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Isomorphisms of generalized Cayley graphs*

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Abstract

In this paper, we investigate the isomorphism problems of the generalized Cayley graphs, which are generalizations of the traditional Cayley graphs. We find that there are two types of natural isomorphisms for the generalized Cayley graphs. We also study the GCI-groups among the generalized Cayley graphs, and the Cayley regressions of some groups. We mainly showed that, for an odd prime power n, Z_{2n} (resp. D_{2n}) is a restricted GCI-group if D_{2n} (resp. Z_{2n}) is a CI-group. We also obtain that the cyclic group of order 2^n is a 4-quasi-Cayley regression if and only if $n = 3$.

Keywords: Generalized Cayley graph, natural isomorphism, GCI-group, Cayley regression.

Math. Subj. Class.: 05C25, 20D20

1 Introduction

Let G be a finite group, $S \subseteq G$ be a subset and $\alpha \in \mathrm{Aut}(G)$. If G, S, α satisfy the following three conditions:

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(i) $\alpha^2 = 1$;

(ii) if $g \in G$, then $\alpha(g^{-1})g \notin S$;

(iii) if $g, h \in G$ and $\alpha(g^{-1})h \in S$, then $\alpha(h^{-1})g \in S$,

then the structure $\Gamma = \mathrm{GC}(G, S, \alpha)$ is called a *generalized Cayley graph* with $V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid \alpha(g^{-1})h \in S\}$. The *neighborhood* of a vertex $g \in G$ is the set of vertices adjacent to g, denoted by $N(g)$. Then $N(g) = \{\alpha(g)s \mid s \in S\}$.

According to condition (i), α is either the identity of $\mathrm{Aut}(G)$ or an involution. When α is the identity, then the definition of $\mathrm{GC}(G, S, \alpha)$ is just the same as that of Cayley graphs, and thus $\mathrm{GC}(G, S, \alpha) = \mathrm{Cay}(G, S)$. In this case, S is *symmetrical*, i.e., $S = S^{-1} = \{s^{-1} \mid s \in S\}$ and for $\sigma \in \mathrm{Aut}(G)$, we have that σ acts on $V(\Gamma)$ naturally as $V(\Gamma) = G$. Also, if $T = S^\sigma$, then there is a bijection from Γ to $\Gamma^\sigma = \mathrm{Cay}(G, T)$ induced by σ, defined as $\sigma\colon V(\Gamma) \to V(\Gamma^\sigma)$, $g \mapsto g^\sigma$. It follows $\Gamma \cong \Gamma^\sigma$. This kind of isomorphism between Cayley graphs induced by the automorphisms of G is called the *Cayley isomorphism*. It should be mentioned that not all isomorphisms between Cayley graphs are Cayley isomorphisms. In fact, there are pairs of isomorphic Cayley graphs with no Cayley isomorphism between them. This encourages us to investigate the so-called CI-graphs and CI-groups defined below.

Definition 1.1. A Cayley graph $\mathrm{Cay}(G, S)$ is called a CI-graph of G, if for any Cayley graph $\mathrm{Cay}(G, T)$, $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ implies $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$. In this case, S is called a CI-subset. Furthermore, G is called a CI-group if any symmetrical subset not containing the identity is a CI-subset.

For those graphs having particular transitive properties, such as Cayley graphs and bi-Cayley graphs, their isomorphism problems are well studied in the literature (recall that a bi-Cayley graph is a graph which admits a semiregular group of automorphisms with two orbits on the vertices). The isomorphism problem for Cayley graphs was proposed decades ago and has been investigated deeply up to now. It was initiated by Ádám in 1967 who conjectured that any cyclic group is a DCI-group, where a DCI-group satisfies that any subset not containing the identity and not necessarily symmetrical is a CI-subset. Although this conjecture was soon denied by Elspas and Turner [4], it stimulated the study of CI- and DCI-groups. Alspach, Parsons [1] and Babai [3] presented a criteria for CI-graphs. Muzychuk [18, 19] obtained a complete classification of the CI-groups in finite cyclic groups. Li [14] showed that all finite CI-groups are solvable. The isomorphism problem and the automorphism groups for bi-Cayley graphs have also been studied flourishingly; one may refer to [10, 11, 28]. Other related results could be found in [15, 16, 23, 24, 26, 27].

The concept of generalized Cayley graphs was introduced by Marušič et al. [17] when they dealt with the double covering of graphs. Answering a question in [17], the authors in [8] found some vertex-transitive generalized Cayley graphs which are not Cayley graphs. Further, the authors in [25] studied the isomorphism problems of generalized Cayley graphs and found that the alternating group A_n is a restricted GCI-group if and only if $n = 4$.

The present paper can be regarded as the continuance of the above work, and also provides support to the question at the end of [8], where the authors asked for the classification of all generalized Cayley graphs arising from cyclic groups. The structure of this paper is as follows. In Section 2, we give several properties of the generalized Cayely graphs and some lemmas which will be used later. In Section 3, we introduce two types of natural isomorphisms for any generalized Cayley graph. In Section 5, we study the GCI-groups in cyclic groups. We show that when G is a dihedral group of order $2n$ with n an odd prime power, if G is a CI-group, then Z_{2n} is a restricted GCI-group. In Section 6, we study the GCI-groups in dihedral groups. We show that when G is a cyclic group of order $2n$ with n an odd prime power, if G is a CI-group, then D_{2n} is a restricted GCI-group. In Section 7, we study the Cayley regressions, a concept relating to both Cayley graphs and generalized Cayley graphs. We show that the cyclic group Z_{2^n} is a 4-quasi-Cayley regression if and only if $n = 3$. Finally, we propose some questions for future research.

2 Preliminaries

All graphs considered in the paper are simple, finite and undirected. All the automorphisms in the paper that induce generalized Cayley graphs are assumed to be some involutions.

Let G be a finite group that admits an automorphism α of order two. For $g = 1$, we have $\alpha(h^{-1}) \in S$ whenever $h \in S$, by condition (iii), implying $\alpha(S) = S^{-1}$. Let $\omega_\alpha\colon G \to G$ be the mapping defined by $\omega_\alpha(g) = \alpha(g^{-1})g$ for any $g \in G$. Note that ω_α is not necessarily a bijection. Let $\omega_\alpha(G) = \{\omega_\alpha(g) \mid g \in G\}$. We use the same notation and terminology as in [8]. Suppose $s \in S$, then $\alpha(s) \in \alpha(S)$, and thus $\alpha(s) \in S^{-1}$. Therefore $s \in S$ if and only if $\alpha(s^{-1}) \in S$. Let Ω_α be the set containing all elements satisfying $\alpha(g) = g^{-1}$ in $G \setminus \omega_\alpha(G)$, and $\mho_\alpha$ be the set containing all elements in G satisfying $\alpha(g) \neq g^{-1}$. Let $K_\alpha = \{g \in G \mid \alpha(g)g = 1\}$. Then we have

Proposition 2.1 ([25]). *Let* $\mathrm{GC}(G, S, \alpha)$ *be a generalized Cayley graph of* G*. Then*

(1) $S \cap \omega_\alpha(G) = \emptyset$*. Conversely, if* $S \cap \omega_\alpha(G) = \emptyset$*,* α *is an involution in* $\mathrm{Aut}(G)$ *and* $\alpha(S) = S^{-1}$*, then* G, S, α *can induce a generalized Cayley graph.*

(2) $G = K_\alpha \cup \mho_\alpha$ *and* $K_\alpha = \omega_\alpha(G) \cup \Omega_\alpha$*. Furthermore,* $\omega_\alpha(G), \Omega_\alpha, \mho_\alpha$ *are all symmetrical.*

(3) $S = S_1 \cup S_2$*, where* $S_1 \subseteq \Omega_\alpha$ *and* $S_2 \subseteq \mho_\alpha$*.*

Proposition 2.2. *Let* G *be a finite group admitting two automorphisms* α, β *of order two. If* α, β *are conjugate in* $\mathrm{Aut}(G)$*, then* $\mathrm{Cay}(G, \omega_\alpha(G) \setminus \{1\}) \cong \mathrm{Cay}(G, \omega_\beta(G) \setminus \{1\})$*.*

Proof. By Proposition 2.1, we have $\omega_\alpha(G) = \omega_\alpha(G)^{-1}$ and $\omega_\beta(G) = \omega_\beta(G)^{-1}$. Since α, β are conjugate, there exists some $\gamma \in \mathrm{Aut}(G)$ such that $\beta = \gamma\alpha\gamma^{-1} = \alpha^\gamma$. Therefore

$$\begin{aligned}\gamma(\omega_\alpha(G)) &= \{\gamma(\alpha(g^{-1})g) \mid g \in G\} \\ &= \{\gamma\alpha\gamma^{-1}\gamma(g^{-1})\gamma(g) \mid g \in G\} \\ &= \{\beta(\gamma(g)^{-1})\gamma(g) \mid \gamma(g) \in G\} \\ &= \omega_\beta(G).\end{aligned}$$

