

# Strongly regular $m$ -Cayley circulant graphs and digraphs

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## Abstract

The first part of this paper is a survey about strongly regular graphs and digraphs admitting a semiregular cyclic group of automorphisms. In the second part, some new types of such digraphs, called uniform and almost uniform, are studied. By using partial sum families, the form of the parameters is determined and some directed strongly regular graphs derived from these partial sum families with previously unknown parameters are obtained.

*Keywords:*  $m$ -Cayley, circulant, strongly regular graphs, strongly regular digraphs, uniform partial sum families, almost-uniform partial sum families.

*Math. Subj. Class.:* 05Cxx, 05C20, 05C25, 05C50, 05E30

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## 1 Introduction

Strongly regular graphs, which we define below, were introduced by Bose [4] in 1963. They constitute a very important class of graphs; in fact, they are one of the most basic association schemes, more specifically, they are the ones with two classes.

**Definition 1.1.** A graph  $X$  without loops of valency  $k$  and order  $v$  is called a strongly regular graph with parameters  $v, k, \lambda, \mu$  (for short,  $(v, k, \lambda, \mu)$ -SRG) if any two adjacent vertices have exactly  $\lambda$  common neighbours and any two distinct non-adjacent vertices have exactly  $\mu$  common neighbours.

A SRG graph  $X$  is said to be trivial if  $X$  or its complement is a disjoint union of complete graphs.

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Dedicated to the memory of my mother.

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It is well known that if a non-trivial SRG is not a conference graph, that is, if the parameters are not of the form  $k = (v - 1)/2, \mu = (v - 1)/4, \lambda = (v - 5)/4$ , then the eigenvalues of an adjacency matrix are integer numbers, and consequently  $\Delta = \sqrt{\beta^2 + 4\gamma}$ , where  $\beta = \lambda - \mu$  and  $\gamma = k - \mu$ , is also an integer.

The problem of studying what SRGs have nice cyclic automorphism groups has received a considerable attention over the past decades.

A group of automorphisms of a graph is said to be regular if it acts transitively on the set of vertices and all the stabilizers are trivial. A graph is called circulant if it admits a cyclic regular group of automorphisms.

By a classical result of W.G. Bridges and R.A. Mena [5] it is known that the Paley graphs are the only non-trivial circulant strongly regular graphs.

Given integers  $m \geq 1$  and  $n \geq 2$ , an automorphism group of a graph is called  $(m, n)$ -semiregular if it has  $m$  orbits of length  $n$  and no other orbit, and the action is regular on each orbit. An  $m$ -Cayley graph  $X$  is a graph admitting an  $(m, n)$ -semiregular group  $H$  of automorphisms. When  $H$  is abelian, we say that  $X$  is  $m$ -Abelian. If  $H$  is generated by an automorphism  $\rho$  (that is to say, when  $H$  is a cyclic group) and  $m = 1$  (respectively,  $m = 2$ ) we say that  $X$  is  $n$ -circulant (respectively,  $n$ -birculant). Sometimes, when a graph is  $m$ -Cayley over a cyclic group, we will just say that the graph is ‘ $m$ -Cayley circulant’, although that this terminology does not mean that the graph admits a regular group of automorphisms and should not be confused with the definition of circulant graph. Every  $m$ -Cayley graph  $X$  can be represented, following the terminology established by A. Malnič et al. in [18], by an  $m \times m$  array of subsets of  $H$  in the following way. Let  $U_0, \dots, U_{m-1}$  be the  $m$  orbits of  $H$ , and for each  $i$  let  $u_i \in U_i$ . For each  $i$  and  $j$ , let  $S_{i,j}$  be defined by  $S_{i,j} = \{\rho \in H \mid u_i \rightarrow \rho(u_j)\}$ . The family  $(S_{i,j})$  is called the symbol of  $\mathcal{G}$  relative to  $(H; u_0, \dots, u_{m-1})$ .

The notion of strongly regular graph was generalized for directed graphs by Duval in [6]. A directed graph  $X$  without loops, of order  $v$  in which every vertex has both in-degree and out-degree  $k$  is called a *directed strongly regular graph with parameters*  $(v, k, \mu, \lambda, t)$  (for short,  $(v, k, \mu, \lambda, t)$ -DSRG or simply DSRG if we do not specify the parameters) whenever for any vertex  $u$  of  $X$  there are  $t$  undirected edges having  $u$  as an endvertex and for every two different vertices  $u$  and  $w$  of  $X$  the number of paths  $p(u, w)$  of length 2 starting at  $u$  and ending at  $w$  depends only on whether  $uw$  is an arc of  $X$  or not. In particular,

$$p(u, w) = \begin{cases} t & \text{if } u = w \\ \lambda & \text{if } u \neq w \text{ and } uw \in A(X) \\ \mu & \text{if } u \neq w \text{ and } uw \notin A(X) \end{cases}$$

(where  $A(X)$  denotes the arc set of  $X$ ). A directed strongly regular graph will also be referred to as a *strongly regular digraph*.

DSRGs have received a considerable attention in the literature (see, for instance, [7], [8], [9], [11], [12], [13], [14]).

The following relation between the parameters of a DSRG is obvious:

$$k(k - \beta) = \mu v + \gamma, \tag{1.1}$$

where  $\beta = \lambda - \mu$  and  $\gamma = t - \mu$ .

It is well known that  $\beta^2 + 4\gamma$  is a square, unless

$$k = t = (v - 1)/2, \mu = (v - 1)/4, \lambda = (v - 5)/4, \tag{1.2}$$

in which case the graph is undirected and is a conference graph, or

$$k = (v - 1)/2, \mu = (v + 1)/4, \lambda = (v - 3)/4, t = 0. \quad (1.3)$$

We define

$$\Delta = \sqrt{\beta^2 + 4\gamma},$$

which is an integer if the parameters of the digraph are not as indicated above.

A DSRG  $X$  is called trivial if  $k = t$  and  $X$  is trivial as an undirected SRG.

Next, we will review the main results in this paper. In the first part of this paper we will make a review of the study of strongly regular graphs and digraphs which are  $m$ -Cayley over a cyclic group ( $m$ -Cayley circulant SRGs and DSRGs). In Section 2 we will focus on the undirected graphs, and in Section 3 on the more general case of directed graphs.

In the second part we will present some new results on two structures that produce circulant  $m$ -Cayley DSRGs. We study in Sections 4,5 and 6 the first structure, uniform partial sum families, which was proposed in the last section of [2], and we obtain there the general form of the parameters for the circulant case and give an sporadic uniform partial sum family which originates a DSRG with parameters previously undecided. In Section 7, we study almost uniform partial sum families, which are a generalization of uniform partial sum families, and obtain again the form of the parameters for the circulant case. Finally, we use almost uniform partial sum families to obtain three DSRG with parameters previously undecided.

## 2 Semiregular SRGs over cyclic groups

In this section we will focuss on undirected graphs. In particular, we will review some results on strongly regular graphs admitting cyclic semiregular groups of automorphisms.

Probably, the systematic study of circulant SRGs was began by D. Marušič in his seminal paper [21]. There, he obtained the form of the parameters of such graphs for the bicirculant case.

**Proposition 2.1.** *If a non-trivial  $(2n, k, \lambda, \mu)$  bicirculant strongly regular graph exists with prime  $n$  then, up to complementation, the parameters of the graph are*

$$v = 2n = 4s^2 + 4s + 2, k = 2s^2 + s, \lambda = s^2 - 1, \mu = s^2.$$

He determined also some properties of the elements of the symbol. He denoted  $S_{0,0}$ ,  $S_{1,1}$  and  $S_{0,1}$  by  $R$ ,  $S$  and  $T$ , respectively, and he proved that the non-zero elements of the cyclic group  $C_n$  are the disjoint union of  $R$  and  $S$ , and that  $|R| = |S| = s^2 + s$  and  $|T| = s^2$ . He found also examples of such graphs for  $s = 1$  and  $s = 2$ .

