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Optimal Unbiased Estimates of $P\{X < Y\}$ for Some Families of Distributions

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Abstract

In reliability theory, one of the main problems is estimating parameter $R = P\{X < Y\}$. In this paper we shall present UMVUEs for R in different cases i.e. for different distributions of X and Y. Some of them are already existing and some are original.

1 Introduction

In reliability theory the main parameter is the reliability of a system, and its estimation is one of the main goals. The system fails if the applied stress X is greater than strength Y, so R is a measure of system performance. In most cases this parameter is given as $R = P\{X < Y\}$, although for some discrete cases the expression $R = P\{X \leq Y\}$ is also considered.

The problem was first introduced by Birnbaum (1956). Since then numerous papers have been published. Most of results are presented in (Kotz et al., 2003). The vast majority of papers presuppose independence of stress and strength variables, as well as that they come from the same family of, in most cases continuous, distributions. There exists a wide range of applications in engineering, military, medicine and psychology.

The unbiasedness of an estimator is a desired property especially when dealing with relatively small sample sizes, where we cannot count on asymptotic unbiasedness. Since in many cases most popular estimators are biased, it is often important to find the unique minimum variance unbiased estimator (UMVUE).

1.1 UMVUE of R

Let $\mathbf{X} = (X_1, \ldots, X_{n_1})$ and $\mathbf{Y} = (Y_1, \ldots, Y_{n_2})$ be the samples from the distributions of random variables X and Y. Then, using the following theorem we can construct UMVUEs.

Theorem 1 *If $V(\mathbf{X}, \mathbf{Y})$ is any unbiased estimator of parameter θ and T is a complete sufficient statistic for θ, then $E(V(\mathbf{X}, \mathbf{Y})|T)$ is the UMVUE of θ.*

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This theorem is the combination of Rao-Blackwell and Lehmann-Scheffé theorems. Their proofs could be found in (Hogg et al., 2005).

However, in continuous case, the use of this theorem might not be technically convenient. Therefore, the following theorems were proposed to help deriving the UMVUEs (Kotz et al., 2003).

Theorem 2 *Let $\theta_0 \in \Theta$ be an arbitrary value of θ and let T be a complete sufficient statistic for θ. Denote by $g_{\theta_0}(t)$ and $g_{\theta_0}(t|X_1 = x_1, ..., X_k = x_k, Y_1 = y_1, ..., Y_k = y_k)$ the pdf of T and the conditional pdf of T for given $X_j = x_j, Y_j = y_j,\ j = 1, ..., k$, respectively. Then the $UMVUE$ of joint pdf $f_\theta(x_1, ..., x_k, y_1, ..., y_k)$ is of the form*

$$\widehat{f}(x_1, ..., x_k, y_1, ..., y_k) = \\ \prod_{j=1}^{k} f_{\theta_0}(x_j, y_j) \frac{g_{\theta_0}(t|X_1 = x_1, ..., X_k = x_k, Y_1 = y_1, ..., Y_k = y_k)}{g_{\theta_0}(t)}.$$

Theorem 3 *The UMVUE of R is*

$$\widehat{R} = \int\int I(x < y)\widehat{f}(x, y)dxdy,$$

where $\widehat{f}$ is given in theorem 2 for $k = 1$.

2 Existing results

In this section we present a brief summary of existing results obtained for UMVUEs of R for some distributions.

- **Exponential distribution**

  Let X and Y be independent exponentially distributed random variables with densities

$$f_X(x; \alpha) = \alpha e^{-\alpha x},\ x \geq 0,$$

$$f_Y(y; \beta) = \beta e^{-\beta y},\ y \geq 0,$$

  where α and β are unknown positive parameters. The complete sufficient statistics for α and β are $T_X = \sum_{j=1}^{n_1} X_j$ and $T_Y = \sum_{j=1}^{n_2} Y_j$.

