

# Nonlinear maps preserving the elementary symmetric functions

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## Abstract

Let  $\mathcal{M}_n$  be the algebra of all  $n \times n$  complex matrices, and for a natural number  $2 \leq k \leq n$  denote by  $E_k(x)$  the  $k$ th elementary symmetric function on the eigenvalues of  $x \in \mathcal{M}_n$ . For two maps  $\varphi, \psi: \mathcal{M}_n \rightarrow \mathcal{M}_n$ , one of them being surjective, we prove that if  $E_k(\lambda x + y) = E_k(\lambda\varphi(x) + \psi(y))$  for each  $\lambda \in \mathbb{C}$  and  $x, y \in \mathcal{M}_n$ , then  $\varphi = \psi$  on  $\mathcal{M}_n$ , the common value being a linear map from  $\mathcal{M}_n$  into itself. In particular, for  $3 \leq k \leq n$  the general form of  $\varphi$  and  $\psi$  can be computed explicitly.

*Keywords:* Elementary symmetric function, nonlinear, preserver.

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## 1 Introduction and statement of the result

For a natural number  $n$ , let us denote by  $\mathcal{M}_n$  the algebra of all  $n \times n$  matrices over the complex field  $\mathbb{C}$ . By  $I_n \in \mathcal{M}_n$  we shall denote the  $n \times n$  identity matrix. For  $x \in \mathcal{M}_n$ , by  $\text{tr}(x)$  we shall denote its usual trace, and by  $\det(x)$  its determinant. Also, by  $x^t \in \mathcal{M}_n$  we shall denote the transpose of  $x$ .

For  $k \in \{1, \dots, n\}$ , a  $k$ -by- $k$  principal submatrix of  $x \in \mathcal{M}_n$  is the submatrix of  $x$  which lies in the rows and columns of  $x$  indexed by  $J \subseteq \{1, \dots, n\}$  with  $|J| = k$ . Equivalently, we eliminate from the matrix  $x$  the rows and the columns which are not in  $J$ . The determinant of the  $k$ -by- $k$  principal submatrix given by  $J \subseteq \{1, \dots, n\}$  is called a  $k$ -by- $k$  principal minor, and shall be denoted by  $\Delta_J(x)$ . There are  $\binom{n}{k}$  different  $k$ -by- $k$  principal minors, and put

$$E_k(x) = \sum_{|J|=k} \Delta_J(x) \quad (x \in \mathcal{M}_n, k = \overline{1, n}). \quad (1.1)$$

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In particular,  $k = 1$  in (1.1) gives  $E_1(x) = \text{tr}(x)$  and  $k = n$  gives  $E_n(x) = \det(x)$  for each  $x \in \mathcal{M}_n$ .

For  $k \in \{1, \dots, n\}$ , the  $k$ th elementary symmetric function on the complex numbers  $\lambda_1, \dots, \lambda_n$  is

$$S_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \lambda_{i_j}. \tag{1.2}$$

(We have a sum of  $\binom{n}{k}$  products in (1.2).) For  $x \in \mathcal{M}_n$ , if the spectrum  $\sigma(x)$  of  $x$  (taking into account multiplicities) is  $\{\alpha_1, \dots, \alpha_n\}$ , the equality

$$\det(\lambda I_n + x) = \lambda^n + \lambda^{n-1} E_1(x) + \dots + E_n(x) \quad (x \in \mathcal{M}_n, \lambda \in \mathbf{C})$$

implies that

$$E_k(x) = S_k(\sigma(x)) \quad (x \in \mathcal{M}_n, k = \overline{1, n}).$$

Thus, for each  $k \in \overline{1, n}$  and  $x \in \mathcal{M}_n$  we have that  $E_k(x)$  is the  $k$ th elementary symmetric function on the eigenvalues of  $x$ .

