



3 New Experimental Data for the Quarks Mixing Matrix are in Better Agreement with the *Spin-charge-family* Theory Predictions

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Abstract. The *spin-charge-family* theory [1–14] predicts before the electroweak break four - rather than the coupled and observed three - massless families of quarks and leptons. Mass matrices of all the family members demonstrate in this proposal the same symmetry, determined by the scalar fields: There are two SU(2) triplets, the gauge fields of the family groups, and the three singlets, the gauge fields of the three charges (Q, Q' and Y') distinguishing among family members - all with the quantum numbers of the *standard model* scalar Higgs with respect to the weak and the hyper charge [13]: $\pm \frac{1}{2}$ and $\mp \frac{1}{2}$, respectively. Respecting by the *spin-charge-family* theory proposed symmetry of mass matrices and simplifying the study by assuming that mass matrices are hermitian and real and mixing matrices real, we fit the six free parameters of each family member mass matrix to the experimental data of twice three measured masses of quarks and to the measured quarks mixing matrix elements, within the experimental accuracy. Since any 3×3 sub matrix of the 4×4 matrix (either unitary or orthogonal) determine the whole 4×4 matrix uniquely, we are able to predict the properties of the fourth family members provided that the experimental data for the 3×3 sub matrix are enough accurate, which is not yet the case. However, new experimental data [15] fit better to the required symmetry of mass matrices than the old data [16]. The obtained mass matrices are very closed to the democratic ones.

Povzetek. Teorija *spinov-nabojev-družin* [1–14] napoveduje štiri in ne le tri opažene družine kvarkov in leptonov. Simetrija masnih matrik je v tej teoriji enaka za vse člane družine. Določajo jo skalarna polja: Dve tripletni upodobitvi grupe SU(2), ki določata družinska kvantna števila in tri singletne upodobitve grup nabojev (Q, Q' in Y'). Vsa skalarna polja, to je obe tripletni in vsa tri singletna polja, nosijo tudi šibki in hiper naboj kot ju za Higgsov skalarni delec privzame *standardni model* [13]: ali ($\frac{1}{2}$ in $-\frac{1}{2}$), ali pa ($-\frac{1}{2}$ in $\frac{1}{2}$). Prva vrednost v oklepaju velja za šibki in druga za hiper naboj. Avtorja v prispevku upoštevata simetrijo masnih matrik, ki jo za štiri družine predlaga teorija *spinov-nabojev-družin*. Parametre masnih matrik določita iz izmerjenih podatkov. Izračun masnih matrik in mešalnih matrik poenostavita s privzetkom, da so masne metrike hermitske in realne. Mešalne matrike so tedaj ortogonalne. Vsaka matrika ima v tem primeru šest prostih parametrov. Matrični elementi vsake unitarne matrike $n \times n$ so enolično določeni z z matričnimi elementi podmatrike $(n-1) \times (n-1)$, če je $n \geq 4$, pri ortogonalnih matrikah pa za vsak n . Ker pa eksperimentalni podatki nosijo napako, je mešala matrika 4×4 lahko samo približno določena. Tedaj lahko iz eksperimentalnih podatkov za 2 krat po 3 mase in iz mešalne matrike določimo lastnosti četrte družine v okviru eksperimentalne natančnosti. Izmerjeni matrični elementi mešalne matrike nosijo preveliko napako, da bi avtorja lahko napovedala mase četrte družine bolj

natančno kot da so blizu 1 TeV. Vzpodbudno pa je, da se novi eksperimentalni podatki za mešalno matriko kvarkov bolje ujemajo z zahtevano simetrijo masnih matrik kot stari. Avtorja ugotavljata, da so masne matrike zelo blizu edinkam (demokratičnim matrikam).

3.1 Introduction

There are several attempts in the literature to reconstruct mass matrices of quarks and leptons out of the observed masses and mixing matrices and correspondingly to learn more about properties of the fermion families [17–28]. The most popular is the $n \times n$ matrix, close to the democratic one, predicting that $(n - 1)$ families must be very light in comparison with the n^{th} one. Most of attempts treat neutrinos differently than the other family members, relying on the Majorana part, the Dirac part and the "sea-saw" mechanism. Most often are the number of families taken to be equal to the number of the so far observed families, while symmetries of mass matrices are chosen in several different ways [29–31]. Also possibilities with four families are discussed [32–34].

In this paper we follow the *spin-charge-family* theory [1–14], which predicts four families of quarks and leptons and the symmetries of their mass matrices, the same for all the family members.

The mass matrix of each family member is in the *spin-charge-family* theory determined by the scalar fields, which carry besides by the *standard model* required weak and hyper charges [13] ($\pm \frac{1}{2}$ and $\mp \frac{1}{2}$, respectively) also the additional charges: There are two SU(2) triplets, the gauge fields of the family groups, and three singlets, the gauge fields of the three charges (Q, Q' and Y'), which distinguish among family members. These scalar fields cause, after getting nonzero vacuum expectation values [13], the electroweak break. Assuming that the contributions of all the scalar (and in loop corrections also of other) fields to mass matrices of fermions are real and symmetric, we are left with the following symmetry of mass matrices

$$\mathcal{M}^\alpha = \begin{pmatrix} -a_1 - a & e & d & b \\ e & -a_2 - a & b & d \\ d & b & a_2 - a & e \\ b & d & e & a_1 - a \end{pmatrix}^\alpha, \quad (3.1)$$

the same for all the family members $\alpha \in \{u, d, \nu, e\}$. In appendix 3.5.1 the evaluation of this mass matrix is presented and the symmetry commented. The symmetry of the mass matrix Eq.(3.1) is kept in all loop corrections.

A change of phases of the left handed and the right handed basis - there are $(2n - 1)$ free choices - manifests in a change of phases of mass matrices.

The differences in the properties of the family members originate in the different charges of the family members and correspondingly in the different couplings to the corresponding scalar and gauge fields.

We fit (sect. 3.3.1) the mass matrix (Eq. (3.1)) with 6 free parameters of any family member to the so far observed properties of quarks and leptons within the experimental accuracy. That is: *For a pair of either quarks or leptons, we fit twice 6 free parameters of the two mass matrices to twice three so far measured masses and to the corresponding mixing matrix.*

Since we have the same number of free parameters (6 parameters determine in the *spin-charge-family* theory the mass matrix of any family member after the mass matrices are assumed to be real) as there are measured quantities for either quarks or leptons (two times 3 masses and 6 angles of the orthogonal mixing matrix under the simplification that the mixing matrix is real and hermitian), we should predict the fourth family masses and the missing mixing matrix elements ($V_{u_i d_4}, V_{u_4 d_i}, i \in (1, 2, 3)$) uniquely, provided that the measured quantities are accurate. The $n - 1$ sub matrix of any unitary matrix determines the unitary matrix uniquely for $n \geq 4$. The experimental inaccuracy, in particular for leptons and also for some of the matrix elements of the mixing matrix of quarks, is too large to be able to estimate the fourth family masses better than very roughly even for quarks. Yet we found out that our fitting to the experimental data for quarks are better when using the new experimental data for the quarks mixing matrix [15] than the old ones [16], which mainly differ in the second and the third diagonal values. This might be a signal that the *spin-charge-family* theory is the right step beyond the *standard model* (if taking into account also other predictions of this theory [1–14]), although we assume in this calculations the real mass matrices (Eq. (3.1)) and the orthogonal mixing matrices.

We treat all the family members, the quarks and the leptons, equivalently, as required by the *spin-charge-family* theory. We take into account the estimations of the influence of the fourth family masses to the mesons decays of the refs. [43], making also our own estimations (pretty roughly so far, this work is not presented in this paper)¹.

We can say that the so far obtained data do not contradict the prediction of the *spin-charge-family* theory that there are four coupled families of quarks and leptons, the mass matrices of which manifest the symmetry determined by the family groups – the same for all the family members, quarks and leptons. The mass matrices are quite close to the “democratic” ones, in particular for leptons.

Since the mass matrices offer an insight into the properties of the scalar fields, which determine mass matrices (together with other fields), manifesting effectively as the observed Higgs and the Yukawa couplings, we hope to learn about the properties of these scalar fields also from the mass matrices of quarks and leptons.

In sect. 3.2 the procedure to fit free parameters of mass matrices (Eq. (3.1) to the experimental data is discussed.

We comment our studies in sect. 3.4.

¹ M.I.Vysotsky and A.Lenz comment in their papers [43] that the fourth family is excluded provided that one assumes the *standard model* with one scalar field (the scalar Higgs) while extending the number of families from three to four when, in loop corrections, evaluating the decay properties of the scalar Higgs. We have, however, several scalars: Two times three triplets with respect to the family quantum numbers and three singlets, which distinguish among the family members [13], all with the quantum numbers of the scalar Higgs with respect to the weak and hyper charge. These scalar fields determine all the masses and the mixing matrices of quarks and leptons and of the weak gauge fields, what in the *standard model* is achieved by the choice of the scalar Higgs properties and the Yukawa couplings. Our rough estimations of the decay properties of mesons show that the fourth family quarks might have masses close to 1 TeV.

In appendix 3.5 we offer a very brief introduction into the *spin-charge-family* theory, which the reader, accepting the proposed symmetry of mass matrices without knowing the origin of this symmetry, can skip. In Appendix neutrino the old results [11] for leptons are presented *What follows must be carefully checked and corrected. It must go to the discussion section.

