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A catalogue of simplicial arrangements in the real projective plane

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Abstract

An *arrangement* is the complex generated in the real projective plane by a finite family of straight lines that do not form a pencil. The faces of an arrangement are the connected components of the complement of the union of lines. An arrangement is *simplicial* if all its faces are simplices (triangles). The present paper updates the only previous catalogue of simplicial arrangements, published almost 40 years ago, and presented in a very condensed form. The simplicial arrangements often provide optimal solutions for various problems. A problem that seems very hard is to decide whether the collection presented here is a complete listing of simplicial arrangements.

Keywords: Simplicial arrangement.

Math. Subj. Class.: 51M16

1 Introduction

An *arrangement* is the complex generated in the *real projective plane* by a family of straight lines that do not form a pencil. The *vertices* of an arrangement are the intersection points of two or more lines, the *edges* are the segments into which the lines are partitioned by the vertices, and the *faces* are the connected components of the complement of the set of lines generating the arrangement. Figure 1 shows an arrangement generated by five lines; it has five vertices, twelve edges, and eight faces. A *face vector* of an arrangement is a triplet $f = (f_0, f_1, f_2)$, where f_0 is the number of vertices, f_1 the number of edges, and f_2 the number of faces of the arrangement. For the arrangement in Figure 1 we have $f = (5, 12, 8)$. As is well known, Euler's theorem implies that $f_0 - f_1 + f_2 = 1$ for every arrangement.

An arrangement is *simplicial* if all faces are triangles. Clearly, for every simplicial arrangement $2f_1 = 3f_2$. Simplicial arrangements were first introduced by Melchior [8]; an extensive account appeared in [5].

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Throughout the paper we find it convenient to represent the real projective plane in the old-fashioned way as the *extended* Euclidean plane. This consists of all the points of the real Euclidean plane, together with the "ideal points-at-infinity" (each of which corresponds to a pencil of all mutually parallel lines) and with the "line-at-infinity" (that is formed by the totality of the points-at-infinity). This way of representing those projective arrangements that we are interested in avoids the need for an algebraic presentation and makes easily possible the visual verification of the *simplicial* character of the arrangements we are considering.

Three infinite families of (isomorphism classes of) simplicial arrangements are known; we denote them by $\mathcal{R}(0)$, $\mathcal{R}(1)$ and $\mathcal{R}(2)$. Besides these families of simplicial arrangements, only 90 other simplicial arrangements are known; we call them *sporadic* arrangements.

Family $\mathcal{R}(0)$ consists of all *near-pencils*. A near-pencil denoted $\mathcal{A}(n,0)$ consists of $n-1$ lines that have a point in common; the last ("exceptional") line is not incident with that point. Two examples, with $n = 5$, are shown in Figure 1. Near-pencil simplicial arrangements $\mathcal{A}(n,0)$ exist for all $n \geq 3$. For each n, all $\mathcal{A}(n,0)$ are isomorphic; for $n \geq 5$ there are uncountably many projectively inequivalent arrangements $\mathcal{A}(n,0)$.

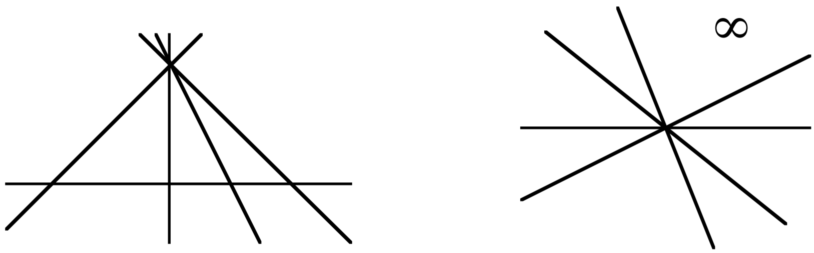

Figure 1: Two illustrations of near-pencils $\mathcal{A}(5,0)$ from the family $\mathcal{R}(0)$. These (as well as all the other diagrams) are set in the *extended Euclidean plane*, which is a convenient model of the real projective plane. In the diagram at left, some of the triangles are the usual Euclidean triangles, while each of the other triangles consists of two components, disjoint in the Euclidean plane but contiguous "at-infinity"; each of these triangles is bounded by a finite segment and by two rays each of two lines. In the diagram at right, the exceptional line is the "line-at-infinity". Each of the triangles has as sides two Euclidean rays and a segment of the line-at-infinity; it has one vertex in the finite plane and two vertices at-infinity. In this illustration, and throughout the catalogue, the inclusion of the line-at-infinity as one of the lines of the arrangement is indicated by the infinity symbol ∞.

The family $\mathcal{R}(1)$ consists of simplicial arrangements denoted $\mathcal{A}(n,1)$; these are most easily described as follows. The arrangement $\mathcal{A}(n,1)$ exists for all even $n = 2m$, with $m \geq 3$. Starting with a regular convex m-gon in the Euclidean plane, $\mathcal{A}(n,1)$ is obtained by taking the m lines determined by the sides of the m-gon together with the m lines of mirror symmetry of that m-gon. The arrangements $\mathcal{A}(8,1)$ and $\mathcal{A}(10,1)$ are illustrated in Figure 2.

The family $\mathcal{R}(2)$ consists of simplicial arrangements denoted $\mathcal{A}(n,1)$, for $n = 4m+1$ with $m \geq 2$. The arrangement $\mathcal{A}(4m+1,1)$ is obtained from $\mathcal{A}(4m,1)$ in the $\mathcal{R}(1)$ family by adjoining the "line-at-infinity" in the extended Euclidean plane model of the projective

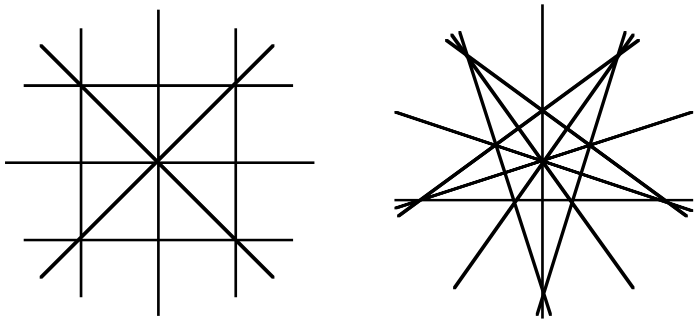

Figure 2: A view of the arrangements $\mathcal{A}(8, 1)$ and $\mathcal{A}(10, 1)$ from the family $\mathcal{R}(1)$.

plane. An example with $m = 3$ is shown in Figure 3.

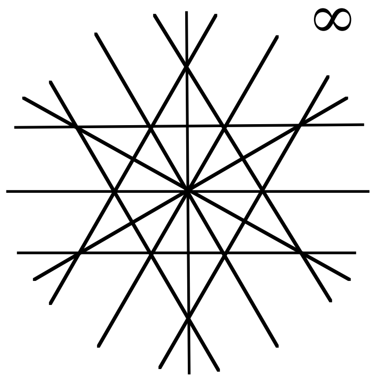

Figure 3: A view of the arrangement $\mathcal{A}(13, 1)$ from the family $\mathcal{R}(2)$.

Sporadic arrangements are rather mysterious. There is no known explanation *why* the ones that exist do exist, or why others do not. In particular, there is no known explanation for the observation that no sporadic simplicial arrangement found so far has more than 37 lines. The restriction cannot be related to the *topology* of the projective plane, since there are several additional infinite families of simplicial arrangements of *pseudolines*, besides many additional sporadic ones.