Frobenius studied in [5] linear maps on  $\mathcal{M}_n$  which preserve the determinant. He proved that if  $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  is a bijective linear map such that  $\det(\phi(x)) = \det(x)$  for each  $x \in \mathcal{M}_n$ , there exist then invertible matrices  $a, b \in \mathcal{M}_n$  satisfying  $\det(ab) = 1$  such that either  $\phi(x) = axb$  for each  $x \in \mathcal{M}_n$ , or  $\phi(x) = ax^t b$  for each  $x \in \mathcal{M}_n$ . One way to generalize this result is to relax the linearity assumption on the map  $\phi$ . In [4], Dolinar and Šemrl proved that we arrive at the same conclusion if we merely suppose that  $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  is a surjective map such that

$$\det(\lambda x + y) = \det(\lambda \phi(x) + \phi(y)) \quad (x, y \in \mathcal{M}_n, \lambda \in \mathbf{C}). \tag{1.3}$$

(In this case, the linearity for the map  $\phi$  is part of the conclusion.) Another way to generalize the result of Frobenius is to consider the elementary symmetric functions  $E_k$  instead of the determinant. The following theorem was proved by Marcus and Purves in [7] for  $4 \leq k < n$  and by Beasley in [1] for  $3 = k < n$ .

**Theorem 1.1** ([7, Theorem 3.1] and [1, Theorem 1.1]). *Let  $3 \leq k < n$  and let  $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  be a linear map such that*

$$E_k(x) = E_k(\phi(x)) \quad (x \in \mathcal{M}_n).$$

*There exist then an invertible matrix  $a \in \mathcal{M}_n$  and  $\eta \in \mathbf{C}$  satisfying  $\eta^k = 1$  such that either*

$$\phi(x) = \eta a x a^{-1} \quad (x \in \mathcal{M}_n),$$

*or*

$$\phi(x) = \eta a x^t a^{-1} \quad (x \in \mathcal{M}_n).$$

The result of Frobenius shows that the conclusion of Theorem 1.1 does not hold if  $k = n$ . The same happens if  $k = 1$  (see, for example, [6, Section 2]) or  $k = 2$  (see, for example, [7, Section 4] or [6, Section 3].) However, if the linear map  $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  preserves both the trace and the determinant, then there exists an invertible matrix  $a \in \mathcal{M}_n$  such that either

$$\phi(x) = a x a^{-1} \quad (x \in \mathcal{M}_n), \tag{1.4}$$

or

$$\phi(x) = ax^t a^{-1} \quad (x \in \mathcal{M}_n). \quad (1.5)$$

(See, for example, [8, Theorem 1] or [9, Theorem 3].) Thus, if  $E_k(x) = E_k(\phi(x))$  for each  $k \in \{1, n\}$  and  $x \in \mathcal{M}_n$ , then  $\phi$  is of the form given by (1.4) or (1.5). Also, if the linear map  $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  satisfies  $E_k(x) = E_k(\phi(x))$  for each  $k \in \{2, n\}$  and  $x \in \mathcal{M}_n$ , there exists then an invertible matrix  $a \in \mathcal{M}_n$  such that either

$$\phi(x) = \eta a x a^{-1} \quad (x \in \mathcal{M}_n), \quad (1.6)$$

or

$$\phi(x) = \eta a x^t a^{-1} \quad (x \in \mathcal{M}_n), \quad (1.7)$$

where  $\eta = 1$  if  $n$  is odd, and  $\eta = -1$  or  $\eta = 1$  if  $n$  is even. (See, for example, [9, Theorem 4].)

The aim of this article is to improve the results of Theorem 1.1 in a way that is similar to [4, Theorem 1.1]. Thus, we eliminate the linearity assumption on the map  $\phi$  and we impose a strengthened preservation condition which is suggested by (1.3). Incidentally, the result holds with a preservation condition which is stated for two maps  $\varphi$  and  $\psi$  on  $\mathcal{M}_n$  instead of a single map  $\phi$ .

**Theorem 1.2.** *Let  $2 \leq k \leq n$  and consider two maps  $\varphi, \psi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ , one of them being surjective, such that*

$$E_k(\lambda x + y) = E_k(\lambda \varphi(x) + \psi(y)) \quad (x, y \in \mathcal{M}_n, \lambda \in \mathbf{C}). \quad (1.8)$$

*Then  $\varphi = \psi$  on  $\mathcal{M}_n$ , the common value being a linear map of  $\mathcal{M}_n$  into itself.*

As a corollary, we obtain the following generalization of Theorem 1.1.

