The possibility of using homogeneous (projective) coordinates in 2D measurement exercises

Možnost uporabe homogenih (projektivnih) koordinat v dvodimenzionalnih merskih nalogah

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Abstract: This work's intention is to present the basic characteristics of projective geometry and the use of homogeneous (projective) coordinates in two-dimensional (further denoted as 2D) measurement exercises. The concept of a projective plane originates from the Euclidean plane, assuming all our given points are ideal and lie upon an ideal line verging towards infinity. The term "ideal point" is taken to mean an intersection of all lines that are parallel in the finite space. By introducing these so-called ideal points and ideal lines, the calculations in 2D measurement exercises – ones that are usually carried out under the rules of Euclidean trigonometry – have been simplified, as the calculations of directional angles and lengths are no longer necessary. As a practical example of the use of projective coordinates, an intersection is presented, such that it can also be used for the Collins Method of Resection, seeing as it is based upon using said intersection twice.

Izvleček: Predstavljene so osnovne značilnosti projektivne geometrije in s tem uporaba homogenih (projektivnih) koordinat v dvodimenzionalnih merskih nalogah. Projektivno ravnino dobimo iz evklidske ravnine, če privzamemo točke in premico v neskončnosti na kateri ležijo vse točke v neskončnosti. Neskončna točka predstavlja presečišče vseh premic, ki so v končnosti med seboj vzporedne. Z uvedbo neskončnih točk in neskončne premice se izračuni v dvodimenzionalnih merskih nalogah, ki se navadno vršijo po pravilih evklidske trigonometrije, poenostavijo saj računanje smernih kotov in dolžin ni potrebno. Kot praktični primer uporabe projektivnih koor-

Scientific paper

dinat je prikazan zunanji urez, ki ga je možno uporabiti tudi pri Collinsovi metodi notranjega ureza, saj temelji na dvakratni uporabi zunanjega ureza.

Key words: homogeneous coordinates of a point, homogeneous coordinates of a line, incidence relation, principle of duality, ideal point, ideal line, intersection, resection

Ključne besede: homogene koordinate točke, homogene koordinate premice, relacija incidence, princip dualnosti, neskončna točka, premica v neskončnosti, zunanji urez, notranji urez

INTRODUCTION

In 2D measurement exercises, all calculations are usually carried out under the rule of Euclidean geometry - where the points, lines and their relationships are defined differently than in a projective plane - in which the coordinates of an unknown point are established through the aid the calculation of so-called lengths and angles of our site. As a reaction to the latter, the article at hand is meant to present the basic relations between points and lines of the projective plane and depict their use in 2D measurement exercises. A practical example of using said projective coordinates would be an intersection, in which the coordinates of a new point are calculated from the measured angles of given points of an existing triangulation network. In what follows, the coordinates of a point and the formulas of lines on a Euclidean plane shall be marked using upper case, whereas the coordinates of points and formulas of lines on the projective plane will be denoted using lower case letters.

POINT AND LINE

In projective geometry, a point is defined as a set of three coordinates that equal the set $(y\ x\ \omega)$ and therefore obviously also as an ordered set of three numbers $(y\ x\ \omega)$ - which do not all equal zero at the same time, since then $(\lambda y\ \lambda x\ \lambda\omega)$ would be the same point for any given $\lambda \neq 0$.

For example, (2 3 6) is our exemplary point, and (⅓ ½ 1) is another of the numerous ways to mark that exact same point, bearing in mind the principle that an unlimited number of sets of three numbers $(y\ x\ \omega)$ may correspond to each point, but only one point may correspond to each ordered set. Furthermore, from non-homogeneous coordinates of any given point, we beget an infinite number of sets of homogeneous coordinates of that same point:

$$\lambda \neq 0 \Rightarrow \begin{pmatrix} Y \\ X \end{pmatrix} \rightarrow \lambda \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} y \\ x \\ \omega \end{pmatrix} \tag{1}$$

And from homogeneous coordinates of that certain point, we can get one single ordered pair of numbers:

$$\lambda \neq 0 \Rightarrow \lambda \begin{pmatrix} y \\ x \\ \omega \end{pmatrix} = \begin{pmatrix} \lambda y \\ \lambda x \\ \lambda\omega \end{pmatrix} = \begin{pmatrix} \frac{\lambda y}{\lambda\omega} \\ \frac{\lambda x}{\lambda\omega} \\ \frac{\lambda\omega}{\lambda\omega} \end{pmatrix} = \begin{pmatrix} \frac{\lambda y}{\lambda\omega} \\ \frac{\lambda x}{\lambda\omega} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda y}{\lambda\omega} \\ \frac{\lambda x}{\lambda\omega} \end{pmatrix} = \begin{pmatrix} Y \\ X \end{pmatrix} \tag{2}$$

A line is defined in almost the same way as a point, the one difference being that the line is treated as a set of all points that equal the set of three (*v u w*). For example, if (3 2 -2) defines a line, then (-3/2 -1 1) is the notation for that same line. In other words, a line is an ordered set of three numbers, denoted as (*v u w*), which do not equal zero all at the same time, in which case (μv μu μw) would be the same line for any given $\mu \neq 0$ [1].

Incidence relation

Given point *P* and line *l* - where *P* and *l* are short for ($y\ x\ \omega$) and (*v u w*), respectively - we say that the two are incident to one another (incidence relation being descriptive of the relation simply summed up as the point lying on the line i.e. the line going "through" the point) in the case, the following is true: $\{P \cdot l\} = \{l \cdot P\} = yv + xu + \omega w = vy + ux + w\omega = 0$ [2].

The principle of duality

The axiom of geometry says that there is exactly one line that is in incidence with two different points P_1 and P_2, i.e. exactly one line goes through both different points. If we exchange the term "line" with the term "point" in the above axiom, we get a geometrical theorem that states there is exactly one point that is in incidence with two different lines l_1 and l_2 - in other words, two different lines intersect at exactly one point in space. The afore-mentioned axiom and theorem are said to be mutually dual, i.e. points and lines are mutually dual spatial elements, while the "running" and intersecting of lines are mutually dual operations[3].