The present compilation of data about simplicial arrangements arose from several aims. First, there has been no detailed published account of the known simplicial arrangements beyond the paper [5], published more that a third of a century ago. The collection in which the paper appeared is not widely available, and I have to admit that the presentation there is not "user-friendly". Moreover, although the number of sporadic arrangements is still the same as quoted there, two changes have occurred. One pair of arrangements — denoted $\mathcal{A}_2(17)$ and $\mathcal{A}_7(17)$ in the paper — have been found to be isomorphic, as reported in [7] and [1, p. 64]. On the other hand, an additional arrangement $\mathcal{A}(16, 7)$ was found, as indicated in [6, pages 7 and 9]. The complete list of the presently known (isomorphism classes of) simplicial arrangements is given in the table below, and the 90 sporadic members (as well as selected other arrangements) are illustrated by one or more diagrams at the end of this catalog. For each arrangement we also list in the table several data, as well any comments that may seem appropriate.

I conjecture that the present list is a complete enumeration of isomorphism classes of sporadic arrangements; the language throughout the paper is simplified by assuming that this is a fact, even though it is the most attractive open problem concerning simplicial arrangements.

The second reason for this collection is the possibility that a better presentation than the one in [5] may generate more interest in the challenging topic. One way of improving the presentation is by noticing that the simplicial arrangements form a partially ordered set, with respect to the relation in which lines of one are a subset of lines of another. The interesting aspect is that there is only a small number of arrangements that are maximal with respect to the number of lines. This is visible from the Hasse diagram shown in Figure 4. Indeed, from their construction it is clear that none of the arrangements in the families $\mathcal{R}(0)$, $\mathcal{R}(1)$ and $\mathcal{R}(2)$ is maximal, while the diagram shows there are only ten sporadic maximal ones. In the diagram, the maximal arrangements are indicated by bold-framed numerals. The numerals with shaded backgrounds indicate "pseudo-minimal" sporadic simplicial arrangements, that is, arrangements that do not contain as subarrangements any sporadic arrangements. A few arrangements from the families $\mathcal{R}(1)$ and $\mathcal{R}(2)$ are also shown in cases where this seemed appropriate.

An additional reason is that the simplicial arrangements, or arrangements that are nearly simplicial, appear as examples or counterexamples in many contexts of combinatorial geometry and its applications. This is illustrated, among others, in [6], [7], [1], [2, p. 825], [3, p. 86], [4, Section 5.4].

The list of simplicial arrangements that follows includes all the arrangements with at most 37 lines. The sporadic arrangement denoted $\mathcal{A}(n,k)$ coincides with the arrangement denoted $\mathcal{A}_k(n)$ in [5], except that $\mathcal{A}(17,7)$ is $\mathcal{A}_9(17)$ because $\mathcal{A}_7(17)$ has been shown to be isomorphic to $\mathcal{A}_2(17)$ and was deleted. The first column, n, denotes the number of lines, the second column the notation for the isomorphism class of the arrangement. The third column shows the f-vector $f = (f_0, f_1, f_2)$, the number of vertices, edges, and faces of the arrangement. The fourth column, $t = (t_2, t_3, t_4, \dots)$, list the values of t_j, that is the number of vertices incident with precisely j of the lines. The fifth column, $r = (r_2, r_3, r_4, \dots)$, similarly gives the value of r_j, the number of lines each of which is incident with precisely j of the vertices. Some of the t- and r-vectors contain many consecutive zeros; in order to save space, the presence of p consecutive zeros has been indicated by 0^p. The last column serves to indicate which of the arrangements are non-sporadic, as well as which are maximal (signaled by M) or pseudominimal (indicated by m).

As is visible from the list, most of the arrangements are characterized by their f- and t-vectors. In a majority of the other cases, the r-vectors distinguish between non-isomorphic arrangements. In only four cases are these vectors not sufficient. In such cases a criterion that describes the difference is explained following the diagrams of the arrangements in question.

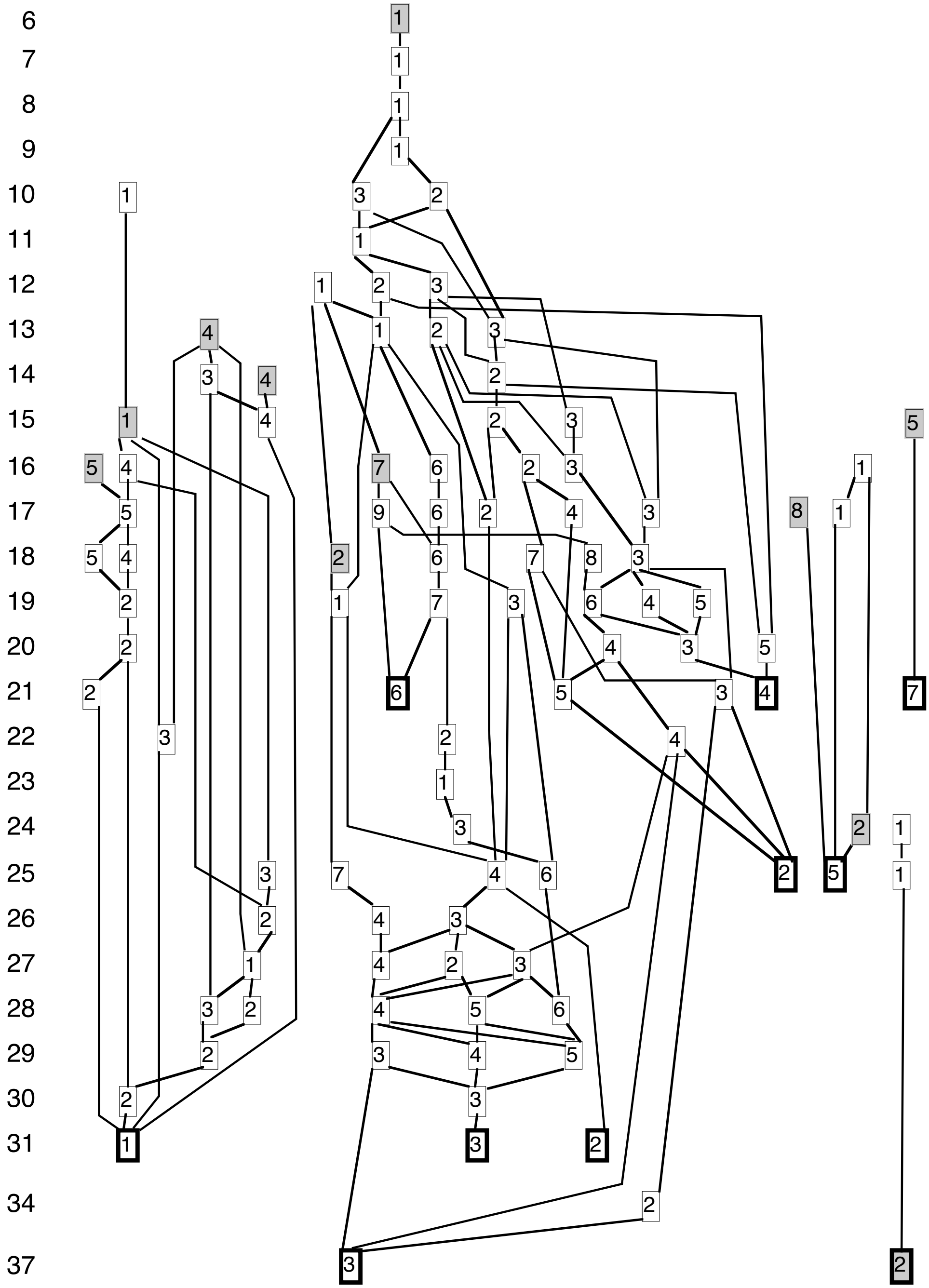

Figure 4: A Hasse diagram of the partially ordered set of sporadic simplicial arrangements, together with a few arrangements from the families $\mathcal{R}(1)$ and $\mathcal{R}(2)$. The arrangement $\mathcal{A}(n,k)$ is indicated by the entry k in row n.

