

# A-trails of embedded graphs and twisted duals\*

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## Abstract

Kotzig showed that every connected 4-regular plane graph has an *A*-trail—an Eulerian circuit that turns either left or right at each vertex. However, this statement is not true for Eulerian plane graphs and determining if an Eulerian plane graph has an *A*-trail is NP-hard. The aim of this paper is to give a characterization of Eulerian embedded graphs having an *A*-trail. Andersen et al. showed the existence of orthogonal pairs of *A*-trails in checkerboard colourable 4-regular graphs embedded on the plane, torus and projective plane. A problem posed in their paper is to characterize Eulerian embedded graphs (not necessarily checkerboard colourable) which contain two orthogonal *A*-trails. In this article, we solve this problem in terms of twisted duals. Several related results are also obtained.

*Keywords:* Embedded graphs, twisted duals, Eulerian, *A*-trails, checkerboard colourable.

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## 1 Introduction

A *cellularly embedded graph* is a graph  $G$  embedded in a surface  $\Sigma$  such that every connected component of  $\Sigma - G$  is a 2-cell, called a *face*. We use the term *embedded graph* loosely to mean any of three equivalent representations of graphs in surfaces: cellularly embedded graphs, ribbon graphs and arrow presentations. We shall move from one to another

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freely and refer the reader to [5, 6, 10] for details. A *quasi-tree* is an embedded graph with exactly one boundary component (or face). A *bouquet* is an embedded graph with exactly one vertex. They are geometric duals to each other. A *quasi-tree bouquet* is a bouquet that is also a quasi-tree.

In this article, all graphs will be finite, connected, but not necessarily simple. A graph is said to be *Eulerian* if the degree of each of its vertices is even. A graph is *bipartite* if its vertex set can be partitioned into two nonempty subsets  $X$  and  $Y$  so that every edge has one end in  $X$  and one end in  $Y$ . We denote a bipartite graph  $G$  with bipartition  $(X, Y)$  by  $G[X, Y]$ . Note that bipartite graphs might have multiple edges but not loops. A *star* is a bipartite graph  $G[X, Y]$  with  $|X| = 1$  or  $|Y| = 1$ . A *parallel graph*, also known as a generalized theta graph especially for 3 or more edges, is a special star  $G[X, Y]$  with  $|X| = 1$  and  $|Y| = 1$ . A *walk* in a graph between vertices  $v_0$  and  $v_k$  is a sequence of vertices and edges  $(v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ , where  $v_{i-1}$  and  $v_i$  are the endvertices of the edge  $e_i$ . A *trail* is a walk with no repeated edge and *circuit* is a trail with  $v_0 = v_k$ . An *A-trail* [7] in an Eulerian embedded graph  $G$  is an eulerian circuit such that every two consecutive edges in the circuit are adjacent in the rotation of the common vertex. A *Petrie walk* in an embedded graph  $G$  is such a walk that when traveling along it, we alternately turn to the left edge and to the right edge of the current edge in the cyclic rotation around the common vertex. It is obvious that an Eulerian Petrie walk is a special kind of  $A$ -trails.

Kotzig [7] showed that every 4-regular plane graph has an  $A$ -trail, and sufficient conditions were discovered for the existence of  $A$ -trails in 2-connected plane graphs in [2]. The existence of  $A$ -trails has been studied almost exclusively in the case of graphs embedded in the plane, the projective plane and the torus. In this paper, we shall characterize general Eulerian embedded graphs having an  $A$ -trail in terms of twisted duals. Let  $G = (V(G), E(G))$  be an embedded graph. We denote by  $G^*$  and  $G^\times$  the *geometric dual* and *Petrieal* [14] of  $G$ , respectively. Let  $A \subseteq E(G)$ . We denote by  $G^{\delta(A)}$  and  $G^{\tau(A)}$  the partial dual [4] and partial Petrieal of  $G$  with respect to  $A$ , respectively. Particularly, if  $A = E(G)$ , then  $G^{\delta(E(G))} = G^*$  and  $G^{\tau(E(G))} = G^\times$ . Partial duality and partial Petriality are further combined together to form twisted duality [1, 5]. The twisted duality has the scope to develop the understanding of a wide variety of graph theoretical problems. We refer the reader to [5, 6] for the details. Petrie walks have some very interesting properties, see [8, 12]. They also play an important role in the design of CMOS VLSI circuits, where it is convenient if the graph representing a circuit has an Eulerian Petrie walk. Žitnik [13] gave a characterization of 4-regular plane graphs with Eulerian Petrie walks. Since Eulerian Petrie walks are a special kind of  $A$ -trails, we also give a characterization of Eulerian embedded graphs having an Eulerian Petrie walk. We first obtain the following theorem.

**Theorem 1.1.** *Let  $G$  be an Eulerian embedded graph. Then  $G$  has an  $A$ -trail if and only if there exists  $B \subseteq E(G)$  such that the underlying graph of  $(G^{\tau(B)})^*$  is a star. In particular,  $G$  has an Eulerian Petrie walk if and only if the underlying graph of  $(G^\times)^*$  is a star.*

Andersen, Bouchet and Jackson [3] characterized the 4-regular plane graphs which contain two orthogonal  $A$ -trails, that is to say two  $A$ -trails for which no subtrail of length 2 appears in both  $A$ -trails. They also discussed the corresponding problem for checkerboard colourable graphs embedded in the projective plane and the torus. And they posed the following problem.

**Problem 1.2** ([3]). We do not have any characterization of 4-regular graphs in the projective plane and the torus having two orthogonal  $A$ -trails which we know to be valid also for

graphs with no 2-face colouring.

In this paper, we give a general definition of two orthogonal  $A$ -trails in Eulerian embedded graphs. We say that two  $A$ -trails are *orthogonal* if these two trails have different transitions at each vertex with degree greater than 2. We consider the above problem for general Eulerian embedded graphs which may have high genus and are not necessarily checkerboard colourable and characterize the Eulerian embedded graphs which have two orthogonal  $A$ -trails or Eulerian Petrie walks in terms of twisted duals as follows.

