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15 Massless and Massive Representations in the *Spinor Technique*

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Abstract. This contribution uses the technique [1] for representing spinors and the definition of the discrete symmetries [3] to illustrate on a toy model [2] properties of massless and massive solutions of spinors. It might help to solve the problem about representations of Dirac, Weyl and Majorana, presented in the ref. [4] in this proceedings.

Povzetek. Prispevek uporablja tehniko [1] predstavitve spinorjev in definicijo diskretnih simetrij [3] za ilustracijo lastnosti brezmasnih in masivnih rešitev spinorjev v preprostem modelu. Prispeva lahko k rešitvi problema upodobitev Diracovih, Weylovih in Majoraninih spinorjev predstavljenega v prispevku [4] v tem zborniku.

15.1 Introduction

We study in a toy model defined in $d = (5 + 1)$, presented in the refs. [2], massless and massive positive and negative energy solutions of the equations of motion, and look for, by taking into account the definition of the discrete symmetry operators in the second quantized picture ($\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$, presented in the paper [3]) the antiparticle states to the particle ones. We present the representations in the *spinor technique* [1].

In this toy model the $\mathcal{M}^{5+1}$ manifold is assumed to break into $\mathcal{M}^{3+1}$ $\times$ an almost S^2 sphere due to the zweibein in $d = (5, 6)$.

We first study massless solutions in $d = (3 + 1)$ assuming that the extra dimensions bring no contribution to the masses in $d = (3 + 1)$. We correspondingly solve the Weyl equation in $d = (5 + 1)$ and present the representations and comment on particle and antiparticle states.

Requiring that there is only one massless of a particular handedness and mass protected solution in $d = (3 + 1)$, what is achieved by a particular choice of the spin connection fields on this (almost) S^2 sphere, what consequently forces the rest of solutions to be massive, we comment on the corresponding particle and antiparticle states. In these two cases the spin in $d = (5, 6)$ is a conserved quantity.

Assuming nonzero vacuum expectation values of the spin connection fields [1–3], the gauge fields of S^{56} with indices $d = (5, 6)$, which manifest as scalars in $d = (3+1)$ and carry the $U(1)$ charge S^{56}, all the spinors become massive and no charge is the conserved quantity any longer. The Weyl equation in $d = (5 + 1)$ manifests

$d = (3+1)$ as the Dirac equation for massive states. We present representations and comment on particle and antiparticle states also in this case.

15.2 Massless solutions

Let us look for the solutions of the Weyl equations $\gamma^a p_a \psi = 0$ in $d = (5+1)$ for a particular choice of the coordinate system: $p^a = (p^0, 0, 0, |p^3|, 0, 0)$. Then the Weyl equations read

$$(-2iS^{03}p^0 = p^3)\psi \,. \tag{15.1}$$

In Table I, taken from the paper [3], the solutions of Eq. 15.1 are presented, using the technique of the refs. [1]. We found for the basic states

$$\begin{aligned}
\Psi_1 &= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\, |vac>_{fam}, \\
\Psi_2 &= \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}\, |vac>_{fam}, \\
\Psi_3 &= \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\, |vac>_{fam}, \\
\Psi_4 &= \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]}\, |vac>_{fam}\,,
\end{aligned} \tag{15.2}$$

where $|vac>_{fam}$ is defined so that there are $2^{\frac{d}{2}-1}$ family members (this is, however, not a second quantized vacuum). All the basic states are eigenstates of the Cartan subalgebra (of the Lorentz transformation Lie algebra), for which we take: S^{03}, S^{12}, S^{56}, with the eigenvalues, which can be read from Eq. (15.2) if taking $\frac{1}{2}$ times the numbers $\pm i$ or ± 1 in the parentheses of nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$: $S^{ab} \stackrel{56}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}$, $S^{ab} \stackrel{56}{[k]} = \frac{k}{2} \stackrel{ab}{[k]}$.