He considered also non-trivial tricirculant SRGs over a cyclic group  $C_n$  of prime order and proved that, for such a graph or its complement, the elements of the symbol  $S_{0,0}$ ,  $S_{1,1}$  and  $S_{2,2}$  form a partition of the set of non-zero elements of  $C_n$ .

His results were generalized by De Resmini and Jungnickel in [22]. They proved that the form of the parameters of a non-trivial bicirculant SRG over a cyclic group of order  $n$  was the one indicated in Proposition 2.1 if  $n$  is odd or not divisible by  $\Delta$ . They found also one such graph for  $s = 4$ . Unfortunately, the technique that they used to construct the graph, which is based on the existence of certain difference families over cyclic groups, was not fruitful to get examples with  $s \geq 5$ .

This was further generalized by K.H. Leung and S.L. Ma in [17]. They found the general form of the parameters of non-trivial bicirculant SRGs, as stated in the following proposition, where  $c = |S_{0,1}|, d = |S_{0,0}|$ :

**Proposition 2.2.** *Up to complementation, the parameters for any non-trivial bicirculant SRG over a cyclic group of order  $n$  are the following:*

1.  $(n; c, d; \lambda, \mu) = (2m^2 + 2m + 1; m^2, m^2 + m; m^2 - 1, m^2)$  where  $m \geq 1$ .
2.  $(n; c, d; \lambda, \mu) = (2m^2; m^2, m^2 - m; m^2 - m; m^2 - m)$  where  $m \geq 2$ .
3.  $(n; c, d; \lambda, \mu) = (2m^2; m^2, m^2 + m; m^2 + m; m^2 + m)$  where  $m \geq 3$ .
4.  $(n; c, d; \lambda, \mu) = (2m^2; m^2 \pm m, m^2; m^2 \pm m; m^2 \pm m)$  where  $m \geq 2$ .

Besides the graphs found by Marušič and by De Resmini and Jungnickel, Leung and Ma found one example for  $m = 2$  in family 2 and one example for  $m = 2$  in family 4 of the previous proposition. They also proved the non-existence of bicirculants SRG over cyclic groups of order  $2m^2$  where  $m = p^r u, p$  is a prime congruent to 3 modulo 4,  $p$  and  $u$  are relatively prime and  $u^2 < p^r$ .

More bicirculant SRGs were found by A. Malnič et al. in [19]. Concretely, they obtained bicirculant SRGs with parameters of the same form as in Proposition 2.1 for  $s = 3, 4$  and 5. The graphs that they found were the first known pairs of complementary bicirculant SRGs which are vertex-transitive but not edge-transitive.

K. Kutnar et al. studied tricirculant SRGs in [16]. They proved that, under certain appropriate conditions, the elements on the symbol of a tricirculant SRG over a cyclic group  $C_n$  satisfy that  $S_{0,0}, S_{1,1}, S_{2,2}$  form a partition of  $C_n - \{0\}$  and the parameters of the graph can be determined. They denoted  $S_{0,0}, S_{1,1}, S_{2,2}, S_{0,1}, S_{1,2}$  and  $S_{2,0}$  by  $A, B, C, R, S$  and  $T$ , respectively (the other elements in the symbol can be easily determined from these ones), and they denoted by  $TCay(C_n; A, B, C; R, S, T)$  the associated tricirculant. In the next proposition, for a subset  $D \subseteq C_n$ , we set  $\overline{D} = C_n \setminus (S \cup \{0\})$ .

**Proposition 2.3.** *Let  $X = TCay(C_n; (A, B, C; R, S, T))$  be a non-trivial  $(3n, k, \lambda, \mu)$ -strongly regular tricirculant, where  $|A| + |B| + |C| \leq |\overline{A}| + |\overline{B}| + |\overline{C}|$ . Assume  $A \cup B \cup C \neq \emptyset$ . If  $n$  is a prime or  $n$  is coprime to  $6\Delta$  then there exists an integer  $s$  such that the following two statements hold.*

1. *If the cardinalities of  $A, B$  and  $C$  are not all equal, then*

$$(3n, k, \lambda, \mu) = (3(12s^2 + 9s + 2), (4s + 1)(3s + 1), s(4s + 3), s(4s + 1)).$$

*If in addition  $|A| = |C| \neq |B|$  (which is equivalent to  $|R| = |S| \neq |T|$ ), then*

$$|A| = 2s(1 + 2s), |B| = (4s + 1)(s + 1), |R| = s(4s + 1) \text{ and } |T| = (1 + 2s)^2.$$

2. *If  $|A| = |B| = |C|$  then*

$$(3n, k, \lambda, \mu) = (3(3s^2 - 3s + 1), s(3s - 1), s^2 + s - 1, s^2).$$

*In this case  $|A| = s(s - 1)$  and  $|R| = |S| = |T| = s^2$ .*

When  $|A| = |C| \neq |B|$  they proved that, for  $s = -1$ , exactly one tricirculant SRG exists up to isomorphisms, and for  $s = 1$  no such graph exists.

When  $|A| = |B| = |C|$  they proved that for  $s = 2$  exactly one tricirculant SRG exists up to isomorphisms, and for  $s = 3$  exactly four exist, and that for  $s = -2$  no such graph exists.

### 3 Semiregular DSRGs over cyclic groups

DSRGs admitting a semiregular group of automorphisms were studied by Martínez and Araluze in [20] by using the concept of partial sum family. In the next definition, the third identity holds in the group ring  $\mathbb{Z}[H]$ . As usual, we will identify a subset of  $H$  with the sum in  $\mathbb{Z}[H]$  of its elements.

**Definition 3.1.** Let  $H$  be a group of order  $n$  and let  $m$  be an integer with  $m \geq 1$ . A family  $\mathfrak{S} = \{S_{i,j}\}$ , with  $0 \leq i, j < m$ , of subsets of  $H$  is a  $(m, n, k, \mu, \lambda, t)$ -partial sum family (for short,  $(m, n, k, \mu, \lambda, t)$ -PSF, or simply PSF if we do not specify the parameters) if it satisfies:

1. for every  $i$  it holds that  $0 \notin S_{i,i}$ , where  $0$  is the identity of  $H$ .
2. for every  $i$  it holds that  $\sum_{j=0}^{m-1} |S_{i,j}| = \sum_{j=0}^{m-1} |S_{j,i}| = k$
3. for every  $i$  and  $j$  it holds that  $\sum_{l=0}^{m-1} S_{l,j}S_{i,l} = \delta_{i,j}\gamma\{0\} + \beta S_{i,j} + \mu H$ , where  $\delta_{i,j}$  is the Kronecker delta, and where  $\gamma = t - \mu$  and  $\beta = \lambda - \mu$ .

The symbol notation, presented in the introduction for undirected graphs, is valid also for directed graphs. Martínez and Araluze proved that the existence of a  $(m, n, k, \mu, \lambda, t)$ -partial sum family over a group  $H$  is equivalent to the existence of a  $(mn, k, \mu, \lambda, t)$ -DSRG which admits a group of automorphisms isomorphic to  $H$  acting semiregularly and with  $m$  orbits (in fact, the elements of the PSF form the symbol of the digraph).