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3.2 Procedure used to fit free parameters of mass matrices to experimental data

This part repeats in many points the ref. [11]

Matrices, following from the *spin-charge-family* theory, might not be hermitian (appendix 3.7). We, however, simplify our study, presented in this paper, by assuming that the mass matrix for any family member, that is for quarks and leptons, is real and symmetric. We take the simplest phases up to signs, which depend on the choice of phases of the basic states, as discussed in appendices 3.5.1².

The matrix elements of mass matrices, with the loop corrections in all orders taken into account, manifesting the symmetry of Eq. (3.1), are in this paper taken as free parameters. Due to this symmetry, required by the family quantum numbers of the scalar fields [13], there 6 parameters having $(n - 1) \cdot (2 - 2)/2$ complex phases. Assuming, to simplify the calculations, that mass matrices are real, there are correspondingly 6 free real parameter for the mass matrix for u and d quarks and for ν and e leptons.

Let us first briefly overview properties of mixing matrices, a more detailed explanation of which can be found in subsection 3.2.1 of this section.

Let M^α , α denotes the family member ($\alpha = u, d, \nu, e$), be the mass matrix in the massless basis (with all loop corrections taken into account). Let $V_{\alpha\beta} = S^\alpha S^{\beta\dagger}$, where α represents either the u-quark and β the d-quark, or α represents the ν -lepton and β the e-lepton, denotes a (in general unitary) mixing matrix of a particular pair: the quarks one or the leptons one.

For $n \times n$ matrix ($n = 4$ in our case) it follows:

- i. If a known sub matrix $(n - 1) \times (n - 1)$ of an unitary matrix $n \times n$ with $n \geq 4$ is extended to the whole unitary matrix $n \times n$, the n^2 unitarity conditions determine $(2(2(n - 1) + 1))$ real unknowns completely. If the sub matrix $(n - 1) \times (n - 1)$ of an unitary matrix is made unitary by itself, then we loose the information of the last row and last column.
- ii. If the mixing matrix is assumed to be orthogonal, then the $(n - 1) \times (n - 1)$ sub matrix contains all the information about the $n \times n$ orthogonal matrix to which it belongs and the $n(n + 1)/2$ conditions determine the $2(n - 1) + 1$ real unknowns

² In the ref. [9] we made a similar assumption, except that we allow that the symmetry on the tree level of mass matrices might be changed in loop corrections. We got in that study dependence of mass matrices and correspondingly mixing matrices for quarks on masses of the fourth family.

completely for any n .

If the sub matrix of the orthogonal matrix is made orthogonal by itself, then we lose all the information of the last row and last column.

We make in this paper, to simplify the present study, several assumptions [39], as it has been already written in the introduction. In what follows we present the procedure used in our study and repeat the assumptions.

1. If the mass matrix M^α is hermitian, then the unitary matrices S^α and T^α , introduced in appendix 3.7 to diagonalize a non hermitian mass matrix, differ only in phase factors depending on phases of basic vectors and manifesting in two diagonal matrices, $F^{\alpha S}$ and $F^{\alpha T}$, corresponding to the left handed and the right handed basis, respectively. For hermitian mass matrices we therefore have: $T^\alpha = S^\alpha F^{\alpha S} F^{\alpha T \dagger}$. By changing phases of basic vectors we can change phases of $(2n - 1)$ matrix elements.
2. We take the diagonal matrices M_d^α and the mixing matrices $V_{\alpha\beta}$ from the available experimental data. The mass matrices M^α in Eq. (3.1) have, if they are hermitian and real, 6 free real parameters ($a^\alpha, a_1^\alpha, a_2^\alpha, b^\alpha, e^\alpha, d^\alpha$), $\alpha = (u, d, \nu, e)$.
3. We limit the number of free parameters of the mass matrix of each family member α by taking into account n relations among free parameters, in our case $n = 4$, determined by the invariants

$$\begin{aligned} I_1^\alpha &= - \sum_{i=1,4} m_i^\alpha, & I_2^\alpha &= \sum_{i>j=1,4} m_i^\alpha m_j^\alpha, \\ I_3^\alpha &= - \sum_{i>j>k=1,4} m_i^\alpha m_j^\alpha m_k^\alpha, & I_4^\alpha &= m_1^\alpha m_2^\alpha m_3^\alpha m_4^\alpha, \\ \alpha &= u, d, \nu, e, \end{aligned} \quad (3.2)$$

which are expressions appearing at powers of $\lambda_\alpha, \lambda_\alpha^4 + \lambda_\alpha^3 I_1 + \lambda_\alpha^2 I_2 + \lambda_\alpha I_3 + \lambda_\alpha^0 I_4 = 0$, in the eigenvalue equation. The invariants are fixed, within the experimental accuracy of the data, by the observed masses of quarks and leptons and by the fourth family mass, if we make a choice of it for a chosen m_4^α . Correspondingly there are $(6 - 4)$ free real parameters left for each mass matrix, after a choice is made for the mass of the fourth family member.

4. The diagonalizing matrices S^α and S^β , each depending on the reduced number of free parameters, are for real and symmetric mass matrices orthogonal. They follow from the procedure

$$\begin{aligned} M^\alpha &= S^\alpha M_d^\alpha T^{\alpha \dagger}, & T^\alpha &= S^\alpha F^{\alpha S} F^{\alpha T \dagger}, \\ M_d^\alpha &= (m_1^\alpha, m_2^\alpha, m_3^\alpha, m_4^\alpha), \end{aligned} \quad (3.3)$$

provided that S^α and S^β fit the experimentally observed mixing matrices $V_{\alpha\beta}^\dagger$ within the experimental accuracy and that M^α and M^β manifest the symmetry presented in Eq. (3.1). We keep the symmetry of the mass matrices accurate. One can proceed in two ways.

$$\begin{aligned} \text{A. : } & S^\beta = V_{\alpha\beta}^\dagger S^\alpha, & \text{B. : } & S^\alpha = V_{\alpha\beta} S^\beta, \\ \text{A. : } & V_{\alpha\beta}^\dagger S^\alpha M_d^\beta S^{\alpha \dagger} V_{\alpha\beta} = M^\beta, & \text{B. : } & V_{\alpha\beta} S^\beta M_d^\alpha S^{\beta \dagger} V_{\alpha\beta}^\dagger = M^\alpha. \end{aligned} \quad (3.4)$$

In the case A. one obtains from Eq. (3.3), after requiring that the mass matrix M^α has the desired symmetry, the matrix S^α and the mass matrix $M^\alpha (= S^\alpha \mathbf{M}_d^\alpha S^{\alpha\dagger})$, from where we get the mass matrix $M^\beta = V_{\alpha\beta}^\dagger S^\alpha \mathbf{M}_d^\beta S^{\alpha\dagger} V_{\alpha\beta}$. In case B. one obtains equivalently the matrix S^β , from where we get $M^\alpha (= V_{\alpha\beta} S^\beta \mathbf{M}_d^\alpha S^{\beta\dagger} V_{\alpha\beta}^\dagger)$. We use both ways iteratively taking into account the experimental accuracy of masses and mixing matrices.

5. Under the assumption of the present study that the mass matrices are real and symmetric, the orthogonal diagonalizing matrices S^α and S^β form the orthogonal mixing matrix $V_{\alpha\beta}$, which depends on at most $6 (= \frac{n(n-1)}{2})$ free real parameters (appendix 3.7). Since, due to what we have explained at the beginning of this section, the experimentally measured matrix elements of the 3×3 sub matrix of the 4×4 mixing matrix (if not made orthogonal by itself) determine (within the experimental accuracy) the 4×4 mixing matrix, also the fourth family masses are determined, again within the experimental accuracy. We must not forget, however, that the assumption of the real and symmetric mass matrices, leading to orthogonal mixing matrices, might not be an acceptable simplification, since we do know that the 3×3 sub matrix of the mixing matrix has one complex phase, while the unitary 4×4 has three complex phases. (In the next step of study, with hopefully more accurate experimental data, we shall relax conditions on hermiticity of mass matrices and correspondingly on orthogonality of mixing matrices.) We expect that too large experimental inaccuracy leave the fourth family masses in the present study quite undetermined, in particular for leptons.
6. We study quarks and leptons equivalently. The difference among family members originate on the tree level in the eigenvalues of the operators ($Q^\alpha, Q'^\alpha, Y'^\alpha$), which in loop corrections together with other contributors in all orders contribute to all mass matrix elements and cause the difference among family members³.

Let us conclude. If the mass matrix of a family member obeys the symmetry required by the *spin-charge-family* theory, which in a simplified version (as it is taken in this study) is real and symmetric, the matrix elements of the mixing matrices of quarks and leptons are correspondingly real, each of them with $\frac{n(n-1)}{2}$ free parameters. These six parameters of each mixing matrix are, within the experimental inaccuracy, determined by the three times three experimentally determined sub matrix. After taking into account three so far measured masses of each family member, the six parameters of each mass matrix reduce to three. Twice three free parameters are within the experimental accuracy correspondingly determined by the 3×3 sub matrix of the mixing matrix. The fourth family masses are correspondingly determined - within the experimental accuracy.

Since neither the measured masses nor the measured mixing matrices are determined accurately enough to reproduce the 4×4 mixing matrices, we can in the best case expect that the masses and mixing matrix elements of the fourth family will be determined only within some quite large intervals.

³ There are also Majorana like terms contributing in higher order loop corrections [7] which might strongly influence in particular the neutrino mass matrix.

3.2.1 Submatrices and their extensions to unitary and orthogonal matrices

In this part well known properties of $n \times n$ matrices, extended from $(n-1) \times (n-1)$ submatrices are discussed. We make a short overview of the properties, needed in this paper, although all which will be presented here, is the knowledge on the level of text books.