For different points $P_1(y_1\ x_1\ \omega_1)$ and $P_2(y_2\ x_2\ \omega_2)$; $y_1 \neq \lambda \cdot y_2$, $x_1 \neq \lambda \cdot x_2$, $\omega_1 \neq \lambda \cdot \omega_2$ there is always an ordered set of three elements, (*v u w*), for which the following need most definitely apply:

$$\begin{aligned} y_1 v + x_1 u + \omega_1 w &= 0 \\ y_2 v + x_2 u + \omega_2 w &= 0 \end{aligned} \tag{3}$$

And for two different lines, in our particular example l_1 $(v_1\ u_1\ w_1)$ and l_2 $(v_2\ u_2\ w_2)$; $v_1 \neq \mu \cdot v_2$, $u_1 \neq \mu \cdot u_2$, $w_1 \neq \mu \cdot w_2$; there is always an ordered set of three elements, $(y\ x\ \omega)$, for which the following need apply:

$$\begin{aligned} v_1 y + u_1 x + w_1\, \omega &= 0 \\ v_2 y + u_2 x + w_2\, \omega &= 0 \end{aligned} \tag{4}$$

Thus, homogeneous systems of equations are combined with the following expression:

$$\begin{vmatrix} \begin{Bmatrix} P \\ l \end{Bmatrix} \\ \begin{Bmatrix} P \\ l \end{Bmatrix}_1 \\ \begin{Bmatrix} P \\ l \end{Bmatrix}_2 \end{vmatrix} = \begin{vmatrix} \begin{Bmatrix} y \\ v \end{Bmatrix} & \begin{Bmatrix} x \\ u \end{Bmatrix} & \begin{Bmatrix} \omega \\ w \end{Bmatrix} \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_1 & \begin{Bmatrix} u \\ x \end{Bmatrix}_1 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_2 & \begin{Bmatrix} u \\ x \end{Bmatrix}_2 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 \end{vmatrix} = 0 \Rightarrow \begin{cases} \begin{vmatrix} P \\ l_1 \\ l_2 \end{vmatrix} = \begin{vmatrix} y & x & \omega \\ v_1 & u_1 & w_1 \\ v_2 & u_2 & w_2 \end{vmatrix} = 0 \\ \begin{vmatrix} l \\ P_1 \\ P_2 \end{vmatrix} = \begin{vmatrix} v & u & w \\ y_1 & x_1 & \omega_1 \\ y_2 & x_2 & \omega_2 \end{vmatrix} = 0 \end{cases} \tag{5}$$

Then we are able to beget the formula of the point or line:

$$\begin{vmatrix} \begin{Bmatrix} y \\ v \end{Bmatrix} & \begin{Bmatrix} x \\ u \end{Bmatrix} & \begin{Bmatrix} \omega \\ w \end{Bmatrix} \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_1 & \begin{Bmatrix} u \\ x \end{Bmatrix}_1 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_2 & \begin{Bmatrix} u \\ x \end{Bmatrix}_2 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 \end{vmatrix} = \begin{Bmatrix} y \\ v \end{Bmatrix} \begin{vmatrix} \begin{Bmatrix} u \\ x \end{Bmatrix}_1 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 \\ \begin{Bmatrix} u \\ x \end{Bmatrix}_2 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 \end{vmatrix} - \begin{Bmatrix} x \\ u \end{Bmatrix} \begin{vmatrix} \begin{Bmatrix} v \\ y \end{Bmatrix}_1 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_2 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 \end{vmatrix} + \begin{Bmatrix} \omega \\ w \end{Bmatrix} \begin{vmatrix} \begin{Bmatrix} v \\ y \end{Bmatrix}_1 & \begin{Bmatrix} u \\ x \end{Bmatrix}_1 \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_2 & \begin{Bmatrix} u \\ x \end{Bmatrix}_2 \end{vmatrix} = 0 \tag{6}$$

Each set of three $\left(\begin{Bmatrix} y \\ v \end{Bmatrix} \begin{Bmatrix} x \\ u \end{Bmatrix} \begin{Bmatrix} \omega \\ w \end{Bmatrix} \right)$ therefore represents a solution to the system under question.

The coordinates of the point or line follow:

$$\begin{Bmatrix} y \\ v \end{Bmatrix} : \begin{Bmatrix} x \\ u \end{Bmatrix} : \begin{Bmatrix} \omega \\ w \end{Bmatrix} = \begin{vmatrix} \begin{Bmatrix} u \\ x \end{Bmatrix}_1 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 \\ \begin{Bmatrix} u \\ x \end{Bmatrix}_2 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 \end{vmatrix} : \begin{vmatrix} \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 & \begin{Bmatrix} v \\ y \end{Bmatrix}_1 \\ \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 & \begin{Bmatrix} v \\ y \end{Bmatrix}_2 \end{vmatrix} : \begin{vmatrix} \begin{Bmatrix} v \\ y \end{Bmatrix}_1 & \begin{Bmatrix} u \\ x \end{Bmatrix}_1 \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_2 & \begin{Bmatrix} u \\ x \end{Bmatrix}_2 \end{vmatrix} \tag{7}$$

A point or a line pertaining to a projective plane can be multiplied by any number λ or μ , as long as the value is not equal to zero, which then gives us the actual coordinates of this same point or line:

$$\left(\begin{Bmatrix} y \\ v \end{Bmatrix} : \begin{Bmatrix} x \\ u \end{Bmatrix} : \begin{Bmatrix} \omega \\ w \end{Bmatrix} \right) = \begin{Bmatrix} \lambda \\ \mu \end{Bmatrix} \left(\begin{vmatrix} \begin{Bmatrix} u \\ x \end{Bmatrix}_1 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 \\ \begin{Bmatrix} u \\ x \end{Bmatrix}_2 & \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 \end{vmatrix} \begin{vmatrix} \begin{Bmatrix} w \\ \omega \end{Bmatrix}_1 & \begin{Bmatrix} v \\ y \end{Bmatrix}_1 \\ \begin{Bmatrix} w \\ \omega \end{Bmatrix}_2 & \begin{Bmatrix} v \\ y \end{Bmatrix}_2 \end{vmatrix} \begin{vmatrix} \begin{Bmatrix} v \\ y \end{Bmatrix}_1 & \begin{Bmatrix} u \\ x \end{Bmatrix}_1 \\ \begin{Bmatrix} v \\ y \end{Bmatrix}_2 & \begin{Bmatrix} u \\ x \end{Bmatrix}_2 \end{vmatrix} \right) \tag{8}$$