It follows that $\gamma(\omega_\alpha(G) \setminus \{1\}) = \omega_\beta(G) \setminus \{1\}$. Hence the result follows. □

Theorem 2.3. *Let* G *be a finite group admitting an automorphism* α *of order two,* $S \subseteq G$ *such that* $S \cap \omega_\alpha(G) = \emptyset$*. Let* $\Phi(g) = \alpha(g)Sg^{-1}$*. If* S *is symmetrical and* $\Phi(g) = S$ *for any* $g \in G$*, then* $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(G, S)$*.*

Proof. Let $\Gamma_1 = \mathrm{GC}(G, S, \alpha)$ and $\Gamma_2 = \mathrm{Cay}(G, S)$. Let $\phi\colon V(\Gamma_1) \to V(\Gamma_2)$, $x \mapsto x^{-1}$ be a bijection between these two graphs. For any $\{g, h\} \in E(\Gamma_1)$, there exists some $s \in S$ such that $h = \alpha(g)s$. $\{g, h\}^\phi = \{g^{-1}, h^{-1}\}$. Note that $gh^{-1} = gs^{-1}\alpha(g)^{-1} = \alpha(\alpha(g))s^{-1}\alpha(g)^{-1}$. Since S is symmetrical and $\Phi(g) = S$ for any $g \in G$, we have $\alpha(\alpha(g))s^{-1}\alpha(g)^{-1} \in S$. This implies $\{g, h\}^\phi \in E(\Gamma_2)$, and thus $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(G, S)$. □

Theorem 2.3 can be regarded as a criteria to judge whether some generalized Cayley graphs are Cayley graphs or not.

Theorem 2.4. *Let G be any finite group admitting an automorphism α of order 2. Then we have $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(G, S)$, where $S = \mho_\alpha, \Omega_\alpha$ or $G \setminus \omega_\alpha(G)$.*

Proof. By proposition 2.1, $G = \omega_\alpha(G) \cup \Omega_\alpha \cup \mho_\alpha$. For any $g \in G$, $G = \alpha(g)Gg^{-1}$. For any $x \in \omega_\alpha(G)$, there exists some $h \in G$ such that $x = \alpha(h^{-1})h$. So $\alpha(g)xg^{-1} = \alpha(g)\alpha(h^{-1})hg^{-1} = \alpha((hg^{-1})^{-1})hg^{-1}$, and hence $\omega_\alpha(G) = \alpha(g)\omega_\alpha(G)g^{-1}$. As a result, $\Omega_\alpha \cup \mho_\alpha = \alpha(g)\Omega_\alpha g^{-1} \cup \alpha(g)\mho_\alpha g^{-1}$.

For any $s \in \Omega_\alpha$, assume that $\alpha(g)sg^{-1} \in \mho_\alpha$, then $\alpha(\alpha(g)sg^{-1})^{-1} \in \mho_\alpha$ and $\alpha(g)sg^{-1} \neq \alpha(\alpha(g)sg^{-1})^{-1}$. Since $\alpha(\alpha(g)sg^{-1})^{-1} = \alpha(g)\alpha(s^{-1})g^{-1}$, we have that $s \neq \alpha(s^{-1})$, which is a contradiction as $s \in \Omega_\alpha$. This means $\alpha(g)sg^{-1} \in \Omega_\alpha$. Thus, $\Omega_\alpha = \alpha(g)\Omega_\alpha g^{-1}$ and $\mho_\alpha = \alpha(g)\mho_\alpha g^{-1}$. By Theorem 2.3, we get the result. □

Let $\mathrm{Fix}(\alpha) = \{g \in G \mid \alpha(g) = g\}$. So $\mathrm{Fix}(\alpha) \leq G$ and we have the following lemma.

Lemma 2.5 ([8]). $|\omega_\alpha(G)| = \frac{|G|}{|\mathrm{Fix}(\alpha)|}$.

Note that some references also use $C_G(\alpha)$ to denote $\mathrm{Fix}(\alpha)$. Those papers mainly investigate the properties of the finite groups which admit involutory automorphisms; one can refer to [2, 13, 21, 22]. Although those problems are not considered in this paper, we borrow the following well-known result.

Lemma 2.6 ([7]). *Let G be a finite group of odd order admitting an automorphism ϕ of order two. Then the following statements hold.*

(1) $G = FK = KF$, $F \cap K = 1$, and $|K| = |G : F|$, where $F = C_G(\phi)$ and $K = K_\phi$;

(2) Two elements of K conjugate in G are conjugate by an element of F;

(3) If H is a subgroup of F, then $N_G(H) = C_G(H)N_F(K)$.

By Lemmas 2.5 and 2.6, we get

Proposition 2.7. *Let G be a group of odd order admitting an automorphism α of order two. Then $\Omega_\alpha = \emptyset$.*

Proof. By Lemmas 2.5 and 2.6, $|K_\alpha| = |\omega_\alpha(G)| = \frac{|G|}{|\mathrm{Fix}(\alpha)|}$. As $K_\alpha = \omega_\alpha(G) \cup \Omega_\alpha$, we obtain $\Omega_\alpha = \emptyset$. □

Remark 2.8. By Proposition 2.7, for any generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$, if $|G|$ is odd, $S \subseteq \mho_\alpha$. We present an alternative proof avoiding Lemmas 2.5 and 2.6. If $\Omega_\alpha \neq \emptyset$, assume that $\mho_\alpha = \emptyset$. Then G is an abelian group of odd order by Proposition 2.1. Thus α is a fixed-point-free automorphism of G. Then $K_\alpha = \omega_\alpha(G) = G$ according to [7, Lemma 10.1.1], which is a contradiction. This implies that $\mho_\alpha \neq \emptyset$. Since the S in $\mathrm{GC}(G, S, \alpha)$ are choosen from Ω_α and $\mho_\alpha$. Therefore $|S|$ must be odd, which is a contradiction as, there are no regular graphs of odd order with odd valency. This implies $\Omega_\alpha = \emptyset$.

It is well known that a finite group G of odd order is solvable by Feit-Thompson Theorem [5]. From above, we can see that the classification of $\mathrm{GC}(G, S, \alpha)$ of finite group G of odd order seems to be more clear as the elements of S can only be chosen from $\mho_\alpha$ since $\Omega_\alpha = \emptyset$.

In [8], Hujdurović et al. defined the following set

$$\mathrm{Aut}(G, S, \alpha) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S, \alpha\varphi = \varphi\alpha\}.$$

Moreover, one sees that $\mathrm{Aut}(G, S, \alpha) = \mathrm{Aut}(G, S) \cap C_{\mathrm{Aut}(G)}(\alpha)$, where $\mathrm{Aut}(G, S) = \mathrm{Aut}(G, S, 1)$.

Proposition 2.9. *Let S be the set as in (3) of Proposition 2.1. Then* $\mathrm{Aut}(G, S, \alpha) = \mathrm{Aut}(G, S_1, \alpha) \cap \mathrm{Aut}(G, S_2, \alpha) = \mathrm{Aut}(G, S_1) \cap \mathrm{Aut}(G, S_2) \cap C_{\mathrm{Aut}(G)}(\alpha)$. *Furthermore, the couples of the form like $\{s, \alpha(s^{-1})\}$ are imprimitive blocks of* $\mathrm{Aut}(G, S, \alpha)$.

Proof. For any $s \in S_1$ and $s' \in S_2$, if there exists some $\varphi \in \mathrm{Aut}(G, S, \alpha)$ such that $s = \varphi(s')$, then $\alpha\varphi(s') = \alpha(s)$. Since $\alpha\varphi = \varphi\alpha$ and $s = \alpha(s^{-1})$, $\varphi\alpha(s'^{-1}) = s$. This implies $\alpha(s') = s'^{-1}$, which is a contradiction as $s' \in S_2$. Hence $\varphi(S_1) = S_1$ and $\varphi(S_2) = S_2$ for any $\varphi \in \mathrm{Aut}(G, S, \alpha)$.