**Theorem 1.3.** *Let  $G$  be an embedded graph. Then  $G$  has two orthogonal  $A$ -trails if and only if there exists  $B \subseteq E(G)$  such that the underlying graph of  $(G^{\tau(B)})^*$  is a parallel graph. In particular,  $G$  has two orthogonal Eulerian Petrie walks if and only if the underlying graph of  $(G^\times)^*$  is a parallel graph.*

In the case of 4-regular embedded graphs, we have the following theorem.

**Theorem 1.4.** *Let  $H$  be a 4-regular embedded graph. Then  $H$  has two orthogonal  $A$ -trails if and only if there exists  $D \subseteq E(H)$  such that  $H^{\tau(D)}$  is a medial graph of a quasi-tree bouquet. Particularly,  $H$  has two orthogonal Eulerian Petrie walks if and only if  $H^\times$  is a medial graph of a quasi-tree bouquet.*

## 2 Preliminaries

In this section we introduce the notions and the tools, which we will need in further Sections 3 and 4. We use standard notations  $V(G)$ ,  $E(G)$  and  $F(G)$  to denote the sets of vertices, edges, and faces, respectively, of a cellularly embedded graph  $G$  and  $v(G) = |V(G)|$ ,  $e(G) = |E(G)|$  and  $f(G) = |F(G)|$ , respectively. We denote by  $d(v)$  the degree of a vertex  $v$  in  $G$ , i.e. the number of half-edges incident with  $v$ . We give a brief review of ribbon graphs referring the reader to [5, 6] for further details.

**Definition 2.1** ([6]). A *ribbon graph*  $G = (V(G), E(G))$  is a (possibly non-orientable) surface with boundary, represented as the union of two sets of topological discs, a set  $V(G)$  of vertices, and a set  $E(G)$  of edges such that

1. the vertices and edges intersect in disjoint line segments, we call them *common line segments* as in [9];
2. each such common line segment lies on the boundary of precisely one vertex and precisely one edge;
3. every edge contains exactly two such common line segments.

Let  $G$  be a ribbon graph,  $v \in V(G)$  and  $e \in E(G)$ . By deleting the common line segments from the boundary of  $v$ , we obtain  $d(v)$  disjoint line segments, called *vertex line segments*. By deleting common line segments from the boundary of  $e$ , we obtain two disjoint line segments, called *edge line segments*. See Figure 1 for an example of these concepts. We think of each edge line segment having two *half-edge line segments*. It is obvious that every edge disc contains four half-edge line segments which correspond to the four flags incident on that edge. For any vertex line segment, there are exactly two half-edge line segments incident with it as shown in Figure 2.

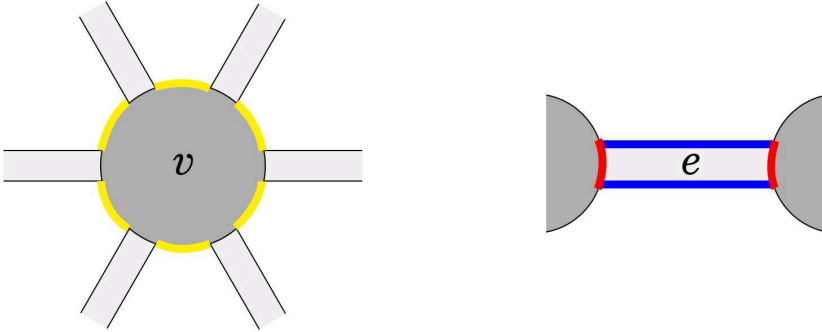


Figure 1: Vertex line segments (yellow), common line segments (red) and edge line segments (blue).

Let  $G$  be a ribbon graph and  $A \subseteq E(G)$ . Then the *partial Petrial*,  $G^{\tau(A)}$ , of  $G$  with respect to  $A$  is the ribbon graph obtained from  $G$  by adding a half-twist to each of the edges in  $A$ . Let  $Orb_{(\tau)}(G) = \{G^{\tau(A)} \mid A \subseteq E(G)\}$  denote the set of all partial Petrials of  $G$ . Let  $H$  be an arrow presentation and  $B \subseteq E(H)$ . Then the *partial dual*,  $H^{\delta(B)}$ , of  $H$  with respect to  $B$  is the arrow presentation obtained as follows. For each  $e \in B$ , suppose  $\alpha$  and  $\beta$  are the two arrows labelled  $e$  in the arrow presentation of  $H$ . Draw a line segment with an arrow on it directed from the tail of  $\beta$  to the head of  $\alpha$ , and a line segment with an arrow on it directed from the head of  $\beta$  to the tail of  $\alpha$ . Label both of these arrows  $e$  and delete  $\alpha$  and  $\beta$  and the arcs containing them. This process is illustrated locally at a pair of arrows in Figure 3. Let  $Orb_{(\delta)}(H) = \{H^{\delta(B)} \mid B \subseteq E(H)\}$  denote the set of all partial duals of  $H$ .

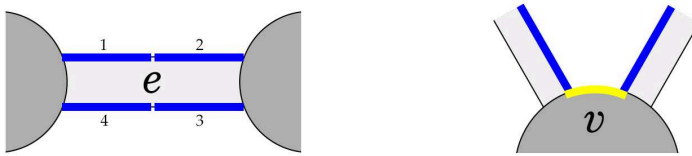


Figure 2: Four half-edge line segments of  $e$ , one of the vertex line segments of  $v$  and its incident two half-edge line segments.

$$\text{Let } \mathcal{B} = \langle \delta, \tau \mid \delta^2, \tau^2, (\delta\tau)^3 \rangle.$$

**Definition 2.2** ([5]). Let  $G$  be a ribbon graph. The ribbon graph  $H$  is called a *twisted dual* (briefly, *twdual*) of  $G$  if it can be written in the form

$$H = G^{\prod_{i=1}^6 \xi_i(A_i)},$$

where the  $A_i$ 's partition  $E(G)$  and the  $\xi_i$ 's are the six elements of  $\mathcal{B}$ .