For a point, this would be:

$$\begin{pmatrix} y & x & \omega \end{pmatrix} = \mu \begin{pmatrix} \begin{vmatrix} u_1 & w_1 \\ u_2 & w_2 \end{vmatrix} & \begin{vmatrix} w_1 & v_1 \\ w_2 & v_2 \end{vmatrix} & \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} \end{pmatrix}, \quad (9)$$

And for a line:

$$\begin{pmatrix} v & u & w \end{pmatrix} = \lambda \begin{pmatrix} \begin{vmatrix} x_1 & \omega_1 \\ x_2 & \omega_2 \end{vmatrix} & \begin{vmatrix} \omega_1 & y_1 \\ \omega_2 & y_2 \end{vmatrix} & \begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} \end{pmatrix}, \quad (10)$$

The ideal point and line

The parallels *(coefficient of site $k = k_1 = k_2$)* l_1 and l_2 with formulas $Y = k_1X + n_1$ and $Y = k_2X + n_2$ are given. For X, we enter values $X = \frac{x}{\omega}$, and for Y values $Y = \frac{y}{\omega}$, then multiplying the equations with ω, which in turn gives us the formulas of lines using homogeneous coordinates:

$$y = kx + n_1\omega \text{ and } y = kx + n_2\omega \quad (11)$$

Considering that the lines are parallel, we are interested in the set of three, (v u w), such that it must correspond to both formulas.

By subtracting the equations we get: $\omega(n_1 - n_2) = 0$.

As $n_1 \neq n_2$, then $\omega = 0$ and the equations $y = kx + n_1\omega$ and $y = kx + n_2\omega$ are reduced into $y = xk$.

Since we are dealing with homogeneous coordinates, we can say that $x = 1$. From this, we come to the conclusion that $y = k$. Thus, we obtain a set of three, (k 1 0), which does indeed correspond to both equations.

If the lines l_1 and l_2 are parallel to the y-axis, then the formulas of the lines in homogeneous coordinates have the form $x = \omega x_1$ and $x = \omega x_2$. In this case, the set of three (0 1 0) corresponds to both formulas.

To summarize, the set of three, (k 1 0), corresponds to a formula of the forms $y = kx + n_1\omega$ and $y = kx + n_2\omega$ only when $k = k_1 = k_2$, or when the lines are parallel and the coefficient of site equals k. The set of three (0 1 0) corresponds to all formulas of the form $x = \omega x_1$ that describe the parallels of the y-axis.

A bouquet of parallel lines (all parallel lines are of the same class and form organised heaps of parallel lines denoted as "bouquets") defines a point P_∞ in projective plane that has been defined as an ideal point. The bouquet P_∞ consists of all lines that are parallel to a certain line l. The equation pertaining to line l is $Y = kX + n$, or $X = X_P$ if it is parallel to the y rather than the x-axis.

The line l belongs to bouquet P_∞ exactly when the set of three (k 1 0) corresponds to the equation of line l in its homogeneous coordinates, and to pencil P_∞ exactly when what corresponds to this equation is this set of three: (0 1 0). Consequentially, we can have the set of three (k 1 0) in the former case, and the set of three (0 1 0) in the latter case for homogeneous coordinates of the ideal point P_∞.

Since we may multiply homogeneous coordinates with any number that is different than zero, we may say that the set of three (y x 0) represents the homogeneous coordinates of one ideal point, where y and x are any given elements different from zero. In this way, we have adjusted our homogeneous coordinates (y x ω) so that they befit each and every point of our projective plane. The point with such coordinates also lies in the Euclidean plane if $\omega \neq 0$, and is an ideal point when $\omega = 0$ [4].

Two ideal points define the ideal line l_∞:

$$\begin{vmatrix} v_\infty & u_\infty & w_\infty \\ y_1 & x_1 & 0 \\ y_2 & x_2 & 0 \end{vmatrix} = 0 \tag{12}$$

The solution of the system are the very coordinates of our ideal line l_∞:

$$\begin{pmatrix} v_\infty & u_\infty & w_\infty \end{pmatrix} = \mu \left(\begin{vmatrix} x_1 & 0 \\ x_2 & 0 \end{vmatrix} \; \begin{vmatrix} 0 & y_1 \\ 0 & y_2 \end{vmatrix} \; \begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} \right) = \mu \left(0 \quad 0 \quad \begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} \right) \tag{13}$$

or, if:

$$\left(\mu \neq 0 \wedge \begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} \neq 0 \wedge \mu = \frac{1}{\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix}} \right) \Rightarrow \begin{pmatrix} v_\infty & u_\infty & w_\infty \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \tag{14}$$

The set of three (0 0 1) represents the coordinates of an ideal line such that all ideal points lie on it.

THE IDEAL LINE AND ANGLE OF SITE

An ideal point represents the intersection of a group of all lines that are parallel to one another in finite space. A specific ideal point upon an ideal line belongs to each group of parallel lines that in finite space represent a so-called »angle of site« between the lines of a certain class and the positive end of the x-axis.

The coordinates of the line that goes through the points $P_1(y_1\ x_1\ \omega_1)$ and $P_2(y_1 + d\cos\varphi\ \ x_1 + d\sin\varphi\ \omega_2)$, where φ is the so-called angle of site are enclosed within the line $\overline{P_1P_2}$ and the positive side of x-axis, and d represents the distance between the points P_1 and P_2, which would be:

$$(v\ u\ w) = \mu\,(x_1(\omega_2 - \omega_1) - d\cos\varphi\omega_1\ \ y(\omega_1 - \omega_2) + d\sin\varphi\omega_2\ \ d(y_1\cos\varphi - x_1\sin\varphi)) \qquad (15)$$

If :

$$\omega_2 = \omega_1 = 1 \Rightarrow (v\ u\ w) = \mu\,(-d\cos\varphi\quad d\sin\varphi\quad d(y_1\cos\varphi - x_1\sin\varphi)) \qquad (16)$$

or, when:

$$(\omega_2 = \omega_1 = 1 \wedge d \neq 0 \wedge d \neq \infty \wedge \mu = \tfrac{1}{d}) \Rightarrow (v\ u\ w) = (-\cos\varphi\ \sin\varphi\ (y_1\cos\varphi - x_1\sin\varphi)) \qquad (17)$$

The set $(v\ u\ w) = (-\cos\varphi\ \sin\varphi\ (y_1\cos\varphi - x_1\sin\varphi))$ represents the coordinates of the line that is notated using polar coordinates.

