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The 4-girth-thickness of the complete graph

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Abstract

In this paper, we define the 4-girth-thickness $\theta(4, G)$ of a graph G as the minimum number of planar subgraphs of girth at least 4 whose union is G. We prove that the 4-girth-thickness of an arbitrary complete graph K_n, $\theta(4, K_n)$, is $\lceil \frac{n+2}{4} \rceil$ for $n \neq 6, 10$ and $\theta(4, K_6) = 3$.

Keywords: Thickness, planar decomposition, girth, complete graph.

Math. Subj. Class.: 05C10

1 Introduction

A finite graph G is *planar* if it can be embedded in the plane without any two of its edges crossing. A planar graph of order n and girth g has size at most $\frac{g}{g-2}(n-2)$ (see [6]), and an acyclic graph of order n has size at most $n-1$, in this case, we define its girth as ∞. The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G; i.e. the minimum number of planar subgraphs into which the edges of G can be partitioned.

The thickness was introduced by Tutte [11] in 1963. Since then, exact results have been obtained when G is a complete graph [1, 3, 4], a complete multipartite graph [5, 12, 13] or a hypercube [9]. Also, some generalizations of the thickness for the complete graph K_n have been studied such that the outerthickness θ_o, defined similarly but with outerplanar instead of planar [8], and the S-thickness θ_S, considering the thickness on a surfaces S instead of the plane [2]. See also the survey [10].

We define the *g-girth-thickness* $\theta(g, G)$ of a graph G as the minimum number of planar subgraphs of girth at least g whose union is G. Note that the 3-girth-thickness $\theta(3, G)$ is the usual thickness and the ∞-girth-thickness $\theta(\infty, G)$ is the *arboricity number*, i.e. the minimum number of acyclic subgraphs into which $E(G)$ can be partitioned. In this paper, we obtain the 4-girth-thickness of an arbitrary complete graph of order $n \neq 10$.

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2 The exact value of $\theta(4, K_n)$ for $n \neq 10$

Since the complete graph K_n has size $\binom{n}{2}$ and a planar graph of order n and girth at least 4 has size at most $2(n-2)$ for $n \geq 3$ and $n-1$ for $n \in \{1,2\}$ then the 4-girth-thickness of K_n is at least

$$\left\lceil \frac{n(n-1)}{2(2n-4)} \right\rceil = \left\lceil \frac{n+1}{4} + \frac{1}{2n-4} \right\rceil = \left\lceil \frac{n+2}{4} \right\rceil$$

for $n \geq 3$ and also $\left\lceil \frac{n+2}{4} \right\rceil$ for $n \in \{1,2\}$, we have the following theorem.

Theorem 2.1. *The* 4*-girth-thickness* $\theta(4, K_n)$ *of* K_n *equals* $\left\lceil \frac{n+2}{4} \right\rceil$ *for* $n \neq 6, 10$ *and* $\theta(4, K_6) = 3$.

Proof. Figure 1 displays equality for $n \leq 5$.

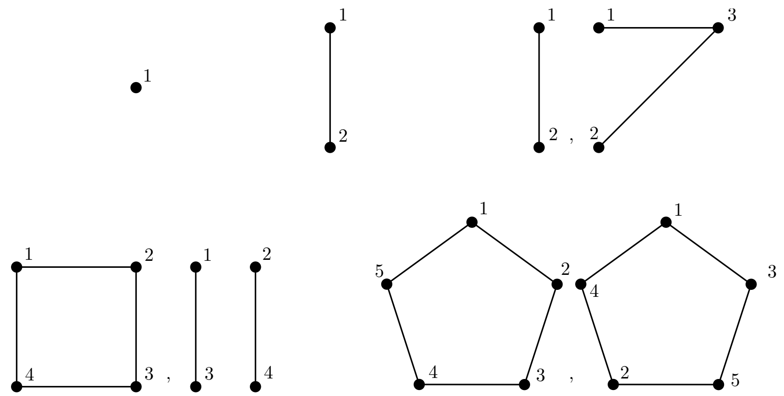

Figure 1: $\theta(4, K_n) = \left\lceil \frac{n+2}{4} \right\rceil$ for $n = 1, 2, 3, 4, 5$.