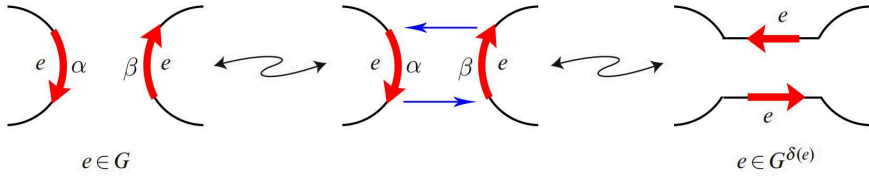


Figure 3: Taking the partial dual of an edge in an arrow presentation.

If  $G$  is cellularly embedded in  $\Sigma$ , we construct its *medial graph*  $G_m$  in the embedded surface by placing a vertex on each of its edges, and for each face  $f$  with boundary  $e_1, e_2, \dots, e_{d(f)}$ , drawing  $d(f)$  edges  $\{e_1, e_2\}, \dots, \{e_{d(f)}, e_1\}$  inside the face  $f$  along the boundary of  $f$ . It is obvious that  $G_m$  is also cellularly embedded in  $\Sigma$ . In particular, the medial graph of an isolated vertex is a free loop.  $G$  is *checkerboard colourable* if the faces of  $G$  can be properly 2-coloured. A *checkerboard colouring* of  $G$  is a particular proper 2-colouring of the faces of  $G$ . Throughout we will use red and blue to refer to the two colours used in a checkerboard colouring. It is obvious that there is a correspondence between checkerboard colouring and *face boundary colouring*, so we shall move from one to another freely. Note that in a checkerboard coloured 4-regular embedded graph  $G$ , the *redface graph*  $G_R$  of  $G$  is the embedded graph constructed by placing one vertex in each red face and adding an edge between two of these vertices whenever the corresponding faces meet at a vertex of  $G$ . The *blueface graph*  $G_B$  is constructed analogously by placing vertices in the blue faces.

A *Petrie walk* in an embedded graph  $G$  is such a walk that when traveling along it, we alternately turn to the left edge and to the right edge of the current edge in the cyclic rotation around the common vertex. We shall only consider closed Petrie walks, for which this condition holds also for the last and the first edge of the walk. Petrie walks are sometimes also called *left-right paths*. An example of a Petrie walk is shown in Figure 4, where the dotted curve indicates the order of edges in the walk. Note that the boundary components of faces of the Petrie dual are exactly Petrie walks of the original embedding of  $G$  as shown in Figure 5.

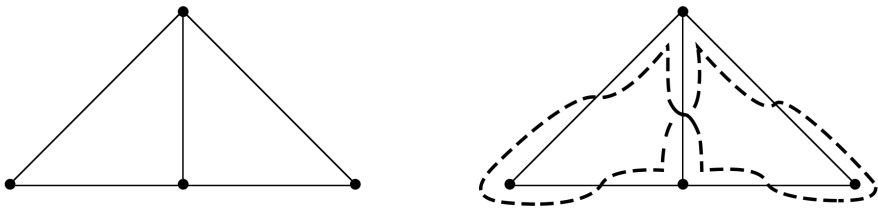


Figure 4: The curve representing a Petrie walk.

Let  $v$  be a vertex of  $G$ . A *transition* at  $v$  is a partition of the half-edges incident to  $v$  into

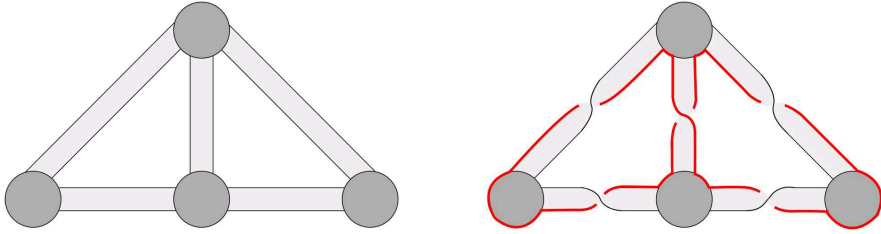


Figure 5: A Petrie walk corresponding to a face of the Petrie dual of the original embedded graph.

pairs. A transition  $T$  at  $v$  is *smooth* if  $T$  only pairs half-edges adjacent in the cyclic order at  $v$  given by the embedding of  $G$ . A *transition system* of  $G$  is a choice of a transition at every vertex of  $G$ . A *smooth transition system* is a transition system such that every transition is smooth. It is obvious that we can induce a circuit decomposition of  $G$  by a transition system  $T$ . Similarly, any circuit decomposition  $C$  of  $G$  recovers a transition system of  $G$  by pairing half-edges traversed consecutively in a circuit of  $C$ . A circuit decomposition of  $G$  is smooth if it induces a smooth transition system. In particular, an Eulerian circuit that induces a smooth transition system is called an *A-trail* [11]. Note that there are precisely two disjoint smooth transitions for any vertex  $v$  of  $G$  with  $d(v) \geq 4$  and two edges are consecutive if this is indicated by a curve as shown in Figure 6. We say that two *A-trails*

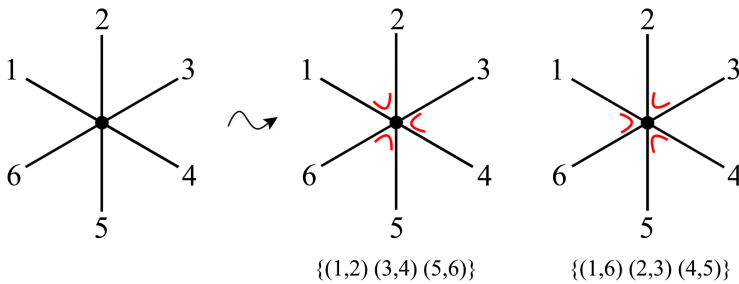


Figure 6: Performing exactly two smooth transitions locally at a vertex.

(or smooth transition systems) of an Eulerian embedded graph  $G$  are *orthogonal* if the two trails (or smooth transition systems) have different smooth transitions at each vertex  $v$  of  $G$  with  $d(v) \geq 4$ .