The first two positive energy solutions ($\psi_i^{\mathbf{pos}}, i = (1,2)$) and the last two negative energy solutions ($\psi_i^{\mathbf{neg}}, i = (3,4)$) correspond to $p^3 = |p^3|$. These all are the solutions of the Weyl equations

$$(\Gamma^{(3+1)} \frac{p^0}{|p^0|} = \frac{2\vec{p}\cdot\vec{S}}{|p^0|})\,\psi \,. \tag{15.3}$$

for the choice $\vec{p} = (p^1, p^2, p^3)$ (in our case is $(0, 0, p^3)$), presented in all text books. Here $\vec{S} = (S^{23}, S^{31}, S^{12})$, $S^{ab} = \frac{i}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, and $\Gamma^{((d-1)+1)}$ (in usual notation is for $d = (3+1)$ named γ^5) determines handedness for fermions in any d. For $d = (5+1)$ $\Gamma^{(5+1)} = \Pi_a \gamma^a$ in ascending order, equal also to $\Gamma^{(3+1)}(-2S^{56})$. For

ψ^{pos}_i	positive energy state	$\frac{p^0}{\lvert p^0\rvert}$	$\frac{p^3}{\lvert p^3\rvert}$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	S^{56}	$\frac{2p^3 S^{12}}{\lvert p^0\rvert}$
ψ^{pos}_1	$\overset{03}{(+i)}\,\overset{12}{(+)}\mid\overset{56}{(+)}\;e^{-i\lvert p^0\rvert x^0+i\lvert p^3\rvert x^3}$	$+1$	$+1$	$+1$	$+1$	$\frac{1}{2}$	1
ψ^{pos}_2	$\overset{03}{(+i)}\,\overset{12}{[-]}\mid\overset{56}{[-]}\;e^{-i\lvert p^0\rvert x^0+i\lvert p^3\rvert x^3}$	$+1$	$+1$	$+1$	-1	$-\frac{1}{2}$	-1
ψ^{pos}_3	$\overset{03}{[-i]}\,\overset{12}{[-]}\mid\overset{56}{(+)}\;e^{-i\lvert p^0\rvert x^0-i\lvert p^3\rvert x^3}$	$+1$	-1	-1	$+1$	$\frac{1}{2}$	1
ψ^{pos}_4	$\overset{03}{[-i]}\,\overset{12}{(+)}\mid\overset{56}{[-]}\;e^{-i\lvert p^0\rvert x^0-i\lvert p^3\rvert x^3}$	$+1$	-1	-1	-1	$-\frac{1}{2}$	-1
ψ^{neg}_i	negative energy state	$\frac{p^0}{\lvert p^0\rvert}$	$\frac{p^3}{\lvert p^3\rvert}$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	S^{56}	$\frac{2p^3 S^{12}}{\lvert p^0\rvert}$
ψ^{neg}_1	$\overset{03}{(+i)}\,\overset{12}{(+)}\mid\overset{56}{(+)}\;e^{i\lvert p^0\rvert x^0-i\lvert p^3\rvert x^3}$	-1	-1	$+1$	$+1$	$\frac{1}{2}$	-1
ψ^{neg}_2	$\overset{03}{(+i)}\,\overset{12}{[-]}\mid\overset{56}{[-]}\;e^{i\lvert p^0\rvert x^0-i\lvert p^3\rvert x^3}$	-1	-1	$+1$	-1	$-\frac{1}{2}$	1
ψ^{neg}_3	$\overset{03}{[-i]}\,\overset{12}{[-]}\mid\overset{56}{(+)}\;e^{i\lvert p^0\rvert x^0+i\lvert p^3\rvert x^3}$	-1	$+1$	-1	$+1$	$\frac{1}{2}$	-1
ψ^{neg}_4	$\overset{03}{[-i]}\,\overset{12}{(+)}\mid\overset{56}{[-]}\;e^{i\lvert p^0\rvert x^0+i\lvert p^3\rvert x^3}$	-1	$+1$	-1	-1	$-\frac{1}{2}$	1

Table 15.1. Four positive energy states and four negative energy states, the solutions of Eq. (15.1), half have $\frac{p^3}{|p^3|}$ positive and half negative. $p^a = (p^0, 0, 0, p^3, 0, 0)$, $\Gamma^{(5+1)} = -1$, $\Gamma^{((d-1)+1)}$ defines the handedness in d-dimensional space-time, S^{56} defines the charge in $d = (3+1)$, $\frac{2p^3 S^{12}}{|p^0|}$ defines the helicity. Nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ operate on the vacuum state $|vac>_{fam}$ not written in the table. Table is taken from [3].