  The UMVUE of R was derived by Tong (1974; 1977), and it is given by

$$\widehat{R} = \begin{cases} \sum_{i=0}^{n_1-2} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-i-1)\Gamma(n_2+i+1)} \left(\frac{T_Y}{T_X}\right)^{i+1}, & \text{if } T_Y \leq T_X \\ \sum_{i=0}^{n_2-1} (-1)^i \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1+i)\Gamma(n_2-i)} \left(\frac{T_X}{T_Y}\right)^{i}, & \text{if } T_Y > T_X. \end{cases}$$

- **Normal distribution**

Let X and Y be normally distributed independent random variables with densities

$$f_X(x;\mu_1,\sigma_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}},\ x \in \mathbb{R},$$

$$f_Y(y;\mu_2,\sigma_2) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}},\ y \in \mathbb{R},$$

where μ_1, σ_1^2, μ_2 and σ_2^2 are unknown parameters. The complete sufficient statistics for (μ_1, σ_1^2, μ_2, σ_2^2) are $(\bar{X}, S_X^2, \bar{Y}, S_Y^2)$.

The UMVUE of R was derived by Downtown (1973) and it is given by

$$\widehat{R} = \left[B\left(\frac{1}{2}, \frac{n_1-2}{2}\right) B\left(\frac{1}{2}, \frac{n_2-2}{2}\right)\right]^{-1} \int_{\Omega} (1-u^2)^{\frac{n_1-4}{2}} (1-v^2)^{\frac{n_2-4}{2}} du dv,$$

where $B(a,b)$ is the beta function and

$$\Omega = \left\{ (u,v) \in [-1,1] \times [-1,1] | -u\frac{S_X(n_1-1)}{\sqrt{n_1}} + v\frac{S_Y(n_2-1)}{\sqrt{n_2}} + (\bar{Y}-\bar{X}) > 0 \right\}.$$

- **Gamma distribution**

Let X and Y be independent gamma distributed random variables with densities

$$f_X(x;\alpha_1,\sigma_1) = \frac{x^{\alpha_1-1}}{\Gamma(\alpha_1)\sigma_1^{\alpha_1}} e^{-\frac{x}{\sigma_1}},\ x \geq 0,$$

$$f_Y(y;\alpha_2,\sigma_2) = \frac{y^{\alpha_2-1}}{\Gamma(\alpha_2)\sigma_2^{\alpha_2}} e^{-\frac{y}{\sigma_2}},\ y \geq 0,$$

where α_1 and α_2 are known integer values and σ_1 and σ_2 are unknown positive parameters. The complete sufficient statistics for σ_1 and σ_2 are $T_X = \sum_{j=1}^{n_1} X_j$ and $T_Y = \sum_{j=1}^{n_2} Y_j$.

The UMVUE of R was derived by Constantine et al. (1986), and it is given by

$$\widehat{R} = \begin{cases} 1 - \sum_{k=0}^{(n_2-1)\alpha_2-1} \frac{B(\alpha_1+\alpha_2+k,(n_1-1)\alpha_1)}{B(\alpha_1,(n_1-1)\alpha_1)B(\alpha_2,(n_2-1)\alpha_2)} \\ \qquad \times \binom{(n_2-1)\alpha_2-1}{k} \frac{(-1)^k}{\alpha_2+k} \left(\frac{T_X}{T_Y}\right)^{\alpha_2+k}, & \text{if } T_Y \leq T_X \\ \sum_{k=0}^{(n_1-1)\alpha_1-1} \frac{B(\alpha_2+\alpha_1+k,(n_2-1)\alpha_2)}{B(\alpha_2,(n_2-1)\alpha_2)B(\alpha_1,(n_1-1)\alpha_1)} \\ \qquad \times \binom{(n_1-1)\alpha_1-1}{k} \frac{(-1)^k}{\alpha_1+k} \left(\frac{T_Y}{T_X}\right)^{\alpha_1+k}, & \text{if } T_Y > T_X. \end{cases}$$

- **Gompertz distribution**

Let X and Y be independent Gompertz distributed random variables with densities

$$f_X(x;c,\lambda_1) = \lambda_1 e^{cx} e^{\frac{-\lambda_1(e^{cx}-1)}{c}}, x > 0,$$
$$f_Y(y;c,\lambda_2) = \lambda_2 e^{cy} e^{\frac{-\lambda_2(e^{cy}-1)}{c}}, y > 0,$$

where c is a known positive value and λ_1 and λ_2 are unknown positive parameters. The complete sufficient statisitcs for λ_1 and λ_2 are

$$W_X = \frac{1}{c}\sum_{j=1}^{n_1}(e^{cX_j}-1), W_Y = \frac{1}{c}\sum_{j=1}^{n_2}(e^{cY_j}-1).$$