They found 17 new DSRGs by using partial sum families. 14 of them had cyclic groups as groups of automorphisms, which are the kind of digraphs that we are considering in this paper. More specifically, they found:

Four PSFs with parameters  $(2, 15, 13, 6, 5, 8)$ ,  $(2, 17, 14, 6, 5, 12)$ ,  $(2, 17, 15, 7, 6, 9)$ ,  $(2, 20, 17, 8, 6, 11)$  which produce bicirculant digraphs.

Five PSFs with parameters  $(3, 9, 10, 4, 3, 6)$ ,  $(3, 11, 11, 4, 3, 4)$ ,  $(3, 13, 10, 3, 1, 6)$ ,  $(3, 14, 14, 5, 4, 5)$ ,  $(3, 15, 14, 4, 5, 6)$  which produce tricirculant digraphs.

Four PSFs with parameters  $(4, 7, 7, 2, 1, 2)$ ,  $(4, 8, 9, 3, 1, 6)$ ,  $(4, 8, 10, 3, 3, 7)$ ,  $(4, 11, 10, 2, 3, 4)$  which produce tetracirculant digraphs.

One PSF with parameters  $(5, 7, 8, 2, 1, 4)$  which produce a pentacirculant digraph.

Finally, Araluze et al. found in [2] the form of the possible parameters of bicirculant DSRGs. They called the partial sum families partial sum quadruples in this case when  $m = 2$ .

**Proposition 3.2.** Let  $\mathfrak{S}$  be a non-trivial PSQ in a cyclic group  $H$ . Then, up to complementation, the parameters of  $\mathfrak{S}$  are of the following form, where  $U = k - t$  and  $s, f$  are positive integers, and where  $\mathbf{q} = |S_{1,0}|$  and  $\mathbf{r} = |S_{0,0}|$ :

1.  $n = s(2f + 1)$ ,  $\mathbf{q} = sf$ ,  $\mathbf{r} = sf$ ,  $k = 2sf$ ,  $\mu = sf$ ,  $\lambda = s(f - 1)$ ,  $t = sf$ .
2.  $n = s(2f + 1)$ ,  $\mathbf{q} = s(f + 1)$ ,  $\mathbf{r} = sf$ ,  $k = s(2f + 1)$ ,  $\mu = s(f + 1)$ ,  $\lambda = sf$ ,  $t = s(f + 1)$ .
3.  $n = 4s$ ,  $\mathbf{q} = 2s$ ,  $\mathbf{r} = 2s - 1$ ,  $k = 4s - 1$ ,  $\mu = s$ ,  $\lambda = 3s - 2$ ,  $t = 3s - 1$ .
4.  $n = 2s^2 + 2s + 1 + 2U$ ,  $\mathbf{q} = s^2 + U$ ,  $\mathbf{r} = s^2 + s + U$ ,  $k = 2s^2 + s + 2U$ ,  $\mu = s^2 + U$ ,  $\lambda = s^2 - 1 + U$  and  $t = 2s^2 + s + U$ .
5.  $n = 2s^2 + 2U$ ,  $\mathbf{q} = s^2 + U$ ,  $\mathbf{r} = s^2 + s + U$ ,  $k = 2s^2 + s + 2U$ ,  $\mu = s^2 + s + U$ ,  $\lambda = s^2 + s + U$  and  $t = 2s^2 + s + U$ , where  $2s$  divides  $s^2 + U$ .

6.  $n = 2s^2 + 2U, \mathbf{q} = s^2 + U, \mathbf{r} = s^2 - s + U, k = 2s^2 - s + 2U, \mu = s^2 - s + U, \lambda = s^2 - s + U$   
and  $t = 2s^2 - s + U$ , where  $2s$  divides  $s^2 + U$ .
7.  $n = 2s^2 + 2U, \mathbf{q} = s^2 \pm s + U, \mathbf{r} = s^2 + U, k = 2s^2 \pm s + 2U, \mu = s^2 \pm s + U, \lambda = s^2 \pm s + U$   
and  $t = 2s^2 \pm s + U$ , where  $s$  divides  $U$ .
8.  $n = 4s^2, \mathbf{q} = 2s^2 \pm 2s, \mathbf{r} = 2s^2 \pm s, k = 4s^2 \pm 3s, \mu = (s \pm 1)(2s \pm 1), \lambda = (s \pm 1)(2s \pm 1)$   
and  $t = s^2 + (s \pm 1)(2s \pm 1)$

Families 4,5,6 and 7 correspond with the ones found by Leung and Ma in 2.2 (in fact, they correspond to the particular case  $U = 0$ ). Araluze et al. proved that PSQs with parameters as in families 1,2 and 3 exists for every  $s$  and  $f$ . They found several sporadic examples for the other families, two of which produced DSRGs not isomorphic to any known ones with that parameters.

Finally, we will mention a result about bicirculant association schemes. Let us recall first the definition of association scheme.

An association scheme of class  $s$  is a pair  $\mathfrak{X} = (X, \mathfrak{R})$ , where  $\mathfrak{R} = \{R_0, \dots, R_s\}$  is a partition of  $X^2$  that satisfies:

1.  $R_0 = \{(x, x) \mid x \in X\}$ .
2. For every  $i = 0, \dots, n$ , we have  $\{(y, x) \mid (x, y) \in R_i\} \in \mathfrak{R}$ .
3. For every  $i, j, k = 0, \dots, n$ , there exists a non-negative integer  $p_{i,j}^k$  such that  $|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = p_{i,j}^k$  whenever  $(x, y) \in R_k$ .

The cardinality of  $X$  is called the order of the association scheme. The  $R_i$  are called the basic relations of the association scheme, and the  $(X, R_i)$  are called the basic digraphs. It is an easy consequence of part 3 of the definition that all the basic digraphs  $(X, R_i)$  are regular. The degree of  $R_i$  is defined as the degree of  $(X, R_i)$ . The association schemes of class one are called trivial.

The linear span  $\mathbb{C}[\mathfrak{R}]$  of the adjacency matrices of the  $R_i$  is called the Bose-Mesner algebra of the association scheme. The vector space  $\mathbb{C}^X$  is a left  $\mathbb{C}[\mathfrak{R}]$ -module in a natural way. Since  $\mathbb{C}[\mathfrak{R}]$  is a semisimple algebra, the mentioned vector space  $\mathbb{C}^X$  decomposes as a direct sum of irreducible  $\mathbb{C}[\mathfrak{R}]$ -submodules, which are called the irreducible representations of  $\mathfrak{X}$ .

An association scheme  $\mathfrak{X} = (X, \{R_0, \dots, R_s\})$  is said to be primitive if all the  $R_i$  are strongly connected.

An automorphism of an association scheme  $\mathfrak{X} = (X, \{R_0, \dots, R_s\})$  is a bijection of  $X$  that is an automorphism of all the digraphs corresponding to the relations  $R_i$ . An association scheme  $\mathfrak{X} = (X, \{R_0, \dots, R_s\})$  is said to be bicirculant if there exists a cyclic group of automorphisms of  $\mathfrak{X}$  which acts semiregularly on  $X$  and has 2 orbits.

I. Kovács et al. obtained in [15] the following result:

**Proposition 3.3.** *Let  $\mathfrak{X}$  be a primitive bicirculant scheme on  $2p^e$  points,  $p > 2$  a prime. Then*

1.  $\mathfrak{X}$  is of class at most two; and
2. if the class is exactly 2, then  $2p^e = (2s + 1)^2 + 1$  for some natural number  $s$ , and the degrees of basic digraphs of  $\mathfrak{X}$  are  $1, s(2s + 1), (s + 1)(2s + 1)$ , and the multiplicities of the irreducible representations of  $\mathfrak{X}$  in its standard module are  $1, p^e, p^e - 1$ .