**Corollary 1.3.** *Let  $3 \leq k < n$  and consider two maps  $\varphi, \psi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ , one of them being surjective, such that (1.8) holds. There exist then an invertible matrix  $a \in \mathcal{M}_n$  and  $\eta \in \mathbf{C}$  satisfying  $\eta^k = 1$  such that either*

$$\varphi(x) = \psi(x) = \eta a x a^{-1} \quad (x \in \mathcal{M}_n),$$

or

$$\varphi(x) = \psi(x) = \eta a x^t a^{-1} \quad (x \in \mathcal{M}_n).$$

Of course, if we suppose that (1.8) holds for  $k \in \{1, n\}$ , then  $\varphi = \psi$  on  $\mathcal{M}_n$ , the common value being a linear map of the form (1.4) or (1.5), and if we suppose that (1.8) holds for  $k \in \{2, n\}$ , then  $\varphi = \psi$  on  $\mathcal{M}_n$ , the common value being a linear map of the form (1.6) or (1.7).

Since Theorem 1.2 also holds for  $k = n$ , we obtain a different proof for the following slight generalization of [4, Theorem 1.1]. (See also [2, Theorem 1] and [3, Theorem 1].)

**Corollary 1.4.** *Consider two maps  $\varphi, \psi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ , one of them being surjective, such that*

$$\det(\lambda x + y) = \det(\lambda \varphi(x) + \psi(y)) \quad (x, y \in \mathcal{M}_n, \lambda \in \mathbf{C}).$$

*There exist then invertible matrices  $a, b \in \mathcal{M}_n$  satisfying  $\det(ab) = 1$  such that either*

$$\varphi(x) = \psi(x) = a x b \quad (x \in \mathcal{M}_n),$$

or

$$\varphi(x) = \psi(x) = a x^t b \quad (x \in \mathcal{M}_n).$$

## 2 Preliminary lemmas

Let  $2 \leq k \leq n$ . For  $x, y \in \mathcal{M}_n$ , consider the complex polynomial (with respect to  $\lambda$ ) given by

$$\lambda \mapsto E_k(\lambda x + y).$$

This section is devoted to the study of these polynomials. As a general property, let us observe that its degree is always at most  $k$ , the coefficient of  $\lambda^k$  being exactly  $E_k(x)$ . In particular, the degree of the polynomial is also bounded by the rank of the matrix  $x$ . Also, if we fix  $x \in \mathcal{M}_n$ , then the coefficient of  $\lambda^{k-1}$  is linear with respect to  $y \in \mathcal{M}_n$ . Indeed, this comes from (1.1) and the fact that for  $J \subseteq \{1, \dots, n\}$  with  $|J| = k$  we have

$$\Delta_J(\lambda x + y) = \lambda^k \Delta_J(x) + \lambda^{k-1} \text{tr}(\text{adj}(x_J)y_J) + \dots + \Delta_J(y),$$

where  $x_J$  (respectively,  $y_J$ ) is the principal submatrix of  $x$  (respectively,  $y$ ) corresponding to  $J$ , and for  $z \in \mathcal{M}_k$  by  $\text{adj}(z) \in \mathcal{M}_k$  we have denoted the (classical) adjoint of the matrix  $z$ , obtained from its cofactors.

This coefficient will play an important role in our approach to prove Theorem 1.1, and shall be studied thoroughly in this section.

**Lemma 2.1.** *Let  $2 \leq k \leq n$ , and let  $1 \leq i, j \leq n$ , with  $i \neq j$ . There exists then a matrix  $x_0 \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_0 + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $y_{ij}$ .*

*Proof.* Suppose, without loss of generality, that  $i = 1$  and  $j = 2$ . Put then  $J_0 = \{1, \dots, k\}$ , and let

$$x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & & \\ 0 & I_{k-2} & & 0 \\ 0 & 0 & 0 & 0_{n-k} \end{bmatrix} \in \mathcal{M}_n.$$

For  $y \in \mathcal{M}_n$ , let us observe that

$$\begin{aligned} \Delta_{J_0}(\lambda x_0 + y) &= \det \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1k} \\ -\lambda + y_{21} & y_{22} & \dots & y_{2k} \\ \vdots & \vdots & \lambda I_{k-2} + (y_{st})_{3 \leq s,t \leq k} & \\ y_{k1} & y_{k2} & & \end{bmatrix} \\ &= 0 \cdot \lambda^k + y_{12} \lambda^{k-1} + \dots \end{aligned}$$