Let $\Delta = \{s, \alpha(s^{-1})\}$ be a couple in S_2. For any $\varphi \in \mathrm{Aut}(G, S, \alpha)$, $\Delta^\varphi \subseteq S_2$. If $\Delta \cap \Delta^\varphi \neq \emptyset$, then $s = \varphi(s)$ or $s = \varphi\alpha(s^{-1})$. If $s = \varphi(s)$, then $\alpha(s^{-1}) = \varphi\alpha(s^{-1})$. If $s = \varphi\alpha(s^{-1})$, then $\alpha(s^{-1}) = \varphi(s)$. This implies that $\Delta = \Delta^\varphi$. Thus Δ is an imprimitive block. □

Let $\mathrm{GC}(G, S, \alpha)$ be a generalized Cayley graph of G. Under the condition of Proposition 2.9, $S \cap S^{-1} = (S_1 \cup S_2) \cap (S_1 \cup S_2)^{-1} = (S_1 \cap S_1^{-1}) \cup (S_1 \cap S_2^{-1}) \cup (S_2 \cap S_1^{-1}) \cup (S_2 \cap S_2^{-1})$. Note that $S_1 \cap S_2^{-1} = S_2 \cap S_1^{-1} = \emptyset$, it follows that $S \cap S^{-1} = (S_1 \cap S_1^{-1}) \cup (S_2 \cap S_2^{-1})$. Since $S_1 \subseteq \Omega_\alpha$, and Ω_α is symmetrical, so $S_1 \cap S_1^{-1} \subseteq \Omega_\alpha$. Similarly, $S_2 \cap S_2^{-1} \subseteq \mho_\alpha$. Let $T = S \cap S^{-1}$. It follows that $\mathrm{GC}(G, T, \alpha)$ is still a generalized Cayley graph of G. We call $\mathrm{GC}(G, T, \alpha)$ the *induced* generalized Cayley graph of $\mathrm{GC}(G, S, \alpha)$. Note that $T^{-1} = T$, this encourages us to consider the Cayley graph $\mathrm{Cay}(G, T)$, called the *induced* Cayley graph of $\mathrm{GC}(G, S, \alpha)$. Next we consider $\mathrm{Aut}(G, S, \alpha)$, $\mathrm{Aut}(G, T, \alpha)$ and $\mathrm{Aut}(G, T)$.

Proposition 2.10. $\mathrm{Aut}(G, S, \alpha) \leq \mathrm{Aut}(G, T, \alpha) \leq \mathrm{Aut}(G, T)$. *Furthermore,* $\mathrm{Aut}(G, S, \alpha) < \mathrm{Aut}(G, T, \alpha)$ *if S is not symmetrical;* $\mathrm{Aut}(G, T, \alpha) = \mathrm{Aut}(G, T)$ *if* $\alpha \in Z(\mathrm{Aut}(G))$.

Proof. For any $\varphi \in \mathrm{Aut}(G, S, \alpha)$, we have $\varphi(S) = S$ and $\varphi(S^{-1}) = S^{-1}$, thus $\varphi(T) = T$, $\varphi \in \mathrm{Aut}(G, T, \alpha)$. If S is not symmetrical, we have $\alpha \notin \mathrm{Aut}(G, S, \alpha)$ as $\alpha(S) = S^{-1} \neq S$, but $\alpha \in \mathrm{Aut}(G, T, \alpha)$ as $\alpha(T) = T$. $\mathrm{Aut}(G, T, \alpha) \leq \mathrm{Aut}(G, T)$ is obvious by the definition. Since $\mathrm{Aut}(G, T, \alpha) = \mathrm{Aut}(G, T) \cap C_{\mathrm{Aut}(G)}(\alpha)$, we get the result. □

Finally, we introduce a lemma about the connectivity of the generalized Cayley graph.

Lemma 2.11. *Let G be a group, $A \subseteq G$ and $\alpha \in \mathrm{Aut}(G)$ of order 2. The generalized Cayley graph $X = \mathrm{GC}(G, A, \alpha)$ is connected if and only if A is a left generating set for $(G, *)$, where $f * g = \alpha(f)g$ for all $f, g \in G$.*

3 Two basic types of isomorphisms

In this section, we will introduce two types of natural isomorphisms of generalized Cayley graphs for any finite group. First, we introduce the first type of natural isomorphism found by A. Hujdurović et al.

Theorem 3.1 ([9]). $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S^\beta, \alpha^\beta)$ *for any $\beta \in \mathrm{Aut}(G)$, where $\alpha^\beta = \beta\alpha\beta^{-1}$.*

Remark 3.2. From Theorem 3.1, one can see that if α, γ are conjugate, then there is a generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$ if and only if there is a generalized Cayley graph $\mathrm{GC}(G, S^\beta, \gamma)$ with $\gamma = \alpha^\beta$ such that these two graphs are isomorphic. Hence, if we intend to study all the generalized Cayley graphs of some group G, we only need to study the generalized Cayley graphs related to the representatives of the conjugacy classes of elements in $\mathrm{Aut}(G)$.

Corollary 3.3. $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S^{-1}, \alpha)$.

Proof. Let $\beta = \alpha$. Then $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, \alpha(S), \alpha^\alpha)$ by Theorem 3.1. Note that $\alpha(S) = S^{-1}$, this completes the proof. □

Next, we introduce the second type of natural isomorphism.

Theorem 3.4. *Let $\mathrm{GC}(G, S, \alpha)$ be a generalized Cayley graph. Then $\mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$ is also a generalized Cayley graph of G for any $g \in G$. Furthermore, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$.*

Proof. For any $x \in G$, if $\alpha(x^{-1})x \in \alpha(g)Sg^{-1}$, $\alpha(g^{-1})\alpha(x^{-1})xg \in S$, that is, $\alpha((xg)^{-1})xg \in S$, which conflicts with condition (ii). If $\alpha(x^{-1})y \in \alpha(g)Sg^{-1}$, then we have $\alpha((xg)^{-1})yg \in S$. Thus $\alpha((yg)^{-1})xg \in S$ by condition (iii). It follows that $\alpha(y^{-1})x \in \alpha(g)Sg^{-1}$. Therefore, $\mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$ is also a generalized Cayley graph of G for any $g \in G$.

Let $\Gamma = \mathrm{GC}(G, S, \alpha)$ and $\Gamma_g = \mathrm{GC}(G, \alpha(g)Sg^{-1}, \alpha)$. Let $\theta\colon V(\Gamma) \to V(\Gamma_g)$, $a \mapsto ag^{-1}$. So θ is a bijection. For any $\{a, b\} \in E(\Gamma)$, $\alpha(a^{-1})b \in S$. Since

$$\alpha((ag^{-1})^{-1})(bg^{-1}) = \alpha(g)(\alpha(a^{-1})b)g^{-1} \in \alpha(g)Sg^{-1},$$

we have $\{ag^{-1}, bg^{-1}\} \in E(\Gamma_g)$. Therefore $\{a, b\} \in E(\Gamma)$ if and only if $\{a, b\}^\theta \in E(\Gamma_\alpha)$. Thus they are isomorphic. □

According to Theorem 3.1, $\Gamma \cong \Gamma^\beta$ for any $\beta \in \mathrm{Aut}(G)$, we call the mapping $x \mapsto x^\beta$ the *the first basic type of isomorphism* of Γ. By Theorem 3.4, $\Gamma \cong \Gamma_g$ for any $g \in G$, we call the mapping $x \mapsto xg^{-1}$ *the second basic type of isomorphism* of Γ.

For any $g \in G$, $R(g)\colon x \mapsto xg$ is a permutation of G. Set $R(H) = \{R(h) \mid S = \alpha(h)Sh^{-1}\}$.

Theorem 3.5. *Let $\Gamma = \mathrm{GC}(G, S, \alpha)$ be a generalized Cayley graph. Then $R(H) \leq \mathrm{Aut}(\Gamma)$.*

Proof. For any $\{a,b\} \in E(\Gamma)$, it suffices to show that $\{a,b\}^{R(h)} \in E(\Gamma)$ for any $R(h) \in R(H)$. Since $\{a,b\} \in E(\Gamma)$, $\alpha(a^{-1})b \in S = \alpha(h)Sh^{-1}$. It follows that $\alpha((ah)^{-1})bh \in S$, which implies that $\{ah, bh\} \in E(\Gamma)$. Thus $R(h) \in \mathrm{Aut}(\Gamma)$. For any $R(h), R(h') \in R(H)$, $S = \alpha(h)Sh^{-1}$ and $S = \alpha(h')Sh'^{-1}$. Therefore $S = \alpha(h'^{-1}h)S(h'^{-1}h)^{-1}$, thus $R(h'^{-1}h) \in R(H)$. This implies that $R(H) \leq \mathrm{Aut}(\Gamma)$. □

4 GCI, restricted GCI and strongly GCI groups

Similarly to the CI-groups in Cayley graphs and BCI-groups in bi-Cayley graphs, we propose the following definitions relating to generalized Cayley graphs.

