An algebraic proof of the Erdős-Ko-Rado theorem for intersecting families of perfect matchings

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Abstract

In this paper we give a proof that the largest set of perfect matchings, in which any two contain a common edge, is the set of all perfect matchings that contain a fixed edge. This is a version of the famous Erdős-Ko-Rado theorem for perfect matchings. The proof given in this paper is algebraic, we first determine the least eigenvalue of the perfect matching derangement graph and then use properties of the perfect matching polytope to prove the result.

Keywords: Perfect matching derangement graph, independent sets, Erdős-Ko-Rado theorem.

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1 Introduction

A perfect matching in the complete graph K_{2k} is a set of k vertex disjoint edges. Two perfect matchings intersect if they contain a common edge. In this paper we use an algebraic method to prove that the natural version of the Erdős-Ko-Rado (EKR) theorem holds for perfect matchings. This theorem shows that the largest set of perfect matchings, with the property that any two intersect, is the set of the all perfect matchings that contain a specific edge.

The algebraic method in this paper is similar to the proof that the natural version of the EKR theorem holds for permutations in [4]. In this paper we determine the least eigenvalue for the perfect matching derangement graph. This, with the Delsarte-Hoffman bound, implies that a maximum intersecting set of perfect matchings corresponds to a facet in the perfect matching polytope. The characterization of the maximum set of intersecting perfect matchings follows from the characterization of the facets of this polytope.

Meagher and Moura [8] proved a version of the EKR theorem holds for intersecting uniform partitions using a counting argument [8]. This result includes the EKR theorem for perfect matchings. It is interesting that the counting argument in [8] is straight-forward, except for the case of perfect matchings; in this case a more difficult form of the counting method is necessary.

2   Perfect matchings

A *perfect matching* is a set of vertex disjoint edges in the complete graph K_{2k}. This is equivalent to a partition of a set of size $2k$ into k-disjoint classes, each of size 2. The number of perfect matchings in K_{2k} is

$$\frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} = (2k-1)(2k-3)\cdots 1.$$

For an odd integer n define

$$n!! = n(n-2)(n-4)\cdots 1.$$

With this notation, there are $(2k-1)!!$ perfect matchings.

We say that two perfect matchings are intersecting if they both contain a common edge. Further, a set of perfect matchings is intersecting if the perfect matchings in the set are pairwise intersecting. If e represents a pair from $\{1,\ldots,2k\}$, then e is an edge of K_{2k}. Define S_e to be the set of all perfect matchings that include the edge e. For any edge e, the set S_e is intersecting. Any set S_e, where e is a pair from $\{1,\ldots,2k\}$, is called a *canonically intersecting* set of perfect matchings. For every e

$$|S_e| = (2k-3)!! = (2k-3)(2k-5)\cdots 1.$$

The main result of this paper can be stated as follows.

Theorem 2.1. *The largest set of intersecting perfect matchings in K_{2k} has size $(2k-3)!!$. The only sets that meet this bound are the canonically intersecting sets of perfect matchings.*

3   Perfect matching derangement graph

One approach to proving EKR theorems for different objects is to define a graph where the vertices are the objects and two objects are adjacent if and only if they are not intersecting (see [4, 9, 15] for just a few examples of where this is done). This is the approach that we take with the perfect matchings.

We use the standard graph notation. A *clique* in a graph is a set of vertices in which any two are adjacent; a *coclique* is a set of vertices in which no two are adjacent. If X is a graph, then $\omega(X)$ denotes the size of the largest clique, and $\alpha(X)$ is the size of the largest coclique. A graph is *vertex transitive* if its automorphism group is transitive on the vertices.

For a vertex-transitive graph, there is a relationship between the maximum clique size and maximum coclique size known as the *clique-coclique bound*. The next result is this bound.

Theorem 3.1. *Let X be a vertex-transitive graph, then*

$$\alpha(X)\,\omega(X) \leq |V(X)|. \qquad \square$$

The *eigenvalues* of a graph are the eigenvalues of the adjacency matrix of the graph. Similarly, the eigenvectors and eigenspaces of the graph are the eigenvectors and eigenspaces of the adjacency matrix.

Define the *perfect matching derangement graph $M(2k)$* to be the graph whose vertices are all perfect matchings on K_{2k} and vertices are adjacent if and only if they have no common edges. Theorem 2.1 is equivalent to the statement that the size of the maximum coclique in $M(2k)$ is $(2k-3)!!$ and that only the canonically intersecting sets meet this bound.