Also, if  $|J| = k$  and  $J \neq J_0$ , then the degree of  $\lambda \mapsto \Delta_J(\lambda x_0 + y)$  with respect to  $\lambda$  is at most  $k - 2$ , since we have at most  $k - 2$  appearances of  $\lambda$  in the principal submatrix of  $\lambda x_0 + y$  corresponding to  $J$ . We use now (1.1) to finish the proof.  $\square$

The remaining of this section is devoted to prove that the same type of statement as the one of Lemma 2.1 also holds for  $i = j$ . This requires a little extra work.

**Lemma 2.2.** *Let  $2 \leq k \leq n$ , and let  $J_0 \subseteq \{1, \dots, n\}$  with  $|J_0| = k - 1$ . There exists then a matrix  $x_0 \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_0 + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\sum_{j \notin J_0} y_{jj}$ .*

*Proof.* Suppose, without loss of generality, that  $J_0 = \{1, \dots, k - 1\}$ . Let then

$$x_0 = \begin{bmatrix} I_{k-1} & 0 \\ 0 & 0_{n-k+1} \end{bmatrix} \in \mathcal{M}_n.$$

For  $y \in \mathcal{M}_n$ , let us observe that for each  $j \in \{1, \dots, n\} \setminus J_0$  we have that

$$\Delta_{J_0 \cup \{j\}}(\lambda x_0 + y) = 0 \cdot \lambda^k + y_{jj} \lambda^{k-1} + \dots.$$

Also, if  $|J| = k$  and  $|J \cap J_0| \leq k - 2$ , then the degree of  $\lambda \mapsto \Delta_J(\lambda x_0 + y)$  with respect to  $\lambda$  is at most  $k - 2$ , since we have at most  $k - 2$  appearances of  $\lambda$  in the principal submatrix of  $\lambda x_0 + y$  corresponding to  $J$ . We use again (1.1) to finish the proof.  $\square$

**Corollary 2.3.** *Let  $2 \leq k \leq n$ , and let  $1 \leq i, j \leq n$ , with  $i \neq j$ . There exist then matrices  $x_1, x_2 \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_1 + y) - E_k(\lambda x_2 + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $y_{jj} - y_{ii}$ .*

*Proof.* Consider  $J \subseteq \{1, \dots, n\} \setminus \{i, j\}$  such that  $|J| = k - 2$ . We apply then Lemma 2.2 to  $J \cup \{i\}$  to find a matrix  $x_1 \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_1 + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\sum_{t \notin (J \cup \{i\})} y_{tt}$ , and we apply the same lemma to  $J \cup \{j\}$  to find a matrix  $x_2 \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_2 + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\sum_{t \notin (J \cup \{j\})} y_{tt}$ . To finish the proof, observe that  $(\sum_{t \notin (J \cup \{i\})} y_{tt}) - (\sum_{t \notin (J \cup \{j\})} y_{tt}) = y_{jj} - y_{ii}$ .  $\square$

**Lemma 2.4.** *Let  $2 \leq k \leq n$ , and let  $j \in \{1, \dots, n\}$ . There exist then  $q > 0$ ,  $m, p \geq 0$  in  $\mathbf{Z}$  and matrices  $x_0, x_1, \dots, x_{m+p} \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial*

$$\lambda \mapsto \sum_{i=0}^m E_k(\lambda x_i + y) - \left( \sum_{i=m+1}^{m+p} E_k(\lambda x_i + y) \right)$$

*is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $q \cdot y_{jj}$ .*

*Proof.* Consider  $J \subseteq \{1, \dots, n\} \setminus \{j\}$  such that  $|J| = k - 1$ . We apply Lemma 2.2 to  $J$  to find a matrix  $x_0 \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_0 + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\sum_{t \notin J} y_{tt} = y_{jj} + \sum_{t \notin (J \cup \{j\})} y_{tt}$ . For each  $t \notin (J \cup \{j\})$  in  $\{1, \dots, n\}$ , we apply Corollary 2.3 to  $t \neq j$  to find matrices  $x_t^{(1)}, x_t^{(2)} \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial  $\lambda \mapsto E_k(\lambda x_t^{(1)} + y) - E_k(\lambda x_t^{(2)} + y)$  is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $y_{jj} - y_{tt}$ . To finish the proof, observe that