Any $n \times n$ complex matrix has $2n^2$ free parameters. The $n + 2n(n-1)/2$ unitarity requirements reduce the number of free parameters to n^2 ($= 2n^2 - (n + 2n(n-1)/2)$). Let us assume a $(n-1) \times (n-1)$ known sub matrix of the unitary matrix. The sub matrix can be extended to the unitary matrix by $(2 \times [2(n-1) + 1])$ real parameters of the last column and last row. The n^2 unitarity conditions on the whole matrix reduce the number of unknowns to $(2(2n-1) - n^2)$. For $n = 4$ and higher the $(n-1) \times (n-1)$ sub matrix contains all the information about the unitary $n \times n$ matrix.

The ref. [37] proposes a possible extension of an $(n-1) \times (n-1)$ unitary matrix $V_{(n-1)(n-1)}$ into $n \times n$ unitary matrices V_{nn} .

The choice of phases of the left and the right basic states which determine the unitary matrix (like this is the case with the mixing matrices of quarks and leptons) reduces the number of free parameters for $(2n-1)$. Correspondingly is the number of free parameters of such an unitary matrix equal to $n^2 - (2n-1)$, which manifests in $\frac{1}{2}n(n-1)$ real parameters and $\frac{1}{2}(n-1)(n-2)$ ($= n^2 - \frac{1}{2}n(n-1) - (2n-1)$) phases (which determine the number of complex parameters).

Any real $n \times n$ matrix has n^2 free parameters which the $\frac{1}{2}n(n+1)$ orthogonality conditions reduce to $\frac{1}{2}n(n-1)$. The $(n-1) \times (n-1)$ sub matrix of this orthogonal matrix can be extended to this $n \times n$ orthogonal matrix with $[2(n-1) + 1]$ real parameters. The $\frac{1}{2}n(n+1)$ orthogonality conditions reduce these $[2(n-1) + 1]$ free parameters to $(2n-1 - \frac{1}{2}n(n+1))$, which means that the $(n-1) \times (n-1)$ sub matrix of an $n \times n$ orthogonal matrix determine properties of its $n \times n$ orthogonal matrix completely. Any $(n-1) \times (n-1)$ sub matrix of an orthogonal matrix contains all the information about the whole matrix for any n . Making the sub matrix of the orthogonal matrix orthogonal by itself one loses the information about the $n \times n$ orthogonal matrix.

3.2.2 Free parameters of mass matrices after taken into account invariants

It is useful for numerical evaluation purposes to take into account for each family member its mass matrix invariants (sect. 3.2), expressible with three within the experimental accuracy known masses, while we keep the fourth one as a free parameter. We shall make a choice of $a^\alpha = \frac{1}{4} I_1^\alpha$ (Eqs. (3.1, 3.6)) instead of the fourth family mass.

We shall skip in this section the family member index α and introduce new parameters as follows

$$a, b, \quad f = d + e, \quad g = d - e, \quad q = \frac{a_1 + a_2}{\sqrt{2}}, \quad r = \frac{a_1 - a_2}{\sqrt{2}}. \quad (3.5)$$

After choosing as a free parameter $a = \frac{I_1}{4}$ (Eq. (3.6)), which is indeed the fourth family mass - summed together with the three known (from the experiment)

masses in I_1 - the four invariants of Eq. (3.2) reduce the number of free parameters to 2. The four invariants therefore relate six parameters leaving three of them undetermined. There are for each pair of family members the measured mixing matrix elements, assumed in this paper to be orthogonal and correspondingly determined by six parameters, which then fixes these two times 3 parameters. The (accurately enough) measured 3×3 sub matrix of the (assumed to be orthogonal) 4×4 mixing matrix namely determines these 6 parameters within the experimental accuracy.

Using the starting relation among the invariants I_i , $i \in (1, 2, 3, 4)$ and replacing new parameters (a, b, f, g, q, r) from Eq. (3.5) we obtain

$$\begin{aligned}
 a &= \frac{I_1}{4}, \\
 I'_2 &= -I_2 + 6a^2 - q^2 - r^2 - 2b^2 = f^2 + g^2, \\
 I'_3 &= -\frac{1}{2b}(I_3 - 2aI_2 + 4a^2) = f^2 - g^2, \\
 I'_4 &= I_4 - aI_3 + a^2I_2 - 3a^4 \\
 &= \frac{1}{4}(q^2 - r^2)^2 + (q^2 + r^2)b^2 + \frac{1}{2}(q^2 - r^2) \cdot (\pm) \cdot [\pm] 2gf \\
 &\quad + b^2(f^2 + g^2) + \frac{1}{4}(2gf)^2.
 \end{aligned} \tag{3.6}$$

We eliminate, using the first two equations, the parameters f and g , expressing them as functions of I'_2 and I'_3 , which depend, for a particular family member, on the three known masses, the parameter a and the three parameters r, q and b . We are left with the four free parameters (a, b, q, r) and the below relation among these parameters

$$\begin{aligned}
 &\left\{-\frac{1}{2}(q^4 + r^4) + (-2b^2 + \frac{1}{2}(-I_2 + 6a^2 - 2b^2))(q^2 + r^2)\right. \\
 &+ \left.(I'_4 - \frac{1}{4}((-I_2 + 6a^2 - 2b^2)^2 + I_3'^2) + b^2(-I_2 + 6a^2 - 2b^2))\right\}^2 \\
 &= -\frac{1}{4}(q^2 - r^2)^2((-I_2 + 6a^2 - 2b^2 - (q^2 + r^2))^2 - I_3'^2),
 \end{aligned} \tag{3.7}$$

which reduces the number of free parameters to 3. These 3 free parameters must be determined, together with the corresponding three parameters of the partner, from the measured mixing matrix.

We eliminate one of the 4 free parameters in Eq. (3.7) by solving the cubic equation for, let us make a choice, q^2

$$\alpha q^6 + \beta q^4 + \gamma q^2 + \delta = 0. \tag{3.8}$$

Parameter $(\alpha, \beta, \gamma, \delta)$ depend on the 3 free remaining parameters (a, b, r) and the three, within experimental accuracy, known masses.

To reduce the number of free parameters from the starting 6 in Eq. (3.1) to the 3 left after taking into account invariants of each mass matrix, we look for the solution of Eq (3.8) for all allowed values for (a, b, r) . We make a choice for a in the interval of (a_{\min}, a_{\max}) , determined by the requirement that a , which solves

the equations, is a real number. Allowing only real values for parameters f and g we end up with the equation

$$-I_2 + 6a^2 - 2b^2 - (q^2 + r^2) > \left| \frac{I_3 + 8a^3 - 2aI_2}{2b} \right|, \quad (3.9)$$

which determines the maximal positive b for $q = 0 = r$ and also the minimal positive value for b . For each value of the parameter a the interval (b_{\min}, b_{\max}) , as well as the interval $(r_{\min} = 0, r_{\max})$, follow when taking into account experimental values for the three lower masses.

Trying to fit the free parameters to the experimental values of the 3×3 submatrix to the mixing matrix we minimize the uncertainty defined in Eq. (3.10)

$$\sigma = \sqrt{\sum_{(i,j)=1}^3 \left(\frac{V_{u_i d_j \text{ exp}} - V_{u_i d_j \text{ cal}}}{\sigma_{V_{u_i d_j \text{ exp}}}} \right)^2},$$

$$\delta V_{u_i d_j} = \left| \frac{V_{u_i d_j \text{ exp}} - V_{u_i d_j \text{ cal}}}{\sigma_{V_{u_i d_j \text{ exp}}}} \right|, \quad (3.10)$$

where expressions $\sigma_{V_{u_i d_j \text{ exp}}}$ stay for the experimental uncertainties, presented in Eqs. (3.11, 3.12).

3.3 Numerical results

Taking into account the assumptions and the procedure explained in sect. 3.2 and in the ref. [39] we are looking for the 4×4 in this paper taken to be real and symmetric mass matrices for quarks and leptons, obeying the symmetry of Eq. (3.1) and manifesting observed properties - masses and mixing matrices - of the so far observed three families of quarks in agreement with the experimental limits for the appearance of the fourth family masses and mixing matrix elements to the lower three families, as presented in the refs. [16,15,43]. We also take into account our so far made rough estimations of possible contributions of the fourth family members to the decay of mesons. More detailed estimations are in progress. The results for leptons, presented in Appendix 3.6 are the old ones, taking from [11]. They are added only for the comparison.

We hope that we shall be able to learn from the mass matrices of quarks and leptons also about the properties of the scalar fields, which cause masses of quarks and leptons, manifesting effectively so far as the measured Higgs and Yukawa couplings.

We take the 3×3 measured mixing matrices for quarks and leptons and the measured masses, all with the experimental inaccuracy. We extend the measured nine mixing matrix elements for each pair to the corresponding 4×4 mass matrix, by taking into account the unitarity of the 4×4 matrix, in our case indeed the orthogonality of the 4×4 matrices. We then look for twice 4×4 mass matrices with the symmetry of Eq. (3.1), and correspondingly for the fourth family masses, for quarks and leptons.

We perform the calculations for quarks with the old [16] and new [15] experimental data for the quarks mixing matrix, to see, whether or not the more accurate values fit better into by the *spin-charge-family* theory predicted symmetry of mass matrices (Eq. (3.1)). We present in appendix 3.6 also one trial for the lepton mass matrices. Since the experimental data for the mixing matrix and masses are for leptons known so inaccurate, the results do not tell much.