2 The list of simplicial arrangements with $n \leq 37$

n	$\mathcal{A}(n,0)$	$f=(f_0,f_1,f_2)$	$t=(t_2,t_3,t_4,\dots)$	$r=(r_2,r_3,r_4,\dots)$	Notes
3	$\mathcal{A}(3,0)$	$(3,6,4)$	(3)	(3)	$\mathcal{R}(0)$
4	$\mathcal{A}(4,0)$	$(4,9,6)$	$(3,1)$	$(3,1)$	$\mathcal{R}(0)$
5	$\mathcal{A}(5,0)$	$(5,12,8)$	$(4,0,1)$	$(4,0,1)$	$\mathcal{R}(0)$
6	$\mathcal{A}(6,0)$	$(6,15,10)$	$(5,0,0,1)$	$(5,0,0,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(6,1)$	$(7,18,12)$	$(3,4)$	$(0,6)$	$\mathcal{R}(1)$
7	$\mathcal{A}(7,0)$	$(7,18,12)$	$(6,0,0,0,1)$	$(6,0,0,0,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(7,1)$	$(9,24,16)$	$(3,6)$	$(0,4,3)$	m
8	$\mathcal{A}(8,0)$	$(8,21,14)$	$(7,0^4,1)$	$(7,0^4,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(8,1)$	$(11,30,20)$	$(4,6,1)$	$(0,2,6)$	$\mathcal{R}(1)$
9	$\mathcal{A}(9,0)$	$(9,24,16)$	$(8,0^5,1)$	$(8,0^5,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(9,1)$	$(13,36,24)$	$(6,4,3)$	$(0,0,9)$	$\mathcal{R}(2)$
10	$\mathcal{A}(10,0)$	$(10,27,18)$	$(9,0^6,1)$	$(9,0^6,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(10,1)$	$(16,45,30)$	$(5,10,0,1)$	$(0,0,5,5)$	$\mathcal{R}(1)$
	$\mathcal{A}(10,2)$	$(16,45,30)$	$(6,7,3)$	$(0,0,6,3,1)$	
	$\mathcal{A}(10,3)$	$(16,45,30)$	$(6,7,3)$	$(0,1,3,6)$	
11	$\mathcal{A}(11,0)$	$(11,30,20)$	$(10,0^7,1)$	$(10,0^7,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(11,1)$	$(19,54,36)$	$(7,8,4)$	$(0,0,4,4,3)$	
12	$\mathcal{A}(12,0)$	$(11,33,22)$	$(11,0^8,1)$	$(11,0^8,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(12,1)$	$(22,63,42)$	$(6,15,0,0,1)$	$(0,0,3,3,6)$	$\mathcal{R}(1)$
	$\mathcal{A}(12,2)$	$(22,63,42)$	$(8,10,3,1)$	$(0,0,3,3,6)$	
	$\mathcal{A}(12,3)$	$(22,63,42)$	$(9,7,6)$	$(0,0,3,3,6)$	
13	$\mathcal{A}(13,0)$	$(12,36,24)$	$(12,0^9,1)$	$(12,0^9,1)$	$\mathcal{R}(0)$
	$\mathcal{A}(13,1)$	$(25,72,48)$	$(9,12,3,0,1)$	$(0,0,3,0,10)$	$\mathcal{R}(2)$
	$\mathcal{A}(13,2)$	$(25,72,48)$	$(12,4,9)$	$(0,0,3,0,10)$	
	$\mathcal{A}(13,3)$	$(25,72,48)$	$(10,10,3,2)$	$(0,0,1,4,8)$	
	$\mathcal{A}(13,4)$	$(27,78,52)$	$(6,18,3)$	$(0^4,13)$	m
14	$\mathcal{A}(14,0)$	$(14,39,26)$	$(13,0^{10},1)$	$(13,0^{10},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(14,1)$	$(29,84,56)$	$(7,21,0,0,0,1)$	$(0,0,0,7,0,7)$	$\mathcal{R}(1)$
	$\mathcal{A}(14,2)$	$(29,84,56)$	$(11,12,4,2)$	$(0,0,1,4,4,4,1)$	
	$\mathcal{A}(14,3)$	$(30,87,58)$	$(9,16,4,1)$	$(0^4,11,3)$	
	$\mathcal{A}(14,4)$	$(29,84,56)$	$(10,14,4,0,1)$	$(0,0,0,4,6,4)$	m
15	$\mathcal{A}(15,0)$	$(15,42,28)$	$(14,0^{11},1)$	$(14,0^{11},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(15,1)$	$(31,90,60)$	$(15,10,0,6)$	$(0^4,15)$	m
	$\mathcal{A}(15,2)$	$(33,96,64)$	$(13,12,6,2)$	$(0,0,1,4,2,4,4)$	
	$\mathcal{A}(15,3)$	$(34,99,66)$	$(12,13,9)$	$(0^4,9,3,3)$	
	$\mathcal{A}(15,4)$	$(33,96,64)$	$(12,14,6,0,1)$	$(0^4,10,4,1)$	
	$\mathcal{A}(15,5)$	$(34,99,66)$	$(9,22,0,3)$	$(0^4,9,3,3)$	m
16	$\mathcal{A}(16,0)$	$(16,45,30)$	$(15,0^{12},1)$	$(15,0^{12},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(16,1)$	$(37,108,72)$	$(8,28,0^4,1)$	$(0,0,0,4,4,0,8)$	$\mathcal{R}(1)$
	$\mathcal{A}(16,2)$	$(37,108,72)$	$(14,15,6,1,1)$	$(0,0,1,2,4,2,7)$	
	$\mathcal{A}(16,3)$	$(37,108,72)$	$(15,13,6,3)$	$(0^4,10,0,6)$	
	$\mathcal{A}(16,4)$	$(36,105,70)$	$(15,15,0,6)$	$(0^4,10,5,0,0,1)$	
	$\mathcal{A}(16,5)$	$(37,108,72)$	$(14,16,3,4)$	$(0,0,0,2,4,8,0,2)$	m