Although this result of I. Kovács et al. is mainly applied to the study of primitive permutation groups, it has an interpretation in terms of the kind of objects in which we are interested in this paper, because association schemes of class two are DSRGs, so that they proved that non-trivial primitive bicirculant schemes of certain orders are in fact DSRGs, and they obtained additional restrictions on that orders.

#### 4 Uniform partial sum families

Uniform partial sum families were introduced in [2] as a kind of digraphs which generalizes what happens in family 4 of Proposition 3.2 and in part 2 of Proposition 2.3.

**Definition 4.1.** Let  $H$  be a group of order  $n \geq 2$  and let  $m \geq 1$  be an integer. Then an  $(m, n, k, \mu, \lambda, t)$ -partial sum family  $\mathfrak{S} = \{S_{i,j}\}$ , with  $0 \leq i, j < m$ , of subsets of  $H$  is uniform if it satisfies the following conditions:

1. The cardinalities of the ‘diagonal’ blocks  $S_{i,i}$  are all equal.
2. The cardinalities of the ‘non-diagonal’ blocks  $S_{i,j}$ ,  $i \neq j$ , are all equal.
3. The ‘diagonal’ blocks  $\{S_{i,i} : i \in \mathbb{Z}_m\}$  form a partition of  $H - \{0\}$ , where 0 is the identity element of the group.

The following two propositions give infinite families, found in [2], of uniform PSFs:

**Proposition 4.2.** *If  $H$  is a group of order  $ef + 1$  and  $A_0, \dots, A_{e-1}$  is a partition of  $H - \{0\}$  and  $|A_i| = f$  for every  $i$ , then  $\mathfrak{S} = \{S_{i,j}\}$ , where  $S_{i,j} = A_j$ , is a uniform  $(e, ef + 1, ef, f, f - 1, f)$ -PSF in  $H$ .*

**Proposition 4.3.** *If  $H$  is a group of order  $ef + 1$  and  $A_0, \dots, A_{e-1}$  is a partition of  $H - \{0\}$  with  $|A_i| = f$  for every  $i$ , then  $\mathfrak{S} = \{S_{i,j}\}$ , where*

$$S_{i,j} = \begin{cases} A_i & \text{if } i = j, \\ \{0\} \cup A_j & \text{otherwise,} \end{cases}$$

*is a uniform  $(e, ef + 1, e(f + 1) - 1, f + 1, e + f - 2, e + f - 1)$ -PSF in  $H$ .*

We let  $A = |S_{0,0}|$ , and  $B = |S_{0,1}|$ . Thus,  $|S_{i,i}| = A$  for every  $i$ , and  $|S_{i,j}| = B$  for every distinct  $i$  and  $j$ .

**Lemma 4.4.**  $B = (2k - \beta \pm \Delta)/(2m)$  and  $A = k + (1 - m)B$ .

*Proof.* Using the second part of Definition 3.1 we obtain

$$A + (m - 1)B = k,$$

and applying the trivial character in the third part of the same definition with  $i = 0, j = 1$  we obtain

$$2AB + (m - 2)B^2 = \beta B + \mu n.$$

Using (1.1) with the DSRG associated to the PSF we get

$$k(k - \beta) = \mu mn + \gamma.$$

Now, the result follows from the three previous identities and the definition of  $\Delta$ . □

### 5 Form of the parameters

Now we will derive the parameters of uniform PSFs when the group  $H$  is cyclic. We will need first a lemma.

**Lemma 5.1.** *Let  $\mathfrak{S}$  be a PSF in an abelian group  $H$ , and let  $\chi$  be a non-trivial character of  $H$ . Then,*

$$\sum_{l=0}^{m-1} \chi(S_{l,l}) = m(\beta - \Delta)/2 + i\Delta \text{ for some } i \in \{0, \dots, m\}.$$

*Proof.* By using Definition 3.1, we have that the matrix  $A_\chi = (\chi(S_{i,j}))$  satisfies  $A_\chi^2 = \beta A_\chi + \gamma I_m$ , and therefore its trace  $\sum_{l=0}^{m-1} \chi(S_{l,l})$  is the sum of  $m$  roots of the polynomial  $x^2 - \beta x - \gamma$ . Since those roots are  $\frac{1}{2}(\beta \pm \Delta)$ , the result follows.  $\square$

**Proposition 5.2.** *If an  $m$ -circulant  $(m, n, k, \mu, \lambda, t)$ -uniform partial sum family exists with  $m \geq 3$ , then the parameters have one of the following forms:*

1.

$$n = sm - rm + r^2m + 1, k = sm + r^2m - r, t = r^2m - r + s,$$

$$\lambda = rm + s + r^2 - 2r - 1, \mu = s + r^2$$

$$A = s - r + r^2, B = s + r^2$$

with  $r$  an integer and  $s$  a non-negative integer.

2.

$$n = \frac{2i(i-1)mr^2}{s}, k = \frac{-2r^2m^2i + 2m^2r^2i^2 - sri - s + smri}{sm}$$

$$t = \frac{-2sr^2i^2 - 4sri + 2sr^2mi - 2s + 4m^2r^2i^2 - 4m^2r^2i + 4smri + s^2}{2m^2s},$$

$$\lambda = \frac{4m^2r^2i^2 - 4m^2r^2i - 2smr + s^2 + 2rm^2s - 4sm}{2m^2s}$$

$$\mu = \frac{(s - 2rm + 2rmi)(s + 2rmi)}{2m^2s}$$

$$A = \frac{-2r^2mi + 2mr^2i^2 - s}{sm}, B = \frac{ri(s - 2rm + 2rmi)}{sm}$$

with  $r$  an integer,  $s$  a non-negative integer and  $i \in \{2, \dots, m\}$ , and where  $2m$  divides  $s$ .

*Proof.* By the previous lemma, for every non-trivial character  $\chi$  of  $H$  it holds that

$$\sum_{l=0}^{m-1} \chi(S_{l,l}) = m(\beta - \Delta)/2 + i\Delta \text{ for some } i \in \{0, \dots, m\}.$$

By part 3 of Definition 4.1 we have that the previous sum equals  $-1$ , and hence we deduce that

$$\beta = ((m - 2i)\Delta - 2)/m, \quad (5.1)$$

where  $\Delta$  is a non-negative integer. Now, from the definition of  $\Delta$  we obtain

$$\gamma = -\frac{(\Delta i + 1)(\Delta i + 1 - \Delta m)}{m^2}. \quad (5.2)$$

Putting  $U = k - t$ , we deduce from (5.2) that

$$k = U + \mu - \frac{(\Delta i + 1)(\Delta i + 1 - \Delta m)}{m^2}. \quad (5.3)$$

From (1.1),(5.1),(5.2) and (5.3) we get

$$\begin{aligned} n &= (Um^2 + m - 1 + \mu m^2 - 2\Delta i + m\Delta i + \Delta m - \Delta m^2 + \Delta^2 im - \Delta^2 i^2) \\ &= (Um^2 - 1 + m + \mu m^2 - 2\Delta i + \Delta m + m\Delta i + \Delta^2 im - \Delta^2 i^2)/(m^5 \mu) \end{aligned} \quad (5.4)$$

Let us suppose that the plus sign holds in Lemma 4.4. Since  $A = (n - 1)/m$ , we obtain from that lemma that

$$n = \frac{m\Delta i + m - m^2\Delta i + Um^2 - 1 + \mu m^2 - 2\Delta i + \Delta m + \Delta^2 im - \Delta^2 i^2}{m^2}. \quad (5.5)$$

From (5.4) and (5.5) we find  $\mu$  as

$$\begin{aligned} \mu &= \frac{1}{2m^2(-1+m)} (-m^2\Delta i + 2\Delta^2 mi - 2\Delta^2 i^2 + 4m\Delta i - 4\Delta i - 2 - m^2 + 3m \\ &\quad - 2m^2\Delta - Um^3 + 2m\Delta + 2Um^2 + m^3\Delta i + \Delta^2 i^2 m - \Delta^2 m^2 i \\ &\quad \pm m((Um^2 + 3m\Delta i + \Delta^2 im - 2\Delta i - \Delta^2 i^2 + m - 1 - m^2\Delta i)^2 \\ &\quad + 4im^2\Delta^2(m-1)(-1+i))^{1/2} \end{aligned} \quad (5.6)$$

Since  $\beta$  is an integer,  $i = 0$  is not possible (see (5.1) and use that  $m \geq 3$ ). Let us suppose that  $i = 1$ .

