynomial of the stated form is a proper square root of an automorphism of $\mathbb{Z}_n$, and distinct pairs (k, s) correspond to disconjugate skew morphisms.

Conversely, suppose that φ is a proper square root of an automorphism of $\mathbb{Z}_n$ of skew-type $k > 1$. By Lemma 4.1, $k^2 \mid n$, $|\varphi| = 2k\ell$ for some positive integer ℓ, and the power function of φ is given by $\pi(x) \equiv 1 + 2xu\ell \pmod{2k\ell}$ for some $u \in \mathbb{Z}_k^*$. Set $s = \varphi(1)$. By Lemma 3.1, we have

$$2 \equiv \pi(1) + \pi(\varphi(1)) \equiv (1 + 2u\ell) + (1 + 2su\ell) \equiv 2 + 2(1+s)u\ell \pmod{2k\ell},$$

which implies $2(1+s)u\ell \equiv 0 \pmod{2kl}$. Since $u \in \mathbb{Z}_k^*$, we obtain $s \equiv -1 \pmod{k}$.

Since φ^2 is an automorphism of $\mathbb{Z}_n$, $\varphi^2(x) \equiv rx \pmod{n}$ for some r coprime to n. By Lemma 4.1, $r^\ell \equiv 1 + vn/k \pmod{n}$ for some $v \in \mathbb{Z}_k^*$. Then

$$\begin{aligned}\varphi(x) \equiv& \varphi(x-1) + \varphi^{\pi(x-1)}(1) \equiv \varphi(x-1) + \varphi^{2\ell u(x-1)+1}(1) \\ \equiv& \varphi(x-1) + \varphi^{2\ell u(x-1)}(s) \equiv \varphi(x-1) + sr^{\ell u(x-1)} \\ \equiv& \varphi(x-1) + s\Big(1 + \frac{vn}{k}\Big)^{u(x-1)} \pmod{n}.\end{aligned}$$

By induction we obtain

$$\varphi(x) \equiv s\sum_{i=1}^{x}\Big(1 + \frac{vn}{k}\Big)^{u(i-1)} \pmod{n}, \quad x \in \mathbb{Z}_n.$$

Since $k^2 \mid n$, for any positive integer j, we have

$$\Big(1 + \frac{vn}{k}\Big)^j \equiv 1 + \frac{jvn}{k} + \sum_{i=2}^{j}\binom{j}{i}\Big(\frac{vn}{k}\Big)^i \equiv 1 + \frac{jvn}{k} \pmod{n}.$$

Thus,

$$\begin{aligned}\varphi(x) \equiv& s\sum_{i=1}^{x}\Big(1+\frac{vn}{k}\Big)^{u(i-1)} \equiv s\sum_{i=1}^{x}\Big(1+\frac{uvn(i-1)}{k}\Big)\\ \equiv& s\Big(x+\frac{uvnx(x-1)}{2k}\Big) \equiv sx-\frac{uvnx(x-1)}{2k} \pmod{n}.\end{aligned}$$

It follows that

$$r=\varphi^2(1)=\varphi(s)\equiv s^2-\frac{uvns(s-1)}{2k} \pmod{n}. \tag{5.3}$$

Hence, $r\equiv s^2 \pmod{n/k}$ and by Lemma 4.1 (e), s has multiplicative order 2ℓ in $\mathbb{Z}_{n/k}$.

Since

$$\begin{aligned}1+\frac{vn}{k} \equiv& r^\ell \equiv \Big(s^2-\frac{s(s-1)uvn}{2k}\Big)^\ell\\ \equiv& s^{2\ell}-\binom{\ell}{1}s^{2(\ell-1)}\frac{s(s-1)uvn}{2k}+\sum_{i=2}^{\ell}\binom{\ell}{i}s^{2(\ell-i)}\Big(-\frac{s(s-1)uvn}{2k}\Big)^i\\ \equiv& s^{2\ell}-\frac{s^{2(\ell-1)}s(s-1)\ell uvn}{2k} \equiv s^{2\ell}-\frac{s(s-1)\ell uvn}{2k} \pmod{n},\end{aligned}$$