The number of vertices in $M(2k)$ is $(2k-1)!!$. The degree of $M(2k)$, denoted by $d(2k)$, is the number of perfect matchings that do not contain any the edges from some fixed perfect matching. This number can be calculated using the principle of inclusion-exclusion:

$$d(2k) = \sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(2k-2i-1)!!. \qquad (3.1)$$

In practice, this formula can be tricky to use, but we will make use of the following simple lower bound on $d(2k)$.

Lemma 3.2. *For any k*

$$d(2k) > (2k-1)!! - \binom{k}{1}(2k-3)!!.$$

Proof. For any $i \in \{0, \dots, k-1\}$

$$\binom{k}{i}(2k-2i-1)!! = \frac{i+1}{k-i}\binom{k}{i+1}(2k-2i-1)(2k-2(i+1)-1)!!$$

$$> \binom{k}{i+1}(2k-2(i+1)-1)!!$$

(since $\frac{i+1}{k-i}(2k-2i-1) > 1$ for these values of i). This implies that the terms in Equation 3.1 are strictly decreasing in absolute value. Since it is an alternating sequence, the first two terms give a lower bound on $d(2k)$. $\qquad \square$

Next we give some simple properties of the perfect matching derangement graph, including a simple proof of the bound in Theorem 2.1 that uses the clique-coclique bound.

Theorem 3.3. *Let $M(2k)$ be the perfect matching derangement graph.*

1. *The graph $M(2k)$ is vertex transitive, and $\mathrm{Sym}(2k)$ is a subgroup of the automorphism group of $M(2k)$.*

2. *The size of a maximum clique in $M(2k)$ is $2k-1$.*

 3. *The size of a maximum coclique in* $M(2k)$ *is* $(2k-3)!!$.

Proof.   It is clear that the group $\mathrm{Sym}(2k)$ acts transitively on the perfect matchings and, though this action, each permutation in $\mathrm{Sym}(2k)$ gives an automorphism of $M(2k)$.

Let C be a clique in $M(2k)$. For every perfect matching in C, the element 1 is matched with a different element of $\{2, 3, \ldots, 2k\}$. Thus the size of C is no more than $2k-1$. A 1-factorization of the complete graph on $2k$ vertices is a clique of size $\frac{1}{k}\binom{2k}{2}$ in $M(2k)$. Since a 1-factorization of K_{2k} exists for every k, the size of the maximum clique is exactly $\frac{1}{k}\binom{2k}{2} = 2k-1$.

Since $M(2k)$ is vertex transitive, the clique-coclique bound, Theorem 3.1, holds so

$$\alpha(M(2k)) \leq \frac{(2k-1)!!}{\frac{1}{k}\binom{2k}{2}} = (2k-3)!!.$$

Since the size of any canonically intersecting set meets this bound, we have that

$$\alpha(M(2k)) = (2k-3)!!.$$

<div align="right">□</div>

4   Perfect matching association scheme

We have noted that the group $\mathrm{Sym}(2k)$ acts on the set of perfect matchings. Under this action, the stabilizer of a single perfect matching is isomorphic to the wreath product of $\mathrm{Sym}(2)$ and $\mathrm{Sym}(k)$. This is a subgroup of $\mathrm{Sym}(2k)$ and is denoted by $\mathrm{Sym}(2) \wr \mathrm{Sym}(k)$. Thus the set of perfect matchings in K_{2k} correspond to the set of cosets

$$\mathrm{Sym}(2k)/(\mathrm{Sym}(2) \wr \mathrm{Sym}(k)).$$

This implies that the action of $\mathrm{Sym}(2k)$ on the perfect matchings is equivalent to the action of $\mathrm{Sym}(2k)$ on the cosets $\mathrm{Sym}(2k)/(\mathrm{Sym}(2) \wr \mathrm{Sym}(k))$. This action produces a permutation representation of $\mathrm{Sym}(2k)$. We will not give much detail on the representation theory of the symmetric group, rather we will simply state the results that we need a refer the reader to any standard text on the representation theory of the symmetric group, such as [1, 3, 7, 12].

Each irreducible representation of $\mathrm{Sym}(2k)$ corresponds to an integer partition $\lambda \vdash 2k$; these representations will be written as χ_λ and the character will be denoted by χ_λ. The $\mathrm{Sym}(2k)$-module will be denoted by V_λ. Information about the representation is contained in the partition. For example, the dimension of the representation can be found just from the partition using the *hook length formula*.