Continued on next page

n	$\mathcal{A}(n,0)$	$f=(f_0,f_1,f_2)$	$t=(t_2,t_3,t_4,\dots)$	$r=(r_2,r_3,r_4,\dots)$	Notes
	$\mathcal{A}(16,6)$	$(37,108,72)$	$(15,12,9,0,1)$	$(0^4,7,6,3)$	
	$\mathcal{A}(16,7)$	$(38,111,74)$	$(12,19,6,0,1)$	$(0,0,0,3,3,2,8)$	m
17	$\mathcal{A}(17,0)$	$(17,48,32)$	$(16,0^{13},1)$	$(16,0^{13},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(17,1)$	$(41,120,80)$	$(12,24,4,0,0,0,1)$	$(0^4,8,0,9)$	$\mathcal{R}(2)$
	$\mathcal{A}(17,2)$	$(41,120,80)$	$(16,16,7,0,2)$	$(0,0,1,0,6,0,10)$	
	$\mathcal{A}(17,3)$	$(41,120,80)$	$(18,12,7,4)$	$(0^4,8,0,9)$	
	$\mathcal{A}(17,4)$	$(41,120,80)$	$(16,16,7,0,2)$	$(0,0,1,0,6,0,10)$	
	$\mathcal{A}(17,5)$	$(41,120,80)$	$(16,18,1,6)$	$(0^4,6,8,1,0,2)$	
	$\mathcal{A}(17,6)$	$(42,123,82)$	$(16,15,10,0,1)$	$(0^4,6,3,7,0,1)$	
	$\mathcal{A}(17,7)$	$(43,126,84)$	$(13,22,7,0,1)$	$(0^4,6,0,10,0,1)$	was $\mathcal{A}_9(17)$
	$\mathcal{A}(17,8)$	$(43,126,84)$	$(14,20,7,2)$	$(0,0,0,0,1,8,8)$	m
18	$\mathcal{A}(18,0)$	$(18,51,34)$	$(17,0^{14},1)$	$(17,0^{14},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(18,1)$	$(46,135,90)$	$(9,36,0^5,1)$	$(0^4,9,0,0,9)$	$\mathcal{R}(1)$
	$\mathcal{A}(18,2)$	$(46,135,90)$	$(18,18,6,3,1)$	$(0^4,3,3,12)$	m
	$\mathcal{A}(18,3)$	$(46,135,90)$	$(19,16,6,5)$	$(0^4,6,2,6,3,1)$	
	$\mathcal{A}(18,4)$	$(46,135,90)$	$(18,19,3,6)$	$(0^4,3,9,3,0,3)$	
	$\mathcal{A}(18,5)$	$(46,135,90)$	$(18,19,3,6)$	$(0^4,3,9,3,0,3)$	
	$\mathcal{A}(18,6)$	$(47,138,92)$	$(18,16,12,0,1)$	$(0^4,5,2,7,2,2)$	
	$\mathcal{A}(18,7)$	$(46,135,90)$	$(18,18,6,3,1)$	$(0^4,3,3,0,6,6)$	
	$\mathcal{A}(18,8)$	$(47,138,92)$	$(16,22,6,2,1)$	$(0^4,6,0,7,4,1)$	
19	$\mathcal{A}(19,0)$	$(19,54,36)$	$(18,0^{15},1)$	$(18,0^{15},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(19,1)$	$(49,144,96)$	$(21,18,6,0,4)$	$(0^4,4,0,15)$	
	$\mathcal{A}(19,2)$	$(51,150,100)$	$(21,18,6,6)$	$(0^4,1,8,6,0,4)$	
	$\mathcal{A}(19,3)$	$(49,144,96)$	$(24,12,6,6,1)$	$(0^4,4,0,15)$	
	$\mathcal{A}(19,4)$	$(51,150,100)$	$(20,20,6,4,1)$	$(0^4,4,4,4,4,3)$	
	$\mathcal{A}(19,5)$	$(51,150,100)$	$(20,20,6,4,1)$	$(0^4,4,4,4,4,3)$	
	$\mathcal{A}(19,6)$	$(51,150,100)$	$(20,20,6,4,1)$	$(0^4,6,0,6,4,3)$	
	$\mathcal{A}(19,7)$	$(52,153,102)$	$(21,15,15,0,1)$	$(0^4,4,3,3,6,3)$	
20	$\mathcal{A}(20,0)$	$(20,57,38)$	$(19,0^{16},1)$	$(19,0^{16},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(20,1)$	$(56,165,110)$	$(10,45,0^6,1)$	$(0^4,5,5,0,0,10)$	$\mathcal{R}(1)$
	$\mathcal{A}(20,2)$	$(56,165,110)$	$(25,15,10,6)$	$(0^5,5,10,0,5)$	
	$\mathcal{A}(20,3)$	$(56,165,110)$	$(21,24,6,4,0,1)$	$(0^4,4,2,4,6,3,1)$	
	$\mathcal{A}(20,4)$	$(56,165,110)$	$(23,20,7,5,1)$	$(0^4,5,1,4,4,6)$	
	$\mathcal{A}(20,5)$	$(55,162,108)$	$(20,26,4,4,0,0,1)$	$(0,0,0,2,2,0,4,12)$	
21	$\mathcal{A}(21,0)$	$(21,60,40)$	$(20,0^{17},1)$	$(20,0^{17},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(21,1)$	$(61,180,120)$	$(15,40,5,0^5,1)$	$(0^4,5,0,5,0,11)$	$\mathcal{R}(2)$
	$\mathcal{A}(21,2)$	$(61,180,120)$	$(30,10,15,6)$	$(0^6,15,0,6)$	
	$\mathcal{A}(21,3)$	$(61,180,120)$	$(24,24,9,0,4)$	$(0^4,6,0,3,0,12)$	
	$\mathcal{A}(21,4)$	$(61,180,120)$	$(22,28,6,4,0,0,1)$	$(0^4,4,0,4,8,4,0,1)$	M
	$\mathcal{A}(21,5)$	$(61,180,120)$	$(26,20,9,4,2)$	$(0^4,5,0,3,4,9)$	
	$\mathcal{A}(21,6)$	$(63,186,124)$	$(25,20,15,2,1)$	$(0^4,1,0,11,0,8,0,1)$	M
	$\mathcal{A}(21,7)$	$(64,189,126)$	$(24,22,15,3)$	$(0^6,12,0,6,3)$	M
22	$\mathcal{A}(22,0)$	$(22,63,42)$	$(21,0^{18},1)$	$(21,0^{18},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(22,1)$	$(67,198,132)$	$(11,55,0^7,1)$	$(0^5,11,0,0,0,11)$	$\mathcal{R}(1)$
	$\mathcal{A}(22,2)$	$(70,207,138)$	$(24,30,12,3,1)$	$(0^4,1,0,6,3,9,0,3)$	