we have

$$s^{2\ell}\equiv 1+\Big(1+\frac{s(s-1)\ell u}{2}\Big)\frac{vn}{k} \pmod{n/k}.$$

By [12, Lemma 1], there exists $c\in\mathbb{Z}_n^*$ such that $c\equiv uv \pmod{k}$. Define $\varphi' := \theta_c\varphi\theta_c^{-1}$, where θ_c is the automorphism of $\mathbb{Z}_n$ taking 1 to c. By Proposition 2.3, φ' is a skew morphism of $\mathbb{Z}_n$. For all $x\in\mathbb{Z}_n$, we have

$$\begin{aligned}\varphi'(x) =& \theta_c\varphi\theta_c^{-1}(x)=\theta_c\varphi(c^{-1}x)\equiv c\Big(sc^{-1}x-\frac{c^{-1}x(c^{-1}x-1)cn}{2k}\Big)\\ \equiv& sx-\frac{x(x-c)n}{2k}\equiv\Big(s+\frac{(c-1)n}{2k}\Big)x-\frac{x(x-1)n}{2k} \pmod{n}.\end{aligned}$$

Let $s'=s+\frac{(c-1)n}{2k}$, then it is easily seen that $s'\equiv -1 \pmod{k}$, $s'\in\mathbb{Z}_n^*$, and s' has multiplicative order 2ℓ in $\mathbb{Z}_{n/k}$. Therefore, up to conjugation we can assume

$$\varphi(x)\equiv sx-\frac{x(x-1)n}{2k} \pmod{n} \quad\text{and}\quad \pi(x)\equiv 1+2w'\ell x \pmod{2k\ell},$$

where $s\equiv -1 \pmod{k}$, $s\in\mathbb{Z}_n^*$, $w'\in\mathbb{Z}_k^*$, and 2ℓ is the multiplicative order of s in $\mathbb{Z}_{n/k}$.

We show that $ww'\equiv 1 \pmod{k}$, that is, w' is the modular inverse of w in $\mathbb{Z}_k$. Noting that the congruence

$$w\equiv\frac{k}{n}(s^{2\ell}-1)-\frac{s(s-1)}{2}\ell \pmod{k}$$

is equivalent to

$$s^{2\ell}-\frac{s(s-1)\ell n}{2k}\equiv 1+\frac{nw}{k} \pmod{n},$$

we have

$$\begin{aligned}
2s-\frac{n}{k} &\equiv \varphi(2)\equiv \varphi(1)+\varphi^{\pi(1)}(1)\\
&\equiv s+\varphi^{2w'\ell}(s)\\
&\equiv s+s\Big(s^2-\frac{s(s-1)n}{2k}\Big)^{\ell w'}\\
&\equiv s+s\Big(s^{2\ell}-\frac{s(s-1)\ell n}{k}\Big)^{w'}\\
&\equiv s+s\Big(1+\frac{nw}{k}\Big)^{w'}\\
&\equiv 2s+\frac{sww'n}{k}\equiv 2s-\frac{nww'}{k} \pmod{n},
\end{aligned}$$

which is reduced to $ww'\equiv 1 \pmod{k}$.

In what follows we consider the particular case that k is even. We have

$$\varphi^2(2)=2\varphi^2(1)\equiv 2s^2-\frac{s(s-1)n}{k}\equiv 2s^2-\frac{2n}{k} \pmod{n}$$

and

$$\begin{aligned}
\varphi^2(2) &\equiv \varphi\Big(2s-\frac{n}{k}\Big)\equiv s\Big(2s-\frac{n}{k}\Big)-\Big(2s-\frac{n}{k}\Big)\big(2s-\frac{n}{k}-1\big)\frac{n}{2k}\\
&\equiv 2s^2-\frac{sn}{k}-\Big(s-\frac{n}{2k}\Big)(2s-1)\frac{n}{k}\\
&\equiv 2s^2-\frac{sn}{k}-\Big(2s^2-s-\frac{sn}{k}+\frac{n}{2k}\Big)\frac{n}{k}\\
&\equiv 2s^2-\frac{2s^2n}{k}-\frac{n^2}{2k^2}\equiv 2s^2-\frac{2n}{k}-\frac{n^2}{2k^2} \pmod{n}.
\end{aligned}$$