For any group G, the trivial representation of G is denoted by $\mathbf{1}_G$ (and the character by 1_G). If χ is a representation of a group $H \leq \mathrm{Sym}(n)$, then $\mathrm{ind}_{\mathrm{Sym}(n)}(\chi)$ is the representation of $\mathrm{Sym}(n)$ induced by χ. Similarly, if χ is a representation of $\mathrm{Sym}(n)$, then $\mathrm{res}_H(\chi)$ is the restriction of χ to H. The permutation representation of $\mathrm{Sym}(2k)$ acting on $\mathrm{Sym}(2k)/(\mathrm{Sym}(2) \wr \mathrm{Sym}(k))$ is the representation induced on $\mathrm{Sym}(2k)$ by the trivial representation on $\mathrm{Sym}(2) \wr \mathrm{Sym}(k)$ which is denoted by $\mathrm{ind}_{\mathrm{Sym}(2k)}(\mathbf{1}_{\mathrm{Sym}(2)\wr\mathrm{Sym}(k)})$ (see [5, Chapter 13] for more details).

For an integer partition $\lambda \vdash k$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, let 2λ denote the partition $(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell)$ of $2k$. It is well-known (see, for example, [13, Example 2.2]) that

decomposition of the permutation representation of $\mathrm{Sym}(2k)$ from its action on the perfect matchings is

$$\mathrm{ind}_{\mathrm{Sym}(2k)}(\mathbf{1}_{\mathrm{Sym}(2)\wr\mathrm{Sym}(k)}) = \sum_{\lambda\vdash k}\chi_{2\lambda}.$$

The multiplicity of each irreducible representation in this decomposition is one, this implies that $\mathrm{ind}_{\mathrm{Sym}(2k)}(\mathbf{1}_{\mathrm{Sym}(2)\wr\mathrm{Sym}(k)})$ is a *multiplicity-free representation*. This implies that the adjacency matrices of the orbitals from the action of $\mathrm{Sym}(2k)$ on the cosets $\mathrm{Sym}(2k)/(\mathrm{Sym}(2)\wr\mathrm{Sym}(k))$ defines an association scheme on the perfect matchings (see [5, Section 13.4] for more details and a proof of this result). This association scheme is known as the *perfect matching scheme*. Each class in this scheme is labelled with a partition $2\lambda = (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell)$. Two perfect matchings are adjacent in a class if their union forms a set of ℓ cycles with lengths $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell$ (this association scheme is described in more detail in [5, Section 15.4] and [10]).

The graph $M(2k)$ is the union of all the classes in this association scheme in which the corresponding partition contains no part of size two. This means that each eigenspace of $M(2k)$ is the union of modules of $\mathrm{Sym}(2k)$; each module in this union is a $\mathrm{Sym}(2k)$-module $V_{2\lambda}$ where $\lambda \vdash k$. If ξ is an eigenvalue of $M(2k)$, and its eigenspace includes the $\mathrm{Sym}(2k)$-module $V_{2\lambda}$, then we say that ξ is the eigenvalue for $V_{2\lambda}$. Conversely, we denote the eigenvalue for $V_{2\lambda}$ by $\xi_{2\lambda}$.

The next lemma contains a formula to calculate the eigenvalue for the $\mathrm{Sym}(2k)$-module $V_{2\lambda}$. This gives considerable information about the eigenvalues of $M(2k)$. For a proof the general form of this formula see [5, Section 13.8], we only state the version specific to perfect matchings. If M denotes a perfect matching and $\sigma \in \mathrm{Sym}(2k)$, we will use M^σ to denote the matching formed by the action of σ on M.

Lemma 4.1. *Let M be a fixed perfect matching in K_{2k}. Let $H \subseteq \mathrm{Sym}(2k)$ be the set of all elements $\sigma \in \mathrm{Sym}(2k)$ such that M and M^σ are not intersecting. The eigenvalue of $M(2k)$ for the $\mathrm{Sym}(2k)$-module $V_{2\lambda}$ is*

$$\xi_{2\lambda} = \frac{d(2k)}{2^k\, k!} \sum_{x\in H} \chi_{2\lambda}(x).$$

The $\mathrm{Sym}(2k)$-module $V_{2\lambda}$ is a subspace of the $\xi_{2\lambda}$-eigenspace and the dimension of this subspace is $\chi_{2\lambda}(1)$. □

This formula can be used to calculate the eigenvalue corresponding to a module for the matching derangement graph, but it is not effective to determine all the eigenvalues for a general matching derangement graph. In Section 6 we will show another way to find some of the eigenvalues.