Most of the representations of embedded graphs are ribbon graphs in this paper, so we introduce the relation between smooth transition systems and ribbon graphs. Let  $G$  be an Eulerian ribbon graph. For every  $v \in V(G)$ , we assign the colours red and blue to all half-edge line segments and vertex line segments of  $v$  such that the colours of vertex line segments are alternating red and blue in the vertex boundary of  $v$  and every vertex line segment and two half-edge line segments incident with it assign the same colour. We call this a *checkerboard colouring of half-edge and vertex line segments* of  $G$  (see Figure 7 for

example). If  $G$  is checkerboard coloured of half-edge and vertex line segments, then for any edge disc, there are exactly two cases as shown in Figure 8. The edge is called *consistent* if two half-edge line segments of one of edge line segments have the same colour, and is called *inconsistent* otherwise. Let  $T$  be a smooth transition system of  $G$ . It is obvious that  $T$  induces a checkerboard colouring of half-edge and vertex line segments for any vertex of  $G$  as shown in Figure 9. We call this a *canonical checkerboard colouring of half-edge and vertex line segments* by  $T$ . An example of a canonical checkerboard colouring of half-edge and vertex line segments by the smooth transition system is given in Figure 10.

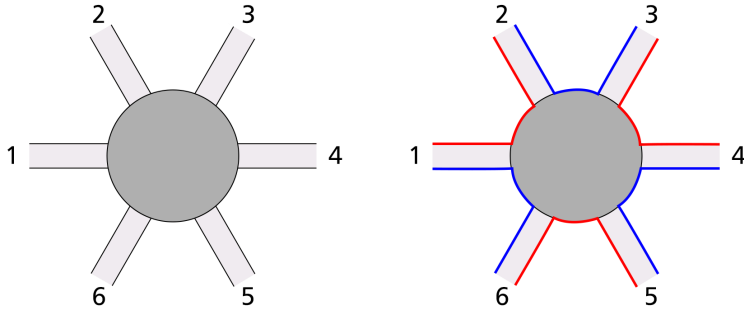


Figure 7: A checkerboard colouring of half-edge and vertex line segments at a vertex.

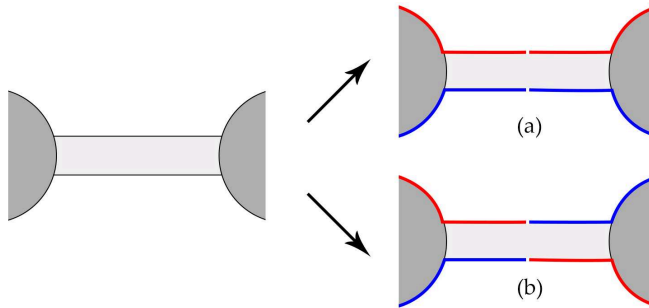


Figure 8: (a) consistent edge (b) inconsistent edge.

### 3 Main results and proofs

Now we give some characterizations of Eulerian embedded graphs having an  $A$ -trail or Eulerian Petrie walk. Kotzig [7] showed that every 4-regular plane graph has an  $A$ -trail. We note that this result can be extended to any 4-regular embedded graph.

**Lemma 3.1.** *Any 4-regular embedded graph always has an  $A$ -trail.*

*Proof.* Let  $T$  be any smooth transition system of a 4-regular embedded graph  $H$ . Note that each smooth transition system corresponds to a specific family of edge-disjoint cycles

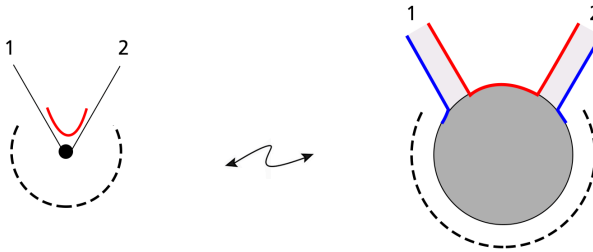


Figure 9: The relation between smooth transition and checkerboard colouring of half-edge and vertex line segments at a vertex.

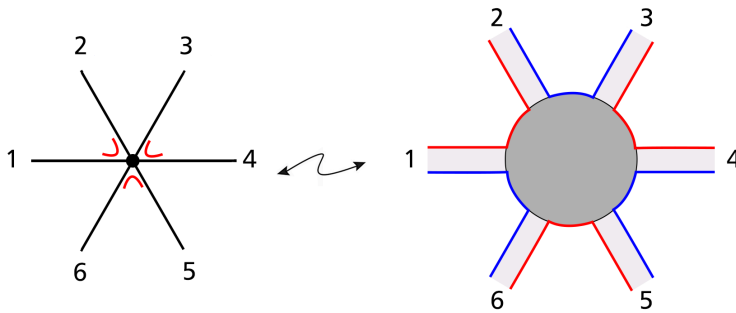


Figure 10: An example of a canonical checkerboard colouring of half-edge and vertex line segments by the smooth transition system.



in  $H$ . The number of edge-disjoint cycles in  $H$  generated by  $T$  will be denoted by  $c(T)$ . If  $c(T) = 1$ , then this completes the proof. Otherwise, there exists  $v \in V(H)$  such that four half-edges 1, 2, 3, 4 incident with  $v$  are not in the same cycle. Assume that the transition at  $v$  is  $(1, 2), (3, 4)$ . Then we change the smooth transition of  $v$  from  $(1, 2), (3, 4)$  to  $(1, 4), (2, 3)$ . Hence, half-edges 1, 2, 3, 4 are in the same edge-disjoint cycle as shown in Figure 11. Repeating the above process, we obtain a smooth transition system which corresponds to an  $A$ -trail.  $\square$

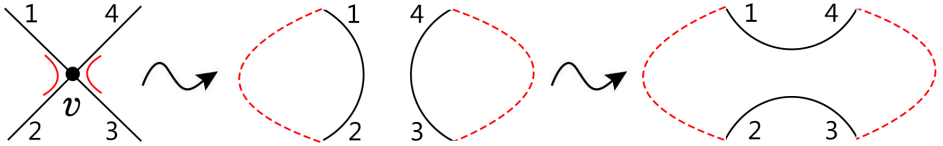


Figure 11: Proof of Lemma 3.1.