Then,

$$\mu = -\frac{Um^2 - 1 + m - 2\Delta + 2\Delta m - \Delta m^2 - \Delta^2 + \Delta^2 m}{m^2} \quad (5.7)$$

or

$$\mu = \frac{Um^2 - 1 + m - 2\Delta + 2\Delta m - \Delta^2 + \Delta^2 m}{m^2(-1+m)}. \quad (5.8)$$

Let us suppose that (5.7) holds. Then we get from (5.4) that  $n = 0$ , which gives a contradiction.

Therefore, (5.8) must hold, and using this and (5.1),(5.2),(5.3) and (5.4) we obtain that the parameters are

$$n = \frac{mU}{m-1} - \Delta + \frac{(\Delta+1)^2}{m}, \tag{5.9}$$

$$k = \frac{Um^2 - \Delta - \Delta^2 + \Delta^2m + \Delta m}{m(m-1)},$$

$$t = \frac{-\Delta^2 + \Delta^2m - \Delta + \Delta m + Um}{m(m-1)},$$

$$\lambda = \frac{Um^2 - 1 + 3m - 2\Delta + 4\Delta m - \Delta^2 + \Delta^2m - 3\Delta m^2 + \Delta m^3 - 2m^2}{m^2(m-1)},$$

$$\mu = \frac{U}{m-1} + \frac{(\Delta+1)^2}{m^2}, \tag{5.10}$$

$$A = \frac{Um^2 - \Delta m^2 + 3\Delta m + 2m + \Delta^2m - 2\Delta - 1 - \Delta^2 - m^2}{m^2(m-1)},$$

$$B = \frac{Um^2 - 1 + m - 2\Delta + 2\Delta m - \Delta^2 + \Delta^2m}{m^2(m-1)}.$$

From (5.2), we have that  $m | (\Delta + 1)^2$ , and then we deduce from (5.9) that  $m - 1 | mU$ . Since  $m - 1$  and  $m$  are coprime, then  $m - 1 | U$ . Now, from (5.10) we deduce that  $m | \Delta + 1$ . Putting  $U = s(m - 1)$  and  $\Delta = rm - 1$  we obtain parameters as in part 1 of the proposition.

Now, let us suppose that  $i > 1$ .

Since  $\mu$  is an integer, the square root in (5.6) is also an integer, and

$$\begin{aligned} \mu = \frac{1}{2m^2(-1+m)} & (-m^2\Delta i + 2\Delta^2mi - 2\Delta^2i^2 + 4m\Delta i - 4\Delta i - 2 - m^2 + 3m \\ & - 2m^2\Delta - Um^3 + 2m\Delta + 2Um^2 + m^3\Delta i + \Delta^2i^2m - \Delta^2m^2i \\ & \pm m(Um^2 + 3m\Delta i + \Delta^2im - 2\Delta i - \Delta^2i^2 + m - 1 - m^2\Delta i + s)) \end{aligned} \tag{5.11}$$

where  $s$  is a non-negative integer and

$$\begin{aligned} U = & (-4m^3\Delta^2i + 4\Delta^2im^2 - 4m^2\Delta^2i^2 + 4m^3\Delta^2i^2 - s^2 - 6sm\Delta i \\ & - 2s\Delta^2im + 4s\Delta i + 2s\Delta^2i^2 - 2sm + 2s + 2sm^2\Delta i)(2sm^2) \end{aligned}$$

Hence,

$$\mu = -1/2 \frac{(-s - 2m\Delta i + 2m^2\Delta i) (-s + 2m\Delta - 2m\Delta i - 2m^2\Delta + 2m^2\Delta i)}{sm^2(-1+m)} \tag{5.12}$$

or

$$\mu = 1/2 \frac{(s - 2m\Delta + 2m\Delta i) (s + 2m\Delta i)}{m^2s} \tag{5.13}$$

Let us suppose that (5.12) holds. Then we get from (5.4) that  $n = -s/(2m(-1 + m))$ , and this contradicts that  $n$  must be positive.

Therefore, (5.13) must hold, and using this and (5.1),(5.2),(5.3) and (5.4) we obtain, by putting  $\Delta = r$ , that the parameters are as in part 2 of the statement. From the form of  $A$  we deduce that  $m$  divides  $s$ , and from the form of  $\mu$  we conclude that 2 divides  $s$ . From this two facts and the expression for  $A$  we obtain that  $2m$  divides  $s$ .

Finally, if the minus sign holds in Lemma 4.4, the parameters have the same form as when the plus sign holds by putting  $\Delta = -r$ , so that  $r$  can have both positive and negative values.  $\square$

By Proposition 4.2,  $m$ -circulant uniform PSFs as stated in part 1 of Proposition 5.2 always exist for  $r = 0$  and, by Proposition 4.3,  $m$ -circulant uniform PSFs as stated in part 1 of Proposition 5.2 always exist for  $r = 1$ .

## 6 The tricirculant case

Now we will consider the case when  $m = 3$ . In this case, we will call the PSF tricirculant.

**Proposition 6.1.** *If a tri-circulant  $(3, n, k, \mu, \lambda, t)$ -uniform partial sum family exists, then either*

1.

$$\begin{aligned} n &= 3r^2 - 3r + 1 + 3s, k = 3r^2 - r + 3s, \mu = r^2 + s, \lambda = r^2 + r - 1 + s, \\ t &= 3r^2 - r + s, A = r^2 - r + s \text{ and } B = r^2 + s. \end{aligned} \quad (6.1)$$

2.

$$\begin{aligned} n &= 9r^2 + 6r + 1, k = 9r^2 + 10r + 2, \mu = 3r^2 + 5r + 2, \lambda = 3r^2 + 4r + 1, \\ t &= 5r^2 + 6r + 2, A = 3r^2 + 2r \text{ and } B = 3r^2 + 4r + 1. \end{aligned} \quad (6.2)$$

3.

$$\begin{aligned} n &= 90r^2 - 60r + 10, k = 90r^2 - 40r + 3, \mu = 30r^2 - 5r, \lambda = 30r^2 - 10r + 1, \\ t &= 80r^2 - 40r + 6, A = 30r^2 - 20r + 3 \text{ and } B = 30r^2 - 10r, \end{aligned} \quad (6.3)$$

with  $r$  an integer and  $s$  a non-negative integer.