Definition 4.1. Let G be a finite group. Let M be the set of all Cayley graphs and N be the set of all generalized Cayley graphs constructed by automorphisms of order two. Then

1. G is called a GCI-group if both of the following are satisfied:
   (i) for any two nontrivial generalized Cayley graphs $\mathrm{GC}(G,S,1)$ and $\mathrm{GC}(G,T,1)$ in M, whenever $\mathrm{GC}(G,S,1) \cong \mathrm{GC}(G,T,1)$, there exists $\delta \in \mathrm{Aut}(G)$ such that $S^\delta = T$.
   (ii) for any two nontrivial generalized Cayley graphs $\mathrm{GC}(G,S,\alpha)$ and $\mathrm{GC}(G,T,\beta)$ in N, whenever $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,T,\beta)$, there exists $\delta \in \mathrm{Aut}(G)$ such that $\beta = \alpha^\delta = \delta\alpha\delta^{-1}$ and $T = \alpha^\delta(g)S^\delta g^{-1}$.
2. G is called a restricted GCI-group if (ii) is satisfied.
3. G is called a strongly GCI-group if for any nontrivial $\mathrm{GC}(G,S,\alpha)$, whenever $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,T,\beta)$, there exists $\delta \in \mathrm{Aut}(G)$ such that $\beta = \alpha^\delta = \delta\alpha\delta^{-1}$ and $T = \alpha^\delta(g)S^\delta g^{-1}$.

Remark 4.2.

1. The definition is based on Theorems 3.1 and 3.4 and Definition 1.1. The two basic types of isomorphisms and their compositions are called the *natural isomorphisms* of generalized Cayley graphs. For instance, $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,S^\gamma,\alpha^\gamma)$ by Theorem 3.1, $\mathrm{GC}(G,S^\gamma,\alpha^\gamma) \cong \mathrm{GC}(G,\alpha^\gamma(g)S^\gamma g^{-1},\alpha^\gamma)$ by Theorem 3.4, then we have $\mathrm{GC}(G,S,\alpha) \cong \mathrm{GC}(G,\alpha^\gamma(g)S^\gamma g^{-1},\alpha^\gamma)$.
2. The word 'nontrivial' in the definition means that the null graph is not considered. In fact, if it is included, for a finite group G which has an automorphism α of order 2, $\mathrm{GC}(G,\emptyset,1)$ and $\mathrm{GC}(G,\emptyset,\alpha)$ are both isomorphic to the null graph. By the definition, G cannot be a strongly GCI-group, otherwise it will make the definition meaningless, thus the null graph is not considered in the definition.
3. If a finite group G has no automorphisms of order two, then we still consider that (ii) is satisfied for G.
4. By definition, strongly GCI-group implies GCI-group, GCI-group implies CI-group and restricted GCI-group. However, restricted GCI does not imply GCI and does not imply CI either. If G is not a restricted GCI-group or a CI-group, then it is not a GCI-group either.

Next we will give some examples of finite groups satisfying special conditions:

Example 4.3. Let $G = Z_4$. Then G is a GCI group by Theorem 5.2. However, let $\alpha\colon x \mapsto -x$ be an involution. Thus $\mathrm{GC}(G, \{1\}, \alpha)$ is a generalized Cayley graph of G. Also, $\mathrm{GC}(G, \{2\}, 1)$ is a generalized Cayley graph of G. Although $\mathrm{GC}(G, \{1\}, \alpha) \cong \mathrm{GC}(G, \{2\}, 1)$ but, α is not conjugate to 1, that means G is not a strongly GCI group. Therefore Z_4 is a GCI but not strongly GCI group.

Let $G = Z_8$. Then G is a CI group [19]. However, Z_{2^n} is a GCI group if and only if it is Z_2 or Z_4 by Theorem 5.2. It follows that G is not a GCI group. Thus Z_8 is a CI but not GCI group.

Though we find example of CI but not restricted GCI groups, like Z_8, we have not found out the example of restricted GCI but not CI groups up to now. Thus we propose the following question:

Question 4.4. Is every restricted GCI group a CI group?

The next theorem is useful to determine whether a group is a restricted GCI-group or not.

Theorem 4.5. *Let G be a finite group admitting two automorphisms α, β of order two. If α, β satisfy the following three conditions:*

(1) α and β are not conjugate;

(2) $|\omega_\alpha(G)| \neq |K_\alpha|$;

(3) $|\omega_\beta(G)| \neq |K_\beta|$,

then G is not a restricted GCI-group.

Proof. Assume $|G| = n$. If these three conditions are satisfied, then n is even by Proposition 2.7. Furthermore, there must exist two generalized Cayley graphs, say $\mathrm{GC}(G, \{s\}, \alpha)$ and $\mathrm{GC}(G, \{s'\}, \alpha)$, which are both isomorphic to $\frac{n}{2}K_2$. But there is no natural automorphism as α and β are not conjugate. Hence G is not a restricted GCI-group. $\square$

To conclude, we give the characterization of strongly GCI-groups.

Theorem 4.6. *A finite group G is a strongly GCI-groups if and only if G is a CI-group and one of the following is true for G:*

(1) G has no involutory automorphisms;

(2) all involutory automorphisms are fixed-point-free.

Proof. First we show the necessity. If G is a strongly GCI-groups, then G must be a CI-group. If not all involutory automorphisms of G are fixed-point-free automorphisms or, as we will show that G has no automorphisms of order two. If there exists some involutory automorphism which is not fixed-point-free, say α, this means $|\operatorname{Fix}(\alpha)| \neq 1$. By Lemma 2.5, we get $\omega_\alpha(G) \neq G$. Since $G = \omega_\alpha(G) \cup \Omega_\alpha \cup \mho_\alpha$ by Proposition 2.1, it follows that $\Omega_\alpha \cup \mho_\alpha \neq \emptyset$. Thus at least one of Ω_α and $\mho_\alpha$, say Ω_α, is not an empty set. According to Theorem 2.4, $\mathrm{GC}(G, \Omega_\alpha, \alpha) \cong \mathrm{GC}(G, \Omega_\alpha, 1)$ which is not a null graph. This is a contradiction to the fact that G is a strongly GCI-group. Therefore G has no automorphisms of order two since otherwise all automorphisms of order two of G are fixed-point-free automorphisms. If G has no automorphisms of order two, then G must be a CI-group as G is a strongly GCI-group.

Next we show the sufficiency. Suppose that all automorphisms of order two of G are fixed-point-free. Let $\alpha \in \mathrm{Aut}(G)$ be such an involution. Then $G = \omega_\alpha(G)$ by Lemma 2.5, so any generalized Cayley graph induced by involutory automorphism is a null graph. □

5 The cyclic GCI groups

Theorem 5.1. *The cyclic group of order p^n with p an odd prime is a GCI-group if and only if it is a CI-group.*

Proof. Let $G = Z_{p^n}$. Then G has only one automorphism of order two, that is $\alpha\colon x \mapsto -x$. Note that $\omega_\alpha(G) = \{\alpha(g^{-1})g \mid g \in G\} = \{2g \mid g \in G\}$, it follows that $S = \emptyset$ as any non-identity of G is a square since $|G|$ is odd. Thus the only generalized Cayley graph of G induced by automorphisms of order two is $\mathrm{GC}(G, \emptyset, \alpha) \cong p^n K_1$. □

Babai [3] classified the CI-groups of cyclic groups of order $2p$ with p a prime. Godsil [6] classified the CI-groups of cyclic groups of order $4p$. Next we will classify the GCI-groups of cyclic groups of even order. We will deal with the problem step by step in this section.

Theorem 5.2. *Let G be a finite cyclic group of order 2^n. Then G is a GCI-group if and only if $n = 1, 2$.*

Proof. Let $G = Z_{2^n} = \{0, 1, \ldots, 2^n - 1\}$. When $n = 1$, $\mathrm{Aut}(G) = 1$, there are no automorphisms of order two in $\mathrm{Aut}(G)$. Therefore G is a GCI-group by Definition 4.1. When $n = 2$, then $\mathrm{Aut}(G) \cong Z_2$, there is a unique element of order two in $\mathrm{Aut}(G)$ since $\mathrm{Aut}(G)$ is cyclic, say $\alpha\colon x \mapsto -x$. If $g \in G$, then $\alpha(g^{-1})g = 2g \notin S$. Hence $S \subseteq \{1, 3\}$. Therefore there are only three generalized Cayley graphs of G, with S being $\{1\}$, $\{3\}$ and $\{1, 3\}$, respectively. Let $\Gamma_1 = \mathrm{GC}(G, \{1\}, \alpha)$, $\Gamma_2 = \mathrm{GC}(G, \{3\}, \alpha)$. Note that $-1 \equiv 3 \pmod 4$, and so $\Gamma_1 \cong \Gamma_2$ by Corollary 3.3.