To prove that $\theta(4, K_6) = 3 > \left\lceil \frac{6+2}{4} \right\rceil = 2$, suppose that $\theta(4, K_6) = 2$. This partition define an edge coloring of K_6 with two colors. By Ramsey's Theorem, some part contains a triangle obtaining a contradiction for the girth 4. Figure 2 shows a partition of K_6 into tree planar subgraphs of girth at least 4.

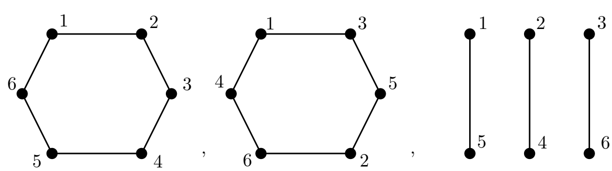

Figure 2: $\theta(4, K_6) = 3$.

For the remainder of this proof, we need to distinguish four cases, namely, when $n = 4k-1$, $n = 4k$, $n = 4k+1$ and $n = 4k+2$ for $k \geq 2$. Note that in each case, the lower bound of the 4-girth thickness require at least $k+1$ elements. To prove our theorem, we exhibit a decomposition of K_{4k} into $k+1$ planar graphs of girth at least 4. The other three cases are based in this decomposition. The case of $n = 4k - 1$ follows because K_{4k-1} is a subgraph of K_{4k}. For the case of $n = 4k + 2$, we add two vertices and some edges to the decomposition obtained in the case of $n = 4k$. The last case follows because K_{4k+1} is a subgraph of K_{4k+2}. In the proof, all sums are taken modulo $2k$.

1. Case $n = 4k$. It is well-known that a complete graph of even order contains a cyclic factorization of Hamiltonian paths, see [7]. Let G be a subgraph of K_{4k} isomorphic to K_{2k}. Label its vertex set $V(G)$ as $\{v_1, v_2, \ldots, v_{2k}\}$. Let $\mathcal{F}_1$ be the Hamiltonian path with edges

$$v_1v_2, v_2v_{2k}, v_{2k}v_3, v_3v_{2k-1}, \ldots, v_{2+k}v_{1+k}.$$

Let $\mathcal{F}_i$ be the Hamiltonian path with edges

$$v_iv_{i+1}, v_{i+1}v_{i-1}, v_{i-1}v_{i+2}, v_{i+2}v_{i-2}, \ldots, v_{i+k+1}v_{i+k},$$

where $i \in \{2, 3, \ldots, k\}$.

Such factorization of G is the partition $\{E(\mathcal{F}_1), E(\mathcal{F}_2), \ldots, E(\mathcal{F}_k)\}$. We remark that the center of $\mathcal{F}_i$ has the edge $e = v_{i+\lceil \frac{k}{2} \rceil}v_{i+\lceil \frac{3k}{2} \rceil}$, see Figure 3.

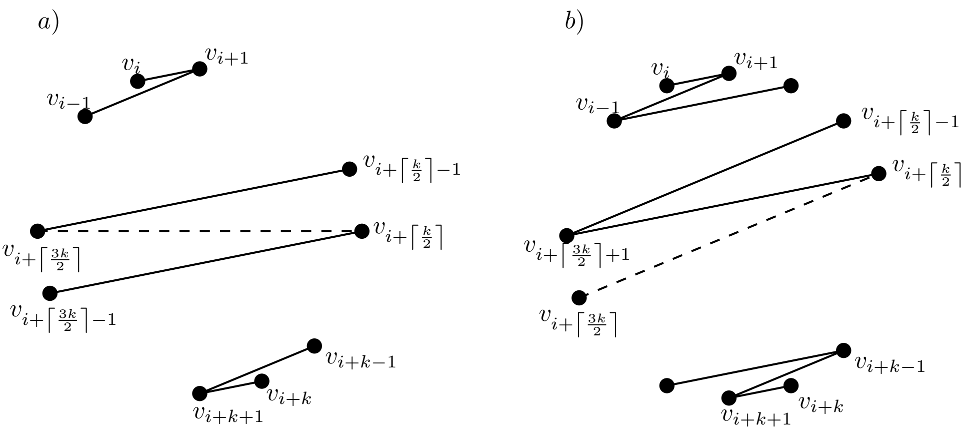

Figure 3: The Hamiltonian path $\mathcal{F}_i$: Left $a)$: The dashed edge e for k odd. Right $b)$ The dashed edge e for k even.