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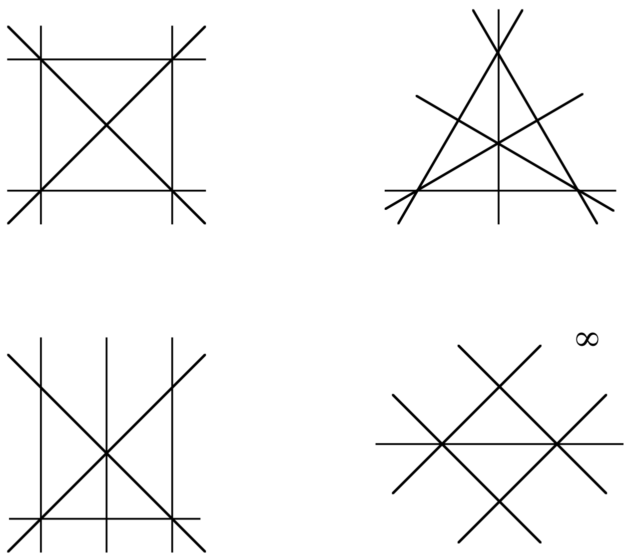
n	$\mathcal{A}(n,0)$	$f=(f_0,f_1,f_2)$	$t=(t_2,t_3,t_4,\dots)$	$r=(r_2,r_3,r_4,\dots)$	Notes
	$\mathcal{A}(22,3)$	$(67,198,132)$	$(27,28,0,12)$	$(0^6,12,0,9,0,1)$	
	$\mathcal{A}(22,4)$	$(67,198,132)$	$(27,25,9,3,3)$	$(0^4,4,0,6,0,6,6)$	
23	$\mathcal{A}(23,0)$	$(23,66,44)$	$(22,0^{19},1)$	$(22,0^{19},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(23,1)$	$(75,222,148)$	$(27,32,10,4,2)$	$(0^4,1,0,6,2,7,4,3)$	
24	$\mathcal{A}(24,0)$	$(24,69,46)$	$(23,0^{20},1)$	$(23,0^{20},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(24,1)$	$(79,234,156)$	$(12,66,0^8,1)$	$(0^5,6,6,0,0,0,12)$	$\mathcal{R}(1)$
	$\mathcal{A}(24,2)$	$(77,228,152)$	$(32,32,0,12,0,0,1)$	$(0^5,4,0,0,20)$	m
	$\mathcal{A}(24,3)$	$(80,237,158)$	$(31,32,9,5,3)$	$(0^4,1,0,6,1,6,6,4)$	
25	$\mathcal{A}(25,0)$	$(25,72,48)$	$(24,0^{21},1)$	$(24,0^{21},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(25,1)$	$(85,252,168)$	$(18,60,6,0^7,1)$	$(0^6,12,0,0,0,13)$	$\mathcal{R}(2)$
	$\mathcal{A}(25,2)$	$(85,252,168)$	$(36,28,15,0,6)$	$(0^4,4,0,3,0,6,0,12)$	M
	$\mathcal{A}(25,3)$	$(91,270,180)$	$(30,40,15,6)$	$(0^8,15,0,10)$	
	$\mathcal{A}(25,4)$	$(85,252,168)$	$(36,30,9,6,4)$	$(0^4,1,0,9,0,3,0,12)$	
	$\mathcal{A}(25,5)$	$(81,240,160)$	$(36,32,0,8,4,0,1)$	$(0^6,5,0,20)$	M
	$\mathcal{A}(25,6)$	$(85,252,168)$	$(36,30,9,6,4)$	$(0^4,1,0,6,0,6,6,6)$	
	$\mathcal{A}(25,7)$	$(85,252,168)$	$(33,34,12,2,3,0,1)$	$(0^4,2,0,4,4,4,0,1)$	
26	$\mathcal{A}(26,0)$	$(26,75,50)$	$(25,0^{22},1)$	$(25,0^{22},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(26,1)$	$(92,273,182)$	$(13,78,0^9,1)$	$(0^6,13,0^4,13)$	$\mathcal{R}(1)$
	$\mathcal{A}(26,2)$	$(96,285,190)$	$(35,40,10,11)$	$(0^8,11,5,10)$	
	$\mathcal{A}(26,3)$	$(92,273,182)$	$(37,36,9,6,3,1)$	$(0^4,1,0,7,2,2,1,8,4,1)$	
	$\mathcal{A}(26,4)$	$(92,273,182)$	$(35,39,10,4,3,0,1)$	$(0^4,1,1,4,4,2,2,7,4,1)$	
27	$\mathcal{A}(27,0)$	$(27,78,52)$	$(26,0^{23},1)$	$(26,0^{23},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(27,1)$	$(101,300,200)$	$(40,40,6,14,1)$	$(0^8,8,8,11)$	
	$\mathcal{A}(27,2)$	$(99,294,196)$	$(39,40,10,6,2,2)$	$(0^4,1,0,5,4,1,2,4,8,2)$	
	$\mathcal{A}(27,3)$	$(99,294,196)$	$(39,40,10,6,2,2)$	$(0^4,1,0,6,2,2,2,5,6,3)$	
	$\mathcal{A}(27,4)$	$(99,294,196)$	$(38,42,9,6,3,0,1)$	$(0^4,1,0,5,4,2,0,7,4,4)$	
28	$\mathcal{A}(28,0)$	$(28,81,54)$	$(27,0^{24},1)$	$(27,0^{24},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(28,1)$	$(106,315,210)$	$(14,91,0^{10},1)$	$(0^6,7,7,0^4,14)$	$\mathcal{R}(1)$
	$\mathcal{A}(28,2)$	$(106,315,210)$	$(45,40,3,15,3)$	$(0^8,6,9,13)$	
	$\mathcal{A}(28,3)$	$(106,315,210)$	$(45,40,3,15,3)$	$(0^8,6,9,13)$	
	$\mathcal{A}(28,4)$	$(106,315,210)$	$(41,44,11,6,2,1,1)$	$(0^4,1,0,4,4,2,1,4,6,6)$	
	$\mathcal{A}(28,5)$	$(106,315,210)$	$(42,42,12,6,1,3)$	$(0^4,1,0,4,4,1,3,1,10,4)$	
	$\mathcal{A}(28,6)$	$(106,315,210)$	$(42,42,12,6,1,3)$	$(0^4,1,0,6,0,3,3,3,6,6)$	
29	$\mathcal{A}(29,0)$	$(29,84,56)$	$(28,0^{25},1)$	$(28,0^{25},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(29,1)$	$(113,336,224)$	$(21,84,7,0^9,1)$	$(0^6,7,0,7,0,0,0,15)$	$\mathcal{R}(2)$
	$\mathcal{A}(29,2)$	$(113,336,224)$	$(50,40,1,14,6)$	$(0^8,5,8,16)$	
	$\mathcal{A}(29,3)$	$(113,336,224)$	$(44,46,13,6,2,0,2)$	$(0^4,1,0,3,4,3,0,4,4,10)$	
	$\mathcal{A}(29,4)$	$(113,336,224)$	$(45,44,14,6,1,2,1)$	$(0^4,1,0,3,4,2,2,1,8,8)$	
	$\mathcal{A}(29,5)$	$(113,336,224)$	$(45,44,14,6,1,2,1)$	$(0^4,1,0,4,2,3,2,2,6,9)$	
30	$\mathcal{A}(30,0)$	$(30,87,58)$	$(29,0^{26},1)$	$(29,0^{26},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(30,1)$	$(121,360,240)$	$(15,105,0^{11},1)$	$(0^7,15,0^5,15)$	$\mathcal{R}(1)$
	$\mathcal{A}(30,2)$	$(116,345,230)$	$(55,40,0,11,10)$	$(0^8,5,5,20)$	
	$\mathcal{A}(30,3)$	$(120,357,238)$	$(49,44,17,6,1,1,2)$	$(0^4,1,0,3,2,4,1,2,4,13)$	
31	$\mathcal{A}(31,0)$	$(31,90,60)$	$(30,0^{27},1)$	$(30,0^{27},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(31,1)$	$(121,360,240)$	$(60,40,0,6,15)$	$(0^8,6,0,25)$	M