Thus,

$$2s^2-\frac{2n}{k}\equiv 2s^2-\frac{2n}{k}-\frac{n^2}{2k^2} \pmod{n},$$

and therefore $2k^2\mid n$. Moreover, if $s>n/2$, then we write $s'=s-n/2$ and define

$$\varphi'(x)\equiv s'x-\frac{x(x-1)n}{2k} \pmod{n}, \qquad x\in\mathbb{Z}_n.$$

It is easily seen that φ' is also a square root of an automorphism of $\mathbb{Z}_n$. We show that φ' is conjugate to φ. Since $2k^2\mid n$, $n=2^ekn_1$ where $e\geq 1$ and $2\nmid n_1$. Note that the number $c:=kn_1+1$ is coprime to n. Let θ_c be the automorphism of $\mathbb{Z}_n$ taking x to cx. Then, for any $x\in\mathbb{Z}_n$,

$$\begin{aligned}
\varphi'\theta_c(x) &\equiv s'cx-\frac{cx(cx-1)n}{2k}\\
&\equiv (s-\frac{n}{2})cx-\frac{\big(cx(x-1)+c(c-1)x^2\big)n}{2k}\\
&\equiv scx-\frac{cx(x-1)n}{2k}+\frac{nx}{2}-\frac{c(c-1)x^2n}{2k}\\
&\equiv scx-\frac{cx(x-1)n}{2k}\equiv \theta_c\varphi(x) \pmod{n}.
\end{aligned}$$

Thus, φ is conjugate to φ', as required. □

Corollary 5.2. *Every smooth proper square root of an automorphism of the cyclic group $\mathbb{Z}_n$ is conjugate to a skew morphism of the form*

$$\varphi(x) \equiv sx - \frac{x(x-1)n}{4} \pmod{n}, \quad x \in \mathbb{Z}_n,$$

with the associated power function given by

$$\pi(x) \equiv 1 + 2\ell x \pmod{4\ell}, \quad x \in \mathbb{Z}_n,$$

where $8 \mid n$, both s and $\frac{2}{n}(s^{2\ell}-1) - \frac{s(s-1)}{2}\ell$ are odd numbers, and the multiplicative order of s in $\mathbb{Z}_{n/2}$ is equal to 2ℓ. In particular, φ has order 4ℓ and skew-type 2.

Proof. By Corollary 3.9, every smooth proper square root of an automorphism has skew-type 2. The result follows immediately from Theorem 1.2. □

Remark 5.3. Note that if φ is proper skew morphism of $\mathbb{Z}_n$ and φ^2 is an involutory automorphism, then $|\varphi| = 4$, and by Theorem 1.2, $k = 2$, $\ell = 1$ and φ is smooth.

Corollary 5.4. *Let φ be a non-smooth skew morphism of the cyclic group $\mathbb{Z}_n$. If φ has skew-type 3, then it is conjugate to a skew morphism of the form*

$$\varphi(x) \equiv sx - \frac{n}{6}x(x-1) \pmod{n}, \qquad x \in \mathbb{Z}_n,$$

where $9 \mid n$, $s \in \mathbb{Z}_n^*$ has multiplicative order 2ℓ in $\mathbb{Z}_{n/3}$, $s \equiv -1 \pmod{3}$ and

$$\frac{3}{n}(s^{2\ell} - 1) - \ell \equiv w' \not\equiv 0 \pmod{3}.$$

Moreover, the order of φ is $m = 6\ell$ and the power function of φ is given by

$$\pi(x) \equiv 1 + \frac{m}{3}w'x \pmod{m}.$$

Proof. Since φ is a non-smooth skew morphism of $\mathbb{Z}_n$ of skew-type 3, the induced skew morphism $\overline{\varphi}$ of $\mathbb{Z}_n/\operatorname{Ker}\varphi$ is an automorphism of the form $\overline{\varphi} = (\overline{0})(\overline{1}, -\overline{1})$. By Lemma 4.3, φ^2 is an automorphism. The result then follows from Theorem 1.2. □

By Theorem 1.2, we have the following special property of a square root of an automorphism of the cyclic group $\mathbb{Z}_n$.

Corollary 5.5. *Let φ be a proper square root of an automorphism of the cyclic group $\mathbb{Z}_n$. Then every subgroup of $\mathbb{Z}_n$ is φ-invariant.*

Proof. Let $H = \langle h \rangle$ be a subgroup of $\mathbb{Z}_n$. If φ and φ' are conjugate by an automorphism of $\mathbb{Z}_n$ and H is φ-invariant, then H is also φ'-invariant. So it suffices to consider the skew morphisms φ given by Theorem 1.2. Let k be the skew-type of φ. For any integer j,

$$\varphi(jh) \equiv sjh - \frac{jh(jh-1)n}{2k} \equiv h\left(sj - \frac{j(jh-1)n}{2k}\right) \pmod{n}.$$

If n is even, $\frac{n}{2k}$ is a positive integer, and if n is odd, then h is also odd and $\frac{j(jh-1)n}{2k}$ is a positive integer. This means that $\varphi(jh) \in H$, and hence H is φ-invariant. □

6 The prime power case

In this section, for the case where $n = p^e$ is a prime power, we enumerate the conjugacy classes of proper square roots of automorphisms of $\mathbb{Z}_n$.