The UMVUE of R was derived by Saracoglu et al. (2009) and it is given by

$$\widehat{R} = \begin{cases} 1 - \sum\limits_{k=0}^{n_2-1}(-1)^k \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1+k)\Gamma(n_2-k)}\left(\frac{W_X}{W_Y}\right)^k, & \text{if } W_X < W_Y \\ \sum\limits_{k=0}^{n_1-1}(-1)^k \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-k)\Gamma(n_2+k)}\left(\frac{W_Y}{W_X}\right)^k, & \text{if } W_X \geq W_Y. \end{cases}$$

- **Generalized Pareto distribution**

Let X and Y be independent random variables from generalized Pareto distribution with densities

$$f_X(x;\alpha_1,\lambda) = \alpha_1\lambda(1+\lambda x)^{-(\alpha_1+1)},\ x > 0,$$
$$f_Y(y;\alpha_2,\lambda) = \alpha_2\lambda(1+\lambda y)^{-(\alpha_2+1)},\ y > 0,$$

where λ is known positive value and α_1 and α_2 are unknown positive parameters. The complete sufficient statistics for parameters α_1 and α_2 are $T_X = \sum\limits_{j=1}^{n_1}\ln(1+X_j)$ and $T_Y = \sum\limits_{j=1}^{n_2}\ln(1+Y_j)$.

The UMVUE of R was derived by Rezaei et al. (2010), and it is given by

$$\widehat{R} = \begin{cases} 1 - \sum\limits_{k=0}^{n_2-1}(-1)^k \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1+k)\Gamma(n_2-k)}\left(\frac{T_X}{T_Y}\right)^k, & \text{if } T_X < T_Y \\ \sum\limits_{k=0}^{n_1-1}(-1)^k \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1-k)\Gamma(n_2+k)}\left(\frac{T_Y}{T_X}\right)^k, & \text{if } T_X \geq T_Y. \end{cases}$$

- **Poisson distribution**

Let X and Y be independent Poisson distributed random variables with mass functions

$$P\{X = x;\lambda_1\} = \frac{e^{-\lambda_1}\lambda_1^x}{x!}, x = 0,1,\ldots,$$
$$P\{Y = y;\lambda_2\} = \frac{e^{-\lambda_2}\lambda_2^y}{y!}, y = 0,1,\ldots,$$

where λ_1 and λ_2 are unknown positive parameters. The complete sufficient statistics for λ_1 and λ_2 are $T_X = \sum\limits_{j=1}^{n_1} X_j$ and $T_Y = \sum\limits_{j=1}^{n_2} Y_j$.

The UMVUE of R was derived by Belyaev and Lumelskii (1988) and it is given by

$$\widehat{R} = \sum_{x=0}^{\min\{T_X, T_Y-1\}} \binom{T_X}{x} \frac{(n_1-1)^{T_X-x}}{n_1^{T_X}} \left(1 - \sum_{y=0}^{x} \binom{T_Y}{y} \frac{(n_2-1)^{T_Y-y}}{n_2^{T_Y}}\right).$$

- **Negative binomial distribution**

  Let X and Y be indepent random variables from negative binomial distributions with mass functions

$$P\{X = x; m_1, p_1\} = \binom{m_1 + x - 1}{x} p_1^x (1-p_1)^{m_1}, \ \ x = 0, 1, \dots,$$

$$P\{Y = y; m_2, p_2\} = \binom{m_2 + y - 1}{y} p_2^y (1-p_2)^{m_2}, \ \ y = 0, 1, \dots,$$

  where m_1 and m_2 are known integer values and p_1 and p_2 are unknown probabilities. The complete sufficient statistics for p_1 and p_2 are $T_X = \sum_{j=1}^{n_1} X_j$ and $T_Y = \sum_{j=1}^{n_2} Y_j$.

  The UMVUE of R was derived by Ivshin and Lumelskii (1995) and it is given by

$$\widehat{R} = \sum_{x=0}^{\min\{T_X, T_Y-1\}} \sum_{y=x+1}^{T_Y} \frac{\binom{m_1+x-1}{x}\binom{T_X-x+m_1(n_1-1)-1}{T_x-x}}{\binom{m_1n_1+T_X-1}{T_X}} \frac{\binom{m_2+y-1}{y}\binom{T_Y-y+m_2(n_2-1)-1}{T_y-y}}{\binom{m_2n_2+T_Y-1}{T_Y}}.$$

3 New results

In this section we shall derive the UMVUE of R for some new distributions. The first model is where stress and strength both have Weibull distribution with known but different shape parameter and unknown rate parameters. As a special case we present the model where stress has exponential and strength has Rayleigh distribution. An example with real data for Weibull model is presented. In the second model, both stress and strength have logarithmic distribution with unknown parameters.