$$\sum_{t \notin J} y_{tt} + \left( \sum_{t \notin (J \cup \{j\})} (y_{jj} - y_{tt}) \right) = q \cdot y_{jj},$$

for some strictly positive integer  $q$ .  $\square$

### 3 Proof of the main result

Let  $2 \leq k \leq n$ . Let us first observe that if  $\varphi, \psi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  satisfy (1.8), dividing by  $\lambda \in \mathbf{C} \setminus \{0\}$  we obtain that  $E_k(x + \mu y) = E_k(\varphi(x) + \mu\psi(y))$  for all  $x, y \in \mathcal{M}_n$  and  $\mu \in \mathbf{C} \setminus \{0\}$ . By continuity, the same holds for  $\mu = 0$ , too. Thus

$$E_k(x + \mu y) = E_k(\varphi(x) + \mu\psi(y)) \quad (x, y \in \mathcal{M}_n, \mu \in \mathbf{C}). \tag{3.1}$$

That is, the same type of equalities as the ones in (1.8) hold, with the role of  $\varphi$  and  $\psi$  interchanged. Thus, without loss of generality, we may suppose for the remaining of the paper that the map  $\varphi$  is surjective. (If not, then  $\psi$  must be surjective, and we work with (3.1) instead of (1.8).)

Another immediate observation is the fact that if  $\varphi$  and  $\psi$  satisfy (1.8), for  $\lambda = 0$  in (1.8) and  $\mu = 0$  in (3.1) we see that  $E_k(y) = E_k(\psi(y))$  for all  $y \in \mathcal{M}_n$ , respectively  $E_k(x) = E_k(\varphi(x))$  for all  $x \in \mathcal{M}_n$ .

As a corollary of Lemma 2.1 and Lemma 2.4, the following result holds.

**Theorem 3.1.** *Suppose  $2 \leq k \leq n$ , and let  $i, j \in \{1, \dots, n\}$ . There exist then a nonzero scalar  $\alpha$ , positive integers  $m$  and  $p$  and matrices  $x_0, x_1, \dots, x_{m+p} \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial*

$$\lambda \mapsto \sum_{s=0}^m E_k(\lambda x_s + y) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right)$$

is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\alpha \cdot y_{ij}$ .

As a direct corollary of Theorem 3.1, we obtain the following test for the equality to  $0 \in \mathcal{M}_n$  in terms of the functions  $E_k$ . (See also [7, Lemma 3.1].)

**Corollary 3.2.** *Suppose  $2 \leq k \leq n$ . Let  $y \in \mathcal{M}_n$  such that*

$$E_k(x + y) = E_k(x) \quad (x \in \mathcal{M}_n). \tag{3.2}$$

Then  $y = 0$ .

*Proof.* Observe that (3.2) gives

$$E_k(\lambda x + y) = \lambda^k E_k(x) \quad (x \in \mathcal{M}_n, \lambda \in \mathbf{C}). \tag{3.3}$$

Let  $i, j \in \{1, \dots, n\}$ . By Theorem 3.1, there exist  $\alpha \neq 0$ , positive integers  $m$  and  $p$  and matrices  $x_0, x_1, \dots, x_{m+p} \in \mathcal{M}_n$  such that, for all  $\lambda \in \mathbf{C}$ ,

$$\sum_{s=0}^m E_k(\lambda x_s + y) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right) = 0 \cdot \lambda^k + (\alpha y_{ij}) \cdot \lambda^{k-1} + \dots$$

Using (3.3), for all  $\lambda \in \mathbf{C}$  we have that

$$\begin{aligned} \sum_{s=0}^m E_k(\lambda x_s + y) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right) &= \lambda^k \left( \sum_{s=0}^m E_k(x_s) - \left( \sum_{s=m+1}^{m+p} E_k(x_s) \right) \right) \\ &= 0. \end{aligned}$$

Thus  $\alpha y_{ij} = 0$ , and therefore  $y_{ij} = 0$ . Since this holds for any  $i$  and  $j$ , we obtain that  $y = 0 \in \mathcal{M}_n$ . □

Theorem 3.1 gives us also linearity for the maps  $\varphi$  and  $\psi$  from the statement of Theorem 1.1.