Lemma 3.1 can not be further generalized to any Eulerian embedded graph and an example is given in Figure 12. Note that taking partial petrial does not affect cyclic order

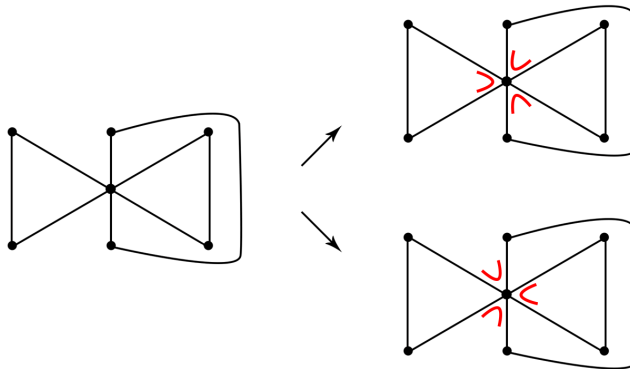


Figure 12: An Eulerian embedded graph which does not have an  $A$ -trail.

of half-edges at vertices, we have the following lemma.

**Lemma 3.2.** *Let  $G$  be an Eulerian embedded graph. Then*

- (1)  $G$  has an  $A$ -trail if and only if every partial Petrial of  $G$  has an  $A$ -trail.
- (2)  $G$  has two orthogonal  $A$ -trails if and only if every partial Petrial of  $G$  has two orthogonal  $A$ -trails.

**Theorem 1.1.** *Let  $G$  be an Eulerian embedded graph. Then  $G$  has an  $A$ -trail if and only if there exists  $B \subseteq E(G)$  such that the underlying graph of  $(G^\tau(B))^*$  is a star. In particular,  $G$  has an Eulerian Petrie walk if and only if the underlying graph of  $(G^\times)^*$  is a star.*

*Proof.* ( $\Rightarrow$ ) Let  $G$  be a ribbon graph. If  $G$  has an  $A$ -trail, then there exists a smooth transition system  $T$  of  $G$  corresponding to the  $A$ -trail. We get a canonical checkerboard colouring of half-edge and vertex line segments according to  $T$ . Suppose that  $B$  is the set of its inconsistent edges. If the  $A$ -trail is an Eulerian Petrie walk, then  $B = E(G)$ . Thus  $G^{\tau(B)}$  obtains a face boundary colouring. It induces a checkerboard colouring with only one red face. Then  $(G^{\tau(B)})^*$  is bipartite with  $|X| = 1$  or  $|Y| = 1$ , hence the underlying graph of  $(G^{\tau(B)})^*$  is a star.

( $\Leftarrow$ ) Since the underlying graph of  $(G^{\tau(B)})^*[X, Y]$  is a star, we assume that  $X = \{v\}$ . Note that  $v$  corresponds to a face of  $G^{\tau(B)}$ . We assign the colour red to this face and colour blue to other faces of  $G^{\tau(B)}$ . This is a checkerboard colouring of  $G^{\tau(B)}$ . Suppose that it is a canonical checkerboard colouring of half-edge and vertex line segments of  $G^{\tau(B)}$ . Let  $T$  be the corresponding smooth transition system. It follows that  $T$  induces an  $A$ -trail of  $G^{\tau(B)}$ . Thus,  $G$  has an  $A$ -trail by Lemma 3.2. If  $B = E(G)$ , then this is a checkerboard colouring of  $G^\times$ . Hence, the boundary of the redface of  $G^\times$  is an Eulerian Petrie walk of  $G$ . □

**Remark 3.3.** According to Theorem 1.1, suppose that  $G$  is an embedded graph whose underlying graph is a star. If  $H \in Orb_{(\tau)}(G^*)$ , then  $H$  has an  $A$ -trail. Particularly, if  $H = (G^*)^\times$ , then  $H$  has an Eulerian Petrie walk.

**Corollary 3.4.** *Let  $H$  be a 4-regular embedded graph. Then  $H$  has an Eulerian Petrie walk if and only if  $H^\times$  is a medial graph of a bouquet.*

*Proof.* By a similar argument as in the proof of Theorem 1.1,  $H^\times$  can obtain a checkerboard colouring with only one red face. Hence, the number of vertices of the redface graph  $(H^\times)_R$  is exactly one, that is a bouquet. Therefore,  $H^\times$  is a medial graph of a bouquet. Conversely, let  $G$  be a bouquet and  $H^\times$  is the medial graph of  $G$ . We give  $G_m$  a checkerboard colouring where the red faces contain the vertices of  $G$ . Then the number of red faces is exactly one. Hence,  $(G_m)^*$  is a star. Thus,  $(G_m)^\times$  has an Eulerian Petrie walk by Remark 3.3. Since  $G_m = H^\times$ , we can see that  $H$  has an Eulerian Petrie walk. □

**Theorem 1.3.** *Let  $G$  be an embedded graph. Then  $G$  has two orthogonal  $A$ -trails if and only if there exists  $B \subseteq E(G)$  such that the underlying graph of  $(G^{\tau(B)})^*$  is a parallel graph. In particular,  $G$  has two orthogonal Eulerian Petrie walks if and only if the underlying graph of  $(G^\times)^*$  is a parallel graph.*

*Proof.* ( $\Rightarrow$ ) Assume that  $T$  and  $T'$  are two orthogonal smooth transition systems recovering from the two orthogonal  $A$ -trails of a ribbon graph  $G$ . Then we get a canonical checkerboard colouring of half-edge and vertex line segments according to  $T$ . Suppose that  $B$  is the set of its inconsistent edges. If the  $A$ -trail is an Eulerian Petrie walk, then  $B = E(G)$ . By the same argument as in the proof of Theorem 1.1, there is a face boundary colouring of  $G^{\tau(B)}$  such that the number of colour red face boundaries is exactly one. Note that  $T'$  corresponds to the blue face boundaries of  $G^{\tau(B)}$ . It follows that the number of colour blue face boundary is also exactly one. Hence, the vertex set of  $(G^{\tau(B)})^*$  can be partitioned into two subsets  $X$  and  $Y$  with  $|X| = |Y| = 1$ . Thus, the underlying graph of  $(G^{\tau(B)})^*$  is a parallel graph.