5 Delsarte-Hoffman bound

In Section 6, we will give an alternate proof of the bound in Theorem 2.1 that uses the eigenvalues of the matching derangement graph. This proof is based on the Delsarte-Hoffman bound, which is also known as the ratio bound. The advantage of this bound is that when equality holds we get additional information about the cocliques of maximum size. This information can be used to characterize all the sets that meet the bound. The Delsarte-Hoffman bound is well-known and there are many references, we offer [5, Theorem 2.4.1] for a proof.

Theorem 5.1. *Let X be a k-regular graph with v vertices and let τ be the least eigenvalue of $A(X)$. Then*

$$\alpha(X) \leq \frac{v}{1 - \frac{k}{\tau}}.$$

If equality holds for some coclique S with characteristic vector v_S, then

$$v_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.   □

If equality holds in the Delsarte-Hoffman bound, we say that the maximum cocliques are *ratio tight*.

The Delsarte-Hoffman bound can be used to prove the EKR theorem for sets. Similar to the situation for the perfect matchings, the group $\mathrm{Sym}(n)$ acts on the subsets of $\{1, \ldots, n\}$ of size k. This action is equivalent to the action of $\mathrm{Sym}(n)$ on the cosets $\mathrm{Sym}(n)/(\mathrm{Sym}(n-k) \times \mathrm{Sym}(k))$. This action corresponds to a permutation representation, namely

$$\mathrm{ind}_{\mathrm{Sym}(n)}(\mathbf{1}_{\mathrm{Sym}(n-k) \times \mathrm{Sym}(k)}) = \sum_{i=0}^{k} \chi_{[n-i,i]}. \tag{5.1}$$

(Details can be found in any standard text on the representation theory of the symmetric group.) This representation is multiplicity free and the orbital schemes from this action is an association scheme better known as the *Johnson scheme*.

The *Kneser graph* $K(n, k)$ is the graph whose vertices are all the k-sets from $\{1, \ldots, n\}$ and two vertices are adjacent if and only if they are disjoint. The Kneser graph is a graph in the Johnson scheme, it is the graph that corresponds to the orbitals of pairs of sets that do not intersect. A coclique in $K(n, k)$ is a set of intersecting k-sets. The Kneser graph is very well-studied and all of its eigenvalues are known (see [6, Chapter 7] or [5, Section 6.6] for a proof).

Proposition 5.2. *The eigenvalues of $K(n, k)$ are*

$$(-1)^i \binom{n - k - i}{r - i}$$

with multiplicities $\binom{n}{i} - \binom{n}{i-1}$ for $i \in \{0, \ldots, k\}$.   □

If we apply the Delsarte-Hoffman bound to $K(n, k)$ we get the following theorem which is equivalent to the standard EKR theorem. The characterization follows from the second statement in the Delsarte-Hoffman bound, see [5, Section 6.6] for details.

Theorem 5.3. *Assume that $n > 2k$. The size of the largest coclique in $K(n, k)$ is $\binom{n-1}{k-1}$, and the only cocliques of this size consist of all k-sets that contain a common fixed element.*   □

To apply the Delsarte-Hoffman bound to $M(2k)$, we first need to determine the value of the least eigenvalue of $M(2k)$. We do not calculate all the eigenvalues of $M(2k)$, rather we calculate the two eigenvalues with the largest absolute value and then show that all other eigenvalues have smaller absolute value.

6 Eigenvalues of the matching derangement graph

In this section we determine the largest and the least eigenvalue of the matching derangement graph. Further, we identify the modules that the eigenvalues are for. First we will use a simple method to show that these two values are eigenvalues of $M(2k)$.

For any edge e in K_{2k}, the partition $\pi = \{S_e, V(M(2k))/S_e\}$ is an equitable partition of the vertices in $M(2k)$. In fact, π is the orbit partition formed by the stabilizer of the edge e in $\text{Sym}(2k)$ (this subgroup is isomorphic to $\text{Sym}(2) \times \text{Sym}(2k-2)$) acting on the set of all vertices of $M(2k)$. Each vertex in S_e is adjacent to no other vertices in S_e (since it is a coclique), and is adjacent to exactly $d(2k)$ vertices in $V(M(2k))/S_e$. A straight-forward counting argument shows that each vertex in $V(M(2k))/S_e$ is adjacent to exactly $d(2k)/(2k-2)$ vertices in S_e, and thus to $d(2k) - d(2k)/(2k-2)$ other vertices in $V(M(2k))/S_e$. This implies that the quotient graph of $M(2k)$ with respect to the partition π is

$$M(2k)/\pi = \begin{pmatrix} 0 & d(2k) \\ \frac{1}{2k-2} d(2k) & \frac{2k-3}{2k-2} d(2k) \end{pmatrix}.$$

The eigenvalues for the quotient graph $M(2k)/\pi$ are

$$d(2k), \qquad -\frac{d(2k)}{2k-2}.$$

Since π is equitable, these are also eigenvalues of $M(2k)$. The next result identifies the modules which the eigenvalues are for.