When $n \geq 3$, then $\mathrm{Aut}(G) \cong Z_2 \times Z_{2^{n-2}}$, and there are only three automorphisms of order two in $\mathrm{Aut}(G)$, say,

$$\alpha\colon x \mapsto -x, \qquad \beta\colon x \mapsto (2^{n-1} - 1)x, \qquad \gamma\colon x \mapsto (2^{n-1} + 1)x.$$

Let $S = \{1, 2^{n-1} + 1\}$. Since $1 \not\equiv 2^{n-1} + 1 \pmod{2^n}$ and they are both odd, we have $S \cap \omega_\alpha(G) = \emptyset$ as $\omega_\alpha(g) = \alpha(g^{-1})g = 2g$ is even for any $g \in G$. Further, $S \cap \omega_\beta(G) = \emptyset$ as $\omega_\beta(g) = \beta(g^{-1})g \equiv 2^{n-1}g + 2g \pmod{2^n}$ is also even for any $g \in G$. Recall that $\beta(-1) = 2^{n-1} + 1$, $\alpha(-1) = 1$ and $\alpha(-(2^{n-1} + 1)) = 2^{n-1} + 1$, hence $\alpha(S) = S^{-1}$ and $\beta(S) = S^{-1}$. Therefore both $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, S, \beta)$ are generalized Cayley graphs of G.

Let $\Gamma_1 = \mathrm{GC}(G, S, \alpha)$. Since $|S| = 2$, the valency of Γ_1 is two. For any $x \in V(\Gamma_1)$, $N(x) = \{\alpha(x) + y \mid y \in S\} = \{-x + 1, -x + 2^{n-1} + 1\}$. Consider the vertex $2^{n-1} + x \pmod{2^n}$, it follows that $x \not\equiv 2^{n-1} + x \pmod{2^n}$. $N(2^{n-1} + x) = \{\alpha(2^{n-1} + x) + y \mid y \in S\} = \{2^{n-1} - x + 1, -x + 1\}$. Thus '$x \to (-x + 1) \to (2^{n-1} + x) \to (2^{n-1} - x + 1) \to x$' is a 4-cycle in Γ_1. Therefore $\Gamma_1 \cong 2^{n-2}C_4$.

Let $\Gamma_2 = \mathrm{GC}(G, S, \beta)$. Since $|S| = 2$, the valency of Γ_2 is two. For any $x \in V(\Gamma_2)$, $N(x) = \{\beta(x) + y \mid y \in S\} = \{(2^{n-1} - 1)x + 1, (2^{n-1} - 1)(x - 1)\}$. We consider the vertex $2^{n-1} + x \pmod{2^n}$. Then $N(2^{n-1} + x) = \{\beta(2^{n-1} + x) + y \mid y \in S\} = \{(2^{n-1} - 1)x - 2^{n-1} + 1, (2^{n-1} - 1)x + 1\}$. Thus '$x \to (2^{n-1} - 1)x + 1 \to (2^{n-1} + x) \to (2^{n-1} - 1)(x - 1) \to x$' is a 4-cycle in Γ_1. Therefore $\Gamma_2 \cong 2^{n-2}C_4$.

From above, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S, \beta) \cong 2^{n-2}C_4$, but α and β are not conjugate in $\mathrm{Aut}(G)$ as $\mathrm{Aut}(G)$ is abelian, hence G is not a restricted GCI-group by Definition 4.1. □

Theorem 5.3. *Let G be a finite cyclic group of order $2^a p^b$ with p an odd prime and $a, b > 0$. If G is a restricted GCI-group, then $a = 1$.*

Proof. Since G is a finite cyclic group of order $2^a p^b$, let $G = G_1 \times G_2$, where $G_1 = Z_{2^a}$ and $G_2 = Z_{p^b}$.

We claim that $a \leq 2$. Now we suppose $a \geq 3$. By Theorem 5.2, $\alpha\colon (g_1, g_2) \mapsto (-g_1, g_2)$ and $\beta\colon (g_1, g_2) \mapsto ((2^{n-1} - 1)g_1, g_2)$ are two different automorphisms of G with order two when $a \geq 3$. Let $S = \{(1, 0), (2^{n-1}, 0)\}$. Then $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, S, \beta)$ are two generalized Cayley graphs of G. According to Theorem 5.2, we have $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S, \beta) \cong 2^{n-2}p^b C_4$. Note that α and β are not conjugate in $\mathrm{Aut}(G)$, it follows that $a \leq 2$.

Assume $a = 2$. Note that $\alpha\colon (g_1, g_2) \mapsto (-g_1, g_2), \beta\colon (g_1, g_2) \mapsto (g_1, -g_2)$ are two automorphisms of G. Furthermore, $\omega_\alpha(G) = \{(0, 0), (2, 0)\}$, $K_\alpha = \{(g_1, 0) \mid g_1 \in G_1\}$. Therefore $\Omega_\alpha = \{(1, 0), (3, 0)\}$. $\omega_\beta(G) = \{(0, g_2) \mid g_2 \in G_2\}$, $K_\beta = \{(0, g_2), (2, g_2) \mid g_2 \in G_2\}$. Thus $\Omega_\beta = \{(2, g_2) \mid g_2 \in G_2\}$. Let $S_1 = \{(1, 0)\}$ and $S_2 = \{(2, 0)\}$. We can see that $\mathrm{GC}(G, S_1, \alpha) \cong \mathrm{GC}(G, S_2, \beta) \cong 2p^b K_2$. However α and β are not conjugate as $\mathrm{Aut}(G)$ is abelian. It follows from above discussion that $a = 1$. □

Theorem 5.4. *Let G be a finite cyclic group of order n, where n is even with at least two different odd prime divisors. Then G is not a restricted GCI-group.*

Proof. Suppose that $n = p_0^{s_0} \cdot p_1^{s_1} \cdots p_k^{s_k}$, where $p_0 = 2$, p_i, p_j are different odd primes for any $i, j \in \{1, \dots, k\}$ and $s_t \geq 1$ is an integer for any $t \in \{0, 1, \dots, k\}$, $k \geq 2$. It follows that G can be decomposed into the direct product of some cyclic groups, say

$$G = G_0 \times \cdots \times G_k = Z_{2^{s_0}} \times Z_{p_1^{s_1}} \times \cdots \times Z_{p_k^{s_k}}, \text{ where } G_i = Z_{p_i^{s_i}},\ i = 0, 1, \dots, k.$$

Let

$$\alpha\colon (x_0, x_1, \dots, x_k) \mapsto (x_0, -x_1, \dots, x_k)$$

and

$$\beta\colon (x_0, x_1, x_2, \dots, x_k) \mapsto (x_0, x_1, -x_2, \dots, x_k).$$

Since $k \geq 2$, then such α, β can not appear in $\mathrm{Aut}(G)$. Obviously $\omega_\alpha(G) = \{(0, x_1, 0, \dots, 0) \mid x_1 \in G_1\}$ and $\omega_\beta(G) = \{(0, 0, x_2, \dots, 0) \mid x_2 \in G_2\}$. Let $g_i \in G_i$, $i \in \{0, 1, \dots, k\}$ and g_0 the element of order two. Then $(g_0, g_1, 0, \dots, 0) \in \Omega_\alpha$ and $(g_0, 0, g_2, 0, \dots, 0) \in \Omega_\beta$. Therefore $\mathrm{GC}(G, \{(g_0, g_1, 0, \dots, 0)\}, \alpha)$ and $\mathrm{GC}(G, \{(g_0, 0, g_2, 0, \dots, 0)\}, \beta)$ are both generalized Cayley graphs of G. In fact, they are both isomorphic to $\frac{n}{2}K_2$, but α and β are not conjugate in $\mathrm{Aut}(G)$. Thus G is not a restricted GCI-group by Theorem 4.5. □

Theorem 5.5. *Let $G = Z_{2n}$, where n is an odd prime power. Then G is not a strongly GCI-group.*

Proof. Let $G = \langle a, b \mid a^n = b^2 = 1, ab = ba\rangle$. It can be checked that the mapping $\alpha\colon a \mapsto a^{-1}, b \mapsto b$ is the only automorphism of G of order two. Also $\Omega_\alpha = \{a^i b \mid i \in \{1, \dots, n\}\}$ and $\mho_\alpha = \emptyset$ by direct computation. Let $\mathrm{GC}(G, S, \alpha)$ be any generalized Cayley graph of G. Then $S \subseteq \Omega_\alpha$. Let $H = \langle a', b' \mid a'^n = b'^2 = 1, b'a'b' = a'^{-1}\rangle$ and $\varphi\colon a^s \mapsto a'^s, a^t b \mapsto a'^{-t} b$. It follows that φ is a bijection from G to H. Furthermore,

$\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(H, \varphi(S))$. Let $S = \{ab, a^2b\}$. Then $\varphi(S) = \{a^{-1}b, a^{-2}b\}$, this implies $\mathrm{Cay}(H, \varphi(S)) \cong C_{2n}$ as $\langle \varphi(S) \rangle = H$. Let $T = \{ab, a^{-1}b\}$. Then $\langle T \rangle = G$, therefore $\mathrm{Cay}(G, T) \cong C_{2n}$. Thus $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, T, 1) \cong C_{2n}$, which means that G is not a strongly GCI-group from Definition 4.1. □

Theorem 5.6. *Let $G = Z_{2n}$, $H = D_{2n}$, where n is an odd prime power. Then G is a restricted GCI-group if H is a CI-group.*