( $\Leftarrow$ ) Since the underlying graph of  $(G^{\tau(B)})^*$  is a parallel graph with vertices  $v$  and  $w$ . Note that the number of face boundaries of  $G^{\tau(B)}$  are two, which correspond to  $v$  and  $w$ , respectively. We give one colour red and another colour blue. Hence, this is a checkerboard

colouring of  $G^{\tau(B)}$ . Let  $T$  and  $T'$  be the corresponding two orthogonal smooth transition systems to the red face boundary and the blue face boundary, respectively. It is obvious that  $T$  and  $T'$  induce two orthogonal  $A$ -trails of  $G^{\tau(B)}$ . Thus,  $G$  has two orthogonal  $A$ -trails by Lemma 3.2. If  $B = E(G)$ , then this is a checkerboard colouring of  $G^\times$ . Note that the redface and blueface of  $G^\times$  are both Eulerian Petrie walks of  $G$ . Hence,  $G$  has two orthogonal Eulerian Petrie walks.  $\square$

**Remark 3.5.** According to Theorem 1.3, suppose that  $G$  is an embedded graph whose underlying graph is a parallel graph. If  $H \in \text{Orb}_{(\tau)}(G^*)$ , then  $H$  has two orthogonal  $A$ -trails. In particular, if  $H = (G^*)^\times$ , then  $H$  has two orthogonal Eulerian Petrie walks.

**Theorem 1.4.** *Let  $H$  be a 4-regular embedded graph. Then  $H$  has two orthogonal  $A$ -trails if and only if there exists  $D \subseteq E(H)$  such that  $H^{\tau(D)}$  is a medial graph of a quasi-tree bouquet. Particularly,  $H$  has two orthogonal Eulerian Petrie walks if and only if  $H^\times$  is a medial graph of a quasi-tree bouquet.*

*Proof.* ( $\Rightarrow$ ) By the same argument as in the proof of Theorem 1.3, there exists  $D \subseteq E(H)$  such that there is a checkerboard colouring of  $H^{\tau(D)}$  which the number of colour red face and blue face are both exactly one, that is, the number of vertices of  $(H^{\tau(D)})_R$  and  $(H^{\tau(D)})_B$  are both exactly one. Note the number of face of  $(H^{\tau(D)})_R$  is also one, since  $(H^{\tau(D)})_R$  and  $(H^{\tau(D)})_B$  are geometric duals. Therefore,  $(H^{\tau(D)})_R$  is a quasi-tree bouquet. Hence,  $H^{\tau(D)}$  is a medial graph of a quasi-tree bouquet.

( $\Leftarrow$ ) Let  $G$  be a quasi-tree bouquet and  $H^{\tau(D)}$  be the medial graph of  $G$ . Since  $G$  and  $G_m$  embedded in the same surface, we have  $v(G) - e(G) + f(G) = v(G_m) - e(G_m) + f(G_m)$  by Euler characteristic. Note that  $v(G_m) = e(G)$ ,  $e(G_m) = 2v(G_m) = 2e(G)$ . Hence,  $f(G_m) = v(G) + f(G) = 2$ . Thus,  $(G_m)^*$  is a parallel graph since  $G_m$  is checkerboard colourable and  $f(G_m) = 2$ . Then  $G_m$  has two orthogonal  $A$ -trails by Theorem 1.3, that is,  $H^{\tau(D)}$  has two orthogonal  $A$ -trails. Hence,  $H$  has two orthogonal  $A$ -trails by Lemma 3.2. If  $D = E(H)$ , then  $(H^\times)^*$  is a parallel graph. It follows that  $H$  has two orthogonal Eulerian Petrie walks by Remark 3.5.  $\square$

**Remark 3.6.** According to Theorem 1.4, suppose that  $G$  is a quasi-tree bouquet. If  $H \in \text{Orb}_{(\tau)}(G_m)$ , then  $H$  has two orthogonal  $A$ -trails. In particular, if  $H = (G_m)^\times$ , then  $H$  has two orthogonal Eulerian Petrie walks.

**Lemma 3.7.** *Let  $G$  be an embedded graph. Then  $E(G)$  can be partitioned into two edge disjoint spanning quasi-trees if and only if  $G$  is the partial dual of a quasi-tree bouquet.*

*Proof.* Suppose  $A$  and  $A^c$  are the edge sets of two spanning quasi-trees which partition  $E(G)$ , then  $G^{\delta(A)}$  is a bouquet since the vertex boundaries of  $G^{\delta(A)}$  correspond to the face boundaries of  $(V(G), A)$  which is a spanning quasi-tree. Thus,  $G^{\delta(A^c)}$  is also a bouquet by the similar discussion. Note that  $G^{\delta(A)} = (G^{\delta(A^c)})^*$ . Hence,  $G^{\delta(A)}$  is a quasi-tree bouquet, that is,  $G$  is the partial dual of a quasi-tree bouquet. Conversely, there exists  $A \subseteq E(G)$  such that  $G^{\delta(A)}$  is a quasi-tree bouquet. Then  $G^{\delta(A^c)}$  is also a bouquet. It follows that  $(V(G), A)$  and  $(V(G), A^c)$  are both spanning quasi-trees, that is,  $E(G)$  can be partitioned into two edge disjoint spanning quasi-trees.  $\square$

**Lemma 3.8** ([5]). *Let  $G$  be an embedded graph. Then*

$$\text{Orb}_{(\delta)}(G) = \{H \mid G_m \text{ and } H_m \text{ are partial Petrials}\}.$$