3.1 Weibull model

Let X and Y be independent random variables from Weibull distribution with densities

$$f_X(x; \alpha_1, \sigma_1) = \alpha_1 \sigma_1^{\alpha_1} x^{\alpha_1 - 1} e^{-(\sigma_1 x)^{\alpha_1}}, \ x \geq 0,$$
$$f_Y(y; \alpha_2, \sigma_2) = \alpha_2 \sigma_2^{\alpha_2} y^{\alpha_2 - 1} e^{-(\sigma_2 y)^{\alpha_2}}, \ y \geq 0.$$

The Weibull distribution is one of the most used distribution in modeling life data. Many researchers have studied the reliability of Weibull model. Most of them did not consider unbiased estimators (e.g. Kundu and Gupta, 2006), and recently the case with common known shape parameter α has been studied in (Amiri et al., 2013).

We consider the case where shape parameters α_1 and α_2 are known positive values, while rate parameters σ_1 and σ_2 are unknown positive parameters.

The complete sufficient statistics for parameters σ_1 and σ_2 are $T_X = \sum_{j=1}^{n_1} X_j^{\alpha_1}$ and $T_Y = \sum_{j=1}^{n_2} Y_j^{\alpha_2}$. Since X^{α_1} and Y^{α_2} have exponential distributions with rate parameters $\sigma_1^{\alpha_1}$ and $\sigma_2^{\alpha_2}$, both statistics have gamma distribution, i.e. T_X has $\Gamma(n_1, \sigma_1^{-\alpha_1})$ and T_Y has $\Gamma(n_2, \sigma_2^{-\alpha_2})$. Similarly, for $k \leq \min(n_1, n_2)$, $T_X - \sum_{j=1}^{k} X_j^{\alpha_1}$ has $\Gamma(n_1 - k, \sigma_1^{-\alpha_1})$ and $T_Y - \sum_{j=1}^{k} Y_j^{\alpha_2}$ has $\Gamma(n_2 - k, \sigma_2^{-\alpha_2})$. Using this and transformation of random variables $(X_1, \dots, X_k, \sum_{j=k+1}^{n_1} X_j^{\alpha_1})$ to $(X_1, \dots, X_k, T_X)$ we get, for $\sigma_1 = 1$,

$$g(t_X|X_1 = x_1, ..., X_k = x_k) = \frac{(t_X - \sum_{j=1}^{k} x_j^{\alpha_1})^{n_1-k-1}}{\Gamma(n_1 - k)} e^{-(t_X - \sum_{j=1}^{k} x_j^{\alpha_1})} I\{t_X \geq \sum_{j=1}^{k} x_j^{\alpha_1}\}.$$

Using theorem 2, we get that

$$\widehat{f}(x_1, ..., x_k) = \alpha_1^k \prod_{j=1}^{k} x_j^{\alpha_1 - 1} \frac{(t_X - \sum_{j=1}^{k} x_j^{\alpha_1})^{n_1-k-1}\Gamma(n_1)}{t_X^{n_1-1}\Gamma(n_1 - k)} I\{t_X \geq \sum_{j=1}^{k} x_j^{\alpha_1}\}.$$

For $k = 1$ we obtain that

$$\widehat{f}(x) = \alpha_1(n_1 - 1)x^{\alpha_1 - 1} \frac{(t_X - x^{\alpha_1})^{n_1-2}}{(t_X)^{n_1-1}} I\{t_X \geq x^{\alpha_1}\}.$$

Analogously we get that

$$\widehat{f}(y) = \alpha_2(n_2 - 1)y^{\alpha_2 - 1} \frac{(t_Y - y^{\alpha_2})^{n_2-2}}{(t_Y)^{n_2-1}} I\{t_Y \geq y^{\alpha_2}\}.$$