Proof. Let $G = \langle a, b \mid a^n = b^2 = 1, ab = ba \rangle$ and $H = \langle a', b' \mid a'^n = b'^2 = 1, b'a'b' = a'^{-1} \rangle$. It is easy to see that $\alpha\colon a \mapsto a^{-1}, b \mapsto b$ is the only automorphism of G of order two. Then we have $\Omega_\alpha = \{a^i b \mid i \in \{1, \ldots, n\}\}$ and $\mho_\alpha = \emptyset$ by direct computation. Let $\mathrm{GC}(G, S, \alpha)$ be any generalized Cayley graph of G. Then $S \subseteq \Omega_\alpha$. Let $\varphi\colon a^s \mapsto a'^s, a^t b \mapsto a'^{-t} b$. Obviously φ is a bijection from G to H. Furthermore, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(H, \varphi(S))$. Assume that $\mathrm{GC}(G, S_1, \alpha) \cong \mathrm{GC}(G, S_2, \alpha)$, then $\mathrm{Cay}(H, \varphi(S_1)) \cong \mathrm{Cay}(H, \varphi(S_2))$. Since H is a CI-group, there exists some $\gamma \in \mathrm{Aut}(H)$ such that $\gamma(\varphi(S_1)) = \varphi(S_2)$. Without loss of generality, assume that there exist k, l satisfying $(k, n) = 1$ and $1 \leq l \leq n$ such that γ is the mapping $a' \mapsto a'^k, b' \mapsto a'^l b'$. Let $\delta\colon a \mapsto a^k, b \mapsto b$. Then $\delta \in \mathrm{Aut}(G)$. Since n is some odd prime power, there must exist some $1 \leq m \leq n$ such that $a^l = a^{2m}$. Therefore there exist $\delta \in \mathrm{Aut}(G)$ and $g = a^{-m}$ such that $S_2 = \alpha(g) S_1^\delta g^{-1}$. Hence G is a restricted GCI-group. □

Theorem 5.7. *Let G be a finite cyclic group of odd order n, where n has at least two different prime divisors. Then G is not a strongly GCI-group.*

Proof. Let $G = G_1 \times G_2 \times \cdots \times G_s$ where $G_i = Z_{p_i^{k_i}}$, p_i is some odd prime. Let $\alpha\colon (g_1, g_2, \ldots, g_s) \mapsto (-g_1, g_2, \ldots, g_s)$. Then $\omega_\alpha(G) = G_1$, and thus $G \setminus \omega_\alpha(G) \neq \emptyset$. By Theorem 2.4, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{GC}(G, S, 1)$, where $S = G \setminus \omega_\alpha(G)$. It follows that G is not a GCI-group. □

6 The GCI-groups in dihedral groups

Theorem 6.1. *Let $G = D_{2n}$ ($n \geq 3$) be a dihedral group. If G is a restricted GCI-group, then n is some odd prime power.*

Proof. Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be a GCI-group. Assume first that n is even. Let $\alpha\colon a \mapsto a^{-1}, b \mapsto b$. Then $\alpha \in \mathrm{Aut}(G)$ is of order two. $\omega_\alpha(G) = \{\alpha(g^{-1})g \mid g \in G\} = \{a^i \in G \mid i \text{ is even}\}$. $K_\alpha = \{\alpha(g)g = 1 \mid g \in G\} = \{a^i, b, a^{\frac{n}{2}} b \mid a^i \in G\}$. It follows that $\Omega_\alpha(G) = \{a^i, b, a^{\frac{n}{2}} b \mid i \text{ is odd}\}$. This implies $\mathrm{GC}(G, \{a\}, \alpha)$ and $\mathrm{GC}(G, \{b\}, \alpha)$ are always generalized Cayley graphs of G. Note that they are both isomorphic to nK_2. Furthermore, $\alpha(g) a^\gamma g^{-1} = a^{-2i+j}$ if $g = a^i$ and $a^\gamma = a^j$, $\alpha(g) a^\gamma g^{-1} = a^{-2i-j}$ if $g = a^i b$ and $a^\gamma = a^j$. It follows that $\alpha(g) a^\gamma g^{-1} \in \langle a \rangle$. Since G is a GCI-group, $\alpha(g) a^\gamma g^{-1} = b$ for some $g \in G$ and $\gamma \in \mathrm{Aut}(G)$, but this is impossible. Thus n is not even.

Assume n is odd and has at least two different prime factors, say $n = p_1^{r_1} \cdots p_t^{r_t}$ is the prime decomposition and $t \geq 2$. By [20, Lemma 3.4], $\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_t)$, where $G_i = \langle a_i, b \rangle$ and $\langle a \rangle = \langle a_1 \rangle \times \cdots \times \langle a_t \rangle$. It can be checked that there must exist two automorphisms $\alpha\colon a_1 \mapsto a_1^{-1}, a_i \mapsto a_i, b \mapsto b$ and $\beta\colon a_2 \mapsto a_2^{-1}, a_j \mapsto a_j$, $b \mapsto b$ in $\mathrm{Aut}(G)$. Notice that each is of order two, and they are not conjugate in $\mathrm{Aut}(G)$ as they belong to $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$ respectively, and $\mathrm{Aut}(G)$ is the direct product of these $\mathrm{Aut}(G_i)$. Furthermore, $b \in \Omega_\alpha(G) \cap \Omega_\beta(G)$. Thus $\mathrm{GC}(G, \{b\}, \alpha)$ and

$\mathrm{GC}(G, \{b\}, \beta)$ are two generalized Cayley graphs of G which are isomorphic to nK_2. However α and β are not conjugate. Thus n is some odd prime power. □

Theorem 6.2. *Let $G = D_{2n}$, $H = Z_{2n}$, with n odd prime power. Then G is a restricted GCI-group if H is a CI-group.*

Proof. Let $G = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ and $H = Z_{2n} = \{0, 1, \ldots, 2n-1\}$. We will show first that any two automorphisms of G of order two are conjugate. Let $\alpha\colon a \mapsto a^i, b \mapsto a^j b$ and $\beta\colon a \mapsto a^k, b \mapsto a^l b$ be two automorphisms of order two. Then $i, k = -1$. Let $\gamma\colon a \mapsto a^s, b \mapsto a^t b$ with $(n, s) = 1$. Then $\gamma \in \mathrm{Aut}(G)$. We can see that $\gamma^{-1}\colon a \mapsto a^r, b \mapsto a^{-rt} b$ with $rs \equiv 1 \pmod{n}$. It follows that $\gamma\alpha\gamma^{-1}\colon a \mapsto a^{-1}$, $b \mapsto a^{2t+sj} b$. For any j, l, the equation $2t + sj \equiv l \pmod{n}$ has a solution. It follows that α, β are conjugate.

According to the Remark 3.2, it suffices to consider the isomorphisms of the generalized Cayley graphs induced by the same automorphisms. Without loss of generality, we consider $\alpha\colon a \mapsto a^{-1}, b \mapsto b$. Let $s = \frac{n-1}{2}$ and $I = \{1, \ldots, s\}$. Then $\omega_\alpha(G) = \{\alpha(g^{-1})g \mid g \in G\} = \langle a \rangle$. $K_\alpha = \{b\} \cup \langle a \rangle$. Thus $\Omega_\alpha = \{b\}$ and $\mho_\alpha = \cup_{i \in I}\{a^i b, a^{-i} b\}$.

Let $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, T, \alpha)$ be any two isomorphic generalized Cayley graphs. We divide the proof into two cases.

Case 1: $\Omega_\alpha \subseteq S$.

If $\Omega_\alpha \subseteq S$, then $\Omega_\alpha \subseteq T$. Suppose $S = \cup_{i \in I_1 \subseteq I}\{a^i b, a^{-i} b\} \cup \Omega_\alpha$ and $T = \cup_{i \in I_2 \subseteq I}\{a^i b, a^{-i} b\} \cup \Omega_\alpha$. Let $\varphi\colon G \to H, a^s \mapsto 2s, a^t b \mapsto n - 2t$. Then φ is a bijection from G to H. Furthermore, $\mathrm{GC}(G, S, \alpha) \cong \mathrm{Cay}(H, \varphi(S))$ and $\mathrm{GC}(G, T, \alpha) \cong \mathrm{Cay}(H, \varphi(T))$, where $\varphi(S) = \cup_{i \in I_1 \subseteq I}\{n - 2i, n + 2i\} \cup \{n\}$ and $\varphi(T) = \cup_{i \in I_2 \subseteq I}\{n - 2i, n + 2i\} \cup \{n\}$. Since H is a CI-group, there exists some automorphism $\phi \in \mathrm{Aut}(H)$ such that $\varphi(T) = \phi(\varphi(S))$. Since n is the unique involution in H, $\phi(n) = n$ and $\phi(n - 2i) = \phi(n) - 2\phi(i) = n - 2\phi(i)$ for any $i \in I_1 \subseteq I$. This implies that ϕ can induce an automorphism $\overline{\phi}$ of G with rules $a^i \mapsto a^{\phi(i)}, b \mapsto b$. As $\overline{\phi}\alpha = \alpha\overline{\phi}$, there exist $\overline{\phi}$ and $g = 1 \in G$ such that the isomorphism between $\mathrm{GC}(G, S, \alpha)$ and $\mathrm{GC}(G, T, \alpha)$ is a natural isomorphism.