We need a technical result from number theory.

Proposition 6.1 ([3, 24]). *Suppose that $n = p^e$, where p is a prime and $e \geq 1$. Then*

(a) *if $p > 2$, then $\mathbb{Z}_{p^e}^* \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{p^{e-1}}$ is cyclic of order $p^{e-1}(p-1)$. In particular, for each i, $1 \leq i \leq e-1$, an element of the form $1 + up^{e-i}$ in $\mathbb{Z}_{p^e}^*$ has order p^i if and only if $p \nmid u$,*

(b) *if $p = 2$, then $\mathbb{Z}_{2^e}^*$ is trivial if $e = 1$, $\mathbb{Z}_{2^e}^* \cong \mathbb{Z}_2$ if $e = 2$, and $\mathbb{Z}_{2^e}^* \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{e-2}}$ if $e \geq 3$. In particular, in the last case for each i, $2 \leq i \leq e-1$, an element of the form $\pm 1 + u2^i$ in $\mathbb{Z}_{2^e}^*$ has order 2^{e-i} if and only if $2 \nmid u$.*

Let $N(p^e)$ denote the number of conjugacy classes of proper square roots of automorphisms of $\mathbb{Z}_{p^e}$. Then $N(p^e)$ is determined in the following theorem.

Theorem 6.2. *Suppose that p is a prime and $e \geq 1$. If $p \neq 2$, then*

$$N(p^e) = \begin{cases} \frac{1}{p-1}(p^{\frac{e}{2}} - 1)^2, & \text{if } e \text{ is even} \\ \frac{1}{p-1}(p^{\frac{e+1}{2}} - 1)(p^{\frac{e-1}{2}} - 1), & \text{if } e \text{ is odd,} \end{cases}$$

while if $p = 2$, then

$$N(2^e) = \begin{cases} 0, & \text{if } e < 3 \\ 1, & \text{if } e = 3 \\ 2^{e-1} - 3 \cdot 2^{\frac{e-2}{2}}, & \text{if } e > 3 \text{ is even} \\ 2^{e-1} - 2^{\frac{e+1}{2}}, & \text{if } e > 3 \text{ is odd.} \end{cases}$$

Proof. Denote $n = p^e$ and $k = p^f$. Then for fixed prime p and integer $e \geq 1$, by Theorem 1.2, $N(p^e)$ is equal to the number of pairs (f, s) which satisfy the following conditions:

(a) $2 \leq 2f \leq e$ and $s \in \mathbb{Z}_{p^e}^*$ if $p \neq 2$, and $2 \leq 2f \leq e-1$ and $s \in \mathbb{Z}_{2^{e-1}}^*$ if $p = 2$,

(b) $s \equiv -1 \pmod{p^f}$, s has multiplicative order 2ℓ in $\mathbb{Z}_{p^{e-f}}$ and $p \nmid w$, where

$$w = p^{f-e}(s^{2\ell} - 1) - \frac{1}{2}s(s-1)\ell.$$

For each admissible value of the parameter f, let $N(p^e, p^f)$ denote the number of admissible values of the parameter s. In what follows, we first determine $N(p^e, p^f)$, and then determine $N(p^e)$. We divide the proof into two cases according to the parity of p.

Case (A). $p \neq 2$.

Since $s \equiv -1 \pmod{p^f}$, we may write $s = tp^h - 1$ where $1 \leq f \leq h \leq e$ and $t \in \mathbb{Z}_{p^{e-h}}^*$. Then $s^2 = 1 + tp^h(tp^h - 2)$. According to the multiplicative order 2ℓ of s in $\mathbb{Z}_{p^{e-f}}$, we distinguish two subcases as follows.