Denote $M = \min\{t_X^{\frac{1}{\alpha_1}}, t_Y^{\frac{1}{\alpha_2}}\}$. Using the independence of samples and the theorem 3, we obtain

$$\begin{aligned}
\widehat{R} &= \int_0^\infty \int_0^\infty I\{x < y\}\widehat{f}(x)\widehat{f}(y)dxdy \\
&= \int_0^M \frac{\alpha_1(n_1 - 1)(n_2 - 1)x^{\alpha_1-1}(t_X - x^{\alpha_1})^{n_1-2}}{t_X^{n_1-1}t_Y^{n_2-1}} dx \int_x^{t_Y^{\frac{1}{\alpha_2}}} \alpha_2 y^{\alpha_2-1}(t_Y - y^{\alpha_2})^{n_2-2}dy \\
&= \int_0^M \frac{\alpha_1(n_1 - 1)x^{\alpha_1-1}}{t_X^{n_1-1}t_Y^{n_2-1}}(t_X - x^{\alpha_1})^{n_1-2}(t_Y - x^{\alpha_2})^{n_2-1}dx.
\end{aligned}$$

Now applying the binomial formula we obtain that the UMVUE of R is

$$\widehat{R} = \begin{cases} \sum_{r=0}^{n_1-2}\sum_{s=0}^{n_2-1} \frac{(-1)^{r+s}\alpha_1(n_1-1)}{\alpha_1(r+1)+\alpha_2 s}\binom{n_1-2}{r}\binom{n_2-1}{s}\frac{T_X^{\frac{\alpha_2 s}{\alpha_1}}}{T_Y^s}, & \text{if } T_X^{\frac{1}{\alpha_1}} \le T_Y^{\frac{1}{\alpha_2}} \\ \sum_{r=0}^{n_1-2}\sum_{s=0}^{n_2-1} \frac{(-1)^{r+s}\alpha_1(n_1-1)}{\alpha_1(r+1)+\alpha_2 s}\binom{n_1-2}{r}\binom{n_2-1}{s}\frac{T_Y^{\frac{\alpha_1(r+1)}{\alpha_2}}}{T_X^{r+1}}, & \text{if } T_X^{\frac{1}{\alpha_1}} > T_Y^{\frac{1}{\alpha_2}}. \end{cases} \tag{3.1}$$

3.1.1 Exponential-Rayleigh model

As a special case of Weibull model we have a model where X has exponential and Y has Rayleigh distribution with densities

$$f_X(x;\alpha) = \alpha e^{-\alpha x},\ x \ge 0,$$

$$f_Y(y;\beta) = 2\beta^2 y e^{-\beta^2 y^2},\ y \ge 0,$$

where α and β are unknown positive parameters. The complete sufficient statistics for α and β are $T_X = \sum_{j=1}^{n_1} X_j$ and $T_Y = \sum_{j=1}^{n_2} Y_j^2$.

The UMVUE of R is

$$\widehat{R} = \begin{cases} \sum_{r=0}^{n_1-2}\sum_{s=0}^{n_2-1} \frac{(-1)^{r+s}(n_1-1)}{(r+1)+2s}\binom{n_1-2}{r}\binom{n_2-1}{s}\left(\frac{T_X^2}{T_Y}\right)^s, & \text{if } T_X \le \sqrt{T_Y} \\ \sum_{r=0}^{n_1-2}\sum_{s=0}^{n_2-1} \frac{(-1)^{r+s}(n_1-1)}{(r+1)+2s}\binom{n_1-2}{r}\binom{n_2-1}{s}\left(\frac{\sqrt{T_Y}}{T_X}\right)^{r+1}, & \text{if } T_X > \sqrt{T_Y}. \end{cases}$$

3.1.2 Numerical example

Here we present an example with real data. We wanted to compare daily wind speeds in Rotterdam and Eindhoven. We obtained two samples of 30 randomly chosen daily wind speeds (in 0.1 m/s) from the period of April 1st 2010 to April 1st 2014 taken from the website of Royal Netherlands Meteorological Institute. The first sample is from Rotterdam and the second one is from Eindhoven:

Rotterdam (X): 48, 15, 27, 18, 40, 26, 84, 19, 35, 32, 55, 29, 45, 51, 47, 66, 38, 13, 39, 28, 50, 36, 15, 74, 53, 85, 18, 58, 18, 48.

Eindhoven (Y): 44, 25, 43, 35, 20, 59, 25, 38, 26, 15, 37, 16, 35, 17, 34, 27, 40, 37, 33, 17, 51, 50, 33, 52, 25, 21, 34, 39, 23, 60.