Lemma 6.1. *The eigenvalue of $M(2k)$ for the $\text{Sym}(2k)$-module $V_{[2k]}$ is $d(2k)$, and the eigenvalue of $M(2k)$ for the $\text{Sym}(2k)$-module $V_{[2k-2,2]}$ is $-d(2k)/(2k-2)$.*

Proof. The first statement is clear using the formula in Lemma 4.1.

To prove the second statement, we will consider the equitable partition π defined above. The partition π is the orbit partition of $\text{Sym}(2) \times \text{Sym}(2k-2)$ acting on the perfect matchings. Let $H = \text{Sym}(2) \times \text{Sym}(2k-2)$ and denote the cosets of H in $\text{Sym}(2k)$ by $\{x_0 H = H, x_1 H, \ldots, x_{2k^2-k-1} H\}$.

The $-d(2k)/(2k-2)$-eigenvector of $M(2k)/\pi$ lifts to an eigenvector v of $M(2k)$. A simple calculation shows that the entries of v are $1 - \frac{1}{2k-1}$ or $-\frac{1}{2k-1}$, depending on if the index of the entry is in S_e, or not. This means that $v = v_e - \frac{1}{2k-1}\mathbf{1}$, where v_e is the characteristic vector of S_e.

The group $\text{Sym}(2k)$ acts on the edges of K_{2k}, and for each $\sigma \in \text{Sym}(2k)$, we can define

$$v^\sigma = v_{e^\sigma} - \frac{1}{2k-1}\mathbf{1}.$$

Under this action, the vector v is fixed by any permutation in H. If we define

$$V = \text{span}\{v^\sigma \ : \ \sigma \in \text{Sym}(2k)\},$$

then V is a subspace of the $-d(2k)/(2k-2)$-eigenspace. Moreover, V is invariant under the action of $\text{Sym}(2k)$, so it is also a $\text{Sym}(2k)$-module. To prove this lemma we need to show that V is isomorphic to the $\text{Sym}(2k)$-module $V_{[2k-2,2]}$.

Let W be the $\text{Sym}(2k)$-module for the induced representation $\text{ind}_{\text{Sym}(2k)}(\mathbf{1}_H)$. By Equation 5.1, W is the sum of irreducible modules of $\text{Sym}(2k)$ that are isomorphic to

$M_{[2k]}$, $M_{[2k-1,1]}$ and $M_{[2k-2,2]}$. The vector space W is isomorphic to the vector space of functions $f \in L(\mathrm{Sym}(2k))$ that are constant on H. For each coset xH, let $\delta_{xH}(\sigma)$ be the characteristic function for xH; so $\delta_{xH}(\sigma)$ is defined to be equal to 1 if σ is in xH, and 0 otherwise. Since W is the $\mathrm{Sym}(2k)$-module for the representation induced by $\mathbf{1}_H$, the functions δ_{xH} form a basis for W (see [5, Section 11.13] for details).

Define the map $f : V \to W$ so that

$$f(v^\sigma) = \delta_{\sigma H} - \frac{1}{2k-1} \sum_{i=0}^{2k^2-k-1} \delta_{x_i H}.$$

Since $v^\sigma = v^\pi$ if and only if $\sigma H = \pi H$, this function is well-defined. Further, it is a $\mathrm{Sym}(2k)$-module homomorphism. Thus V is isomorphic to a submodule of W. Since V is not trivial, it must be the $\mathrm{Sym}(2k)$-module $V_{[2k-2,2]}$, since it is the only module (other than the trivial) that is common to both $\mathrm{ind}_{\mathrm{Sym}(2k)}(\mathbf{1}_H)$ and $\mathrm{ind}_{\mathrm{Sym}(2k)}(\mathbf{1}_{\mathrm{Sym}(2)\wr\mathrm{Sym}(k)})$.   □

We have found two of the eigenvalues of $M(2k)$, next we will show that all the remaining eigenvalues are smaller in absolute value. We need the following theorem by Rasala [11] that gives bounds on the dimension of the irreducible representations of $\mathrm{Sym}(n)$.