the choice $p^a = (p^1, p^2, p^3, 0, 0)$ the solutions read

$$
\begin{aligned}
p^0 &= |p^0|,\\
\psi^{pos}_1(\vec{p}) &= \mathcal{N}_1 \left(\overset{03}{(+i)}\,\overset{12}{(+)}\mid\overset{56}{(+)} + \frac{p^1 + ip^2}{|p^0| + p^3}\,\overset{03}{[-i]}\,\overset{12}{[-]}\mid\overset{56}{(+)}\right) e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})},\\
\psi^{pos}_2(\vec{p}) &= \mathcal{N}_2 \left(\overset{03}{[-i]}\,\overset{12}{(+)}\mid\overset{56}{[-]} - \frac{p^1 + ip^2}{|p^0| - p^3}\,\overset{03}{(+i)}\,\overset{12}{[-]}\mid\overset{56}{[-]}\right) e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})},\\
p^0 &= -|p^0|,\\
\psi^{neg}_1(\vec{p}) &= \mathcal{N}_2 \left(\overset{03}{(+i)}\,\overset{12}{(+)}\mid\overset{56}{(+)} - \frac{p^1 + ip^2}{|p^0| - p^3}\,\overset{03}{[-i]}\,\overset{12}{[-]}\mid\overset{56}{(+)}\right) e^{i(|p^0|x^0 + \vec{p}\cdot\vec{x})},\\
\psi^{neg}_2(\vec{p}) &= \mathcal{N}_1 \left(\overset{03}{[-i]}\,\overset{12}{(+)}\mid\overset{56}{[-]} + \frac{p^1 + ip^2}{|p^0| + p^3}\,\overset{03}{(+i)}\,\overset{12}{[-]}\mid\overset{56}{[-]}\right) e^{i(|p^0|x^0 + \vec{p}\cdot\vec{x})},
\end{aligned}
\tag{15.4}
$$

These solutions are obviously possible only if both kinds of "charges" in $d = (3+1)$ are allowed for particles and antiparticles and are not mass protected [2]. We shall see in sect. 15.4 that the first two states in Table 15.1 (ψ^{pos}_1 and ψ^{pos}_2), put on the top of the Dirac sea, describe a particle state, while the second two (ψ^{pos}_4 and ψ^{pos}_3, respectively) are the corresponding antiparticle states put on the top of the Dirac sea. The antiparticle states are obtained by emptying the negative energy states (ψ^{neg}_3 and ψ^{neg}_4, respectively).

For a particular choice of the spin connection one gets (normalizable) massless solutions of only one handedness and one charge and mass protected [2], say: the right handed ones of the "charge" $S^{56} = \frac{1}{2}$. In Table 15.1 is this state presented by a particle state ψ_1^{pos} put on the top of the Dirac sea. Its antiparticle state is ψ_4^{pos} put on the top of the Dirac sea. It is obtained by emptying the negative energy solution ψ_3^{neg}.

15.3 Massive solutions

To find the massive states we must solve the Weyl equation in which we allow the scalar fields, the gauge fields of S^{56}, that is $f^{\sigma}_{s}\omega_{56\sigma}$, with $s = (5,6)$ and $\sigma = ((5),(6))$, to have non zero vacuum expectation values. These scalar fields are then, analogously as there is the Higgs scalar in the *standard model* but carrying in our case only the "hyper" charge S^{56}, responsible for the spinors masses. The charge, which is the spin in $d = (5,6)$, is now not a conserved quantity any longer.

The Weyl equation in $(5+1)$, leading to the massive Dirac equation in $d = (3+1)$, reads [2]

$$(\gamma^m p_m + \overset{56}{(+)}\, p_{0+} + \overset{56}{(-)}\, p_{0-})\psi = 0\,,$$
$$p_{0\pm} = p^5_0 \mp p^6_0\,,\quad p_{0s} = f^{\sigma}_{s}(p_\sigma - \frac{1}{2}S^{ab}\omega_{ab\sigma}) = p_s - S^{56} i\omega_{56s}\,. \tag{15.5}$$