*Proof.* Using the same notations as in the proof of Proposition 5.2, we have  $2 + 2i\Delta \equiv 0 \pmod{3}$ , and hence  $i = 1$  or  $i = 2$ . If  $i = 1$ , by following the proof of that proposition, we can see that the parameters are as in (6.1). Let us suppose that  $i = 2$ . We will assume that the plus sign holds in Lemma 4.4, because when the minus sign holds, it is easy to prove that the parameters have the same form, changing the sign of  $r$ . We have from (5.11) that  $(9U + 2(\Delta - 1)^2)^2 + 144\Delta^2 = ((9U + 2(\Delta - 1)^2) + s)^2$  for some positive integer  $s$ , and hence  $2(9U + 2(\Delta - 1)^2)s + s^2 = 144\Delta^2$ . Since  $U \geq 0$ , we deduce that  $s \leq 39$  and, since 6 must divide  $s$ , then the possible values that  $s$  can take are 6, 12, 18, 24, 30, 36. Now, having into account that  $\Delta \equiv 1 \pmod{3}$  and the expression for  $n$  in Proposition 5.2 we have that for  $s = 18, 36$  we obtain a contradiction with the fact that  $n$  is an integer, and for

$s = 6, 24$  we obtain a contradiction with the fact that 3 divides  $n - 1$ . For  $s = 12$ , by putting  $\Delta = 1 + 3r$  we obtain parameters as in part (6.2), and for  $s = 30$ , by putting  $\Delta = 1 + 3u$  we have  $U = \frac{2u^2+8u-7}{5}$  and hence  $u \equiv -2 \pmod{5}$ . Putting now  $u = -2 + 5r$  we obtain parameters as in (6.3).  $\square$

We can eliminate some values of  $r$  in family (6.3). In the next proposition, Gröbner bases are used (for definitions and results on Gröbner bases, we refer the reader to [3]).

**Proposition 6.2.** *If a tricirculant uniform PSF exists with parameters as in family (6.3) of the previous proposition, then  $r \equiv 1 \pmod{2}$ .*

*Proof.* If  $\chi$  is any non-trivial character of  $H$  then we obtain from Definition 3.1, by applying the character  $\chi$ , that

$$\sum_{l=0}^2 \chi(S_{i,l})\chi(S_{l,j}) - \delta_{i,j}\gamma - \beta\chi(S_{i,j}) = 0 \text{ for every } i, j. \tag{6.4}$$

By putting the indeterminate  $T_{i,j}$  instead of  $\chi(S_{i,j})$  and calculating a Gröbner basis for the polynomials obtained from the lefthandsides in (6.4) with respect to the lexicographic order with

$$T_{0,0} > T_{1,1} > T_{2,2} > T_{0,1} > T_{1,0} > T_{0,2} > T_{2,0} > T_{1,2} > T_{2,1} > r,$$

we conclude that

$$\begin{aligned} &\chi(S_{1,1})^2 - \chi(S_{1,1}) - 6 + 35r + \chi(S_{1,2})\chi(S_{2,1}) \\ &- 50r^2 + \chi(S_{0,1})\chi(S_{1,0}) + 5\chi(S_{1,1})r = 0. \end{aligned} \tag{6.5}$$

Now we take the character  $\chi$  that takes the generator of the cyclic group  $H$  to  $\theta^{n/2}$ , where  $\theta$  is a primitive  $n$ -th root of unity. Since  $A$  is an odd number and  $B$  is an even number, we have that the  $\chi(S_{i,j})$  are integer numbers such that  $\chi(S_{i,j})$  is odd if  $i = j$  and even in other case. Now, we obtain from (6.5) that

$$(\chi(S_{1,1}) - 1)(\chi(S_{1,1}) + r) \equiv 2(r^2 + 1) \pmod{4}.$$

Suppose that  $r$  is even. Then the above implies that  $\chi(S_{1,1}) \equiv 3 \pmod{4}$ . We show below that this leads to a contradiction, hence  $r$  must be indeed odd.

By the choice of the character  $\chi$  we get for every  $i \in \{0, 1, 2\}$ ,

$$\chi(S_{i,i}) = -|S_{i,i}| + 2|S_{i,i} \cap \ker(\chi)| = -30r^2 + 20r - 3 + 2|S_{i,i} \cap \ker(\chi)|.$$

The group  $\ker(\chi)$  is of order  $n/2 = 45r^2 - 30r + 5$ , which is odd. Using this and that  $\cup_{i=0}^2 (S_{i,i} \cap \ker(\chi)) = \ker(\chi) - \{0\}$ , we see that at least one of the numbers  $|S_{i,i} \cap \ker(\chi)|$  must be even. We may set the indices at the beginning so that  $|S_{1,1} \cap \ker(\chi)|$  is even, and hence above we find  $\chi(S_{1,1}) \equiv 1 \pmod{4}$ , a contradiction.  $\square$

Apart from the infinite families of uniform PSFs showed in Section 4, we will analyze some sporadic examples for the tricirculant case.

When  $s = 0$  in the first family in Proposition 6.1, that is, when the graph is undirected, Kutnar et al. found in [16] PSFs for  $r = 1, 2$  and 3.

For  $r = -1, s = 2$ , a PSF with the parameters and structure indicated in Proposition 6.1 was found by the author and A. Araluze in [20].

We have found now for  $r = 2, s = 2$ , by using techniques of combinatorial optimization, a uniform tricirculant PSF which generates a DSRG with new parameters, which appear as an undecided case in Hobart and Brouwer's table [10]. This PSF is

$$S_{0,0} = (3, 4, 7, 9), S_{0,1} = (0, 4, 7, 8, 11, 12), S_{0,2} = (0, 1, 2, 4, 6, 10),$$

$$S_{1,0} = (1, 2, 5, 6, 8, 9), S_{1,1} = (1, 5, 6, 12), S_{1,2} = (2, 3, 5, 6, 7, 8),$$

$$S_{2,0} = (0, 1, 3, 7, 9, 11), S_{2,1} = (0, 5, 6, 7, 8, 11), S_{2,2} = (2, 8, 10, 11).$$

## 7 Almost-uniform partial sum families

In this section we will present a kind of partial sum families that generalizes the uniform ones.

**Definition 7.1.** Let  $H$  be a group of order  $n \geq 2$ ,  $H'$  a subgroup of  $H$  of order  $n'$  and let  $m \geq 1$  be an integer. Then an  $(m, n, k, \mu, \lambda, t)$ -partial sum family  $\mathfrak{S} = \{S_{i,j}\}$ , with  $0 \leq i, j < m$ , with the  $S_{i,j}$  subsets of  $H$ , is almost-uniform with respect to  $H'$  if it satisfies the following conditions:

1. The cardinalities of the 'diagonal' blocks  $S_{i,i}$  are all equal.
2. The cardinalities of the 'non-diagonal' blocks  $S_{i,j}, i \neq j$ , are all equal.
3. The 'diagonal' blocks  $\{S_{i,i} : i \in \mathbb{Z}_m\}$  form a partition of  $H - H'$ .

**Remark 7.2.** Observe that in the particular case when  $H' = \{e\}$  almost uniform PSFs are just uniform PSFs. In what follows, we will study the case when  $H'$  is a proper subgroup of  $H$ .

Let us analyze the form of the parameters of  $m$ -circulant  $(m, n, k, \mu, \lambda, t)$ -almost-uniform partial sum families over a cyclic group  $H$  with respect to a proper and non-trivial subgroup  $H'$ :

**Proposition 7.3.** *If an  $(m, n, k, \mu, \lambda, t)$ -almost-uniform partial sum family exists with  $m \geq 3$  with respect to a proper and non-trivial subgroup  $H'$  of the cyclic group  $H$ , then the parameters have one of the following forms:*

1.

$$n = (im - m + i)imr^2 + (1 - i)mr + ms,$$

$$k = (im - m + i)imr^2 + (-i + m - im)r + ms,$$

$$\mu = (im - m + i)ir^2 - ri + s,$$

$$\lambda = (im - m + i)ir^2 + (m - 3i)r + s,$$

$$t = -ri + i^2r^2m + s, A = (im - m + i)ir^2 + (1 - 2i)r + s,$$

$$\text{and } B = (im - m + i)ir^2 + (1 - i)r + s.$$

2.

$$n = (j^2 - j)r^2 + (-j - 1)r + sm + \frac{r^2j^2}{m},$$

$$k = (j^2 - j)r^2 - rj + sm + \frac{rj(-1 + rj)}{m},$$

$$\mu = s + \frac{rj(rjm - rm + rj - m)}{m^2},$$

$$\lambda = s + r + \frac{rj(-3m - rm + rjm + rj)}{m^2},$$

$$t = s + \frac{rj(-1 + rj)}{m},$$

$$A = s + \frac{rj(rmj - 2m - mr + rj)}{m^2},$$

$$B = s + \frac{rj(-m - mr + rmj + rj)}{m^2},$$

where  $m$  divides  $rj$ .