The intersection of the line given with polar coordinates and the ideal line (0 0 1) is the ideal line, now denoted using polar coordinates:

$$(y_\infty\ x_\infty\ \omega_\infty) = \lambda\left(\begin{vmatrix}\sin\varphi & y_1\cos\varphi - x_1\sin\varphi\\ 0 & 1\end{vmatrix}\ \begin{vmatrix}y_1\cos\varphi - x_1\sin\varphi & -\cos\varphi\\ 1 & 0\end{vmatrix}\ \begin{vmatrix}-\cos\varphi & \sin\varphi\\ 0 & 0\end{vmatrix}\right) \qquad (18)$$

or, if:

$$\lambda = 1 \Rightarrow (y_\infty\ x_\infty\ \omega_\infty) = (\sin\varphi\ \ \cos\varphi\ \ 0) \qquad (19)$$

The notation $(y_\infty\ x_\infty\ \omega_\infty) = (\sin\varphi\ \ \cos\varphi\ \ 0)$ at the same time alos represents standardised coordinates of said ideal line, for which the following is valid:

$$(y_\infty\ x_\infty\ \omega_\infty)\begin{pmatrix}y_\infty\\ x_\infty\\ \omega_\infty\end{pmatrix} = \left(\frac{y_\infty}{\sqrt{y_\infty^{\ 2} + x_\infty^{\ 2}}}\ \ \frac{x_\infty}{\sqrt{y_\infty^{\ 2} + x_\infty^{\ 2}}}\ \ 0\right)\begin{pmatrix}\dfrac{y_\infty}{\sqrt{y_\infty^{\ 2} + x_\infty^{\ 2}}}\\ \dfrac{x_\infty}{\sqrt{y_\infty^{\ 2} + x_\infty^{\ 2}}}\\ 0\end{pmatrix} = 1 \qquad (20)$$

where:

$$\varphi = \tan^{-1}\left(\frac{\sin\varphi}{\cos\varphi}\right) = \frac{\dfrac{y_\infty}{\sqrt{y_\infty^2 + x_\infty^2}}}{\dfrac{x_\infty}{\sqrt{y_\infty^2 + x_\infty^2}}} = \tan^{-1}\frac{y_\infty}{x_\infty} \qquad (21)$$

Should the lines l_1 in l_2 enclose the angle α , and the line l_1 is defined through the ideal point (sin φ_1 cos φ_1 0), then the ideal point of line l_2 is defined as (sin ($\varphi_1 \pm \alpha$) cos ($\varphi_1 \pm \alpha$) 0), where $\varphi_1 \pm \alpha$ is the angle of site of the line l_2:

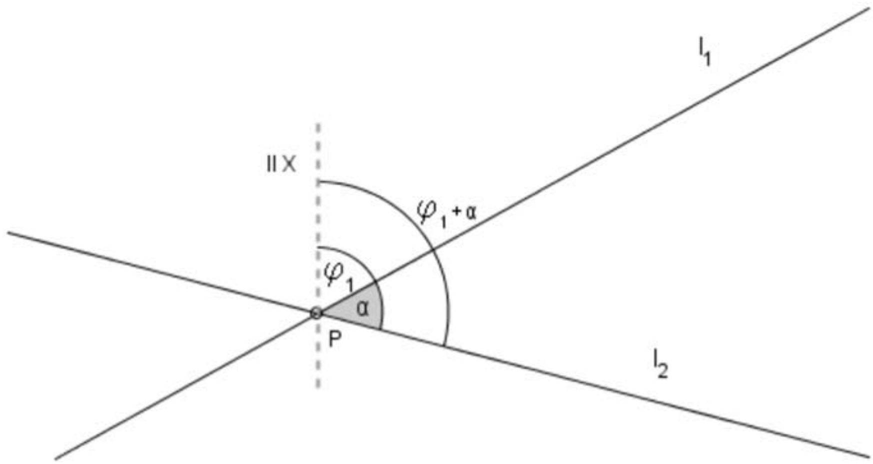

Figure 1. The angle between our two lines and angles of site
Slika 1. Kot med premicama in smerna kota

INTERSECTION

An intersection is both a measurement and calculation method, with the aid of which the coordinates of a new point can be calculated from measured angles or (outer) directions upon given points of the existing triangulation network. A given new point is determined as an intersection of several outer directions that are controllably oriented at each standpoint.

The coordinates of points $L(Y_L X_L)$ and $R(Y_R X_R)$ are given.

Observation:
We are observing the direction from two given points (*L* and *R*) towards the new point *M*. Angle α is observed from the point *L* between points *M-left* and *R-right*, and angle β from point *R* between points *L-left* and *M-right*.

Based on the given and observed information, we must establish the coordinates of the unknown point *M*, which would translate into us looking for X_M and Y_M:

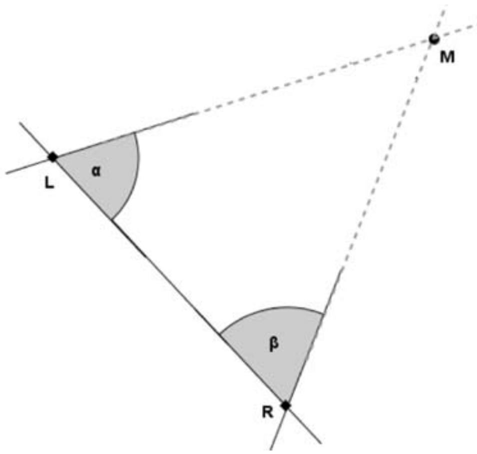

Figure 2. Intersection
Slika 2. Zunanji urez

The connecting line $\overline{LM}$ denotes the line *left*, $\overline{RM}$ the line *right*, and $\overline{LR}$ the line *left-right*. The lines *left* and *left-right* are bisected via one another in point *L* and enclose the angle *α*, while the lines *right* and *left-right* do the same in point *R*, therefore enclosing the angle *β*.