To test the predicting power of our model in dependence of the experimental inaccuracy of masses and mixing matrices, we compare the calculated mass matrices for quarks, obtained when choosing different values for the fourth family masses, among themselves and with the experimental data, the old [16] ones and the new [15] ones.

3.3.1 Numerical results for the observed quarks with mass matrices obeying Eq. (3.1)

We take for the quarks masses the experimental values [16], recalculated to the Z boson mass scale. We take two kinds of the experimental data for the quark mixing matrices, the older data from [16] and the last data [15], with the experimentally declared inaccuracies for the so far measured 3×3 mixing matrix. We assume, as suggested by the *spin-charge-family* theory, that these nine matrix elements belong to the 4×4 unitary mixing matrix. We take into account the experimentally allowed values for the fourth family masses and other limitations, presented in refs. [43,32–34]. We have made also our own rough estimations for the limitations which follow from the meson decays to which the fourth family members participate. Our estimations are still in progress.

A lot of effort was put into the numerical procedure to be sure as much as one can, that we fit the parameters of mass matrices to the experimental values within the experimental inaccuracy, in the best way, that is with the smallest errors.

It is expected that the inaccuracy, mainly due to the quarks mixing matrix, masses do not influence the results so strongly, does not allow to tell much about the fourth family masses. Yet, what we have learned not only supports the predicted symmetry of the *spin-charge-family* theory, but also predicts to what values will the more accurately measured matrix elements of the 3×3 sub matrix of the 4×4 mixing matrix move.

Let us admit that from the so far obtained results we are not yet able to predict the fourth family quarks mass accurately enough, although the results show that the most trustable might be results pushing the fourth family quarks to 1 TeV or above.

The results manifest that the mass matrices are very close to the democratic ones, which is, as expected, more and more the case the higher might be the fourth family masses, and it is true for quarks and leptons.

The calculated 4×4 mixing matrix predicts, in dependence of the fourth family masses, not only the fourth family matrix elements of the mixing matrix, but also the direction in which will the matrix elements of the 3×3 sub matrix move in the future more accurate measurements - under the assumption that the *spin-charge-family* theory is offering the right next step beyond the standard model.

In this paper we do not take yet into account the complex phases of the mass matrix elements and correspondingly of the mixing matrices. Sooner or latter we ought to do that.

We present below two types of the experimental values for the quarks 3×3 mixing matrix, taken as the sub matrix of the 4×4 matrix, the older experimental data [16] and the newer experimental data [15].

We start with the older experimental data [16]

$$|V_{ud}| = \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00415 \pm 0.00049 & |V_{u_1 d_4}| \\ 0.230 \pm 0.011 & 1.006 \pm 0.023 & 0.0409 \pm 0.0011 & |V_{u_2 d_4}| \\ 0.0084 \pm 0.0006 & 0.0429 \pm 0.0026 & 0.89 \pm 0.07 & |V_{u_3 d_4}| \\ |V_{u_4 d_1}| & |V_{u_4 d_2}| & |V_{u_4 d_3}| & |V_{u_4 d_4}| \end{pmatrix}, \quad (3.11)$$

and then repeat all the calculations also with the new experimental data [15]

$$|V_{ud}| = \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & |V_{u_1 d_4}| \\ 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & |V_{u_2 d_4}| \\ 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & |V_{u_3 d_4}| \\ |V_{u_4 d_1}| & |V_{u_4 d_2}| & |V_{u_4 d_3}| & |V_{u_4 d_4}| \end{pmatrix}. \quad (3.12)$$

The matrix elements of the 4×4 quark mixing matrix will be determine in the numerical procedure, which searches for the best fit of the two quarks mass matrices free parameters presented in Eq. (3.1) to the experimental data, taking into account the experimental inaccuracy and unitarity of the mixing matrix, ensuring as much as possible, the best fit.

Let us notice that in the new experimental data differ slightly from the old ones only in the two diagonal matrix elements, $V_{cs} = V_{u_2 d_2}$ and $V_{tb} = V_{u_3 d_3}$, appearing in new data with smaller inaccuracy. The corresponding fourth family mixing matrix elements ($|V_{u_i d_4}|$ and $|V_{u_4 d_j}|$) are accordingly in both cases determined from the unitarity condition for the 4×4 mixing matrix through the fitting procedure, as also all the other matrix elements of the mixing matrix are.

Using first the old experimental data we predict the direction in which new more accurately measured matrix elements should move and then check if this is happening with the new experimental data.

Then we use new experimental data, repeat the procedure in look at what are the new results predicting.

For the quark masses at the energy scale of M_Z we take

$$\begin{aligned} \mathbf{M}_d^u / \text{MeV}/c^2 &= (1.3 + 0.50 - 0.42, 619 \pm 84, 172\,000. \pm 760., \\ &\quad m^{u_4} = 700\,000., 1\,200\,000.), \\ \mathbf{M}_d^d / \text{MeV}/c^2 &= (2.90 + 1.24 - 1.19, 55 + 16 - 15, 2\,900. \pm 90., \\ &\quad m^{d_4} = 700\,000., 1\,200\,000.). \end{aligned} \quad (3.13)$$

We found that the results are not influenced much if changing the masses within the experimental uncertainties.

Experimental values for leptons as well as the obtained mass matrices are presented in appendix 3.6.

Following the procedure explained in sect. 3.2 we look for the mass matrices for the u-quarks and the d-quarks by requiring that the mass matrices reproduce experimental data while manifesting symmetry of Eq. (3.1), predicted by the *spin-charge-family* theory.

We look for several properties of the obtained mass matrices:

- i. We test the influence of the experimentally declared inaccuracy of the 3×3 sub matrices of the 4×4 mixing matrices and of the twice 3 measured masses on the prediction of the fourth family masses.
- ii. We look for how do the old and the new matrix elements of the measured mixing matrix influence the accuracy with which the experimental data are reproduced in the procedure which takes into account the symmetry of mass matrices.
- iii. We look for how different choices for the masses of the fourth family members limit the inaccuracy of particular matrix elements of the mixing matrices or the inaccuracy of the three lower masses of family members.
- iv. We test how close to the democratic mass matrix are the obtained mass matrices in dependence of the fourth family masses.
- v. We look for the predictions of the 4×4 mass matrices with the symmetry presented in Eq. (3.1).

The numerical procedure, used in this contribution, is designed for quarks and leptons. We present in this paper the results for quarks. The results for leptons, presented in appendix 3.6 is only to manifest the general properties of leptons, since the experimental data for leptons are far too non accurate to lead to trustable predictions.

In the next subsection 3.3.1 the numerical results are presented for the 4×4 mass matrices of the u-quarks and the d-quarks as they follow from the by the *spin-charge-family* theory required symmetry after fitting the experimental data.

Mass matrices for quarks In order to test whether or not our results have some experimental support, we use two kinds of the experimental values for the quark mixing matrix, presented in Eqs. (3.11, 3.12), respectively, for several values of the fourth family quark masses.

Searching for mass matrices with the symmetry of Eq. (3.1) to determine the interval for the fourth family quark masses in dependence of the values of the mixing matrix elements within the experimental inaccuracy, we repeat the numerical procedure for data with several values of masses of the fourth family quarks. Here we present results for two of them: for $m_{u_4} = 700 \text{ GeV} = m_{d_4}$ and for $m_{u_4} = 1\,200 \text{ GeV} = m_{d_4}$.

We present below the results for the two experimental matrix elements [16,15] for the quark mixing matrix, first for the data [16] and then for the data [15], in both cases first for $m_{u_4} = 700 \text{ GeV} = m_{d_4}$ and then for $m_{u_4} = 1\,200 \text{ GeV} = m_{d_4}$.

Having results from the fitting procedure when used the old experimental data for the quark mixing matrix, we look for the predictions, which the calculated 3×3 matrix elements of the 4×4 mixing matrix obeying the symmetry of Eq. (3.1) offer, and then check to which extend the predictions agree with new experimental data.

Then we repeat calculations with new experimental data.

- Results for the mass matrices of the two quarks family members, fitted to the mixing matrix elements presented in the ref. [16]. The fit offers the smallest common deviation (Eq. 3.10)) of the sum of all the average values of the nine matrix elements of the 3×3 sub matrix. The masses of quarks and the mixing matrix resulting from diagonalizing the two best fitted mass matrices are also presented.

1. Here $m_{u_4} = 700 \text{ GeV}$ $m_{d_4} = 700 \text{ GeV}$ is chosen.

$$M^u = \begin{pmatrix} 227623. & 131877. & 132239. & 217653. \\ 131877. & 222116. & 217653. & 132239. \\ 132239. & 217653. & 214195. & 131877. \\ 217653. & 132239. & 131877. & 208687. \end{pmatrix}, \quad (3.14)$$

$$M^d = \begin{pmatrix} 175797. & 174263. & 174288. & 175710. \\ 174263. & 175666. & 175710. & 174288. \\ 174288. & 175710. & 175813. & 174263. \\ 175710. & 174288. & 174263. & 175682. \end{pmatrix},$$

$$V_{ud} = \begin{pmatrix} -0.97423 & 0.22531 & -0.003 & 0.01021 \\ 0.22526 & 0.97338 & -0.042 & 0.0016 \\ -0.00663 & -0.04197 & -0.9991 & -0.0004 \\ 0.00959 & -0.00388 & -0.0003 & 0.99995 \end{pmatrix}. \quad (3.15)$$

The corresponding absolute values for the deviations from the average experimental values (Eq.(3.10)) are

$$\delta V_{ud} = \begin{pmatrix} 0.091 & 0.117 & 2.339 \\ 0.431 & 1.418 & 1.348 \\ 2.951 & 0.358 & 1.559 \end{pmatrix}. \quad (3.16)$$

The corresponding total absolute average deviation Eq. (3.10) is 4.5579.