Continued on next page

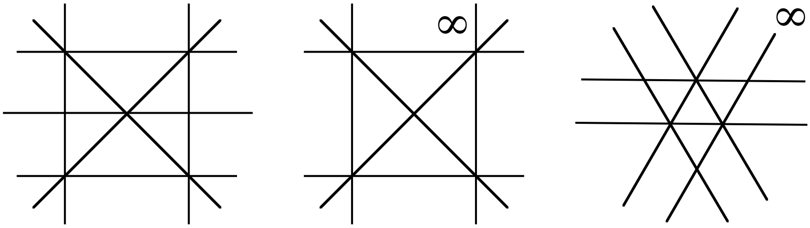
n	$\mathcal{A}(n,0)$	$f=(f_0,f_1,f_2)$	$t=(t_2,t_3,t_4,\ldots)$	$r=(r_2,r_3,r_4,\ldots)$	Notes
	$\mathcal{A}(31,2)$	$(127,378,252)$	$(54,42,21,6,1,0,3)$	$(0^4,1,0,0,0,9,0,6,0,15)$	M
	$\mathcal{A}(31,3)$	$(127,378,252)$	$(54,42,21,6,1,0,3)$	$(0^4,1,0,3,0,6,0,3,0,18)$	M
32	$\mathcal{A}(32,0)$	$(32,93,62)$	$(31,0^{28},1)$	$(31,0^{28},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(32,1)$	$(137,408,272)$	$(16,120,0^{12},1)$	$(0^7,8,8,0^5,16)$	$\mathcal{R}(1)$
33	$\mathcal{A}(33,0)$	$(33,96,63)$	$(32,0^{29},1)$	$(32,0^{29},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(33,1)$	$(145,432,288)$	$(24,112,8,0^{11},1)$	$(0^8,16,0^5,17)$	$\mathcal{R}(2)$
34	$\mathcal{A}(34,0)$	$(34,99,66)$	$(33,0^{30},1)$	$(33,0^{30},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(34,1)$	$(154,459,306)$	$(17,136,0^{13},1)$	$(0^8,17,0^6,17)$	$\mathcal{R}(1)$
	$\mathcal{A}(34,2)$	$(154,459,306)$	$(60,63,18,6,4,0,3)$	$(0^6,3,3,3,0,4,0,6,0,9,6)$	
35	$\mathcal{A}(35,0)$	$(35,102,68)$	$(34,0^{31},1)$	$(34,0^{31},1)$	$\mathcal{R}(0)$
36	$\mathcal{A}(36,0)$	$(36,105,70)$	$(35,0^{32},1)$	$(35,0^{32},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(36,1)$	$(172,513,342)$	$(18,153,0^{14},1)$	$(0^8,9,9,0^6,18)$	$\mathcal{R}(1)$
37	$\mathcal{A}(37,0)$	$(37,108,72)$	$(36,0^{33},1)$	$(36,0^{33},1)$	$\mathcal{R}(0)$
	$\mathcal{A}(37,1)$	$(181,540,360)$	$(27,144,9,0^{13},1)$	$(0^8,9,0,9,0^5,19)$	$\mathcal{R}(2)$
	$\mathcal{A}(37,2)$	$(181,540,360)$	$(72,72,12,24,0^6,1)$	$(0^{10},13,0,0,0,24)$	m,M
	$\mathcal{A}(37,3)$	$(181,540,360)$	$(72,72,24,0,10,0,3)$	$(0^6,3,0,6,0,4,0^3,12,0,12)$	M

3 Illustrations of selected arrangements

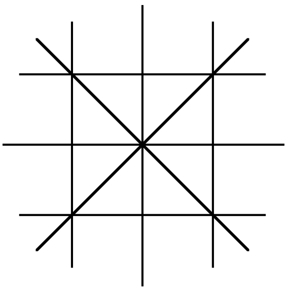


$\mathcal{A}(6,1)$

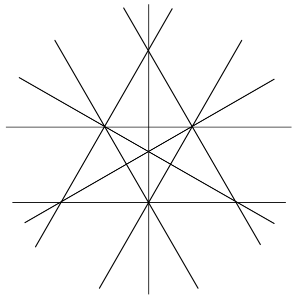
The above are four different presentations of the same simplicial arrangement $\mathcal{A}(6,1)$. Additional ones could be added, but it seems that the ones shown here are sufficient to illustrate the variety of forms in which isomorphic simplicial arrangements may appear. Naturally, in most of the other such arrangements the number of possible appearances would be even greater, making the catalog unwieldy. That is the reason why only one or two possible presentations are shown for most of the other simplicial arrangements. In most cases the form shown is the one with greatest symmetry.