Looking for the solution of Eq. (15.5) by making superposition of states of a particular spin one observes that the term $< p_{0\pm} >= -S^{56} i < (f^{\sigma}_{5} \mp i f^{\sigma}_{6})\omega_{56\sigma} >$ causes on the tree level the mass m of spinors: $< p_{0\pm} >= -iS^{56} 2m$. Solutions of the Weyl equation in $d = (5+1)$ manifest in $d = (3+1)$ as massive states with the mass $2m =< \omega_{56+} >=< \omega_{56-} >$. Let us make a choice of the coordinate system so that $p^a = (p^0, 0, 0, 0, 0, 0)$. One obtains two positive and two negative energy solutions

$$\begin{aligned}
\psi^{pos}_{1\,m} &= \mathcal{N}(\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} - i\, \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]})\, e^{-imx^0}\,,\\
\psi^{pos}_{2\,m} &= \mathcal{N}(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} - i\, \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]})\, e^{-imx^0}\,,\\
\psi^{neg}_{1\,m} &= \mathcal{N}(\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} + i\, \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]})\, e^{imx^0}\,,\\
\psi^{neg}_{2\,m} &= \mathcal{N}(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} + i\, \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]})\, e^{imx^0}\,,
\end{aligned} \tag{15.6}$$

with $m^2 = (p^0)^2$, $m = 2 < (f^{\sigma}_{5} \mp i f^{\sigma}_{6})\omega_{56\sigma} >$ [1]. (To obtain true masses of spinors one must take into account loop corrections in all orders, to which also the dynamical scalar and vector gauge fields contribute.) In this discussion only one family is assumed.

Let us present massive positive and negative energy solutions, the ones which coincide with vectors $\begin{pmatrix}\varphi \\ \frac{\vec\sigma\cdot\vec p}{|p^0|+m}\varphi\end{pmatrix}$, $\varphi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, and $\begin{pmatrix}\frac{\vec\sigma\cdot\vec p}{|p^0|+m}\chi \\ \chi\end{pmatrix}$, $\chi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, with

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[1] One sees that the other two superposition, $\psi^{pos}_3 = \mathcal{N}(\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} + i\, \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]})\; e^{-imx^0}$ and $\psi^{pos}_4 = \mathcal{N}(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} + i\, \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]})\; e^{-imx^0}$ are not the solutions of the Weyl equation and so are not also $\psi^{neg}_3 = \mathcal{N}(\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} - i\, \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]})\; e^{imx^0}$ and $\psi^{neg}_4 = \mathcal{N}(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} - i\, \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]})\; e^{imx^0}$.

$\vec{\sigma} = (S^{23}, S^{31}, S^{12})$, in the usual notation for any $p^m = (p^0, p^1, p^2, p^3)$

$$
\begin{aligned}
\psi^{pos}_{1\,m}(\vec{p}) &= \mathcal{N}(\vec{p}, m)\{\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} - i\frac{|p^0| - p^3 + m}{|p^0| + p^3 + m}\,\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]} \\
&+ \frac{p^1 + ip^2}{|p^0| + p^3 + m}\,(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} + i\,\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]})\} \cdot e^{-i(|p^0| - i\vec{p}\cdot\vec{x})} \\
\psi^{pos}_{2\,m}(\vec{p}) &= \mathcal{N}(\vec{p}, m)\{\frac{p^1 - ip^2}{|p^0| + p^3 + m}\,(\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} + i\,\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]}) \\
&+ \frac{p^0 - p^3 + m}{|p^0| + p^3 + m}\cdot\,\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} - i\,\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}\} \cdot e^{-i(|p^0| - i\vec{p}\cdot\vec{x})}\,, \\
\psi^{neg}_{1\,m}(\vec{p}) &= \mathcal{N}(\vec{p}, m)\{\frac{p^1 + ip^2}{|p^0| + p^3 + m}\,(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} - i\,\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}) \\
&+ \frac{-p^0 + p^3 - m}{|p^0| + p^3 + m}\,\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} - i\,\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]}\} \cdot e^{i(|p^0| + \vec{p}\cdot\vec{x})}\,, \\
\psi^{neg}_{2\,m}(\vec{p}) &= \mathcal{N}(\vec{p}, m)\{\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} + i\frac{|p^0| - p^3 + m}{|p^0| + p^3 + m}\,\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]} \\
&+ \frac{p^1 - ip^2}{|p^0| + p^3 + m}\,(\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} - i\,\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]})\} \cdot e^{i(|p^0| + \vec{p}\cdot\vec{x})}\,. \quad (15.7)
\end{aligned}
$$

The antiparticle states to the particle state $\psi^{pos}_{i\,m}(\vec{p})\,,\, i = (1, 2)$, put on the top of the Dirac sea are the two states $\psi^{pos}_{i\,m}(-\vec{p})\,,\, i = (1, 2)$, respectively, put on the top of the Dirac sea.