The two mass matrices correspond to the diagonal masses

$$\mathbf{M}_d^u/\text{MeV}/c^2 = (1.3, 620.0, 172\,000., 700\,000.),$$

$$\mathbf{M}_d^d/\text{MeV}/c^2 = (2.88508, 55.024, 2\,899.99, 700\,000.). \quad (3.17)$$

2. In the next case $m_{u_4} = 1\,200 \text{ GeV}$ and $m_{d_4} = 1\,200 \text{ GeV}$ are chosen, again fitting the old [16] experimental for quark mixing matrix elements.

$$M^u = \begin{pmatrix} 351916. & 256894. & 257204. & 342714. \\ 256894. & 344411. & 342714. & 257204. \\ 257204. & 342714. & 341900. & 256894. \\ 342714. & 257204. & 256894. & 334395. \end{pmatrix}, \quad (3.18)$$

$$M^d = \begin{pmatrix} 300783. & 299263. & 299288. & 300709. \\ 299263. & 300623. & 300709. & 299288. \\ 299288. & 300709. & 300856. & 299263. \\ 300709. & 299288. & 299263. & 300696. \end{pmatrix},$$

$$V_{ud} = \begin{pmatrix} -0.97425 & 0.22536 & -0.00301 & 0.00474 \\ 0.22534 & 0.97336 & -0.04239 & 0.00212 \\ -0.00663 & -0.04198 & -0.9991 & -0.00021 \\ 0.00414 & -0.00315 & -0.00011 & 0.99999 \end{pmatrix}. \quad (3.19)$$

The corresponding values for the deviations from the average experimental value of the matrix elements of the 3×3 sub matrix are

$$\delta V_{ud} = \begin{pmatrix} 0.003 & 0.226 & 2.335 \\ 0.424 & 1.419 & 1.357 \\ 2.949 & 0.355 & 1.559 \end{pmatrix}. \quad (3.20)$$

The corresponding total average deviation Eq. (3.10) is 4.5595.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned} \mathbf{M}_d^u/\text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 700\,000.), \\ \mathbf{M}_d^d/\text{MeV}/c^2 &= (2.9, 55.0, 2\,900.0, 700\,000.). \end{aligned} \quad (3.21)$$

Let us notice, that while the mass matrices of the u and the d quarks change for a factor of ≈ 1.5 , becoming more "democratic" (that is the matrix elements become more and more equal), when changing the fourth family masses from 700 GeV to 1 200 GeV, the mixing matrix elements of the 3×3 sub matrix do not change a lot (Eqs.(3.15, 3.19)).

Let us now see what does our calculations say. We first make comparison for the old [16] (exp_o) mixing matrix with the calculated ones when the fourth family quark masses are 700 GeV, and 1 200 GeV. Results are presented in Eq. (3.22)

$$|V_{(ud)_{old}}| = \begin{pmatrix} \text{exp}_o & 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00415 \pm 0.00049 \\ \text{old}_1 & 0.97423 & 0.22531 & 0.003 \\ \text{old}_2 & 0.97425 & 0.22536 & 0.00301 \\ \text{exp}_o & 0.230 \pm 0.011 & 1.006 \pm 0.023 & 0.0409 \pm 0.0011 \\ \text{old}_1 & 0.22526 & 0.97338 & 0.042 \\ \text{old}_2 & 0.22534 & 0.97336 & 0.04239 \\ \text{exp}_o & 0.0084 \pm 0.0006 & 0.0429 \pm 0.0026 & 0.89 \pm 0.07 \\ \text{old}_1 & 0.00663 & 0.04197 & 0.9991 \\ \text{old}_2 & 0.00663 & 0.04198 & 0.9991 \end{pmatrix}. \quad (3.22)$$

The calculated mixing predicts:

- i. The matrix element $V_{u_1 d_1}$ should almost not change, $V_{u_1 d_2}$ may slightly rise, and ($V_{u_2 d_3}$ and $V_{u_3 d_3}$) will also rise.
 - ii. The matrix elements ($V_{u_1 d_3}$, $V_{u_2 d_1}$, $V_{u_2 d_2}$, $V_{u_3 d_1}$, $V_{u_3 d_2}$) should lower. Checking the new experimental values one sees that the prediction was in all the cases in agreement with those new experimental data which were done with better accuracy.
- let us repeat the calculations with new experimental data [15] to see how will the new data influence the mass matrices and the mixing matrix elements.

Results for the mass matrices of the two quarks family members, fitted to the new mixing matrix elements [15], which lead to the smallest common deviation for the sum of all the average values of the nine matrix elements of the 3×3 sub matrix, are presented, together with the masses of quarks and the mixing matrix resulting from diagonalizing the two mass matrices. Again the fourth quark masses are first ($m_{u_4} = 700$ GeV, $m_{d_4} = 700$ GeV) and then ($m_{u_4} = 1\,200$ GeV, $m_{d_4} = 1\,200$ GeV)

1. Here $m_{u_4} = 700$ GeV and $m_{d_4} = 700$ GeV is chosen.

$$M^u = \begin{pmatrix} 226521. & 131887. & 132192. & 217715. \\ 131887. & 219347. & 217715. & 132192. \\ 132192. & 217715. & 216964. & 131887. \\ 217715. & 132192. & 131887. & 209790. \end{pmatrix}, \quad (3.23)$$

$$M^d = \begin{pmatrix} 175776. & 174263. & 174288. & 175709. \\ 174263. & 175622. & 175709. & 174288. \\ 174288. & 175709. & 175857. & 174263. \\ 175709. & 174288. & 174263. & 175703. \end{pmatrix},$$

$$V_{ud} = \begin{pmatrix} -0.97423 & 0.22539 & -0.00299 & 0.00776 \\ 0.22534 & 0.97335 & -0.04245 & 0.00349 \\ -0.00667 & -0.04203 & -0.99909 & -0.00038 \\ 0.00677 & -0.00517 & -0.00020 & 0.99996 \end{pmatrix}. \quad (3.24)$$

The corresponding values Eq. (3.10) for the deviations from the average experimental values are

$$\delta V_{ud} = \begin{pmatrix} 0.074 & 0.109 & 2.339 \\ 0.043 & 0.791 & 1.032 \\ 2.291 & 0.753 & 0.685 \end{pmatrix}. \quad (3.25)$$

The corresponding total absolute average deviation Eq. (3.10) is 4.07154.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned} \mathbf{M}_d^u / \text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 700\,000.), \\ \mathbf{M}_d^d / \text{MeV}/c^2 &= (2.9, 55.0, 2\,900.0, 700\,000.). \end{aligned} \quad (3.26)$$

2. Here $m_{u_4} = 1\,200$ GeV $m_{d_4} = 1\,200$ GeV is chosen.

$$M^u = \begin{pmatrix} 354761. & 256877. & 257353. & 342539. \\ 256877. & 350107. & 342539. & 257353. \\ 257353. & 342539. & 336204. & 256877. \\ 342539. & 257353. & 256877. & 331550. \end{pmatrix}, \quad (3.27)$$

$$M^d = \begin{pmatrix} 300835. & 299263. & 299288. & 300710. \\ 299263. & 300714. & 300710. & 299288. \\ 299288. & 300710. & 300765. & 299263. \\ 300710. & 299288. & 299263. & 300644. \end{pmatrix},$$

$$V_{ud} = \begin{pmatrix} 0.97423 & 0.22538 & 0.00299 & 0.00793 \\ -0.22531 & 0.97336 & 0.04248 & -0.00002 \\ 0.00667 & -0.04206 & 0.99909 & -0.00024 \\ -0.00773 & -0.00178 & 0.00022 & 0.99997 \end{pmatrix}. \quad (3.28)$$

The corresponding values for the deviations from the average experimental value for each matrix element are

$$\delta V_{ud} = \begin{pmatrix} 0.07 & 0.097 & 2.329 \\ 0.038 & 0.79 & 1.061 \\ 2.889 & 0.762 & 0.685 \end{pmatrix}. \quad (3.29)$$

The corresponding total average deviation Eq. (3.10) is 4.0724.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned} \mathbf{M}_d^u/\text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 1\,200\,000.), \\ \mathbf{M}_d^d/\text{MeV}/c^2 &= (2.88508, 55.024, 2\,899.99, 1\,200\,000.). \end{aligned} \quad (3.30)$$

Again we notice that the mass matrices of the u and the d quarks change for a factor of ≈ 1.5 when the masses of the fourth family members grow from 700 GeV to 1 200 GeV. The mass matrices become more "democratic". The mixing matrix elements of the 3×3 sub matrix do not change a lot (Eqs.(3.24, 3.28)) with the masses of the fourth family quarks, but they do agree better with the newer [15] than with the older [16] experimental values.

Let us now compare the old [16] (exp_o) and the new [15] (exp_n) mixing matrix elements of the 3×3 sub matrix with the calculated ones for either the old [16] or for the new [15] experimental values, fitting them to the mass matrices of Eq. (3.1), in both cases for $m_{u_4} = m_{d_4} = 700$ GeV and for $m_{u_4} = m_{d_4} = 1\,200$ GeV, to see whether we can learn something out of this comparison.