If $h < e - f$, by Proposition 6.1 we have $\ell = p^{e-f-h}$. Since s has multiplicative ordr 2ℓ in $\mathbb{Z}_{p^{e-f}}$, we have $p^{e-f} \parallel s^{2\ell} - 1$. Since $p \mid \frac{1}{2}s(s-1)\ell$, we have $p \nmid w$.

If $h \geq e - f$, then $\ell = 1$. Recalling that $1 \leq f \leq h \leq e$, we have

$$w \equiv tp^{f+h-e}(tp^h - 2) - \frac{1}{2}(tp^h - 1)(tp^h - 2) \equiv -1 - 2tp^{f+h-e} \pmod{p}.$$

Thus, $p \mid w$ if and only if $h = e - f$ and $p \mid 1 + 2t$, where $t \in \mathbb{Z}^*_{p^f}$, in which case the number of such t is equal to p^{f-1}.

Consequently,

$$N(p^e, p^f) = \sum_{h=f}^{e} \phi(p^{e-h}) - p^{f-1} = 1 + \sum_{h=f}^{e-1} p^{e-h-1}(p-1) - p^{f-1} = p^{e-f} - p^{f-1},$$

where ϕ is the Euler's totient function. Therefore,

$$N(p^e) = \sum_{f=1}^{\lfloor e/2 \rfloor} N(p^e, p^f) = \sum_{f=1}^{\lfloor e/2 \rfloor} (p^{e-f} - p^{f-1}) = \frac{1}{p-1}(p^{\lfloor e/2 \rfloor} - 1)(p^{e-\lfloor e/2 \rfloor} - 1).$$

Note that $\lfloor e/2 \rfloor = e/2$ if e is even, and $\lfloor e/2 \rfloor = (e-1)/2$ if e is odd. The stated formula follows from substitution.

Case (B). $p = 2$.

It is straightforward to check that $N(2^2) = 0$, $N(2^3) = N(2^3, 2^1) = 1$ and $N(2^4) = N(2^4, 2^1) = 2$. In what follows, we assume $e \geq 5$ and distinguish two subcases.

Subcase (a). $s \equiv 1 \pmod{4}$.

Since $s \equiv -1 \pmod{2^f}$, we have $f = 1$. Since $s \in \mathbb{Z}^*_{2^{e-1}}$, we may write $s = 1 + 2^h t$ where $2 \leq h \leq e - 2$ and $t \in \mathbb{Z}^*_{2^{e-h-1}}$. By Proposition 6.1 (b), s has multiplicative order 2^{e-h-1} in $\mathbb{Z}_{2^{e-1}}$, and so $\ell = 2^{e-h-2}$. We have $2 \nmid w$ since

$$2^{e-1} \parallel (s^{2\ell} - 1) \quad \text{and} \quad 2 \mid \frac{1}{2}s(s-1)\ell.$$

Subcase (b). $s \equiv -1 \pmod{4}$.

We may write $s = -1 + 2^h t$, where $2 \leq h \leq e - 1$ and $t \in \mathbb{Z}^*_{2^{e-h-1}}$. Since $s \equiv -1 \pmod{2^f}$, we have $f \leq h$. Recall that s has multiplicative order 2ℓ in $\mathbb{Z}_{2^{e-f}}$.

If $h < e - f - 1$, then $e > f + h + 1 \geq 4$. By Proposition 6.1, s has multiplicative order 2^{e-f-h} in $\mathbb{Z}_{2^{e-f}}$, and hence $\ell = 2^{e-f-h-1}$. We also have $2 \nmid w$ since

$$2^{e-f} \parallel (s^{2\ell} - 1) \quad \text{and} \quad 2 \mid \frac{1}{2}s(s-1)\ell.$$

If $h \geq e - f - 1$, then $\ell = 1$ and hence

$$\begin{aligned} w &\equiv 2^{f-e}\big((-1 + 2^h t)^2 - 1\big) - (-1 + 2^h t)(-1 + 2^{h-1} t) \\ &\equiv (-1 + 2^{h-1} t)(2^{f-e+h+1} t - 2^h t + 1) \\ &\equiv 2^{f-e+h+1} t + 1 \pmod{2}. \end{aligned}$$

It follows that $2 \nmid w$ if and only if $h > e - f - 1$. Therefore the case $h = e - f - 1$ should be excluded.