*Proof of Theorem 1.1.* Let us prove first that (1.8) and the surjectivity of  $\varphi$  implies that  $\psi$  is linear on  $\mathcal{M}_n$ . To see this, consider  $i, j \in \{1, \dots, n\}$  and let us prove that  $\psi_{ij} : \mathcal{M}_n \rightarrow \mathbf{C}$  is linear, where  $\psi_{ij}$  is the  $(i, j)$  entry of the map  $\psi$ . By Theorem 3.1, there exist  $\alpha \neq 0$  in  $\mathbf{C}$ , natural numbers  $m$  and  $p$  and matrices  $x_0, x_1, \dots, x_{m+p} \in \mathcal{M}_n$  such that, for each  $y \in \mathcal{M}_n$  we have that the coefficient of  $\lambda^k$  for the polynomial

$$\lambda \mapsto \sum_{s=0}^m E_k(\lambda x_s + y) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right)$$

is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\alpha y_{ij}$ . Since  $\varphi$  is supposed surjective, let  $w_0, w_1, \dots, w_{m+p} \in \mathcal{M}_n$  such that  $\varphi(w_j) = x_j$  for  $j = 0, \dots, m+p$ . Then for each  $z \in \mathcal{M}_n$ , we have that the coefficient of  $\lambda^k$  for the polynomial

$$\lambda \mapsto \sum_{s=0}^m E_k(\lambda \varphi(w_s) + \psi(z)) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda \varphi(w_s) + \psi(z)) \right)$$

is zero, and the coefficient of  $\lambda^{k-1}$  for the same polynomial is  $\alpha \psi_{ij}(z)$ . Using (1.8), for all  $\lambda \in \mathbf{C}$  we have that

$$\sum_{s=0}^m E_k(\lambda \varphi(w_s) + \psi(z)) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda \varphi(w_s) + \psi(z)) \right)$$

equals

$$\sum_{s=0}^m E_k(\lambda w_s + z) - \left( \sum_{s=m+1}^{m+p} E_k(\lambda w_s + z) \right).$$

The remark at the beginning of Section 2 shows that the coefficient of  $\lambda^{k-1}$  for the polynomial  $\lambda \mapsto \sum_{s=0}^m E_k(\lambda w_s + z) - (\sum_{s=m+1}^{m+p} E_k(\lambda w_s + z))$  is linear with respect to  $z \in \mathcal{M}_n$ . Therefore,  $\psi_{ij}$  is linear with respect to  $z \in \mathcal{M}_n$ .

Thus  $\psi : \mathcal{M}_n \rightarrow \mathcal{M}_n$  is linear and  $E_k(x) = E_k(\psi(x))$  for each  $x \in \mathcal{M}_n$ . If  $\psi(y) = 0$ , then for each  $x \in \mathcal{M}_n$  we have that


$$\begin{aligned} E_k(x) &= E_k(\psi(x)) = E_k(\psi(x) + \psi(y)) = E_k(\psi(x+y)) \\ &= E_k(x+y). \end{aligned}$$

Then Corollary 3.2 gives  $y = 0$ . Thus the linear map  $\psi$  is injective on  $\mathcal{M}_n$ , and therefore bijective. Using (1.8), the linearity of  $\psi$  and the fact that  $E_k(z) = E_k(\psi^{-1}(z))$  for each  $z$ , then for each  $x, y \in \mathcal{M}_n$  we have that

$$\begin{aligned} E_k(x+y) &= E_k(\varphi(x) + \psi(y)) = E_k(\psi^{-1}(\varphi(x) + \psi(y))) \\ &= E_k((\psi^{-1} \circ \varphi)(x) + y). \end{aligned}$$

Denoting  $z = x + y$ , we conclude that  $E_k(z) = E_k(((\psi^{-1} \circ \varphi)(x) - x) + z)$  for each  $x, z \in \mathcal{M}_n$ . Then Corollary 3.2 gives  $(\psi^{-1} \circ \varphi)(x) - x = 0$ , equality which holds for every  $x \in \mathcal{M}_n$ . Thus  $\varphi = \psi$  on  $\mathcal{M}_n$ .  $\square$

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