We present below the old data (exp_o), the new data (exp_n) and both calculated values, each for $m_{u_4} = m_{d_4} = 700$ GeV (old_1, new_1) and $m_{u_4} = m_{d_4} = 1\,200$

GeV (old₂, new₂), putting together all these values in the same matrix.

$$|V_{(ud)}| = \begin{pmatrix} \text{exp}_o & 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00415 \pm 0.00049 \\ \text{exp}_n & 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 \\ \text{old}_1 & 0.97423 & 0.22531 & 0.003 \\ \text{old}_2 & 0.97425 & 0.22536 & 0.00301 \\ \text{new}_1 & 0.97423 & 0.22531 & 0.00299 \\ \text{new}_2 & 0.97423 & 0.22538 & 0.00299 \\ \hline \text{exp}_o & 0.230 \pm 0.011 & 1.006 \pm 0.023 & 0.0409 \pm 0.0011 \\ \text{exp}_n & 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 \\ \text{old}_1 & 0.22526 & 0.97338 & 0.042 \\ \text{old}_2 & 0.22534 & 0.97336 & 0.04239 \\ \text{new}_1 & 0.22534 & 0.97335 & 0.04245 \\ \text{new}_2 & 0.22531 & 0.97336 & 0.04248 \\ \hline \text{exp}_o & 0.0084 \pm 0.0006 & 0.0429 \pm 0.0026 & 0.89 \pm 0.07 \\ \text{exp}_n & 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 \\ \text{old}_1 & 0.00663 & 0.04197 & 0.9991 \\ \text{old}_2 & 0.00663 & 0.04198 & 0.9991 \\ \text{new}_1 & 0.00667 & 0.04203 & 0.99909 \\ \text{new}_2 & 0.00667 & 0.04206 & 0.99909 \end{pmatrix}. \quad (3.31)$$

Comparing the above results and the results for mass matrices and 4×4 mixing matrices one finds:

- i. The old and new experimental data differ mainly in the diagonal matrix elements.
- ii. The old and new experimental data lead in the fitting procedure to quite similar 3×3 sub matrix, while their influence on the fourth family matrix elements are stronger.
- iii. The fourth family masses change the mass matrices considerably, while their influence on the 3×3 sub matrix of the 4×4 mixing matrix is much weaker.
- iv. The prediction (Eq. (3.22)) of the calculated mixing matrix elements, obtained by fitting the symmetry of the mass matrices (Eq. (3.1)) to the experimental data [16], was confirmed by improved experimental data [15]. In all cases are the calculated 3×3 matrix elements closer to the new experimental values than to the old experimental values.
- v. Calculations with new experimental data predict: We expect (Eq. (3.31)) that more accurate experiments will bring a slightly smaller values for $(V_{u_1 d_1}, V_{u_1 d_3}, V_{u_3 d_3})$, smaller $(V_{u_2 d_2}, V_{u_3 d_1})$, $(V_{u_1 d_2}, V_{u_2 d_1})$ will slightly grow and $(V_{u_2 d_3}, V_{u_3 d_2})$ will grow.
- vi. The matrix elements $V_{u_i d_4}$ and $V_{u_4 d_i}$ change considerably with the mass of the fourth family members, and they differ quite a lot also when using new instead of the old experimental data for the mixing matrix.
- vii. Fitting the free parameters of the mass matrices to the new experimental data [15] gives smaller parameter σ (Eq. sigma) than when fitting old experimental data [16]: 4.07154 with respect to 4.5579 for the masses 700 GeV and 4.0724 with respect to 4.5595 for the masses 1 200 GeV, while with the mass σ does not really

change. Only very accurate mixing matrix elements would allow to determine fourth family quarks masses more accurately. Since the choice of the fourth family quark masses does not appreciable influence either the fitting procedure or the obtained 3×3 mixing matrix, and also not the accuracy of the masses of the three lower families, it is difficult to predict the interval for the masses of the fourth family members. For the masses of the fourth family quarks to be close or above 1 TeV speak more other experimental data, like decays of mesons.

An estimation how trustable is the numerical procedure, used to fit free parameters of the quarks mass matrices to the experimental data, can be made by comparing the results for the mixing matrix for several choices of the fourth family masses. The fitting procedure shows up that the 3×3 mixing matrix does not change appreciable, even not for much lower masses from 300 GeV up.

We can conclude: Requiring that the experimental data respect the symmetry of the mass matrices (Eq. (3.1)) (suggested by the *spin-charge-family* theory) the prediction can be made for the change of the matrix elements of the 3×3 sub matrix in future experiments. The masses of the fourth family members are more difficult to predict, since the accuracy of the experimental data for the quark masses and in particular for the mixing matrix should be extremely high to really limit the fourth family masses. For a known fourth family masses the fourth family matrix elements of the mixing matrix are accurate. For masses of the fourth family quarks to be close or above 1 TeV speak more other experimental data, like decays of mesons.

3.4 Discussions and conclusions

One of the most important open questions in the elementary particle physics is: Where do the family originate? Explaining the origin of families would answer the question about the number of families possibly observable at the low energy regime, about the origin of the scalar field(s) and Yukawa couplings and would also explain differences in the fermions properties - the differences in masses and mixing matrices among family members – quarks and leptons.

Assuming that the prediction of the *spin-charge-family* theory that there are four rather than so far observed three coupled families, the mass matrices of which demonstrate in the massless basis the $SU(2) \times SU(2)$ (each of two $SU(2)$ is a subgroup of its own $SO(4)$) symmetry of Eq. (3.1), the same for all the family members - the quarks and the leptons - and simplifying the numerical procedure by the assumption that the mass matrices are symmetric and real and correspondingly the mixing matrices orthogonal, we fit the free parameters of the quarks mass matrices (6 for u-quarks and 6 for d-quarks to twice three masses of quarks and to the mixing matrix 4×4 , extracted from the 3×3 sub matrix elements, fitted to 6 parameters of the orthogonal matrix) to the experimental data. Every unitary $n \times n$ matrix is for $n \geq 4$, through the unitary conditions, uniquely determined by the 3×3 sub matrix.

The numerical procedure, explained in this paper, to fit free parameters to the experimental data within the experimental inaccuracy of masses and in particular of the mixing matrix is very tough. The accurate mixing matrix elements and

masses would completely determine the fourth family masses. The experimental inaccuracies are too large to tell the trustable mass interval, within which the fourth family masses of quarks lie.

In this paper we are not yet able to tell the mass intervals for the fourth family quarks. But since the matrix elements of the 3×3 sub matrix depend very weakly on the fourth family masses, the calculated matrix (from the experimental data under the assumption that the mass matrices manifest the symmetry of Eq. (3.1)) offer the prediction to what values will more accurate measurements move the present experimental data. We checked this prediction by performing calculations with the old matrix elements [16] and then test the prediction on the new ones [15]. The results are presented in Eq. (3.22). Repeating calculations with the new matrix elements for several masses of the fourth family quarks we predict further change of the 3×3 sub matrix elements, presented in Eq. (3.31).

We expect: More accurate experiments will bring a slightly smaller values for (V_{ud}, V_{ub}, V_{tb}) , smaller (V_{cs}, V_{td}) , (V_{us}, V_{cd}) will slightly grow and $(V_{cb}) V_{ts}$ will grow.

The fourth family mixing matrix elements depend, as expected, strongly on the fourth family masses. For chosen masses of the fourth family members their matrix elements can be quite accurately predicted (Eqs. (3.24, 3.28)).

Mass matrices are quite close to the "democratic" ones not only for leptons but also for quarks. With the growing fourth family masses the "democracy" in matrix elements grow (Eqs. (3.23, 3.23)), as expected.

Although we have not study complex mass matrices, we do not expect that the presented results would change considerably after taking into account the complex phases of mass matrices and correspondingly also of the mixing matrices. We estimate the accuracy of our calculations by comparing the results of the calculated 3×3 matrix elements for the interval of the fourth family masses, from 300 GeV to 1 200 GeV. It look very trustable, offering for all these masses only slowly changing matrix elements.

We are concluding: Requiring that the experimental data respect the symmetry of the mass matrices (Eq. (3.1)) (suggested by the *spin-charge-family* theory) the prediction is made for the change of the matrix elements of the 3×3 sub matrix in future more accurate experiments. More (much more) accurate measured 3×3 sub matrix elements in future will determine, following the *spin-charge-family* theory, the fourth family masses and the fourth family matrix elements. However, even with the present experimental data our calculations, respecting the symmetry of the mass matrices (Eq. (3.1)) offer the prediction for the direction to which will more accurately measured matrix elements move.

Since the symmetry of the mass matrices are determined in the *spin-charge-family* theory by two triplet (with respect to the family charges) and tree singlet (with respect to the family members charges (Q, Q', Y)) scalar fields [13,14], all with the weak and the hyper charges as assumed in the *standard model* for the scalar fields, we hope to learn from the properties of the mass matrices and the corresponding mixing matrices more about these scalar fields.

3.5 APPENDIX: A brief presentation of the *spin-charge-family* theory

We present in this section a very brief introduction into the *spin-charge family* theory [1–14]. The reader can skip this appendix taking by the *spin-charge family* theory required symmetry of mass matrices of Eq. (3.1) as an input to the study of properties of the 4×4 mass matrices - with the parameters which depend on charges of the family members - and can come to this part of the paper, if and when would like to learn where do families and scalar fields possibly originate from.

Let us start by directing attention of the reader to one of the most open questions in the elementary particle physics and cosmology: Why do we have families, where do they originate and correspondingly where do scalar fields, manifesting as Higgs and Yukawa couplings, originate? The *spin-charge-family* theory is offering a possible explanation for the origin of families and scalar fields, and in addition for the so far observed charges and the corresponding gauge fields.