**Corollary 3.9.** *Let  $H$  be a checkerboard coloured 4-regular embedded graph. Then  $H$  has two orthogonal  $A$ -trails if and only if the edges of the redface graph  $H_R$  can be partitioned into two edge disjoint spanning quasi-trees.*

*Proof.*  $E(H_R)$  can be partitioned into two edge disjoint spanning quasi-trees if and only if  $H_R$  is the partial dual of a quasi-tree bouquet by Lemma 3.7, that is, there exists  $D \subseteq E(H)$  such that  $H^{\tau(D)}$  is a medial graph of a quasi-tree bouquet by Lemma 3.8, if and only if  $H$  has two orthogonal  $A$ -trails by Theorem 1.4. □

**Corollary 3.10.** *Let  $H$  be a checkerboard coloured 4-regular orientable embedded graph which has two orthogonal  $A$ -trails. Then  $v(H)$  is even.*

*Proof.*  $E(H_R)$  can be partitioned into two edge disjoint spanning quasi-trees by Corollary 3.9. Denote these two spanning quasi-trees by  $G_1 = (V(H_R), E_1), G_2 = (V(H_R), E_2)$ , respectively. Obviously,  $G_1$  and  $G_2$  are both orientable. Then  $v(H_R) - |E_1| + 1 = 2 - 2g(G_1)$  and  $v(H_R) - |E_2| + 1 = 2 - 2g(G_2)$ , where  $g(G_1)$  and  $g(G_2)$  are the genera of  $G_1$  and  $G_2$ , respectively. Hence,  $|E_1| + |E_2| = 2v(H_R) + 2g(G_1) + 2g(G_2) - 2$ . It follows that  $e(H_R)$  is even, that is,  $v(H)$  is even. □

**Remark 3.11.** Andersen, Bouchet and Jackson [3] obtained the same results as Corollaries 3.9 and 3.10 for graphs embedded in the plane, the projective plane and the torus.

### 4 Quasi-tree bouquets

Theorem 1.4 and Lemma 3.7 show that quasi-tree bouquets are an important class of ribbon graphs. In this section, we give a brief characterization of them.

We start by recalling some necessary statements. A ribbon graph is *non-orientable* if it contains a ribbon subgraph that is homeomorphic to a Möbius band, and is *orientable* otherwise. An edge  $e$  of a ribbon graph is a *loop* if it is incident with exactly one vertex. A loop is non-orientable if together with its incident vertex it forms a Möbius band, and is orientable otherwise. Two loops  $e$  and  $f$  are *interlaced* if they are met in the cyclic order  $efef$  when travelling round the boundary of a vertex. A loop  $e$  at the vertex of a bouquet  $G$  is *trivial* if there is no loop in  $G$  which interlaces with  $e$ . The *signed rotation* of a bouquet is a cyclic ordering of the half-edges at the vertex and if the edge is orientable, then we give the same sign to the corresponding two half-edges, and give the different signs otherwise. See Figure 13 for an example. Suppose  $P = p_1p_2 \cdots p_k$  is a *string*. Then we call  $-P = (-p_k) \cdots (-p_2)(-p_1)$  the *inverse* of  $P$ . *Operations 1* and *2* are the moves on bouquets defined in Figure 14 and Figure 15, respectively. *Operation 3* is deleting a pair of interlaced orientable loops and *operation 4* is deleting a trivial non-orientable loop as shown in Figure 16 and Figure 17, respectively. Note that operations 1, 2, 3 and 4 do not change the number of boundary components.

**Theorem 4.1.** *Let  $G$  be a bouquet. Then  $G$  is a quasi-tree bouquet if and only if there is a sequence of operations 1, 2, 3 and 4 which change the signed rotation of  $G$  to be empty.*

*Proof.* The sufficiency is easily verified. To prove the necessity, the result is easily verified when  $E(G) = \emptyset$ , so assume that this is not the case. Then there are following two cases.

**Case 1:** If there exists a non-orientable loop  $r$ , we assume that the signed rotation of  $G$  is  $IrJ(-r)P$ . By a sequence of operation 2, we have  $I(-J)r(-r)P$ . Hence, we get a signed rotation  $I(-J)P$  by operation 4.

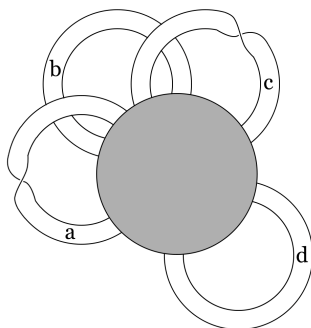


Figure 13: The signed rotation of the bouquet is  $ab(-a)cb(-c)dd$ .

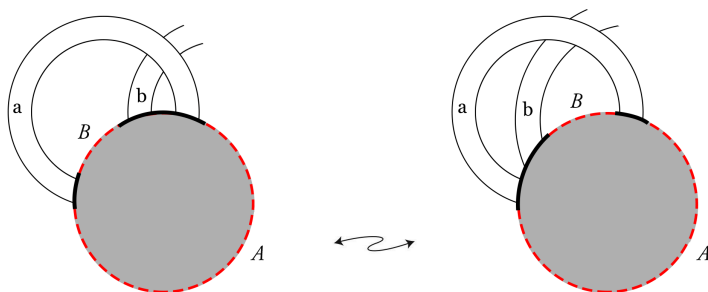


Figure 14: Operation 1. Change the signed rotation from  $AaBba$  to  $AabBa$ .

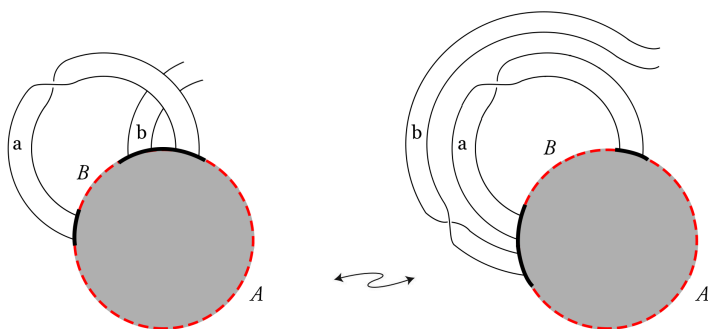


Figure 15: Operation 2. Change the signed rotation from  $AaBb(-a)$  to  $A(-b)aB(-a)$ .