Theorem 6.2. *For $n \geq 15$, the irreducible representations with the seven smallest degrees are given in the following table.*

Representations	Degree
$[n]$, $[1^n]$	1
$[n-1,1]$, $[2,1^{n-2}]$	$n-1$
$[n-2,2]$, $[2,2,1^{n-4}]$	$\frac{1}{2}n(n-3)$
$[n-2,1,1]$, $[3,1^{n-3}]$	$\frac{1}{2}(n-1)(n-2)$
$[n-3,3]$, $[2,2,2,1^{n-6}]$	$\frac{1}{6}n(n-1)(n-5)$
$[n-3,1,1,1]$, $[4,1^{n-4}]$	$\frac{1}{6}(n-1)(n-2)(n-3)$
$[n-3,2,1]$, $[3,2,1^{n-5}]$	$\frac{1}{3}n(n-2)(n-4)$

Next we will bound the size of the other eigenvalues. This bound follows from the straightforward fact that if A is the adjacency matrix of a graph, then the trace of the square of A is equal to both the sum of the squares of the eigenvalues of A, and to twice the number of edges in the graph. The proof of this result closely follows the proof of the least eigenvalue of the derangement graph of the symmetric group by Ellis [2].

Theorem 6.3. *For $\lambda \vdash k$, the absolute value of the eigenvalue of $M(2k)$ for the $\mathrm{Sym}(2k)$-module $V_{2\lambda}$ is strictly less than $d(2k)/(2k-2)$, unless $\lambda = [k]$ or $\lambda = [k-1,1]$.*

Proof. If $2k < 15$, this theorem can be checked by directly calculating all the eigenvalues (this can easily be done using a computational algebra program such as GAP [16]), so we will assume that $2k \geq 16$.

Let A be the adjacency matrix of $M(2k)$ and use $\xi_{2\lambda}$ to denote the eigenvalue for the $\mathrm{Sym}(2k)$-module $V_{2\lambda}$. The sum of the eigenvalues of A^2 is twice the number of edges in $M(2k)$, that is

$$\sum_{\lambda \vdash k} \chi_{2\lambda}(1)\xi_{2\lambda}^2 = (2k-1)!! \, d(2k).$$

From Lemma 6.1 we know the eigenvalues for two of the modules, so this bound can be expressed as

$$\sum_{\substack{\lambda \vdash k \\ \lambda \neq [k],[k-1,1]}} \chi_{2\lambda}(1)\xi_{2\lambda}^2 = (2k-1)!!\, d(2k) - d(2k)^2 - (2k^2-3k)\left(\frac{d(2k)}{2k-2}\right)^2 .$$

Since all the terms in left-hand side of the above summation are positive, any single term is less than the sum. Thus

$$\chi_{2\lambda}(1)\xi_{2\lambda}^2 \leq (2k-1)!!\, d(2k) - d(2k)^2 - (2k^2-3k)\left(\frac{d(2k)}{2k-2}\right)^2 ,$$

(where $\lambda \vdash k$ and $\lambda \neq [k],[k-1,1]$). If $|\xi_{2\lambda}| \geq d(2k)/(2k-2)$, then this reduces to

$$\chi_\lambda(1) \leq \frac{(2k-1)!!(2k-2)^2}{d(2k)} - 6k^2 + 11k - 4.$$

Using the bound in Lemma 3.2, this implies that

$$\chi_\lambda(1) < 2k^2 - k = \frac{(2k)^2 - (2k)}{2}.$$

If $|\xi_{2\lambda}| \geq d(2k)/(2k-2)$, then 2λ must be one of the first four representations in the table of Theorem 6.2. Thus 2λ must be either $[2k]$ or $[2k-2,2]$, which proves the result. □

We restate this result in terms of the least eigenvalue of the matching derangement graph; noting that Theorem 6.3 implies that $V_{[2k-2,2]}$ is the only $\mathrm{Sym}(2k)$-module that has $-d(2k)/(2k-2)$ as its eigenvalue.