By establishing the ideal point L_∞ of the line *left* and the ideal point R_∞ of the line *right*, the lines *left* and *right* have been accurately established. Thus, we have also established their intersection *M*, which represents the coordinates of the point we are seeking.

L_∞ denotes the ideal point on the line *left,*, R_∞ the ideal point on the line *right* and LR_∞ the ideal point on the line *left-right*. Points L_∞ and R_∞ are obtained with the help of the ideal point LR_∞, which is established through the bisection of line *left-right* with the ideal line. When the point LR_∞ is standardised (transformed into a format such as (sin φ cos φ 0)), we can, with the help of the latter, as well as with the help of angles *α* and *β*, determine the points L_∞ and R_∞ on the line l_∞.

1. ***Establishing the line left-right***

- The coordinates of point *L* are $(y_L = Y_L \;\; x_L = X_L \;\; 1)$
- The coordinates of point *R* are $(y_L = Y_R = Y_L + X_L + d \sin \upsilon_L^R \; x_L = X_R = X_L + d \cos \upsilon_L^R \; 1)$

Therefore, the following is true:

$$\begin{vmatrix} v_{LR} & u_{LR} & w_{LR} \\ Y_L & X_L & 1 \\ Y_L + d \cdot \sin\nu_L^R & X_L + d \cdot \cos\nu_L^R & 1 \end{vmatrix} = 0 \tag{22}$$

The formula of the line *left-right* is:

$$v_{LR}\begin{vmatrix} X_L & 1 \\ X_L + d\cdot\cos\nu_L^R & 1\end{vmatrix} - u_{LR}\begin{vmatrix} Y_L & 1 \\ Y_L + d\cdot\sin\nu_L^R & 1\end{vmatrix} + w_{LR}\begin{vmatrix} Y_L & X_L \\ Y_L + d\cdot\sin\nu_L^R & X_L + d\cdot\cos\nu_L^R\end{vmatrix} = 0 \tag{23}$$

or, its coordinates:

$$v_{LR} = \begin{vmatrix} X_L & 1 \\ X_L + d\cos\nu_L^R & 1\end{vmatrix} \tag{24}$$

$$u_{LR} = \begin{vmatrix} 1 & Y_L \\ 1 & Y_L + d\sin\nu_L^R\end{vmatrix} \tag{25}$$

$$w_{LR} = \begin{vmatrix} Y_L & X_L \\ Y_L + d\cdot\sin\nu_L^R & X_L + d\cdot\cos\nu_L^R\end{vmatrix} = \begin{vmatrix} Y_L & X_L \\ Y_L & X_L\end{vmatrix} + \begin{vmatrix} Y_L & X_L \\ d\cdot\sin\nu_L^R & d\cdot\cos\nu_L^R\end{vmatrix} \tag{26}$$

2. *Establishing the ideal point on the line left-right*

The line *left-right* is bisected with the ideal point (0 0 1):

$$\begin{vmatrix} y_{LR_\infty} & x_{LR_\infty} & \omega_{LR\infty} \\ 0 & 0 & 1 \\ \begin{vmatrix} X_L & 1 \\ X_L + d\cdot\cos\nu_L^R & 1\end{vmatrix} & \begin{vmatrix} 1 & Y_L \\ 1 & Y_L + d\cdot\sin\nu_L^R\end{vmatrix} & \begin{vmatrix} Y_L & X_L \\ Y_L + d\cdot\sin\nu_L^R & X_L + d\cdot\cos\nu_L^R\end{vmatrix}\end{vmatrix} = 0 \tag{27}$$

The coordinates of the ideal point are denoted as follows:

$$\begin{aligned} y_{LR_\infty} &= \begin{vmatrix} 0 & 1 \\ \begin{vmatrix} 1 & Y_L \\ 1 & Y_L + d\cdot\sin\nu_L^R\end{vmatrix} & \begin{vmatrix} Y_L & X_L \\ d\cdot\sin\nu_L^R & d\cdot\cos\nu_L^R\end{vmatrix}\end{vmatrix} = -\begin{vmatrix} 1 & Y_L \\ 1 & Y_L + d\cdot\sin\nu_L^R\end{vmatrix} = \\ &= -\begin{vmatrix} 1 & Y_L \\ 1 & Y_L\end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & d\cdot\sin\nu_L^R\end{vmatrix} = -d\cdot\sin\nu_L^R \end{aligned} \tag{28}$$

$$\begin{aligned} x_{LR_\infty} &= \begin{vmatrix} 1 & 0 \\ \begin{vmatrix} Y_L & X_L \\ d\cdot\sin\nu_L^R & d\cdot\cos\nu_L^R\end{vmatrix} & \begin{vmatrix} X_L & 1 \\ X_L + d\cdot\cos\nu_L^R & 1\end{vmatrix}\end{vmatrix} = \begin{vmatrix} X_L & 1 \\ X_L + d\cdot\cos\nu_L^R & 1\end{vmatrix} = \\ &= \begin{vmatrix} X_L & 1 \\ X_L & 1\end{vmatrix} + \begin{vmatrix} 0 & 1 \\ d\cdot\cos\nu_L^R & 1\end{vmatrix} = -d\cdot\cos\nu_L^R \end{aligned} \tag{29}$$

$$\omega_{LR_\infty} = \begin{vmatrix} 0 & 0 \\ \begin{vmatrix} X_L & 1 \\ X_L + d\cdot\sin\nu_L^R & 1\end{vmatrix} & \begin{vmatrix} 1 & Y_L \\ 1 & Y_L + d\cdot\cos\nu_L^R\end{vmatrix}\end{vmatrix} \tag{30}$$

The point $LR_\infty = \lambda(-d \sin \upsilon_L^R - d \cos \upsilon_L^R\ 0)$ is standardised, which means that it is multiplied by factor:

$$\lambda = \frac{1}{\sqrt{y_{LR_\infty}{}^2 + x_{LR_\infty}{}^2 + \omega_{LR_\infty}{}^2}}$$

Consequentially, we transform the notation $LR_\infty = \lambda(-d \sin \upsilon_L^R - d \cos \upsilon_L^R\ 0)$ into $LR_\infty = \lambda(\pm \sin \upsilon_L^R \pm \cos \upsilon_L^R\ 0)$.