Now, consider the complete subgraph G' of K_{4k} such that $G' = K_{4k} \setminus V(G)$. Label its vertex set $V(G')$ as $\{v'_1, v'_2, \ldots, v'_{2k}\}$ and consider the factorization, similarly as before, $\{E(\mathcal{F}'_1), E(\mathcal{F}'_2), \ldots, E(\mathcal{F}'_k)\}$ where $\mathcal{F}'_i$ is the Hamiltonian path with edges

$$v'_iv'_{i+1}, v'_{i+1}v'_{i-1}, v'_{i-1}v'_{i+2}, v'_{i+2}v'_{i-2}, \ldots, v'_{i+k+1}v'_{i+k},$$

where $i \in \{1, 2, \ldots, k\}$.

Next, we construct the planar subgraphs G_1, G_2,...,G_{k-1} and G_k of girth 4, order $4k$ and size $8k - 4$ (observe that $2(4k - 2) = 8k - 4$), and also the matching G_{k+1}, as follows. Let G_i be a spanning subgraph of K_{4k} with edges $E(\mathcal{F}_i) \cup E(\mathcal{F}'_i)$ and

$$v_iv'_{i+1}, v'_iv_{i+1}, v_{i+1}v'_{i-1}, v'_{i+1}v_{i-1}, v_{i-1}v'_{i+2}, v'_{i-1}v_{i+2}, \ldots, v_{i+k+1}v'_{i+k}, v'_{i+k+1}v_{i+k}$$

where $i \in \{1, 2, \dots, k\}$; and let G_{k+1} be a perfect matching with edges $v_j v'_j$ for $j \in \{1, 2, \dots, 2k\}$. Figure 4 shows G_i is a planar graph of girth at least 4.

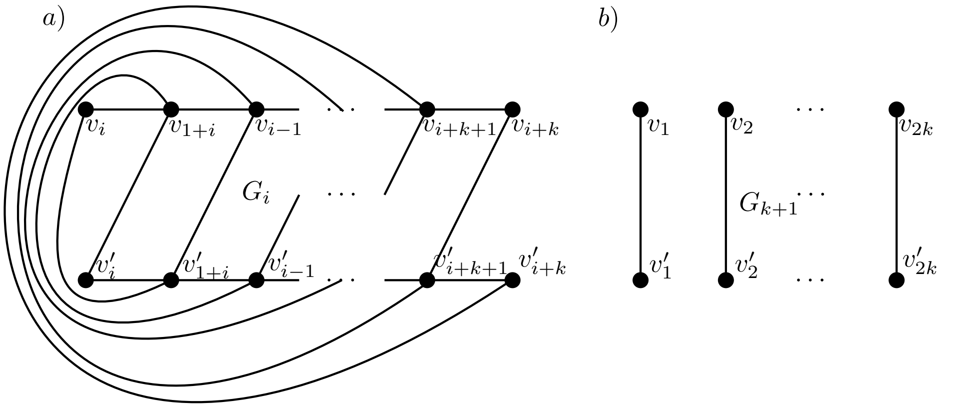

Figure 4: Left $a)$: The graph G_i for any $i \in \{1, 2, \dots, k\}$. Right $b)$ The graph G_{k+1}.

To verify that $K_{4k} = \bigcup_{i=1}^{k+1} G_i$: 1) If the edge $v_{i_1} v_{i_2}$ of G belongs to the factor $\mathcal{F}_i$ then $v_{i_1} v_{i_2}$ belongs to G_i. If the edge is primed, belongs to G'_i. 2) The edge $v_{i_1} v'_{i_2}$ belongs to G_{k+1} if and only if $i_1 = i_2$, otherwise it belongs to the same graph G_i as $v_{i_1} v_{i_2}$. Similarly in the case of $v'_{i_1} v_{i_2}$ and the result follows.