Corollary 6.4. *The smallest eigenvalue of $M(2k)$ is $-d(2k)/(2k-2)$ and the multiplicity of this eigenvalue is $2k^2 - 3k$.* □

7 The $\mathrm{Sym}(2k)$-module $V_{[2k-2,2]}$

Applying the Delsarte-Hoffman bound with the fact that $-d(2k)/(2k-2)$ is the least eigenvalue of $M(2k)$, proves that any canonical coclique is ratio tight since

$$\frac{|V(M(2k))|}{1 - \frac{d}{\tau}} = \frac{(2k-1)!!}{1 - \frac{d(2k)}{-\frac{d(2k)}{2k-2}}} = (2k-3)!!.$$

For S a maximum coclique in $M(2k)$ we will use v_S to denote the characteristic vector of S. The ratio bound implies that $|S| = (2k-3)!!$ and, further, that

$$v_s - \frac{1}{2k-1}\mathbf{1}$$

is a $-d(2k)/(2k-2)$-eigenvector. This vector is called the *balanced* characteristic vector of S, since is it orthogonal to the all ones vector. Since the $\mathrm{Sym}(2k)$-module $V_{[2k-2,2]}$ is the only module for which the corresponding eigenvalue is the least (this follows directly from Theorem 6.3) we have the following result which will be used to determine the structure of the maximum cocliques in $M(2k)$.

Lemma 7.1. *The characteristic vector for any maximum coclique in $M(2k)$ is in the direct sum of the $\mathrm{Sym}(2k)$-modules $V_{[2k]}$ and $V_{[2k-2,2]}$.* □

A perfect matching is a subset of the edges in the complete graph, and thus can be represented as a characteristic vector; this is a vector in $\mathbb{R}^{\binom{2k}{2}}$. Define the *incidence matrix* for the perfect matchings in K_{2k} to be the matrix U whose rows are the characteristic vectors of the perfect matchings of K_{2k}. The columns of U are indexed by the edges in the complete graph and the rows are indexed by the perfect matchings. The column of U corresponding to the edge e is the characteristic vector of S_e.

We will show that the characteristic vector of any maximum coclique of $M(2k)$ is a linear combination of the columns of U.

Lemma 7.2. *The characteristic vectors of the canonical cocliques of $M(2k)$ span the direct sum of the $\mathrm{Sym}(2k)$-modules $V_{[2k]}$ and $V_{[2k-2,2]}$.*

Proof. Let v_e be the characteristic vector of S_e. From Lemma 7.1, the vector $v_e - \frac{1}{2k-1}\mathbf{1}$ is in the $\mathrm{Sym}(2k)$-module $V_{[2k-2,2]}$, and v_e is in the direct sum of the $\mathrm{Sym}(2k)$-modules $V_{[2k]}$ and $V_{[2k-2,2]}$. So all that needs to be shown is that the span of all the vectors v_e has dimension $2k^2 - 3k + 1$, or equivalently, that the rank of U is $2k^2 - 3k + 1$.

Let I denote the $\binom{k}{2} \times \binom{k}{2}$ identity matrix and $A(2k,2)$ the adjacency matrix of the Kneser graph $K(2k,2)$. Then

$$U^T U = (2k-3)!!I + (2k-5)!!A(2k,2).$$

By Proposition 5.2, 0 is an eigenvalue of this matrix with multiplicity $2k-1$. Thus the rank of $U^T U$ (and hence U) is $\binom{2k}{2} - (2k-1) = 2k^2 - 3k + 1$. $\qquad\square$

This result, and the comments at the beginning of this section, imply the following corollary.

Corollary 7.3. *The characteristic vector of a maximum coclique in the perfect matching derangement graph is in the column space of U.* $\qquad\square$

Next we will show that this implies that any maximum coclique is a canonical intersecting set. To do this we will consider a polytope based on the perfect matchings.

8   The perfect matching polytope

The convex hull of the set of characteristic vectors for all the perfect matchings of a graph K_{2k} is called the *perfect matching polytope* of K_{2k}. Let U be the incidence matrix defined in the previous section, then the perfect matching polytope is the convex hull of the rows of U. A *face* of the perfect matching polytope is the convex hull of the rows where Uh achieves its maximum for some vector h. A *facet* is a maximal proper face of a polytope.

If S is a maximum coclique in $M(2k)$, then from Corollary 7.3, we know that $Uh = v_s$ for some vector h. If a vertex of K_{2k} is in S, then the corresponding row of Uh is equal to 1; conversely, if a vertex of K_{2k} is not in S, then the corresponding row of Uh is equal to 0. Thus a maximum intersecting set of perfect matchings is a facet of the perfect matching polytope. In this section, we will give a characterization of the facets of the perfect matching polytope for the complete graph.