Case 2: $\Omega_\alpha \nsubseteq S$.

If $\Omega_\alpha \nsubseteq S$, then $\Omega_\alpha \nsubseteq T$. The rest of the proof is similar to that of Case 1. □

The next result is about the graph structure. Recall that a graph Γ is Hamiltonian if it contains a cycle passing through all vertices of Γ.

Theorem 6.3. *Let $G = D_{2n}$ with n odd prime power. Then any connected generalized Cayley graph of G is Hamiltonian.*

Proof. Let $H = Z_{2n}$ and $\varphi\colon G \to H, a^s \mapsto 2s, a^t b \mapsto n - 2t$ be the bijection from G to H. Then any generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$ of G is isomorphic to the Cayley graph $\mathrm{Cay}(H, \varphi(S))$ of H. Therefore $\mathrm{GC}(G, S, \alpha)$ is connected if and only if $\mathrm{Cay}(H, \delta(S))$ is connected. It is well known that $\mathrm{Cay}(H, \varphi(S))$ is connected if and only if $\langle \varphi(S) \rangle = H$. $\langle \varphi(S) \rangle = H$ if and only if there exist some $a^i b, a^{-i} b \in S$ satisfying $(i, n) = 1$ as $\varphi(a^i b) = n - 2i$. Then there always exists a Hamilton cycle $\mathrm{GC}(G, \{a^i b, a^{-i} b\}, S)$ in a connected generalized Cayley graph of G. This completes the proof. □

7 Cayley regression

First of all, we give the following related definitions.

Definition 7.1. Let G be a finite group.

(1) G is called a *Cayley regression* if any generalized Cayley graph of G is isomorphic to some Cayley graph of G.

(2) G is called an α*-Cayley regression* if any generalized Cayley graph of G induced by $\alpha \in \mathrm{Aut}(G)$ is isomorphic to some Cayley graph of G.

(3) G is called a *quasi-Cayley regression* if any generalized Cayley graph not induced by the automorphism $x \mapsto x^{-1}$ is isomorphic to some Cayley graph of G.

(4) G is called an m*-Cayley regression* if any generalized Cayley graph of G with valency at most m is isomorphic to some Cayley graph of G.

(5) G is called an m*-quasi-Cayley regression* if any generalized Cayley graph not induced by the automorphism $x \mapsto x^{-1}$ of G with valency at most m is isomorphic to some Cayley graph of G.

(6) G is called a *skew Cayley regression* if any generalized Cayley graph of G is isomorphic to some generalized Cayley graph of another finite group.

It is well known that every Cayley graph is also a generalized Cayley graph. But many examples, see [8] for instance, reveal that the converse is not true. Therefore a natural question arises.

Question 7.2. Characterize Cayley regressions.

Remark 7.3. If $\alpha\colon x \mapsto x^{-1}$ is an automorphism of G, then G is abelian. This case is very special as $K_\alpha = G$ and $\mho_\alpha = \emptyset$. In fact, Hujdurović et al. in [9] had already noticed this situation. They studied the relationship between the generalized Cayley graphs induced by involutory automorphism and Cayley graphs. They obtained two families of generalized Cayley graphs induced by involutory automorphisms on $Z_{2^m} \times Z_{2^n}$ and $Z_2 \times Z_2 \times Z_{2k+1}$ respectively (where $m \geq 1, n \geq 2, k \geq 1$) are not vertex-transitive. Therefore we propose the definition of 'quasi-Cayley regression' and 'm-quasi-Cayley regression'. Also, we propose the following problem: Are there finite groups which are quasi-Cayley regressions but not Cayley regressions?

When G is an abelian simple group, then G is a cyclic group of prime order and $\mathrm{Aut}(G)$ is not necessarily a Cayley regression.

Example 7.4. For the prime $p = 61$, obviously, $\mathrm{Aut}(Z_p) \cong Z_{p-1} = Z_{60}$. Let $G = \mathrm{Aut}(Z_p)$, $S = \{\pm 6, \pm 12, 5, 25\}$ and $\alpha(x) = 31x$. By [17, Theorem 4.4], we have $\mathrm{GC}(G, S, \alpha)$ is not a Cayley graph. Thus G is not a Cayley regression.

Theorem 7.5. *Let G be a finite cyclic group of odd order n. Then G is a Cayley regression if and only if n is some odd prime power.*

Proof. The sufficiency is obvious by Theorem 5.1, it suffices to show the necessity. Assume on the contrary that n has at least two different odd prime divisors, say $n = q_1 q_2 m$, where q_1 and q_2 are different prime powers and $(q_1, q_2) = 1$, then we have $G = G_1 \times G_2 \times G_3$, where $|G_1| = q_1$, $|G_2| = q_2$ and $|G_3| = m$. Let $\alpha\colon G \to G$, $(g_1, g_2, g_3) \mapsto (-g_1, g_2, g_3)$. It is easy to see that the order of α is two, so α can induce some generalized Cayley graphs of G. Note that $\omega_\alpha(G) = \{(g_1, 0, 0) \mid g_1 \in G_1\}$. Let $S = \{(0, 1, 0), (0, q_2 - 1, 0)\}$. Then $\Gamma = \mathrm{GC}(G, S, \alpha)$ is a generalized Cayley graph of G. Consider the vertex of the form $(0, g_2, g_3)$ in Γ for any $g_2 \in G_2$, $g_3 \in G_3$. For any fixed g_3, there are q_2 vertices of the form $\{(0, g_2, g_3) \mid g_2 \in G_2\}$ which induce a cycle of length q_2. For any other vertex of the form (g_1, g_2, g_3) with $g_1 \neq 0$, there are $2q_2$ vertices $\{(g_1, g_2, g_3), (-g_1, g_2, g_3) \mid g_2 \in G\}$ which induce a cycle of length $2q_2$. Therefore $\Gamma_1 = mC_{q_2} \cup \frac{(q_1-1)m}{2} C_{2q_2}$, which is not vertex-transitive. Thus $\mathrm{GC}(G, S, \alpha)$ is not a Cayley graph, and hence G is not a Cayley regression. □

Theorem 7.6. *Let* $G = \underbrace{Z_n \times \cdots \times Z_n}_{s}$ *with n odd, $s \geq 2$. Then G is not a Cayley regression.*

Proof. Let $\alpha\colon (i_1, i_2, i_3 \ldots, i_s) \mapsto (i_2, i_1, i_3, \ldots, i_s)$ for all $i_t \in Z_n$. So $\alpha \in \mathrm{Aut}(G)$ and $o(\alpha) = 2$. Therefore α can induce generalized Cayley graphs of G. Let $S = \{(1, 0, \ldots, 0), (0, n-1, 0, \ldots, 0)\}$. It follows that $\mathrm{GC}(G, S, \alpha)$ is a generalized Cayley graph of G.

Consider vertex $(0, \ldots, 0)$, then vertices like $(i, i, 0, \ldots, 0)$ and $(i, i-1, 0, \ldots, 0)$ are in the same cycle with $(0, \ldots, 0)$. Thus $(0, \ldots, 0)$ is in a cycle of length $2n$.

Consider vertex $(0, \frac{n-1}{2}, 0, \ldots, 0)$, then vertices like $(i, \frac{n-1}{2} + i, 0, \ldots, 0)$ and $(\frac{n+1}{2} + i, i, \ldots, 0)$ are in the same cycle with $(0, \frac{n-1}{2}, 0, \ldots, 0)$. It follows that $(0, \frac{n-1}{2}, 0, \ldots, 0)$ is in a cycle of length n.

Therefore $\mathrm{GC}(G, S, \alpha)$ is not vertex-transitive, and thus $\mathrm{GC}(G, S, \alpha)$ is not a Cayley graph. That completes the proof. □

From Theorem 7.5, we see that the cyclic group of odd non prime power order is not an m-Cayley regression for any $m > 0$. So we only consider the cyclic groups of even order in the rest of this section.

Corollary 7.7 ([9])**.** *Let $G = Z_{2n}$. Then $\mathrm{GC}(G, S, \alpha)$ is isomorphic to a Cayley graph on D_{2n}, where $\alpha\colon x \mapsto -x$.*

According to Corollary 7.7, we can see that Z_{2p^n} (with p an odd prime) is a skew-Cayley regression since $\alpha\colon x \mapsto -x$ is the only automorphism of G of order two.