2. Case $n = 4k - 1$. Since $K_{4k-1} \subset K_{4k}$, we have

$$k + 1 \leq \theta(4, K_{4k-1}) \leq \theta(4, K_{4k}) \leq k + 1.$$

3. Case $n = 4k + 2$ (for $k \neq 2$). Let $\{G_1, \dots, G_{k+1}\}$ be the planar decomposition of K_{4k} constructed in the Case 1. We will add the two new vertices x and y to every planar subgraph G_i, when $1 \leq i \leq k + 1$, and we will add 4 edges to each G_i, when $1 \leq i \leq k$, and $4k+1$ edges to G_{k+1} such that the resulting new subgraphs of K_{4k+2} will be planar. Note that $\binom{4k}{2} + 4k + 4k + 1 = \binom{4k+2}{2}$.

   To begin with, we define the graph H_{k+1} adding the vertices x and y to the planar subgraph G_{k+1} and the $4k + 1$ edges

$$\{xy, xv_1, xv'_2, xv_3, xv'_4, \dots, xv_{2k-1}, xv'_{2k}, yv'_1, yv_2, yv'_3, yv_4, \dots, yv'_{2k-1}, yv_{2k}\}.$$

   The graph H_{k+1} has girth 4, see Figure 5.

   In the following, for $1 \leq i \leq k$, by adding vertices x and y to G_i and adding 4 edges to G_i, we will get a new planar graph H_i such that $\{H_1, \dots, H_{k+1}\}$ is a planar decomposition of K_{4k+2} such that the girth of every element is 4. To achieve it, the given edges to the graph H_i will be $v'_j x, xv_{j-1}, v_j y, yv'_{j-1}$, for some odd $j \in \{1, 3, \dots, 2k - 1\}$.

   According to the parity of k, we have two cases:

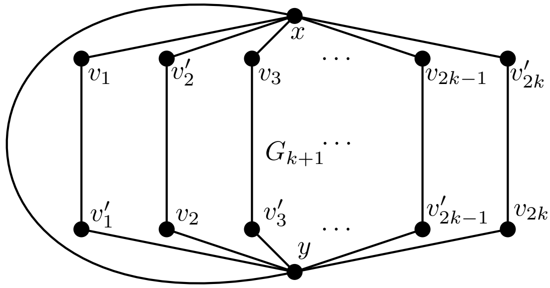

Figure 5: The graph H_{k+1}.

- Suppose k odd. For odd $i \in \{1, 2, \ldots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv'_{i+\lceil \frac{3k}{2} \rceil -1}, xv_{i+\lceil \frac{3k}{2} \rceil}, yv_{i+\lceil \frac{3k}{2} \rceil -1}, yv'_{i+\lceil \frac{3k}{2} \rceil}\}$$

when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv'_{i+\lceil \frac{3k}{2} \rceil -1}, yv_{i+\lceil \frac{3k}{2} \rceil}, xv_{i+\lceil \frac{3k}{2} \rceil -1}, xv'_{i+\lceil \frac{3k}{2} \rceil}\}.$$

Additionally, for even $i \in \{1, 2, \ldots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv'_{i+\lceil \frac{k}{2} \rceil -1}, xv_{i+\lceil \frac{k}{2} \rceil}, yv_{i+\lceil \frac{k}{2} \rceil -1}, yv'_{i+\lceil \frac{k}{2} \rceil}\}$$

when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv'_{i+\lceil \frac{k}{2} \rceil -1}, yv_{i+\lceil \frac{k}{2} \rceil}, xv_{i+\lceil \frac{k}{2} \rceil -1}, xv'_{i+\lceil \frac{k}{2} \rceil}\}.$$

Note that the graph H_i has girth 4 for all i, see Figure 6.