Let S be a subset of the vertices of K_{2k} and define the *boundary* of S to be the set of edges that join a vertex in S to a vertex not in S. The boundary is denoted by ∂S and is also known as an *edge cut*. If S is a subset of the vertices of K_{2k} of odd size, then any

perfect matching in K_{2k} must contain at least one edge from ∂S. If S is a single vertex, then any perfect matching contains exactly one element of ∂S. It is an amazing classical result of Edmonds that these two constraints characterize the perfect matching polytope for any graph. For a proof of this result see Schrijver [14].

Theorem 8.1. *Let X be a graph. A vector x in $\mathbb{R}^{|E(X)|}$ lies in the perfect matching polytope of X if and only if:*

(a) $x \geq 0$;

(b) if $S = \{u\}$ for some $u \in V(X)$, then $\sum_{e \in \partial S} x(e) = 1$;

(c) if S is an odd subset of $V(X)$ with $|S| \geq 3$, then $\sum_{e \in \partial S} x(e) \geq 1$.

If X is bipartite, then x lies in the perfect matching polytope if and only if the first two conditions hold. □

The constraints in Equation (b) define an affine subspace of $\mathbb{R}^{|E(X)|}$. The perfect matching polytope is the intersection of this subspace with affine half-spaces defined by the conditions in Equation (a) and Equation (c); hence the points in a proper face of the polytope must satisfy at least one of these conditions with equality.

For any graph X (that is not bipartite) the vertices of a facet are either the perfect matchings that miss a given edge, or the perfect matchings that contain exactly one edge from ∂S for some odd subset S.

It follows from Theorem 8.1 that every perfect matching in K_{2k} is a vertex in the perfect matching polytope for the complete graph. But we can also determine the vertices of every facet in this polytope.

Lemma 8.2. *In the matching polytope of K_{2k}, the vertices of a facet of maximum size are the perfect matchings that do not contain a given edge.*

Proof. Let F be a facet of the polytope of maximum size. From the above comments, equality holds in at least one of equations

$$\sum_{e \in \partial S} x(e) \geq 1$$

for all $x \in F$. Suppose S is the subset that defines such an equation, then S is an odd subset of the vertices in K_{2k} for which $\sum_{e \in \partial S} x(e) = 1$ for all $x \in F$.

Let s be the size of S. Each perfect matching with exactly one edge in ∂S consists of the following: a matching of size $(s-1)/2$ covering all but one vertex of S; an edge joining this missed vertex of S to a vertex in $\overline{S}$; and a matching of size $(2k-s-1)/2$ covering all but one vertex in $\overline{S}$. Hence there are

$$(s-2)!! \, s(2k-s) \, (2k-s-2)!! = s!!(2k-s)!!$$

such perfect matchings. We denote this number by $N(s)$ and observe that

$$\frac{N(s-2)}{N(s)} = \frac{2k-s+2}{s}.$$

Hence for all s such that $3 \leq s \leq k$ we see that the values $N(s)$ are strictly decreasing, so the maximum size of a set of such vertices is $N(3) = 3(2k-3)!!$.

On the other hand, the number of perfect matchings in K_{2k} that do not contain a given edge is

$$(2k-1)!! - (2k-3)!! = (2k-2)(2k-3)!!.$$

Since this is always larger than $N(3)$, the lemma follows. □

We now have all the tools to show that any maximum intersecting set of perfect matchings is the set of all matchings that contain a fixed edge.

Theorem 8.3. *The largest coclique in $M(2k)$ has size $(2k-3)!!$. The only cocliques that meet this bound are the canonically intersecting sets of perfect matchings.*

Proof. Let S be a maximum coclique in $M(2k)$ and let v_S be the characteristic vector of S. Then $|S| = (2k-3)!!$, by the Delsarte-Hoffman bound and Corollary 6.4. The Delsarte-Hoffman bound, along with Theorem 6.3, further imply that the vector $v_S - \frac{1}{2k-1}\mathbf{1}$ is in the $\mathrm{Sym}(2k)$-module $V_{[2k-2,2]}$.

By Lemma 7.2, $v_S - \frac{1}{2k-1}\mathbf{1}$ is a linear combination of the balanced characteristic vectors of the canonical cocliques. This also implies that v_S is a linear combination of the characteristic vectors of the canonical cocliques. So there exists a vector x such that $Ux = v_S$ (where U is the matrix defined in Section 7).

Finally, by Lemma 8.2, $\overline{S}$ is a face of maximal size an it consists of all the perfect matching that avoid a fixed edge. This implies that S is a canonical coclique of $M(2k)$. □

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