From the above discussion, we obtain

$$N(2^e, 2^1) = \sum_{h=2}^{e-2} \phi(2^{e-h-1}) + \sum_{h=2}^{e-1} \phi(2^{e-h-1}) - \phi(2) = 2^{e-2} - 2,$$

and for $f > 1$,

$$N(2^e, 2^f) = \sum_{h=f}^{e-f-2} \phi(2^{e-h-1}) + \sum_{h=e-f}^{e-1} \phi(2^{e-h-1}) = 2^{e-f-1} - 2^{f-1}.$$

Consequently, for $e \geq 5$, we get

$$\begin{aligned} N(2^e) &= \sum_{f=1}^{\lfloor \frac{e-1}{2} \rfloor} N(2^e, 2^f) = 2^{e-2} - 2 + \sum_{f=2}^{\lfloor \frac{e-1}{2} \rfloor} (2^{e-f-1} - 2^{f-1}) \\ &= 2^{e-2} - 2 + (2^{\lfloor \frac{e-1}{2} \rfloor - 1} - 1)(2^{e-1-\lfloor \frac{e-1}{2} \rfloor)} - 2). \end{aligned}$$

Note that $\lfloor \frac{e-1}{2} \rfloor = (e-2)/2$ if e if even, and $\lfloor \frac{e-1}{2} \rfloor = (e-1)/2$ if e is odd. The result follows from substitution for $\lfloor \frac{e-1}{2} \rfloor$ in the above formula, as required. □

Remark 6.3. By Theorem 1.2, one can enumerate the conjugacy classes of proper square roots of automorphisms of $\mathbb{Z}_n$ for any positive integer n in the following steps:

(a) Find the set of all positive integers k satisfying that k^2 divides n if k is odd, and $2k^2$ divides n if k is even. Denote this set by $A(n)$.

(b) For any $k \in A(n)$, find the set of all s satisfying (i) $s \equiv -1 \pmod{k}$ and (ii) $s \in \mathbb{Z}_n^*$ if k is odd, and $s \in \mathbb{Z}_{n/2}^*$ if k is even. Denote this set by $S(n,k)$.

(c) For any $s \in S(n,k)$, calculate the smallest positive integer ℓ such that $s^{2\ell} \equiv 1 \pmod{n/k}$ and check whether $\frac{k}{n}(s^{2\ell} - 1) - \frac{1}{2}s(s-1)\ell$ is coprime to k or not. Let $A(n,k)$ be the set of all $s \in S(n,k)$ satisfying that $\frac{k}{n}(s^{2\ell} - 1) - \frac{1}{2}s(s-1)\ell$ is coprime to k.

(d) Now (k,s) is admissible for proper square root of automorphism of $\mathbb{Z}_n$ if and only if $k \in A(n)$ and $s \in A(n,k)$. The number $N(n)$ of the conjugacy classes of proper square roots of automorphisms of $\mathbb{Z}_n$ is $\sum_{k \in A(n)} |A(n,k)|$.

Using the method above, we obtain $N(18) = 2$, $N(24) = 2$, $N(40) = 2$ and $N(72) = 16$. In each case the parameters (n,k,s) are given below (details are omitted):

(n,k)	$(18,3)$	$(24,2)$	$(40,2)$	$(72,2)$	$(72,3)$	$(72,6)$
s	$11,17$	$7,11$	$11,19$	$7,11,19,23,31,35$	$11,17,29,35,47,53,65,71$	$23,35$

We close the paper by attaching a full list of conjugacy classes of proper square roots of automorphisms of $\mathbb{Z}_n$ for some small values of n.

Table 1: Proper square roots of automorphisms of $\mathbb{Z}_n$.