- Suppose k even. Similarly that the previous case, for odd $i \in \{1, 2, \ldots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv_{i+\lceil \frac{3k}{2} \rceil +1}, xv'_{i+\lceil \frac{3k}{2} \rceil}, yv'_{i+\lceil \frac{3k}{2} \rceil +1}, yv_{i+\lceil \frac{3k}{2} \rceil}\}$$

when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv_{i+\lceil \frac{3k}{2} \rceil +1}, yv'_{i+\lceil \frac{3k}{2} \rceil}, xv'_{i+\lceil \frac{3k}{2} \rceil +1}, xv_{i+\lceil \frac{3k}{2} \rceil}\}.$$

On the other hand, for even $i \in \{1, 2, \ldots, k\}$, we define the graph H_i adding the vertices x and y to the planar subgraph G_i and the 4 edges

$$\{xv_{i+\lceil \frac{k}{2} \rceil}, xv'_{i+\lceil \frac{k}{2} \rceil -1}, yv'_{i+\lceil \frac{k}{2} \rceil}, yv_{i+\lceil \frac{k}{2} \rceil -1}\}$$

when $\lceil \frac{k}{2} \rceil$ is even, otherwise

$$\{yv_{i+\lceil \frac{k}{2} \rceil}, yv'_{i+\lceil \frac{k}{2} \rceil -1}, xv'_{i+\lceil \frac{k}{2} \rceil}, xv_{i+\lceil \frac{k}{2} \rceil -1}\}.$$

Note that the graph H_i has girth 4 for all i, see Figure 7.

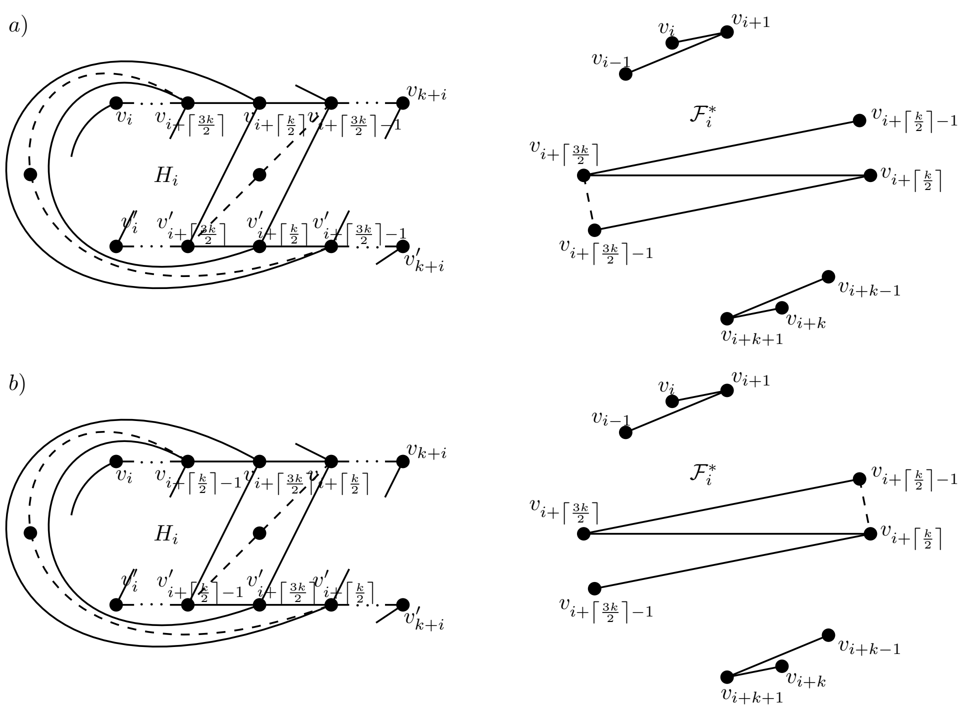

Figure 6: The graph H_i when k is odd and its auxiliary graph $\mathcal{F}_i^*$. Above $a)$ When i is odd. Botton $b)$ When i is even.