15.4 Second quantized solutions

Let us now pay attention on the second quantized picture, discussions of which we already started in the above two sections. Following the proposal from the ref. [3] the discrete symmetries are in cases of the Kaluza-Klein kind for d even, as it is in our toy model, defined as follows

$$
\begin{aligned}
\mathcal{C}_{\mathcal{N}}\,\psi(x^0, \vec{x}) &= \Gamma^{(3+1)}\,\gamma^2\,K\,\psi(x^0, x^1, x^2, x^3, x^5, -x^6, x^7, -x^8, \ldots, x^{d-1}, -x^d) \\
&= \Gamma^{(3+1)}\,\gamma^2\,K\,I_{6,8,\cdots,d}\,\psi(x^0, \vec{x})\,, \\
\mathcal{T}_{\mathcal{N}}\,\psi(x^0, \vec{x}) &= \Gamma^{(3+1)}\,\gamma^1\,\gamma^3\,K\,\psi(-x^0, x^1, x^2, x^3, -x^5, x^6, -x^7, \ldots, -x^{d-1}, x^d) \\
&= \Gamma^{(3+1)}\,\gamma^1\,\gamma^3\,K\,I_{x^0}\,I_{5,7,\cdots,d-1}\,\psi(x^0, \vec{x})\,, \\
\mathcal{P}^{d-1}_{\mathcal{N}}\,\psi(x^0, \vec{x}) &= \gamma^0\,\Gamma^{(3+1)}\,\Gamma^{(d)}\,\psi(x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^{d-1}, x^d) \\
&= \gamma^0\,\Gamma^{(3+1)}\,\Gamma^{(d)}\,I_{\vec{x}_3}\,\psi(x^0, \vec{x})\,. \quad (15.8)
\end{aligned}
$$

$I_{\vec{x}_3}$ reflects (x^1, x^2, x^3), $I_{6,8,\cdots,d}$ reflects $(x^6, x^8, \cdots, x^d)$, I_{x^0} reflects the time component x^0 and $I_{5,7,\cdots,d-1}$ reflects $(x^5, x^7, \cdots, x^{d-1})$. It is $\mathcal{C}_{\mathcal{N}} \cdot \mathcal{P}^{d-1}_{\mathcal{N}}$, which transforms in our case positive energy states into the corresponding negative energy states, staying within the same Weyl.

One finds

$$\begin{aligned}\{\mathcal{C}_{\mathcal{N}}, \gamma^a p_a\}_+ &= 0\,,\\ \{\mathcal{P}_{\mathcal{N}}, \gamma^a p_a\}_- &= 0\,,\\ \{\mathcal{T}_{\mathcal{N}}, \gamma^a p_a\}_+ &= 0\,,\\ \{\mathcal{C}_{\mathcal{N}}\,\mathcal{P}_{\mathcal{N}}, \gamma^a p_a\}_+ &= 0\,,\\ \{\mathcal{C}_{\mathcal{N}}\,\mathcal{P}_{\mathcal{N}}\,\mathcal{T}_{\mathcal{N}}, \gamma^a p_a\}_- &= 0\,. \end{aligned} \tag{15.9}$$

In even dimensional spaces namely neither $\mathcal{C}_{\mathcal{N}}$ nor $\mathcal{P}^{d-1}_{\mathcal{N}}$ transforms states within the same Weyl representation, it is only $\mathcal{C}_{\mathcal{N}}\ \mathcal{P}^{d-1}_{\mathcal{N}}$, which does this and it is correspondingly a good symmetry, keeping the states within one Weyl representation.

To obtain an antiparticle state to a chosen particle state above the Dirac sea we must accordingly apply on a particle state the operator $\mathcal{C}_{\mathcal{N}}\ \mathcal{P}^{d-1}_{\mathcal{N}}$ and then empty the obtained negative energy state. Emptying the corresponding negative energy state and putting it on the top of the Dirac sea determines the antiparticle state to the starting particle state.