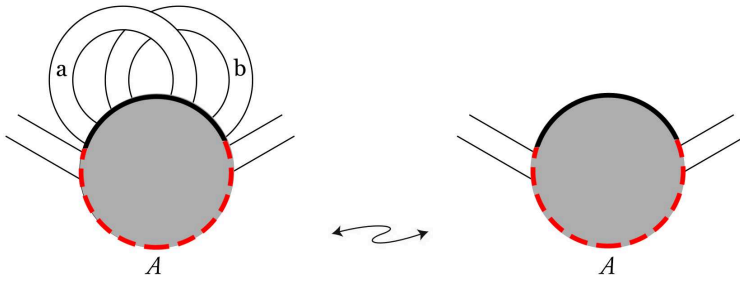


Figure 16: Operation 3. Change the signed rotation from  $Aabab$  to  $A$ .

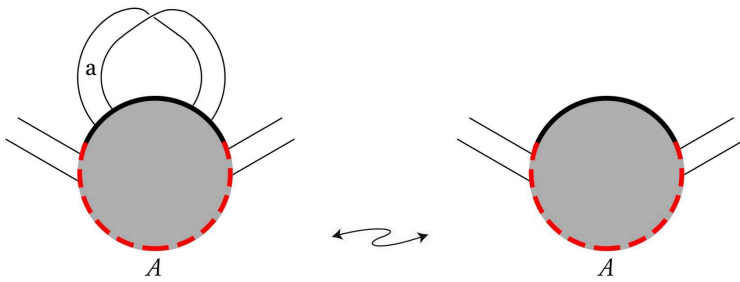


Figure 17: Operation 4. Change the signed rotation from  $Aa(-a)$  to  $A$ .

**Case 2:** Otherwise, there exists a pair of interlaced orientable loops  $e, f$ . Assume that the signed rotation of  $G$  is  $IeJfPeQfM$ . By a sequence of operation 1, we have

$$IeJfPeQfM \Leftrightarrow IefPeQfJM \Leftrightarrow IefeqPfJM \Leftrightarrow IQPefefJM.$$

Then by operation 3, we get a signed rotation  $IQPJM$ .

Note that both Case 1 and Case 2 induce a shorter signed rotation. By repeating the above operations, we can reduce the signed rotation of  $G$  until it is empty.  $\square$

**Remark 4.2.** A bouquet with signed rotation  $AxyB(-x)(-y)C$  is not a quasi-tree bouquet since  $AxyB(-x)(-y)C \Leftrightarrow Axx(-B)y(-y)C$ .

We now present an algorithm to get the number of boundary components of any bouquet  $G$  in terms of signed rotations.

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**Algorithm 1** Calculate the number of boundary components of any bouquet.

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- 1: **Input** The signed rotation  $R$  of a bouquet  $G$ .
  - 2: Let  $a := 1$ .
  - 3: **Step 1.** If  $R = \emptyset$ , then stop and output the number of boundary components of  $G$  is  $a$ .
  - 4: **Step 2.** Otherwise, there are three cases.
  - 5: **if** there exists  $i$  such that  $R = AiiB$  **then**
  - 6:   Let  $R := AB, a := a + 1$ . Return to Step 1.
  - 7: **else if** there exists  $r$  such that  $R = IrJ(-r)P$  **then**
  - 8:   Let  $R := I(-J)P, a := a$ . Return to Step 1.
  - 9: **else**
  - 10:   Find a pair of interlaced loops  $e, f$  such that  $R = IeJfPeQfM$ .  
       Let  $R := IQPJM, a := a$ . Return to Step 1.
- 

**Example 4.3.** A bouquet with signed rotation 13214234 is not a quasi-tree bouquet. Since there is a pair of interlaced loops 1, 3, we get a shorter signed rotation 4224. A bouquet with signed rotation

$$182(-1)34325(-4)756867$$

is a quasi-tree bouquet. Since

$$182(-1)34325(-4)756867 \xrightarrow{1,(-1)} (-2)(-8)34325(-4)756867 \xrightarrow{(-2),2} (-3)(-4)(-3)85(-4)756867 \xrightarrow{(-3),(-4)} 85756867 \xrightarrow{8,7} 6565 \xrightarrow{6,5} \emptyset.$$

**Proposition 4.4.** Let  $G$  be an orientable bouquet. If  $G$  is a quasi-tree bouquet, then  $e(G)$  is even.

*Proof.* It follows immediately from Euler’s formula.  $\square$

**Proposition 4.5.** Let  $G$  be a quasi-tree bouquet. If there exists  $A \subseteq E(G)$  such that  $(V(G), A)$  and  $(V(G), A^c)$  are plane graphs, then  $|A| = |A^c|$ .

*Proof.* Note that the vertex boundaries and face boundaries of  $G^{\delta(A)}$  correspond to the face boundaries of  $(V(G), A)$  and  $(V(G), A^c)$ , respectively. It follows that  $v(G^{\delta(A)}) = |A| + 1$  and  $f(G^{\delta(A)}) = |A^c| + 1$ , that is,  $G^{\delta(A)}$  is a plane graph. Then  $E(G^{\delta(A)})$  can be partitioned into two edge disjoint spanning trees by Lemma 3.7. Hence,

$$e(G^{\delta(A)}) = 2(v(G^{\delta(A)}) - 1) = 2|A| = e(G) = |A| + |A^c|,$$

that is,  $|A| = |A^c|$ . □

**Example 4.6.** A bouquet  $G$  with signed rotation 121324356465 is not a quasi-tree bouquet. This follows from the fact that  $(G, \{1, 3, 5, 6\})$  and  $(G, \{2, 4\})$  are plane graphs, but  $|\{1, 3, 5, 6\}| \neq |\{2, 4\}|$ .

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