3.

$$n = \frac{2p(p-1)mr^2}{s}, k = -\frac{r(2rm^2p - 2rm^2p^2 + si + sp - smp)}{sm},$$

$$t = -(-2sr^2mi + 2sr^2i^2 + 4sr^2ip - 2sr^2mp + 2sr^2p^2 + 4m^2r^2p - 4m^2r^2p^2 + 2srm - 4srmp - s^2)/(2m^2s),$$

$$\lambda = -1/2 \frac{4m^2r^2p - 4r^2m^2p^2 + 2srm - s^2 - 2rm^2s + 4srmi}{m^2s},$$

$$\mu = 1/2 \frac{(s + 2rmp)(s - 2rm + 2rmp)}{m^2s},$$

$$A = -\frac{r(-2mrp^2 + 2mrp + si)}{ms}, B = \frac{rp(s - 2mr + 2mrp)}{ms}.$$

*Proof.* We have  $\cup_i S_{i,i} = H - H'$ . If  $\chi$  is a character of  $H$  which is non-trivial on  $H'$ , then  $\sum_i \chi(S_{i,i}) = 0$ . If  $\chi'$  is trivial on  $H'$  but not on  $H$ , then  $\sum_i \chi'(S_{i,i}) = -|H'|$ . Since, by Lemma 5.1, the difference of both sums of characters must be divisible by  $\Delta$ , we have that  $|H'| = i\Delta$  with  $i \in \{1, \dots, m\}$ . Since, again by Lemma 5.1,  $\sum_l \chi(S_{l,l}) = m(\beta - \Delta)/2 + j\Delta$  with  $j \in \{0, \dots, m\}$ , we have

$$\beta = (m - 2j)\Delta/m, \quad (7.1)$$

where  $\Delta$  is a non-negative integer. We have

$$\gamma = (jm - j^2)\Delta^2/m^2. \quad (7.2)$$

Now,

$$\begin{aligned} \gamma &= t - \mu = (jm - j^2)\Delta^2/m^2, k = U + \mu + (jm - j^2)\Delta^2/m^2 \text{ (where } U = k - t), \\ \beta &= (m - 2j)\Delta/m. \end{aligned} \quad (7.3)$$

From these equalities and (1.1) we get

$$\begin{aligned} n &= (Um^2 + \mu m^2 + \Delta m j + \Delta^2 m j - \Delta^2 j^2)(Um^2 + \mu m^2 + \Delta m j \\ &\quad - \Delta m^2 + j \Delta^2 m - j^2 \Delta^2)/(m^5 \mu) \end{aligned} \quad (7.4)$$

Let us suppose that the plus sign holds in Lemma 4.4 Since the PSF is almost-uniform we have  $n - i\Delta = mA$ , and therefore

$$n = \frac{Um^2 + \mu m^2 + \Delta m j - j^2 \Delta^2 + j \Delta^2 m - \Delta m^2 j + i \Delta m^2}{m^2} \quad (7.5)$$

From (7.4) and (7.5) we deduce

$$\begin{aligned} \mu &= (-\Delta m^2 - m^3 U + m j^2 \Delta^2 - m^3 i \Delta + 2 U m^2 - m^2 j \Delta^2 + m^3 \Delta j - \Delta m^2 j \\ &\quad + 2 \Delta m j - 2 j^2 \Delta^2 + 2 j \Delta^2 m \pm m((Um^2 - \Delta m - 2 \Delta m i + i \Delta m^2 + 3 \Delta m j \\ &\quad - j^2 \Delta^2 + j \Delta^2 m - \Delta m^2 j)^2 + 4 m^2 \Delta^2 (i - j + 1)(i - j)(m - 1))^{1/2}) \\ &\quad / (2m^2(m - 1)). \end{aligned}$$

Let us suppose that  $j = i$ . Then,

$$\mu = -\frac{Um^2 + \Delta m i + i \Delta^2 m - i^2 \Delta^2}{m^2} \quad (7.6)$$

or

$$\mu = \frac{Um^2 + \Delta m i - \Delta m^2 + i \Delta^2 m - i^2 \Delta^2}{m^2(m - 1)} \quad (7.7)$$

If (7.6) holds, then we get from (7.4) that  $n = 0$ , which is a contradiction.

Thus, (7.7) must hold, and we obtain

$$n = \frac{Um^2 + \Delta m i - \Delta m + i \Delta^2 m - i^2 \Delta^2}{m(m - 1)} \quad (7.8)$$

$$k = \frac{Um^2 + \Delta i - \Delta m + i\Delta^2 m - i^2 \Delta^2}{m(m-1)} \tag{7.9}$$

$$t = \frac{i\Delta^2 m - i^2 \Delta^2 + Um + \Delta i - \Delta m}{m(m-1)} \tag{7.10}$$

$$\lambda = \frac{Um^2 + 3\Delta mi - 2\Delta m^2 + i\Delta^2 m - i^2 \Delta^2 + \Delta m^3 - 2\Delta m^2 i}{m^2(m-1)} \tag{7.11}$$

$$\mu = \frac{Um^2 + \Delta mi - \Delta m^2 + i\Delta^2 m - i^2 \Delta^2}{m^2(m-1)} \tag{7.12}$$

$$A = \frac{Um^2 + 2\Delta mi - \Delta m + i\Delta^2 m - i^2 \Delta^2 - \Delta m^2 i}{m^2(m-1)} \tag{7.13}$$

$$B = \frac{Um^2 + \Delta mi - \Delta m + i\Delta^2 m - i^2 \Delta^2}{m^2(m-1)} \tag{7.14}$$

From (7.12) we have  $i^2 \Delta^2 \equiv 0 \pmod{m}$  and, from (7.9),  $-i^2 \Delta^2 + i\Delta \equiv 0 \pmod{m}$ . Therefore,  $i\Delta \equiv 0 \pmod{m}$ .

From (7.13),  $(im - i^2)\Delta^2 + (2i - 1)m\Delta \equiv 0 \pmod{m^2}$ . Using this and the previous congruence, we obtain  $m\Delta \equiv 0 \pmod{m^2}$ , and hence

$$\Delta \equiv 0 \pmod{m}. \tag{7.15}$$

From (7.12),

$$U \equiv (i^2 - i)\Delta^2 + (1 - i)\Delta \pmod{m - 1}. \tag{7.16}$$

Now, using the two previous congruences and putting

$$\Delta = rm \text{ and } U = i(i-1)r^2 m^2 + (1-i)rm + s(m-1),$$

we obtain parameters as in part 1 of the proposition.

Now, let us suppose that  $j = i + 1$ .