It is well known that wind speed follows Weibull distribution. To check this we used Kolmogorov-Smirnov test. Since this test requires that the parameters may not be estimated from the testing sample, we estimated them beforehand using some other larger samples from the same populations. We got that X follows Weibull distribution with shape parameter $\alpha = 2.8$ and rate parameter $\sigma = 1/47$ (Kolmogorov-Smirnov test statistics is 0.157 and the p-value is greater than 0.1), while Y follows Weibull distribution with shape parameter $\alpha = 2.6$ and rate parameter $\sigma = 1/41$ (Kolmogorov-Smirnov test statistics is 0.158 and the p-value is greater than 0.1).

Finally, using (3.1) we estimated the probability that the daily wind speed is lower in Rotterdam than in Eindhoven to be $\widehat{r} = 0.32$.

3.2 Logarithmic distribution

Let X and Y be independent random variables from logarithmic distribution with mass functions

$$P\{X=x;p\}=\frac{-1}{\ln(1-p)}\frac{p^x}{x},\quad x=1,2,...$$

$$P\{Y=y;q\}=\frac{-1}{\ln(1-q)}\frac{q^y}{y},\quad y=1,2,...,$$

where p and q are unknown probabilities.

The logarithmic distribution has application in biology and ecology. It is often used for modeling data linked to the number of species.

The complete sufficient statistics for p and q are $T_X=\sum_{j=1}^{n_1}X_j$ and $T_Y=\sum_{j=1}^{n_2}Y_j$. The sum of n independent random variables with logarithmic distributions with the same parameter p has Stirling distribution of the first kind $SDFK(n,p)$ (Johnson et al., 2005), so T_X has $SDFK(n_1,p)$ and T_Y has $SDFK(n_2,q)$ with the following mass functions

$$P\{T_X=x;n_1,p\}=\frac{n_1!|s(x,n_1)|p^x}{x!(-\ln(1-p))^{n_1}},\quad x=n_1,n_1+1,...,$$

$$P\{T_Y=y;n_2,q\}=\frac{n_2!|s(y,n_2)|q^y}{y!(-\ln(1-q))^{n_2}},\quad y=n_2,n_2+1,...,$$

where $s(x,n)$ is Stirling number of the first kind.

An unbiased estimator for R is $I\{X_1<Y_1\}$. Since

$$E(I\{X_1<Y_1\}|T_X=t_X,T_Y=t_Y)=\frac{P\{X_1<Y_1,T_X=t_X,T_Y=t_Y\}}{P\{T_X=t_X,T_Y=t_Y\}}$$

$$=\frac{\sum_{x=1}^{M}\sum_{y=x+1}^{t_Y-n_2+1}P\{X_1=x\}P\{Y_1=y\}P\{\sum_{k=2}^{n_1}X_k=t_X-x\}P\{\sum_{l=2}^{n_2}Y_l=t_Y-y\}}{P\{T_X=t_X\}P\{T_Y=t_Y\}}$$

$$=\sum_{x=1}^{M}\sum_{y=x+1}^{t_Y-n_2+1}\frac{t_X!t_Y!|s(t_X-x,n_1-1)||s(t_Y-y,n_2-1)|}{n_1n_2(t_X-x)!(t_Y-y)!xy|s(t_X,n_1)||s(t_Y,n_2)|},$$

where $M=\min\{t_X-n_1+1,t_Y-n_2\}$, using theorem 1 we get that the UMVUE of R is

$$\widehat{R}=\sum_{x=1}^{\min\{T_X-n_1+1,T_Y-n_2\}}\sum_{y=x+1}^{T_Y-n_2+1}\frac{T_X!T_Y!|s(T_X-x,n_1-1)||s(T_Y-y,n_2-1)|}{n_1n_2(T_X-x)!(T_Y-y)!xy|s(T_X,n_1)||s(T_Y,n_2)|}.$$

4 Conclusion

In this paper we considered the unbiased estimation of the probability $P\{X<Y\}$ when X and Y are two independent random variables. Some known results of UMVUEs for R for some distributions were listed. Two new cases were presented, namely Weibull model with known but different shape parameters and unknown rate parameters and Logarithmic model with unknown parameters. An example using real data was provided.

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