Theorem 7.8. *Let G be a finite cyclic group of order 2^n with $n \geq 3$. Then*

(1) G is a 3-quasi-Cayley regression;

(2) G is a 4-quasi-Cayley regression if and only if $n = 3$.

Proof. Assume that $G = \{0, 1, \ldots, 2^n - 1\}$. By Theorem 5.2, we have that $\alpha\colon x \mapsto (2^{n-1} - 1)x$ and $\beta\colon x \mapsto (2^{n-1} + 1)x$ are the only two automorphisms of G of order two except the automorphism $x \mapsto -x$. Also, the valency of the generalized Cayley of G induced by α or β are even as $x + \alpha(x) \neq 0$ and $x + \beta(x) \neq 0$ for any $x \in G$.

(1) Consider the generalized Cayley graphs induced by α, β respectively. For any $g \in G$,

$$\begin{aligned}\omega_\alpha(G) &= \{\alpha(-g)g \mid g \in G\} = \{(2^{n-1}-1)(-g)+g \mid g \in G\} \\ &= \{2^{n-1}g + 2g \pmod{2^n} \mid g \in G\} = \{g \mid g \equiv 0 \pmod 2\} = K_\alpha.\end{aligned}$$

$$\begin{aligned}\omega_\beta(G) &= \{\beta(-g)g \mid g \in G\} = \{(2^{n-1}+1)(-g)+g \pmod{2^n} \mid g \in G\} \\ &= \{-2^{n-1}g \pmod{2^n} \mid g \in G\} = \{0, 2^{n-1}\} = K_\beta.\end{aligned}$$

It follows that for any generalized Cayley graph $\mathrm{GC}(G, S, \alpha)$, S contains no even integers and, for any generalized Cayley graph $\mathrm{GC}(G, S, \beta)$, 0 and 2^{n-1} are not contained in S.

For any $g \in G$ with g odd,

$$\begin{aligned}\alpha(-g) &= (2^{n-1}-1)(-g) \pmod{2^n} \\ &= g - 2^{n-1}g \pmod{2^n} \\ &= 2^n g + g - 2^{n-1}g \pmod{2^n} \\ &= 2^{n-1}g + g \pmod{2^n} \\ &= 2^{n-1} + g \pmod{2^n}.\end{aligned}$$

This implies that there are 2^{n-2} couples can be included in S, that is, $S_1 = \{1, 2^{n-1}+1\}$, $S_3 = \{3, 2^{n-1}+3\}, \ldots, S_{2^{n-1}-1} = \{2^{n-1}-1, 2^n-1\}$.

For any $g \in G \setminus \omega_\beta(G)$,

$$\begin{aligned}\beta(-g) &= (2^{n-1}+1)(-g) \pmod{2^n} = -g - 2^{n-1}g \pmod{2^n} \\ &= 2^n g - g - 2^{n-1}g \pmod{2^n} = 2^{n-1}g - g \pmod{2^n}.\end{aligned}$$

Then we have

$$\beta(-g) = \begin{cases} 2^{n-1} - g, & \text{if } g \text{ is odd;} \\ 2^n - g, & \text{if } g \text{ is even.} \end{cases}$$

This implies that there are $(2^{n-1}-1)$ couples which could be included in S, they are:

$$\begin{aligned}S_1 &= \{1, 2^{n-1}-1\}, \\ S_3 &= \{3, 2^{n-1}-3\}, \\ &\ldots \\ S_{2^{n-2}-1} &= \{2^{n-2}-1, 2^{n-2}+1\}, \\ S_{2^{n-1}+1} &= \{2^{n-1}+1, 2^n-1\}, \\ &\ldots \\ S_{2^{n-1}+2^{n-2}-1} &= \{2^{n-1}+2^{n-2}-1, 2^{n-1}+2^{n-2}+1\}, \\ T_2 &= \{2, 2^n-2\}, \\ &\ldots \\ T_{2^{n-1}-2} &= \{2^{n-1}-2, 2^{n-1}+2\}.\end{aligned}$$

Let $\Gamma = \mathrm{GC}(G, S, \alpha)$, where $S = S_i$, then $\Gamma \cong 2^{n-2}C_4$ by Theorem 5.2. Let $\Gamma = \mathrm{GC}(G, S, \beta)$. If $S = S_i$, we have $\mathrm{GC}(G, S, \beta) \cong C_{2^n}$ as S_i is the left generating set for $(G, *)$ by Lemma 2.11. If $S = T_i$, then $\mathrm{GC}(G, S, \beta)$ is isomorphic to $2^{n-k}C_{2^k}$

if $2^k i \equiv 0 \pmod{2^n}$; and isomorphic to C_{2^n} otherwise. In conclude, all of the 2-valent generalized Cayley graphs of G induced by α or β are Cayley graphs and, this implies that G is a 3-quasi-Cayley regression.

(2) When $n = 3$, it is easy to check that those 4-valent generalized Cayley graphs induced by α and β, respectively, are all Cayley graphs. Next we construct a family of generalized Cayley graphs which is not vertex-transitive to show the necessity.

Let $S = S_i \cup T_j$, where $i \in \{1, \dots, 2^{n-2} - 1\} \cup \{2^{n-1} + 1, \dots, 2^{n-1} + 2^{n-2} - 1\}$ is odd and $j \in \{2, \dots, 2^{n-1} - 2\}$ is even. If x is odd, then $N(x) = \{2^{n-1} + x + i, x - i, 2^{n-1} + x + j, 2^{n-1} + x - j\}$. If x is even, then $N(x) = \{x + i, 2^{n-1} + x - i, x + j, x - j\}$. Suppose X is the bicirculant such that the vertex set $V(X)$ can be partitioned into to subsets $U = \{u_k \mid k \in Z_{2^{n-1}}\}$ and $V = \{v_k \mid k \in Z_{2^{n-1}}\}$, and there is an automorphism of X such that $\rho(u_k) = u_{k+1}$ and $\rho(v_k) = v_{k+1}$, $k \in Z_{2^{n-1}}$. The edge set $E(X)$ can be partitioned into three subsets:

$$L = \cup_{k \in Z_{2^{n-1}}} \{u_k, u_{k+l} \mid l \in L\},$$
$$M = \cup_{k \in Z_{2^{n-1}}} \{u_k, v_{k+m} \mid m \in M\},$$
$$R = \cup_{k \in Z_{2^{n-1}}} \{v_k, v_{k+r} \mid r \in R\},$$

so we have $L = \{\pm(2^{n-2} + \frac{j}{2})\}$, $M = \{2^{n-2} + \frac{i+1}{2}, -\frac{i-1}{2}\}$, $R = \{\pm\frac{j}{2}\}$. Then $X = BC_{2^{n-1}}[L, M, R]$. Let γ be the mapping as follows:

$$\gamma := \begin{cases} x \mapsto u_{\frac{x-1}{2}}, & \text{if } x \text{ is odd;} \\ x \mapsto v_{\frac{x}{2}}, & \text{if } x \text{ is even.} \end{cases}$$

It follows that $\Gamma \cong X$. Note that $BC_{2^{n-1}}[L, M, R] \cong BC_{2^{n-1}}[aL, aM + b, aR]$ with $a, b \in Z_{2^{n-1}}$ and a invertible [12]. Then $\Gamma \cong BC_{2^{n-1}}[L, M', R]$ with $M' = M + \frac{i-1}{2} = \{0, 2^{n-2} + i\}$. In particular, Γ is connected since $\langle L, M', R\rangle = Z_{2^{n-1}}$. When $j = 2i$, there are no triangles with three vertices of the form $\{u_k, v_{k+2^{n-2}+\frac{i+1}{2}}, v_{k-\frac{i-1}{2}}\}$, but there is a triangle with three vertices as $\{v_{k'}, u_{k'+\frac{i-1}{2}}, u_{k'-2^{n-2}-\frac{i+1}{2}}\}$ since for $n > 3$,

$$k + 2^{n-2} + \frac{i+1}{2} \pm \frac{j}{2} \not\equiv k - \frac{i-1}{2} \pmod{2^{n-1}}$$
$$k' + \frac{i-1}{2} - \left(2^{n-2} + \frac{j}{2}\right) \equiv k' - 2^{n-2} - \frac{i+1}{2} \pmod{2^{n-1}}.$$

This implies that there is no automorphism of X which permutates u_k and $v_{k'}$. So X is not vertex-transitive when $n > 3$. This completes the proof. □

At last, we propose the following questions for further research.

Question 7.9. Classify finite GCI-groups, such as Z_m where m is odd with at least two different prime divisors, abelian groups, dihedral groups and some classes of finite simple groups.

Question 7.10. Characterize the structure of the automorphism group of any generalized Cayley graph.

Question 7.11. Classify Cayley regressions for certain types of group, such as the cyclic groups and the dihedral groups.

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