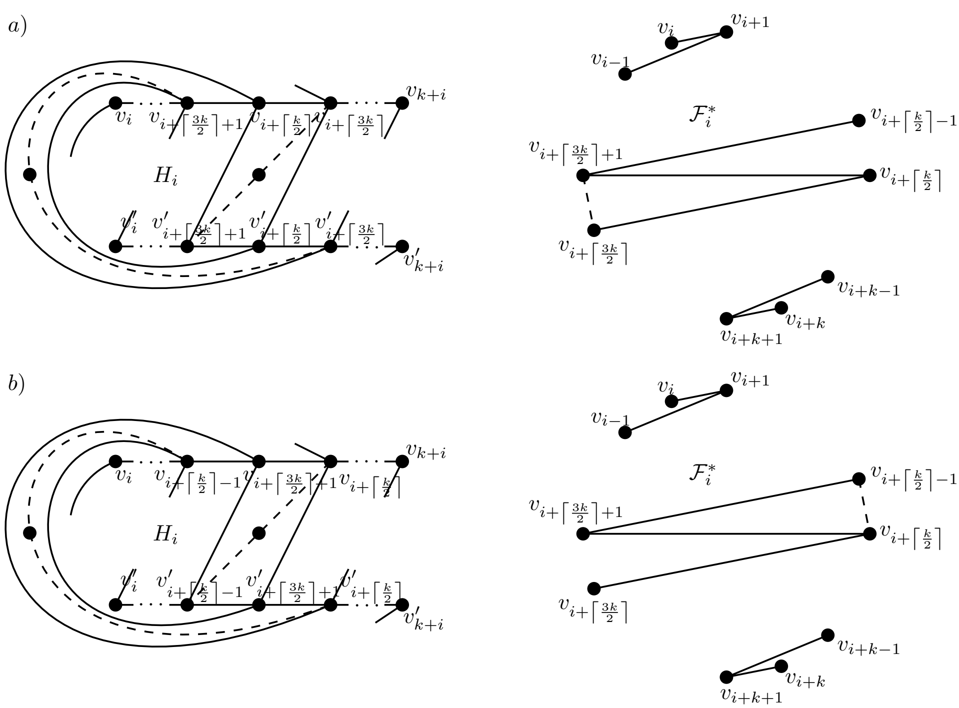

Figure 7: The graph H_i when k is even and its auxiliary graph $\mathcal{F}_i^*$. Above $a)$ When i is odd. Botton $b)$ When i is even.

In order to verify that each edge of the set

$$\{xv'_1, xv_2, xv'_3, xv_3, \ldots, xv'_{2k-1}, xv_{2k}, yv_1, yv'_2, yv_3, yv'_3, \ldots, yv_{2k-1}, yv'_{2k}\}.$$

is in exactly one subgraph H_i, for $i \in \{1, \ldots, k\}$, we obtain the unicyclic graph $\mathcal{F}_i^*$ identifying v_j and v'_j resulting in v_j; identifying x and y resulting in a vertex which is contracted with one of its neighbours. The resulting edge, in dashed, is showed in Figures 6 and 7. The set of those edges are a perfect matching of K_{2k} proving that the added two paths of length 2 in G_i have end vertices v_j and v'_{j-1}, and the other v'_j and v_{j-1}. The election of the label of the center vertex is such that one path is $v_{even}xv'_{odd}$ and $v'_{even}yv_{odd}$ and the result follows.

4. Case $n = 4k + 1$ (for $k \neq 2$). Since $K_{4k+1} \subset K_{4k+2}$, we have

$$k + 1 \leq \theta(4, K_{4k+1}) \leq \theta(4, K_{4k+2}) \leq k + 1.$$

For $k = 2$, Figure 8 displays a decomposition of three planar graphs of girth at least 4 proving that $\theta(4, K_9) = \left\lceil \frac{9+2}{4} \right\rceil = 3$.

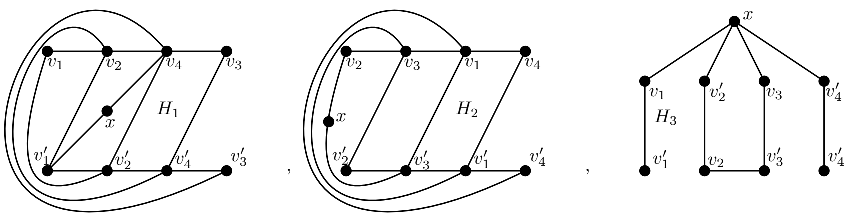

Figure 8: A planar decomposition of K_9 into three subgraphs of girth 4 and 5.

By the four cases, the theorem follows. □

About the case of K_{10}, it follows $3 \leq \theta(4, K_{10}) \leq 4$. We conjecture that $\theta(4, K_{10}) = 4$.

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