$$\lambda = \frac{1}{\sqrt{y_{LR_\infty}{}^2 + x_{LR_\infty}{}^2 + \omega_{LR_\infty}{}^2}} \Rightarrow LR_\infty = \lambda\left(-d \cdot \sin \nu_L^R\ \ -d \cdot \cos \nu_L^R\ \ 0\right) \tag{31}$$

$$\lambda = \frac{1}{\sqrt{\left(d \cdot \sin \nu_L^R\right)^2 + \left(d \cdot \cos \nu_L^R\right)^2}} \Rightarrow LR_\infty = \lambda\left(-d \cdot \sin \nu_L^R\ \ -d \cdot \cos \nu_L^R\ \ 0\right) \tag{32}$$

$$\lambda = \frac{1}{\sqrt{d^2\left(\sin^2 \nu_L^R + \cos^2 \nu_L^R\right)}} \Rightarrow LR_\infty = \lambda\left(-d \cdot \sin \nu_L^R\ \ -d \cdot \cos \nu_L^R\ \ 0\right) \tag{33}$$

$$\lambda = \frac{1}{\sqrt{d^2}} \Rightarrow LR_\infty = \lambda\left(-d \cdot \sin \nu_L^R\ \ -d \cdot \cos \nu_L^R\ \ 0\right) \tag{34}$$

$$LR_\infty = \pm\frac{1}{d}\left(-d \cdot \sin \nu_L^R\ \ -d \cdot \cos \nu_L^R\ \ 0\right) \tag{35}$$

So, finally, the standardised ideal point is:

$$LR_\infty = \left(\mp \sin \nu_L^R\ \ \mp \cos \nu_L^R\ \ 0\right) \tag{36}$$

or:

$$LR_\infty = \left(\mp \sin\left(\nu_L^R + k\pi\right) \mp \cos\left(\nu_L^R + k\pi\right)\ 0\right) \tag{37}$$

3. *Establishing the ideal points on the lines left and right*

(a) Ideal point L_∞ of the line *left*

The line *left-right* encloses, along with the positive side of *x*-axis, the angle υ_L^R, and with the line *left*, the angle α. The angle of site of the line *left* is $\upsilon_L^M = \upsilon_L^R - \alpha$ and the ideal line is of the form:

$$L_\infty = (\ \sin(\upsilon_L^R - \alpha)\ \cos(\upsilon_L^R - \alpha)\ 0) \tag{38}$$

(b) Ideal point R_∞ of the line *right*

The coefficient of site of the line *right* is $\upsilon_L^M = \upsilon_L^R + \beta$ and the ideal line is of the form:

$$R_\infty = (\ \sin(\upsilon_L^R + \beta)\ \cos(\upsilon_L^R + \beta)\ 0) \tag{39}$$

4. ***Establishing the lines left and right***

(a) Establishing the line *left*:

The line *left* goes through the points L_∞ and L:

$$\begin{vmatrix} v_{left} & u_{left} & w_{left} \\ \sin\nu_L^R \cdot \cos\alpha - \cos\nu_L^R \cdot \sin\alpha & \cos\nu_L^R \cdot \cos\alpha + \sin\nu_L^R \cdot \sin\alpha & 0 \\ Y_{left} & X_{left} & 1 \end{vmatrix} = 0 \tag{40}$$

$$v_{left} = \begin{vmatrix} \cos\nu_L^R \cdot \cos\alpha + \sin\nu_L^R \cdot \sin\alpha & 0 \\ X_{left} & 1 \end{vmatrix} = \cos\nu_L^R \cdot \cos\alpha + \sin\nu_L^R \cdot \sin\alpha = \cos\left(\nu_L^R - \alpha\right) \tag{41}$$

$$u_{left} = \begin{vmatrix} 0 & \sin\nu_L^R \cdot \cos\alpha - \cos\nu_L^R \cdot \sin\alpha \\ 1 & 1 \end{vmatrix} = -\sin\nu_L^R \cdot \cos\alpha - \cos\nu_L^R \cdot \sin\alpha = -\sin\left(\nu_L^R - \alpha\right) \tag{42}$$

$$\begin{aligned} w_{left} &= \begin{vmatrix} \sin\nu_L^R \cdot \cos\alpha - \cos\nu_L^R \cdot \sin\alpha & \cos\nu_L^R \cdot \cos\alpha + \sin\nu_L^R \cdot \sin\alpha \\ Y_{left} & X_{left} \end{vmatrix} = \\ &= -Y_{left} \cdot \cos\left(\nu_L^R - \alpha\right) - X_{left} \cdot \sin\left(\nu_L^R - \alpha\right) \end{aligned} \tag{43}$$

Coordinates of the line *left*:

$$\left(v_{left} \;\; u_{left} \;\; w_{left}\right) = \left(\cos\left(\nu_L^R - \alpha\right) \;\; -\sin\left(\nu_L^R - \alpha\right) \;\; -Y_{left} \cdot \cos\left(\nu_L^R - \alpha\right) - X_{left} \cdot \sin\left(\nu_L^R - \alpha\right)\right) \tag{44}$$

(b) Establishing the line *right*

The line *right* is defined by the points R and R_∞:

$$\begin{vmatrix} v_{right} & u_{right} & w_{right} \\ \sin\nu_R^L \cdot \cos\beta + \cos\nu_R^L \cdot \sin\beta & \cos\nu_R^L \cdot \cos\beta - \sin\nu_R^L \cdot \sin\beta & 0 \\ Y_{right} & X_{right} & 1 \end{vmatrix} = 0 \tag{45}$$

$$v_{right} = \begin{vmatrix} \cos\nu_R^L \cdot \cos\beta - \sin\nu_R^L \cdot \sin\beta & 0 \\ X_{right} & 1 \end{vmatrix} = \cos\nu_R^L \cdot \cos\beta - \sin\nu_R^L \cdot \sin\beta = \cos\left(\nu_R^L + \beta\right) \tag{46}$$

$$u_{right} = \begin{vmatrix} 0 & \sin\nu_R^L \cdot \cos\beta + \cos\nu_R^L \cdot \sin\beta \\ 1 & 1 \end{vmatrix} = -\sin\nu_R^L \cdot \cos\beta + \cos\nu_R^L \cdot \sin\beta = -\sin\left(\nu_R^L + \beta\right) \tag{47}$$

$$\begin{aligned} w_{right} &= \begin{vmatrix} \sin\nu_R^L \cdot \cos\beta + \cos\nu_R^L \cdot \sin\beta & \cos\nu_R^L \cdot \cos\beta - \sin\nu_R^L \cdot \sin\beta \\ Y_{right} & X_{right} \end{vmatrix} = \\ &= -Y_{right} \cdot \cos\left(\nu_R^L + \beta\right) + X_{right} \cdot \sin\left(\nu_R^L + \beta\right) \end{aligned} \tag{48}$$