There are, namely, two (only two) kinds of the Clifford algebra objects: One kind, the Dirac γ^a , takes care of the spin in $d = (3 + 1)$, while the spin in $d \geq 4$ (rather than the total angular momentum) manifests in $d = (3 + 1)$ in the low energy regime as the charges. In this part the *spin-charge family* theory is like the Kaluza-Klein theory, unifying spin (in the low energy regime, otherwise the total angular momentum) and charges, and offering a possible answer to the question about the origin of the so far observed charges and correspondingly also about the so far observed gauge fields. The second kind of the Clifford algebra objects, forming the equivalent representations with respect to the Dirac kind, recognized by one of the authors (SNMB), is responsible for the appearance of families of fermions.

There are correspondingly also two kinds of gauge fields, which appear to manifest in $d = (3 + 1)$ as the so far observed vector gauge fields (the number of - obviously non yet observed - gauge fields grows with the dimension) and as the scalar gauge fields. The scalar fields are responsible, after gaining nonzero vacuum expectation values, for the appearance of masses of fermions and gauge bosons. They manifest as the so far observed Higgs [36] and the Yukawa couplings.

All the properties of fermions and bosons in the low energy regime originate in the *spin-charge-family* theory in a simple starting action for massless fields in $d = [1 + (d - 1)]$. Fermions interact with the vielbeins f^α_a and correspondingly with the two kinds of the spin connection fields: with $\omega_{abc} = f^\alpha_c \omega_{ab\alpha}$ which are the gauge fields of $S^{ab} = \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a)$ and with $\tilde{\omega}_{abc} = f^\alpha_c \tilde{\omega}_{ab\alpha}$ which are the gauge fields of $\tilde{S}^{ab} = \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$. α, β, \dots is the Einstein index and a, b, \dots is the flat index. The starting action is the simplest one

$$S = \int d^d x \, E \, \mathcal{L}_f + \int d^d x \, E \, (\alpha R + \tilde{\alpha} \tilde{R}), \quad \mathcal{L}_f = \frac{1}{2} (\bar{\Psi} \gamma^a p_{0a} \Psi) + \text{h.c.}$$

$$p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\}, \quad p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}, \quad (3.32)$$

$$\begin{aligned}
R &= \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\omega_{ab\alpha,\beta} - \omega_{c\alpha\alpha} \omega^c_{b\beta})\} + \text{h.c.}, \\
\tilde{R} &= \frac{1}{2} f^{\alpha[a} f^{\beta b]} (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{c\alpha\alpha} \tilde{\omega}^c_{b\beta}) + \text{h.c.} .
\end{aligned} \tag{3.33}$$

$E = \det(e^a_\alpha)$ and $e^a_\alpha f^\beta_a = \delta^\beta_\alpha$. Fermions, coupled to the vielbeins and the two kinds of the spin connection fields, *manifest* (after several breaks of the starting symmetries) *before the electroweak break four massless families of quarks and leptons*, the left handed fermions are weak charged and the right handed ones are weak chargeless. The vielbeins and the two kinds of the spin connection fields manifest effectively as the observed gauge fields and (those with the scalar indices in $d = (1 + 3)$) as several scalar fields. The mass matrices of the four family members (quarks and leptons) are after the electroweak break expressible on a tree level by the vacuum expectation values of the two kinds of the spin connection fields and the corresponding vielbeins with the scalar indices ([1,2,6,7,12,13]):

i. One kind originates in the scalar fields $\tilde{\omega}_{abc}$, manifesting as the two $SU(2)$ triplets $-\tilde{A}_s^{\tilde{N}_L^i}$, $i = (1, 2, 3)$, $s = (7, 8)$; $\tilde{A}_s^{\tilde{I}^i}$, $i = (1, 2, 3)$, $s = (7, 8)$; – and one singlet $-\tilde{A}_s^4$, $s = (7, 8)$ – contributing equally to all the family members.

ii. The second kind originates in the scalar fields ω_{abc} , manifesting as three singlets $-A_s^Q, A_s^{Q'}, A^{Y'}$, $s = (7, 8)$ – contributing the same values to all the families and distinguishing among family members. Q and Q' are the quantum numbers from the *standard model*, Y' originates in the second $SU(2)$ (a kind of a right handed “weak”) charge.

All the scalar fields manifest, transforming the right handed quarks and leptons into the corresponding left handed ones⁴ and contributing also to the masses of the weak bosons, as doublets with respect to the weak charge. Loop corrections, to which all the scalar and also gauge vector fields contribute coherently, change contributions of the off-diagonal and diagonal elements appearing on the tree level, keeping the tree level symmetry of mass matrices unchanged⁵.

3.5.1 Mass matrices on the tree level and beyond which manifest $SU(2) \times SU(2)$ symmetry

Let us make a choice of a massless basis ψ_i , $i = (1, 2, 3, 4)$, for a particular family member α . And let us take into account the two kinds of the operators, which transform the basis vectors into one another

$$\tilde{N}_L^i, i = (1, 2, 3), \quad \tilde{\tau}_L^i, i = (1, 2, 3), \tag{3.34}$$

⁴ It is the term $\gamma^0 \gamma^s \phi_s^{A^i}$, where $\phi_s^{A^i}$, with $s = (7, 8)$ denotes any of the scalar fields, which transforms the right handed fermions into the corresponding left handed partner [1,7,2,6,12–14]. This mass term originates in $\bar{\psi} \gamma^a p_{0a} \psi$ of the action Eq.(3.32), with $a = s = (7, 8)$ and $p_{0s} = f_s^\sigma (p_\sigma - \frac{1}{2} S^{ab} \tilde{\omega}_{ab\sigma} - \frac{1}{2} S^{st} \omega_{st\sigma})$.

⁵ It can be seen that all the loop corrections keep the starting symmetry of the mass matrices unchanged. We have also started [7,42] with the evaluation of the loop corrections to the tree level values. This estimation has been done so far [42] only up to the first order and partly to the second order.

with the properties

$$\begin{aligned}
\tilde{N}_L^3(\psi_1, \psi_2, \psi_3, \psi_4) &= \frac{1}{2}(-\psi_1, \psi_2, -\psi_3, \psi_4), \\
\tilde{N}_L^+(\psi_1, \psi_2, \psi_3, \psi_4) &= (\psi_2, 0, \psi_4, 0), \\
\tilde{N}_L^-(\psi_1, \psi_2, \psi_3, \psi_4) &= (0, \psi_1, 0, \psi_3), \\
\tilde{\tau}^3(\psi_1, \psi_2, \psi_3, \psi_4) &= \frac{1}{2}(-\psi_1, -\psi_2, \psi_3, \psi_4), \\
\tilde{\tau}^+(\psi_1, \psi_2, \psi_3, \psi_4) &= (\psi_3, \psi_4, 0, 0), \\
\tilde{\tau}^-(\psi_1, \psi_2, \psi_3, \psi_4) &= (0, 0, \psi_1, \psi_2).
\end{aligned} \tag{3.35}$$

This is indeed what the two $SU(2)$ operators in the *spin-charge-family* theory do. The gauge scalar fields of these operators determine, together with the corresponding coupling constants, the off diagonal and diagonal matrix elements on the tree level. In addition to these two kinds of $SU(2)$ scalars there are three $U(1)$ scalars, which distinguish among the family members, contributing on the tree level the same diagonal matrix elements for all the families. In loop corrections in all orders the symmetry of mass matrices remains unchanged, while the three $U(1)$ scalars, contributing coherently with the two kinds of $SU(2)$ scalars and all the massive fields to all the matrix elements, manifest in off diagonal elements as well. All the scalars are doublets with respect to the weak charge, contributing to the weak and the hypercharge of the fermions so that they transform the right handed members into the left handed ones.

With the above (Eq. (3.35) presented choices of phases of the left and the right handed basic states in the massless basis the mass matrices of all the family members manifest the symmetry, presented in Eq. (3.1). One easily checks that a change of the phases of the left and the right handed members, there are $(2n - 1)$ possibilities, causes changes in phases of matrix elements in Eq. (3.1).

3.6 APPENDIX: Mass matrices for leptons

We evaluate 3×3 matrix elements from the data [16]

$$\begin{aligned}
7.05 \cdot 10^{-17} &\leq \Delta(\mathbf{m}_{21}/\text{MeV}/c^2)^2 \leq 8.34 \cdot 10^{-17}, \\
2.07 \cdot 10^{-15} &\leq \Delta(\mathbf{m}_{(31),(32)}/\text{MeV}/c^2)^2 \leq 2.75 \cdot 10^{-15}, \\
0.25 &\leq \sin^2 \theta_{12} \leq 0.37, \quad 0.36 \leq \sin^2 \theta_{23} \leq 0.67, \\
\sin^2 \theta_{13} &< 0.035(0.056), \quad \sin^2 2\theta_{13} = 0.098 \pm 0.013,
\end{aligned} \tag{3.36}$$

which means that $\frac{\pi}{4} - \frac{\pi}{10} \leq \theta_{23} \leq \frac{\pi}{4} + \frac{\pi}{10}$, $\frac{\pi}{4} - \frac{\pi}{10} \leq \theta_{12} \leq \frac{\pi}{4} + \frac{\pi}{10}$, $\theta_{13} < \frac{\pi}{13}$. This reflects in the lepton mixing matrix $V_{\nu e} = S^\nu S^{e\dagger}$

$$|V_{\nu e}| = \begin{pmatrix} 0.8224 & 0.5200 & 0.1552 & |V_{\nu_1 e_4}| \\ 0.3249 & 0.7239 & 0.6014 & |V_{\nu_2 e_4}| \\ 0.4455 & 0.4498 & 0.7704 & |V_{\nu_3 e_4}| \\ |V_{\nu_4 e_1}| & |V_{\nu_4 e_2}| & |V_{\nu_4 e_3}| & |V_{\nu_4 e_4}| \end{pmatrix}, \tag{3.37}$$

determining for each assumed value for any mixing matrix element within the experimentally allowed inaccuracy the corresponding fourth family mixing matrix elements ($|V_{\nu_i e_4}|$ and $|V_{\nu_4 e_j}|$) from the unitarity condition for the 4×4 mixing matrix. The masses of leptons are taken from [15,16] while we take the fourth family masses as free parameters, checking how much does the experimental inaccuracy influence a possible prediction for the fourth family leptons masses and how does this prediction agree with experimentally allowed values [15,16,43] for the fourth family lepton masses.