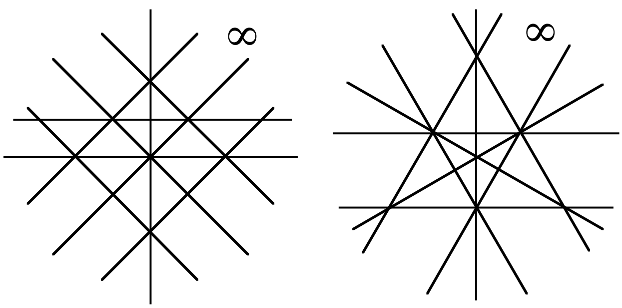
$\mathcal{A}(7,1)$



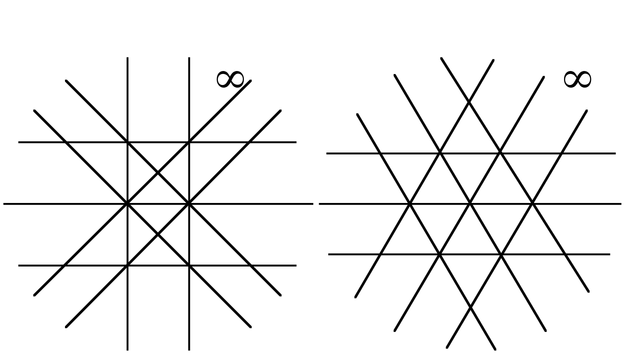
$\mathcal{A}(8,1)$



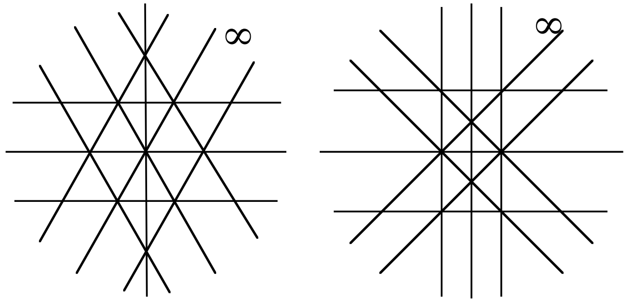
$\mathcal{A}(9,1)$



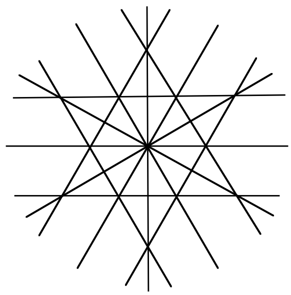
$\mathcal{A}(10,2)$



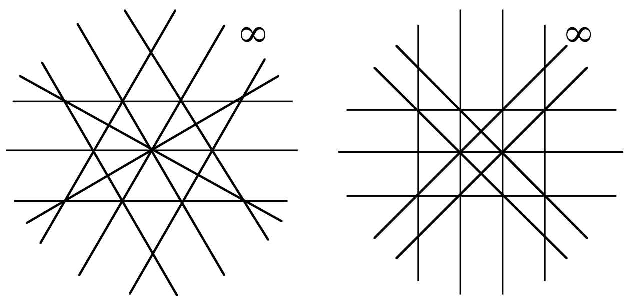
$\mathcal{A}(10,3)$



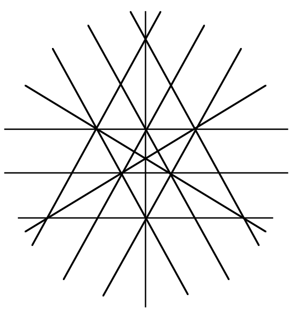
$\mathcal{A}(11,1)$



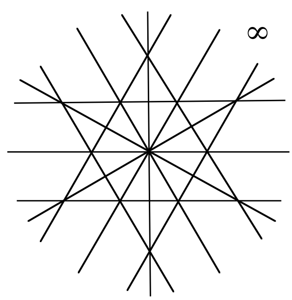
$\mathcal{A}(12,1)$



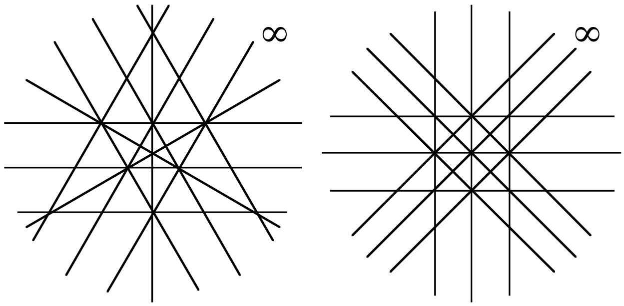
$\mathcal{A}(12,2)$



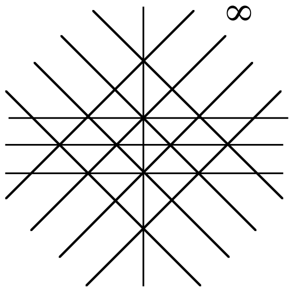
$\mathcal{A}(12,3)$



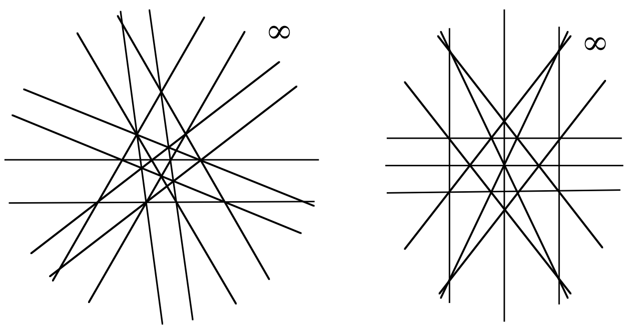
$\mathcal{A}(13,1)$



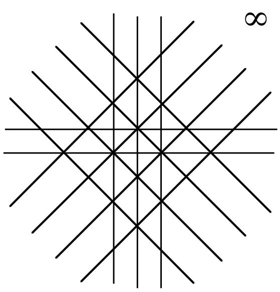
$\mathcal{A}(13,2)$



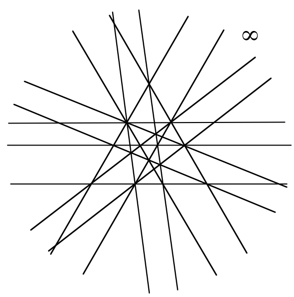
$\mathcal{A}(13,3)$



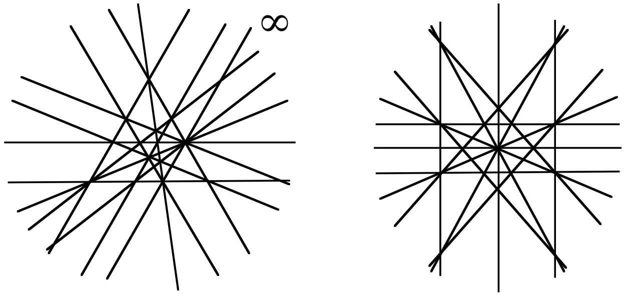
$\mathcal{A}(13,4)$



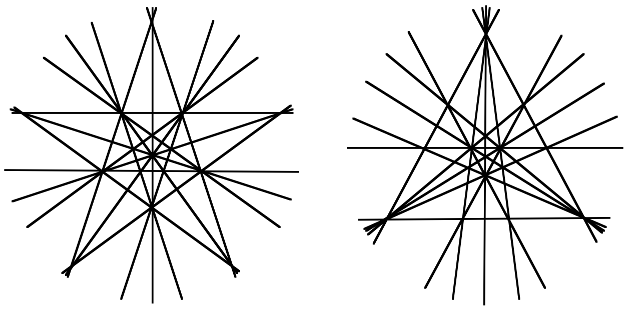
$\mathcal{A}(14,2)$



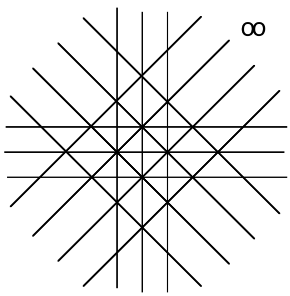
$\mathcal{A}(14,3)$



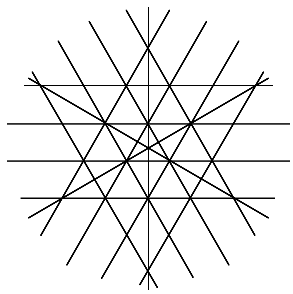
$\mathcal{A}(14,4)$



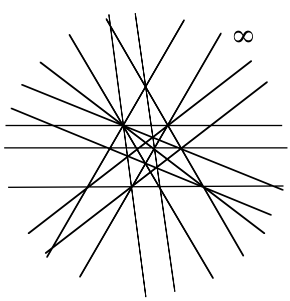
$\mathcal{A}(15,1)$



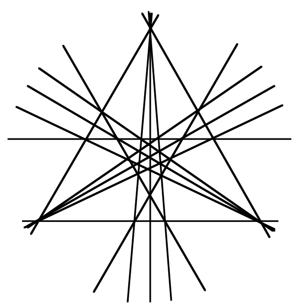
$\mathcal{A}(15,2)$



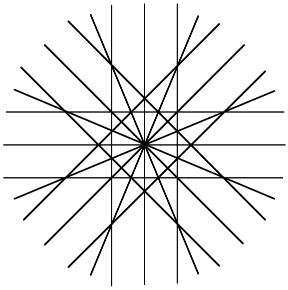
$\mathcal{A}(15,3)$



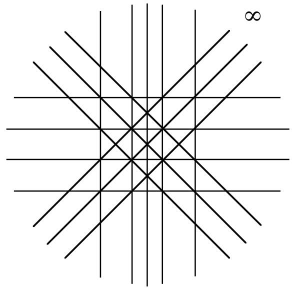
$\mathcal{A}(15,4)$



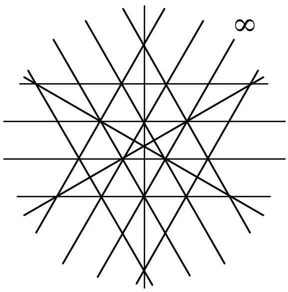
$\mathcal{A}(15,5)$



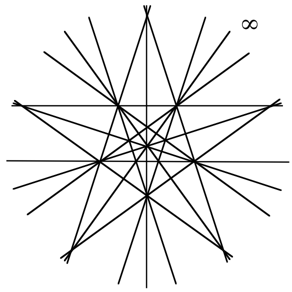
$\mathcal{A}(16, 1)$



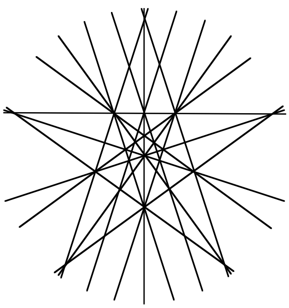
$\mathcal{A}(16, 2)$



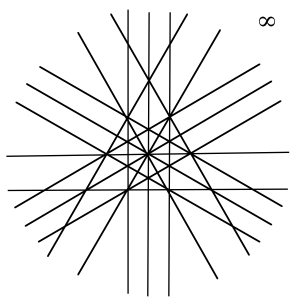
$\mathcal{A}(16, 3)$



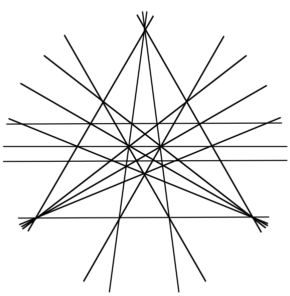