Coordinates of the line *right*:

$$\left(v_{right} \;\; u_{right} \;\; w_{right}\right) = \left(\cos\left(\nu_R^L + \beta\right) \;\; -\sin\left(\nu_R^L + \beta\right) \;\; -Y_{right} \cdot \cos\left(\nu_R^L + \beta\right) + X_{right} \cdot \sin\left(\nu_R^L + \beta\right)\right) \quad (49)$$

5. *Establishing the unknown point M*

The point M is defined as the intersection of the lines $\overline{LM} = \overline{L_\infty L}$ and $\overline{RM} = \overline{R_\infty R}$:

$$\begin{vmatrix} y_M & x_M & \omega_M \\ v_{left} & u_{left} & w_{left} \\ v_{right} & u_{right} & w_{right} \end{vmatrix} = 0 \quad (50)$$

$$M = \left(y_M \;\; x_M \;\; \omega_M\right) = \left(\begin{vmatrix} u_{left} & w_{left} \\ u_{right} & w_{right} \end{vmatrix} \begin{vmatrix} w_{left} & v_{left} \\ w_{right} & v_{right} \end{vmatrix} \begin{vmatrix} v_{left} & u_{left} \\ v_{right} & u_{right} \end{vmatrix} \right)^{[1]} \quad (51)$$

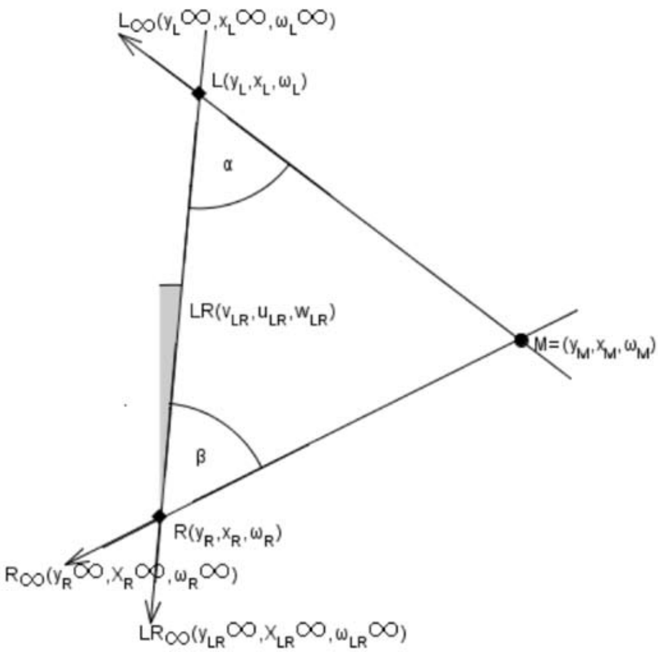

Figure 3. Intersection with projective coordinates
Slika 3. Zunanji urez s projektivnimi koordinatami

CONCLUSIONS

In projective geometry, the lines have their own coordinates inasmuch as points do. A point is defined as an ordered set of three numbers ($y\ x\ \omega$), which are not allowed to simultaneously take on the value of zero, seeing as the latter would make ($\lambda y\ \lambda x\ \lambda\omega$) the same point for any given $\lambda \neq 0$. We can obtain an infinite number of ordered sets of coordinates of a given point from the non-homogeneous coordinates of that same point, just as well as we can obtain only one ordered pair of numbers from homogeneous coordinates.

A line, much like a point, is an ordered set of three numbers, ($v\ u\ w$), which must not all equal zero at the same time, seeing as that would again make ($\mu v\ \mu u\ \mu w$) the same line for any given $\mu \neq 0$. The set ($v\ u\ w$) = (x_1-x_2 y_1-y_2 y_1x_2-y_2x_1) represents the coordinates of the line that are denoted using rectangular coordinates, whilst the set ($v\ u\ w$) = (- cos φ sin φ y_1 cos φ - x_1 sin φ) represents the coordinates of the line that is denoted using polar coordinates.

By using projective coordinates, the establishment of the intersection $R(Y\ X)$ of two lines – the line l_1, defined by the points $A_1(Y_1\ X_1)$ and $A_2(Y_2\ X_2)$, and the line l_2, on which the points $B_1(Y_1\ X_1)$ and $B_2(Y_2\ X_2)$ lie – is very simple, and is defined using the help of nine second order determinants:

First, the coordinates of lines l_1 and l_2 are established, for which six of secondary-order determinants need to be calculated:

$$\lambda = 1 \Rightarrow l_1 = \begin{pmatrix} v_1 & u_1 & w_1 \end{pmatrix} = \left(\begin{vmatrix} X_1 & 1 \\ X_2 & 1 \end{vmatrix} \begin{vmatrix} 1 & Y_1 \\ 1 & Y_2 \end{vmatrix} \begin{vmatrix} Y_1 & X_1 \\ Y_2 & X_2 \end{vmatrix} \right), \tag{52}$$

$$\lambda = 1 \Rightarrow l_2 = \begin{pmatrix} v_2 & u_2 & w_2 \end{pmatrix} = \left(\begin{vmatrix} X_1 & 1 \\ X_2 & 1 \end{vmatrix} \begin{vmatrix} 1 & Y_1 \\ 1 & Y_2 \end{vmatrix} \begin{vmatrix} Y_1 & X_1 \\ Y_2 & X_2 \end{vmatrix} \right) \tag{53}$$

The intersection R, as the intersection of lines l_1 and l_2, is established with the calculation of another three secondary-order determinants:

$$\mu = 1 \Rightarrow R = \begin{pmatrix} y & x & \omega \end{pmatrix} = \left(\begin{vmatrix} u_1 & w_1 \\ u_2 & w_2 \end{vmatrix} \begin{vmatrix} w_1 & v_1 \\ w_2 & v_2 \end{vmatrix} \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} \right), \tag{54}$$

$$R = \begin{pmatrix} y & x & \omega \end{pmatrix} = \begin{pmatrix} \frac{y}{\omega} & \frac{x}{\omega} & 1 \end{pmatrix} = \begin{pmatrix} Y & X & 1 \end{pmatrix} = \begin{pmatrix} Y & X \end{pmatrix} \tag{55}$$

The projective plane, as has already been mentioned, originates from the Euclidean plane when we assume ideal points and an ideal line. Thus, the point with coordinates (y

$x\ \omega$) lies in the Euclidean plane if $\omega \neq 0$, and $\omega = 0$ to render our point ideal.