$$\begin{aligned} \mathbf{M}_d^\nu/\text{MeV}/c^2 &= (1 \cdot 10^{-9}, 9 \cdot 10^{-9}, 5 \cdot 10^{-8}, m^{\nu_4} > 90\,000.), \\ \mathbf{M}_d^e/\text{MeV}/c^2 &= (0.486\,570\,161 \pm 0.000\,000\,042, \\ &102.718\,135\,9 \pm 0.000\,009\,2, 1746.24 \pm 0.20, m^{e_4} > 102\,000). \end{aligned} \quad (3.38)$$

3.6.1 Numerical results for leptons

We present here the old results [11] for leptons, manifesting properties of the lepton mass matrices. These results are less informative than those for quarks, since the experimental results are for leptons mixing matrix much less accurate than in the case of quarks and also masses are known less accurately.

We make a choice of the fourth family masses and take the mixing matrix elements from the old experimental data [16]

We have

$$\begin{aligned} \bullet \quad \mathbf{M}^\nu &= \begin{pmatrix} 14\,021. & 14\,968. & 14\,968. & -14\,021. \\ 14\,968. & 15\,979. & 15\,979. & -14\,968. \\ 14\,968. & 15\,979. & 15\,979. & -14\,968. \\ -14\,021. & -14\,968. & -14\,968. & 14\,021. \end{pmatrix}, \\ \mathbf{M}^e &= \begin{pmatrix} 28\,933. & 30\,057. & 29\,762. & -27\,207. \\ 30\,057. & 32\,009. & 31\,958. & -29\,762. \\ 29\,762. & 31\,958. & 32\,009. & -30\,057. \\ -27\,207. & -29\,762. & -30\,057. & 28\,933. \end{pmatrix}, \end{aligned} \quad (3.39)$$

which leads to the mixing matrix $V_{\nu e}$

$$V_{\nu e 1} = \begin{pmatrix} 0.82363 & 0.54671 & -0.15082 & 0. \\ -0.50263 & 0.58049 & -0.64062 & 0. \\ -0.26268 & 0.60344 & 0.75290 & 0. \\ 0. & 0. & 0. & 1. \end{pmatrix}, \quad (3.40)$$

and the masses

$$\begin{aligned} \mathbf{M}_d^\nu/\text{MeV}/c^2 &= (5 \cdot 10^{-9}, 1 \cdot 10^{-8}, 4.9 \cdot 10^{-8}, 60\,000.), \\ \mathbf{M}_d^e/\text{MeV}/c^2 &= (0.510999, 105.658, 1\,776.82, 120\,000). \end{aligned} \quad (3.41)$$

We did not adapt lepton masses to Z_m mass scale. Zeros (0.) for the matrix elements concerning the fourth family members means that the values are less than 10^{-5} and 1. means that the difference from 1 occurs on the sixth digit at most.

We notice:

- i. The required symmetry, Eq. (3.1), is kept exactly.
- ii. The mass matrices of leptons are very close to the "democratic" matrix.
- iii. The mixing matrix elements among the first three and the fourth family members are very small, what is due to our choice, since the matrix elements of the 3×3 sub matrix of the 4×4 unitary matrix, predicted by the *spin-charge-family* theory are very inaccurately known.

3.7 APPENDIX: Properties of non Hermitian mass matrices

This pedagogic presentation of well known properties of non Hermitian matrices can be found in many textbooks, for example [44]. We repeat this topic here only to make our discussions transparent.

Let us take a non Hermitian mass matrix M^α as it follows from the *spin-charge-family* theory, α denotes a family member (index \pm used in the main text is dropped).

We always can diagonalize a non Hermitian M^α with two unitary matrices, S^α ($S^{\alpha\dagger} S^\alpha = I$) and T^α ($T^{\alpha\dagger} T^\alpha = I$)

$$S^{\alpha\dagger} M^\alpha T^\alpha = \mathbf{M}_d^\alpha = (m_1^\alpha \dots m_i^\alpha \dots m_n^\alpha). \quad (3.42)$$

The proof is added below.

Changing phases of the basic states, those of the left handed one and those of the right handed one, the new unitary matrices $S'^\alpha = S^\alpha F_{\alpha S}$ and $T'^\alpha = T^\alpha F_{\alpha T}$ change the phase of the elements of diagonalized mass matrices \mathbf{M}_d^α

$$\begin{aligned} S'^{\alpha\dagger} M^\alpha T'^\alpha &= F_{\alpha S}^\dagger \mathbf{M}_d^\alpha F_{\alpha T} = \\ &\text{diag}(m_1^\alpha e^{i(\phi_1^{\alpha S} - \phi_1^{\alpha T})} \dots m_i^\alpha e^{i(\phi_i^{\alpha S} - \phi_i^{\alpha T})} \dots m_n^\alpha e^{i(\phi_n^{\alpha S} - \phi_n^{\alpha T})}), \\ F_{\alpha S} &= \text{diag}(e^{-i\phi_1^{\alpha S}}, \dots, e^{-i\phi_i^{\alpha S}}, \dots, e^{-i\phi_n^{\alpha S}}), \\ F_{\alpha T} &= \text{diag}(e^{-i\phi_1^{\alpha T}}, \dots, e^{-i\phi_i^{\alpha T}}, \dots, e^{-i\phi_n^{\alpha T}}). \end{aligned} \quad (3.43)$$

In the case that the mass matrix is Hermitian T^α can be replaced by S^α , but only up to phases originating in the phases of the two basis, the left handed one and the right handed one, since they remain independent.

One can diagonalize the non Hermitian mass matrices in two ways, that is either one diagonalizes $M^\alpha M^{\alpha\dagger}$ or $M^{\alpha\dagger} M^\alpha$

$$\begin{aligned} (S^{\alpha\dagger} M^\alpha T^\alpha)(S^{\alpha\dagger} M^\alpha T^\alpha)^\dagger &= S^{\alpha\dagger} M^\alpha M^{\alpha\dagger} S^\alpha = \mathbf{M}_{dS}^{\alpha 2}, \\ (S^{\alpha\dagger} M^\alpha T^\alpha)^\dagger (S^{\alpha\dagger} M^\alpha T^\alpha) &= T^{\alpha\dagger} M^{\alpha\dagger} M^\alpha T^\alpha = \mathbf{M}_{dT}^{\alpha 2}, \\ \mathbf{M}_{dS}^{\alpha\dagger} &= \mathbf{M}_{dS}^\alpha, \quad \mathbf{M}_{dT}^{\alpha\dagger} = \mathbf{M}_{dT}^\alpha. \end{aligned} \quad (3.44)$$

One can prove that $\mathbf{M}_{dS}^\alpha = \mathbf{M}_{dT}^\alpha$. The proof proceeds as follows. Let us define two Hermitian (H_S^α, H_T^α) and two unitary matrices (U_S^α, H_T^α)

$$\begin{aligned} H_S^\alpha &= S^\alpha \mathbf{M}_{dS}^\alpha S^{\alpha\dagger}, & H_T^\alpha &= T^\alpha \mathbf{M}_{dT}^{\alpha\dagger} T^{\alpha\dagger}, \\ U_S^\alpha &= H_S^{\alpha-1} M^\alpha, & U_T^\alpha &= H_T^{\alpha-1} M^{\alpha\dagger}, \end{aligned} \quad (3.45)$$

It is easy to show that $H_S^{\alpha\dagger} = H_S^\alpha$, $H_T^{\alpha\dagger} = H_T^\alpha$, $U_S^\alpha U_S^{\alpha\dagger} = I$ and $U_T^\alpha U_T^{\alpha\dagger} = I$. Then it follows

$$\begin{aligned} S^{\alpha\dagger} H_S^\alpha S^\alpha &= \mathbf{M}_{dS}^\alpha = \mathbf{M}_{dS}^{\alpha\dagger} = S^{\alpha\dagger} M^\alpha U_S^{\alpha-1} S^\alpha = S^{\alpha\dagger} M^\alpha T^\alpha, \\ T^{\alpha\dagger} H_T^\alpha T^\alpha &= \mathbf{M}_{dT}^\alpha = \mathbf{M}_{dT}^{\alpha\dagger} = T^{\alpha\dagger} M^{\alpha\dagger} U_T^{\alpha-1} T^\alpha = T^{\alpha\dagger} M^{\alpha\dagger} S^\alpha, \end{aligned} \quad (3.46)$$

where we recognized $U_S^{\alpha-1} S^\alpha = T^\alpha$ and $U_T^{\alpha-1} T^\alpha = S^\alpha$. Taking into account Eq. (3.43) the starting basis can be chosen so, that all diagonal masses are real and positive.

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