In this case, by reasoning in a similar way as before, we obtain

$$n = \frac{Um^2 - \Delta m^2 + j\Delta^2 m + \Delta m + \Delta mj - j^2 \Delta^2}{m(m-1)} \tag{7.17}$$

$$k = \frac{Um^2 + j\Delta + j\Delta^2 m - j^2 \Delta^2}{m(m-1)} \tag{7.18}$$

$$\mu = \frac{Um^2 + \Delta mj + j\Delta^2 m - j^2 \Delta^2}{m^2(m-1)} \tag{7.19}$$

$$\lambda = \frac{Um^2 + 3\Delta mj + j\Delta^2 m - j^2 \Delta^2 - \Delta m^2 + \Delta m^3 - 2\Delta m^2 j}{m^2(m-1)} \tag{7.20}$$

$$t = \frac{j\Delta^2 m - j^2 \Delta^2 + Um + j\Delta}{m(m-1)} \tag{7.21}$$

$$A = \frac{Um^2 + j\Delta^2m + 2\Delta mj - j^2\Delta^2 - \Delta m^2j}{m^2(m-1)} \quad (7.22)$$

$$B = \frac{Um^2 + \Delta mj + j\Delta^2m - j^2\Delta^2}{m^2(m-1)} \quad (7.23)$$

From (7.19) we have  $\Delta^2j^2 \equiv 0 \pmod{m}$ , and from (7.18),  $\Delta j - \Delta^2j^2 \equiv 0 \pmod{m}$ , and hence  $\Delta j \equiv 0 \pmod{m}$ .

From (7.19), we have  $U + \Delta j + \Delta^2j - \Delta^2j^2 \equiv 0 \pmod{m-1}$ .

Putting  $U = -\Delta j - \Delta^2j + \Delta^2j^2 + s(m-1)$  and  $\Delta = r$ , we obtain parameters as in part 2 of the proposition.

When  $j \neq i$  and  $j \neq i+1$  we find the parameters following the line of the proof of Proposition 5.2 and putting  $p = j - i$  and  $r = \Delta$ .

If the minus sign holds in Lemma 4.4, we find analogous families of parameters by putting  $r = -\Delta$ , so that  $r$  can be positive or negative.  $\square$

We have found, by using methods of combinatorial optimization, 14 examples of PSFs corresponding to family 1, which are listed in the appendix. For three of the examples, corresponding to  $m = 3, i = 1, r = -1, s = 3, m = 3, i = 1, r = -1, s = 4$  and to  $m = 5, i = 1, r = 1, s = 2$ , the parameters appear as undecided cases in Hobart and Brouwer's table [10]. We wonder if such PSFs exist for  $m = 3, i = 1, r = 1$  or  $r = -1$  and every positive integer  $s$ .

## 8 Appendix

Family 1:

For  $m = 3$ :

1.  $i = 1, r = -1, s = 1$ :

((1), (0, 1, 5), (2, 3, 5)), ((0, 3, 5), (3), (0, 1, 4)),  
((1, 3, 4), (0, 1, 2), (5)))

2.  $i = 1, r = -1, s = 2$ :

((5, 8), (0, 4, 6, 7), (0, 1, 3, 7)), ((1, 2, 3, 5), (2, 4), (0, 5, 6, 7)),  
((2, 3, 6, 8), (0, 3, 4, 7), (1, 7)))

3.  $i = 1, r = -1, s = 3$ :

((5, 9, 11), (1, 2, 4, 9, 10), (0, 1, 3, 8, 9)),  
((3, 5, 8, 9, 11), (1, 2, 10), (0, 1, 3, 7, 9)),  
((0, 2, 3, 4, 11), (4, 5, 7, 8, 9), (3, 6, 7)))

4.  $i = 1, r = -1, s = 4$ :

((6, 9, 12, 13), (0, 1, 3, 4, 11, 12), (6, 7, 10, 11, 13, 14)),  
((1, 4, 5, 8, 10, 14), (1, 7, 11, 14), (0, 2, 5, 6, 9, 11)),  
((1, 2, 3, 4, 7, 9), (0, 7, 8, 9, 10, 13), (2, 3, 4, 8)))

5.  $i = 1, r = 1, s = 1$ :

((5), (3, 4), (2, 3)), ((2, 3), (1), (0, 5)), ((0, 3), (1, 4), (3)))

6.  $i = 1, r = 1, s = 2$ :

$((7, 8), (0, 1, 2), (2, 3, 4)), ((3, 7, 8), (1, 5), (2, 3, 7)),$   
 $((0, 5, 7), (0, 2, 7), (2, 4))$

7.  $i = 1, r = 1, s = 3$ :

$((3, 9, 10), (0, 5, 6, 7), (6, 7, 8, 9)), ((0, 3, 9, 10), (5, 6, 7), (6, 7, 8, 9)),$   
 $((2, 3, 5, 8), (0, 5, 10, 11), (1, 2, 11))$

8.  $i = 1, r = 1, s = 4$ :

$((6, 8, 12, 14), (3, 5, 9, 11, 12), (5, 7, 11, 13, 14)),$   
 $((1, 3, 4, 5, 7), (1, 2, 4, 13), (0, 2, 3, 4, 6)),$   
 $((1, 4, 8, 10, 12), (1, 5, 7, 9, 13), (3, 7, 9, 11))$

For  $m = 4$ :

1.  $i = 1, r = -1, s = 1$ :

$((4, 6, 7), (4, 5), (4, 7), (0, 1)), ((3, 4), (1, 2, 4), (4, 5), (1, 6)),$   
 $((1, 4), (2, 3), (4, 5, 6), (2, 3)), ((0, 1), (2, 7), (5, 6), (2, 3, 4))$

2.  $i = 1, r = 1, s = 1$ :

$((5), (4, 7), (3, 6), (4, 7)), ((1, 2), (1), (0, 7), (0, 1)),$   
 $((1, 2), (0, 1), (7), (0, 1)), ((1, 4), (0, 3), (2, 7), (3))$

3.  $i = 1, r = 1, s = 2$ :

$((1, 2), (3, 4, 5), (4, 5, 6), (2, 6, 10)),$   
 $((2, 4, 9), (5, 7), (1, 6, 8), (2, 6, 10)),$   
 $((6, 7, 8), (9, 10, 11), (10, 11), (0, 4, 8)),$   
 $((0, 2, 7), (3, 5, 10), (4, 6, 11), (4, 8))$

4.  $i = 1, r = 1, s = 3$ :

$((3, 13, 14), (8, 9, 10, 11), (1, 2, 3, 12), (0, 9, 10, 11)),$   
 $((4, 5, 7, 10), (1, 2, 15), (3, 8, 9, 10), (0, 1, 2, 7)),$   
 $((1, 2, 4, 7), (12, 13, 14, 15), (5, 6, 7), (4, 13, 14, 15)),$   
 $((0, 3, 13, 14), (8, 9, 10, 11), (1, 2, 3, 12), (9, 10, 11))$

For  $m = 5$ :

1.  $i = 1, r = 1, s = 1$ :

$((5), (0, 7), (1, 2), (6, 7), (3, 6)), ((3, 8), (3), (4, 5), (0, 9), (6, 9)),$   
 $((4, 9), (6, 9), (1), (5, 6), (2, 5)), ((3, 8), (0, 3), (4, 5), (9), (6, 9)),$   
 $((4, 9), (1, 4), (5, 6), (0, 1), (7))$

2.  $i = 1, r = 1, s = 2$ :

$((2, 13), (0, 2, 4), (7, 9, 11), (0, 2, 4), (10, 12, 14)),$   
 $((3, 13, 14), (1, 5), (7, 8, 12), (0, 1, 5), (0, 10, 11)),$   
 $((1, 5, 6), (3, 7, 8), (10, 14), (3, 7, 8), (2, 3, 13)),$   
 $((2, 6, 13), (0, 4, 8), (0, 7, 11), (4, 8), (3, 10, 14)),$   
 $((3, 10, 14), (1, 5, 12), (4, 8, 12), (1, 5, 12), (7, 11))$

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