In the ref. [3] the second quantized charge conjugation operator $\mathbb{C}_{\mathcal{N}}$ is defined as follows: First one applies on the particle state with positive energy put on the top of the Dirac sea, the operator $\mathcal{C}_{\mathcal{N}}$, which makes a choice of the corresponding negative energy state. Then by *emptying this negative energy state in the Dirac sea one creates an antiparticle with the positive energy and all the properties of the starting single particle state above the Dirac sea, that is with the same* d*-momentum and all the spin degrees of freedom the same, except the* S^{03} *value*, as the starting single particle state [2].

We make now a statement. It is the operator

$$\mathtt{emptying} := \prod_{\gamma^a \in \Im} \gamma^a\ K\,\Gamma^{(3+1)}\,, \tag{15.10}$$

operating on the negative energy state, which *empties the negative energy state* creating the antiparticle state above the Dirac sea.

Let us check on the massless states of Eq. (15.4) what we claimed:
First $\mathcal{C}_{\mathcal{N}}\ \mathcal{P}_{\mathcal{N}}$ applied on $\psi^{pos}_1(\vec{p})$ (this state is put on the top of the Dirac sea) transforms this state into the negative energy state $\psi^{neg}_1(\vec{p})$, then $\prod_{\gamma^a \in \Im} \gamma^a\ K\,\Gamma^{(3+1)}$ applied on $\psi^{\mathbf{neg}}{}_1(\vec{p})$ transforms this state into the positive energy antiparticle state $\psi^{pos}_2(-\vec{p}) = \mathcal{N}_2 \left(\overset{03}{[-i]}\,\overset{12}{(+)} \mid \overset{56}{[-]} + \frac{p^1+ip^2}{|p^0|+p^3}\,\overset{03}{(+i)}\,\overset{12}{[-]} \mid \overset{56}{[-]} \right) e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}$, (put on the top of the Dirac sea), up to phase factors, what can easily be checked. All these states, particle and antiparticle ones, are the solutions of the massless Weyl equation in $d = (3+1)$ (with $p^a = (p^0, p^1, p^2, p^3, 0, 0)$.

The two massless particle/antiparticle pairs are therefore $(\psi^{pos}_1(\vec{p}), \psi^{pos}_2(-\vec{p}))$ and $(\psi^{pos}_2(\vec{p}), \psi^{pos}_1(-\vec{p}))$.

For $p^a = (p^0, 0, 0, p^3, 0, 0)$ we find that the two particle/antiparticle pairs are correspondingly ($\psi^{\mathbf{pos}}_{\mathbf{1}}$ and ψ^{pos}_4) and ($\psi^{\mathbf{pos}}_{\mathbf{2}}$ and ψ^{pos}_3), presented in (Table 15.1).

[2] S^{03} is involved in the boost (contributing in $d = (3+1)$, together with the spin, to handedness) and does not determine the (ordinary) spin.

One checks this by applying the operator $\mathcal{C_N}\,\mathcal{P}^{d-1}_{\mathcal{N}}$ on the state $\psi^{\mathbf{pos}}_{\mathbf{1}}$, transforming this state into the state $\psi^{\mathbf{neg}}_{\mathbf{3}}$, while $emptying$ this negative state generates $\psi^{\mathbf{pos}}_{\mathbf{4}}$. Similarly $\mathcal{C_N}\,\mathcal{P}^{d-1}_{\mathcal{N}}\ \psi^{\mathbf{pos}}_{\mathbf{2}}$ into $\psi^{\mathbf{neg}}_{\mathbf{4}}$, while $emptying$ this negative state generates $\psi^{\mathbf{pos}}_{\mathbf{3}}$.

If only one massless solution, let say the right handed one with respect to $d=(3+1)$, is allowed, as in the case presented in the ref. [2], then the only allowed particle/antiparticle pair is $(\psi^{\mathbf{pos}}_{\mathbf{1}}, \psi^{pos}_{4})$.