n	$\varphi(x)$	$\pi(x)$	$\varphi^2(x)$
8	$6x^2+5x \pmod 8$	$1+2x \pmod 4$	$5x \pmod 8$
9	$3x^2+2x \pmod 9$	$1+2x \pmod 6$	$4x \pmod 9$
9	$3x^2+4x \pmod 9$	$1+2x \pmod 6$	$4x \pmod 9$
16	$12x^2+9x \pmod{16}$	$1+2x \pmod 4$	$9x \pmod{16}$
16	$12x^2+11x \pmod{16}$	$1+2x \pmod 4$	$9x \pmod{16}$
18	$15x^2+2x \pmod{18}$	$1+2x \pmod 6$	$13x \pmod{18}$
18	$15x^2+14x \pmod{18}$	$1+2x \pmod 6$	$7x \pmod{18}$
24	$18x^2+13x \pmod{24}$	$1+2x \pmod 4$	$23x \pmod{24}$
24	$18x^2+17x \pmod{24}$	$1+2x \pmod 4$	$13x \pmod{24}$
27	$9x^2+2x \pmod{27}$	$1+6x \pmod{18}$	$4x \pmod{27}$
27	$9x^2+5x \pmod{27}$	$1+6x \pmod{18}$	$25x \pmod{27}$
27	$9x^2+8x \pmod{27}$	$1+2x \pmod 6$	$10x \pmod{27}$
27	$9x^2+11x \pmod{27}$	$1+6x \pmod{18}$	$13x \pmod{27}$
27	$9x^2+14x \pmod{27}$	$1+12x \pmod{18}$	$7x \pmod{27}$
27	$9x^2+17x \pmod{27}$	$1+4x \pmod 6$	$19x \pmod{27}$
27	$9x^2+20x \pmod{27}$	$1+6x \pmod{18}$	$22x \pmod{27}$
27	$9x^2+23x \pmod{27}$	$1+12x \pmod{18}$	$16x \pmod{27}$
32	$24x^2+11x \pmod{32}$	$1+4x \pmod 8$	$25x \pmod{32}$
32	$24x^2+13x \pmod{32}$	$1+4x \pmod 8$	$25x \pmod{32}$
32	$24x^2+17x \pmod{32}$	$1+2x \pmod 4$	$17x \pmod{32}$
32	$24x^2+19x \pmod{32}$	$1+4x \pmod 8$	$9x \pmod{32}$
32	$24x^2+21x \pmod{32}$	$1+4x \pmod 8$	$9x \pmod{32}$
32	$24x^2+23x \pmod{32}$	$1+2x \pmod 4$	$17x \pmod{32}$
32	$28x^2+11x \pmod{32}$	$1+2x \pmod 8$	$9x \pmod{32}$
32	$28x^2+19x \pmod{32}$	$1+6x \pmod 8$	$25x \pmod{32}$
40	$30x^2+21x \pmod{40}$	$1+2x \pmod 4$	$31x \pmod{40}$
40	$30x^2+29x \pmod{40}$	$1+2x \pmod 4$	$21x \pmod{40}$
64	$48x^2+19x \pmod{64}$	$1+8x \pmod{16}$	$41x \pmod{64}$
64	$48x^2+21x \pmod{64}$	$1+8x \pmod{16}$	$25x \pmod{64}$
64	$48x^2+23x \pmod{64}$	$1+4x \pmod 8$	$17x \pmod{64}$
64	$48x^2+25x \pmod{64}$	$1+4x \pmod 8$	$17x \pmod{64}$
64	$48x^2+27x \pmod{64}$	$1+8x \pmod{16}$	$25x \pmod{64}$
64	$48x^2+29x \pmod{64}$	$1+8x \pmod{16}$	$41x \pmod{64}$
64	$48x^2+33x \pmod{64}$	$1+2x \pmod 4$	$33x \pmod{64}$
64	$48x^2+35x \pmod{64}$	$1+8x \pmod{16}$	$9x \pmod{64}$
64	$48x^2+37x \pmod{64}$	$1+4x \pmod{16}$	$57x \pmod{64}$
64	$48x^2+39x \pmod{64}$	$1+4x \pmod 8$	$49x \pmod{64}$
64	$48x^2+41x \pmod{64}$	$1+4x \pmod 8$	$49x \pmod{64}$
64	$48x^2+43x \pmod{64}$	$1+8x \pmod{16}$	$57x \pmod{64}$
64	$48x^2+45x \pmod{64}$	$1+8x \pmod{16}$	$9x \pmod{64}$
64	$48x^2+47x \pmod{64}$	$1+2x \pmod 4$	$33x \pmod{64}$
64	$56x^2+11x \pmod{64}$	$1+12x \pmod{16}$	$25x \pmod{64}$
64	$56x^2+19x \pmod{64}$	$1+4x \pmod{16}$	$9x \pmod{64}$
64	$56x^2+23x \pmod{64}$	$1+2x \pmod 8$	$17x \pmod{64}$
64	$56x^2+27x \pmod{64}$	$1+12x \pmod{16}$	$57x \pmod{64}$
64	$56x^2+35x \pmod{64}$	$1+4x \pmod{16}$	$41x \pmod{64}$
64	$56x^2+39x \pmod{64}$	$1+6x \pmod 8$	$49x \pmod{64}$

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