Two different ideal points define the ideal line (0 0 1), on which *all* ideal points lie. The notation (sin φ cos φ 0) represents these coordinates of our ideal point, from which we can find the angle which a given line encloses in partnership with the positive side of the *x*-axis.

If the line l_1 and the positive side of the first axis enclose the angle φ_1, and both l_1 and l_2 enclose the angle α, then the ideal point of the line l_2 can be determined as $(\sin(\varphi_1 \pm \alpha)\ \cos(\varphi_1 \pm \alpha)\ 0)$, where $\varphi_2 = \varphi_1 \pm \alpha$ is the angle of site of the line l_2.

Via the introduction of ideal points and an ideal line, we are able to avoid the process of calculation of lengths and angles of site in 2D measurement exercises (the practical example here being the one depicted using intersections). The coordinates of an unknown point may always be established as the intersection of two lines, defined by the given and ideal points.

Povzetek

Možnost uporabe homogenih (projektivnih) koordinat v dvodimenzionalnih merskih nalogah

V projektivni geometriji imajo poleg točk tudi premice koordinate.

Točka je definirana kot urejena trojka števil $(y\ x\ \omega)$, ki niso vse hkrati enake nič, s tem da je $(\lambda y\ \lambda x\ \lambda\omega)$ ista točka za katerikoli $\lambda \neq 0$. Iz nehomogenih koordinat neke točke dobimo neskončno urejenih trojk homogenih koordinat iste točke, iz homogenih koordinat neke točke pa lahko dobimo eno samo urejeno dvojico števil.

Premica je urejena trojka števil $(v\ u\ w)$, ki niso vse hkrati enake nič, s tem da je $(\mu v\ \mu u\ \mu w)$ ista premica za katerikoli $\mu \neq 0$. Trojka $(v\ u\ w) = (x_1 - x_2\ y_1 - y_2\ y_1 x_2 - y_2 x_1)$ predstavlja koordinate premice zapisane s pravokotnimi koordinatami, trojka $(v\ u\ w) = (-\cos\varphi\ \sin\varphi\ y_1 \cos\varphi\ - x_1 \sin\varphi)$ pa koordinate premice zapisane s polarnimi koordinatami.

Z uporabo projektivnih koordinat je določitev presečišča $R(Y\ X)$ dveh premic, premice l_1, ki ju določata točki $A_1(Y_1\ X_1)$ in $A_2(Y_2\ X_2)$ ter premice l_2 na kateri ležita točki $B_1(Y_1\ X_1)$ in $B_2(Y_2\ X_2)$ zelo enostavna, saj se presečišče določi s pomočjo devetih determinant drugega reda:

Najprej se določijo koordinate premic l_1 in l_2 za kar je potrebno rešiti šest determinant drugega reda:

$$\lambda = 1 \Rightarrow l_1 = \begin{pmatrix} v_1 & u_1 & w_1 \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} X_1 & 1 \\ X_2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & Y_1 \\ 1 & Y_2 \end{vmatrix} & \begin{vmatrix} Y_1 & X_1 \\ Y_2 & X_2 \end{vmatrix} \end{pmatrix},$$

$$\lambda = 1 \Rightarrow l_2 = \begin{pmatrix} v_2 & u_2 & w_2 \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} X_1 & 1 \\ X_2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & Y_1 \\ 1 & Y_2 \end{vmatrix} & \begin{vmatrix} Y_1 & X_1 \\ Y_2 & X_2 \end{vmatrix} \end{pmatrix},$$

Presečišče R, kot presek premic l_1 in l_2 pa je odrejeno z rešitvijo še treh determinant drugega reda:

$$\mu = 1 \Rightarrow R = \begin{pmatrix} y & x & \omega \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} u_1 & w_1 \\ u_2 & w_2 \end{vmatrix} & \begin{vmatrix} w_1 & v_1 \\ w_2 & v_2 \end{vmatrix} & \begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix} \end{pmatrix},$$

$$R = \begin{pmatrix} y & x & \omega \end{pmatrix} = \begin{pmatrix} \frac{y}{\omega} & \frac{x}{\omega} & 1 \end{pmatrix} = \begin{pmatrix} Y & X & 1 \end{pmatrix} = \begin{pmatrix} Y & X \end{pmatrix}$$

Projektivno ravnino dobimo iz evklidske ravnine, če privzamemo točke in premico v neskončnosti. Točka s koordinatami ($y\ x\ \omega$) leži v evklidski ravnini, če je $\omega \neq 0$ in je točka v neskončnosti, če je $\omega = 0$.

Dve različni neskončni točki določata premico v neskončnosti (0 0 1) na kateri ležijo vse točke v neskončnosti. Zapis (sin φ cos φ 0) predstavlaj koordinate neskončne točke iz katerih razberemo kot, ki ga neka premica oklepa s pozitivnim delom abscisne osi.

Če oklepa premica l_1 s pozitivnim delom prve osi kot φ_1 s premico l_2 pa kot α lahko neskončno točko premice l_2 določimo kot (sin $(\varphi_1 \pm \alpha)$ $cos(\varphi_1 \pm \alpha)$ 0) pri čemer je $\varphi_2 = \varphi_1 \pm \alpha$ smerni kot premice l_2.

Z uvedbo neskončnih točk in neskončne premice se v merskih dvodimenzionalnih nalogah (praktični primer uporabe prikazan v zunanjem urezu) izognemo računanju dolžin in smernih kotov. Koordinate neznane točke vedno določimo kot presečišče dveh premic, ki jih določat dani točki ter točki v neskončnosti.

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