Before discussing the discrete symmetries of the massive states presented in Eq. (15.5) let us pay attention that $K \overset{(56)}{(+)} = - \overset{(56)}{(+)}$, since $\overset{(56)}{(+)} = \frac{1}{2}(\gamma^5 + i\gamma^6)$ and we make a choice of γ^0, γ^1 real, γ^2 imaginary, γ^3 real, γ^5 imaginary, γ^6 real, and alternating real and imaginary ones we end up in even dimensional spaces with real γ^d. K makes complex conjugation, transforming i into $-i$.

Let us look at discrete symmetries of the massive solutions $\psi^{pos}_{i\,m}(\vec{p}=0)\,, i = 1,2$. We see that $\mathcal{C_N}\ \mathcal{P_N}$ applied on $\psi^{pos}_{1\,m}$ (Eq. (15.6)), put on the top of the Dirac sea, transforms this state into the negative energy state $\psi^{neg}_{4\,m}$. Emptying this negative energy state, that is applying $\prod_{\gamma^a \in \Im} \gamma^a\ K\,\Gamma^{(3+1)}$ on $\psi^{neg}_{4\,m}$, makes the antiparticle state $\psi^{pos}_{1\,m}$ on the top of the Dirac sea. This state does not distinguish from the starting particle state. Massive states have no conserved charge and correspondingly are the particle and antiparticle solutions of the massive Weyl equation for $p^m = (m,0,0,0)$ indistinguishable.

For the general momentum $p^m = (p^0, p^1, p^2, p^3)$ the state $\psi^{neg}_{2\,m}(\vec{p})$ in Eq. (15.7) follows if we apply the discrete operator $\mathcal{C_N}\ \mathcal{P_N}$ on the state $\psi^{pos}_{1\,m}(\vec{p})$. Emptying the state $\psi^{neg}_{2\,m}(\vec{p})$ leads to the antiparticle state above the Dirac sea, which is $\psi^{pos}_{1\,m}(-\vec{p})$. Let us remind the reader again that in this model no charge is the conserved one.

Let us say that the product of the operations $emptying \times \mathcal{C_N}\ \mathcal{P_N}$ leads to $\gamma^0\gamma^5\Gamma^{(3+1)}\, I_{\vec{x}_3}\, I_6 \Gamma^{(6)}$ in our $d=(5+1)$ model. In general even d case we have

$$emptying \times \mathcal{C_N}\,\mathcal{P_N} = \gamma^0 \prod_{\gamma^a \in \Im, a\neq 2} \gamma^a\, \Gamma^{(3+1)}\, I_{\vec{x}_3}\, I_{6,8,\dots,d} \Gamma^{(d)}\,. \tag{15.11}$$

15.5 Conclusions

We discussed in this contribution representations of the Weyl equation in $d = (5+1)$, manifesting as: **i.** massless, **ii.** massive Dirac equation in $d=(3+1)$ in dependence on whether the fields, which manifest in $d=(3+1)$ as the scalar fields, not gain or gain, respectively, nonzero vacuum expectation values which causes the appearance of fermion masses. Using the technique from the ref. [1] for representing spinors, we present the basis and solutions of the equations of motion for the massless and massive states.

We also present the particle/antiparticle pairs in the second quantized picture, using the definition of the discrete symmetries from the ref. [3]. For general $p^a = (p^0, \vec{p}, 0, 0)$ the particle and antiparticle solutions are distinguished by their conserved charges. If no charge is conserved, then particle and the corresponding antiparticle state differ only by having opposite three momentum in $d + (3+1)$ space.

We conclude that *there are in all the cases two particle and two antiparticle states with the positive energy and the corresponding four states in the Dirac sea*. All these states solve the equations of motion, the Weyl or the Dirac ones, in $d = (3+1)$. Not necessarily are all of them either massless or massive. It can happen that under special conditions the number of massless solutions reduces, while the rest of representations belong to massive sector. The number of states enlarges, if spinors carry additional quantum numbers, which are or are not the conserved quantities.

Presenting the spinor basis in all these cases helps to understand what is happening with the degrees of freedom in massless case and after the scalar fields bring masses to the spinors (like the Higgs of the *standard model*.

We shall study in another paper what one of us calls the realistic case, that is the representations of massless and massive states of the *spin-charge-family* theory of N.S.M.B., before and after particular scalar fields cause that spinors manifest masses in $d = (3+1)$.

This might help to clarify the open problems which V. Dvoeglazov put in his contribution in this proceedings.

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