$\mathcal{A}(16, 4)$



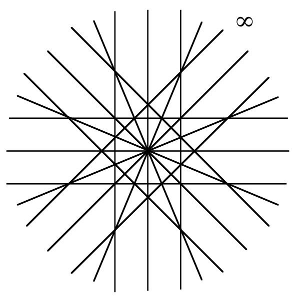
$\mathcal{A}(16, 5)$



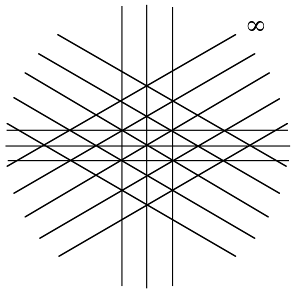
$\mathcal{A}(16, 6)$



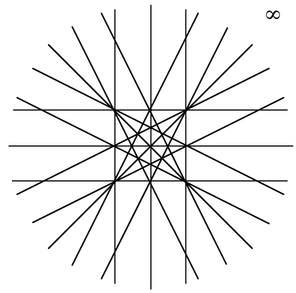
$\mathcal{A}(16, 7)$



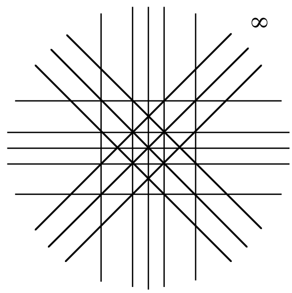
$\mathcal{A}(17, 1)$



$\mathcal{A}(17, 2)$

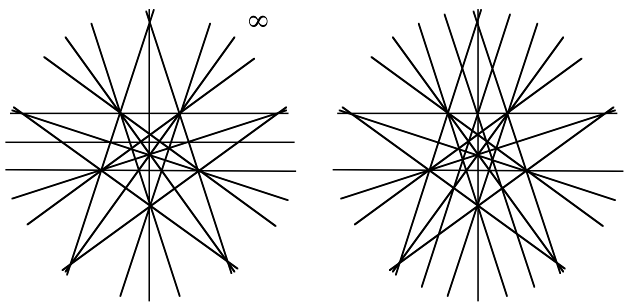


$\mathcal{A}(17, 3)$

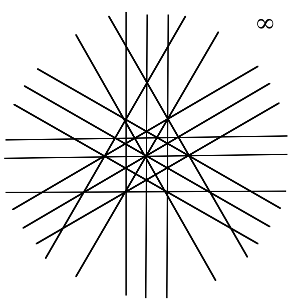


$\mathcal{A}(17, 4)$

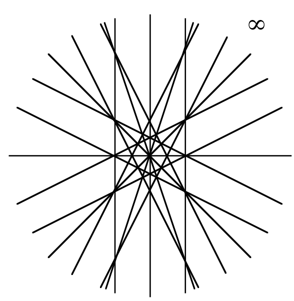
$\mathcal{A}(17, 4)$ has two lines with four quadruple points each, while $\mathcal{A}(17, 2)$ has no such line.



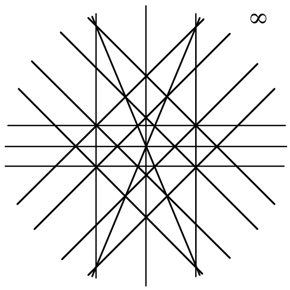
$\mathcal{A}(17, 5)$



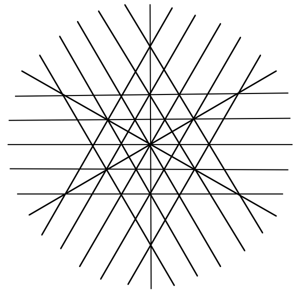
$\mathcal{A}(17, 6)$



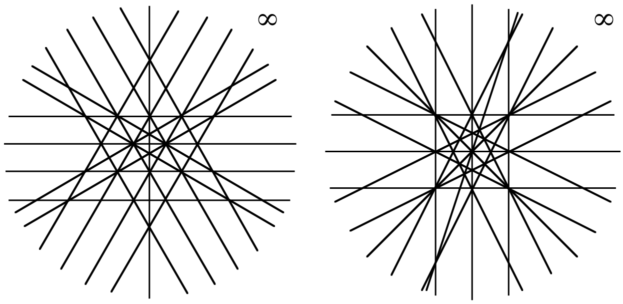
$\mathcal{A}(17, 7)$



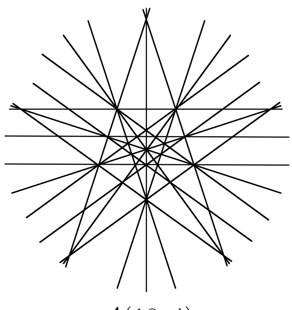
$\mathcal{A}(17, 8)$



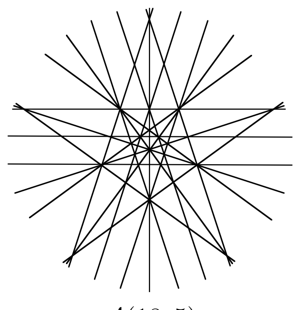
$\mathcal{A}(18, 2)$



$\mathcal{A}(18, 3)$

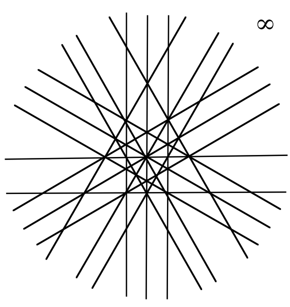


$\mathcal{A}(18, 4)$

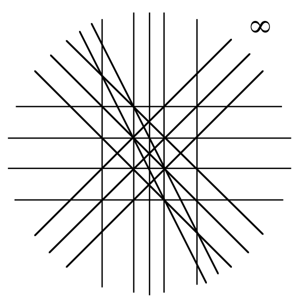


$\mathcal{A}(18, 5)$

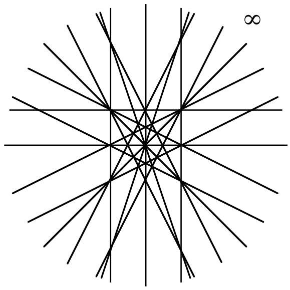
Each of $\mathcal{A}(18, 4)$ and $\mathcal{A}(18, 5)$ contains three quadruple points that determine three lines. These lines determine 4 triangles. In $\mathcal{A}(18, 4)$ there is a triangle that contains three of the quintuple points, while no such triangle exists in $\mathcal{A}(18, 5)$.



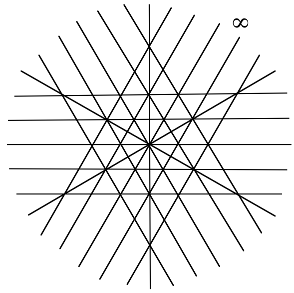
$\mathcal{A}(18, 6)$



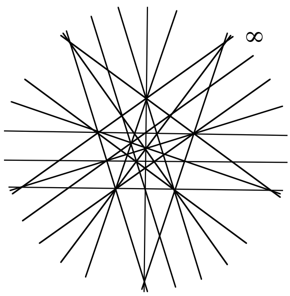
$\mathcal{A}(18, 7)$



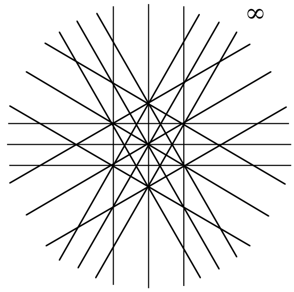
$\mathcal{A}(18,8)$



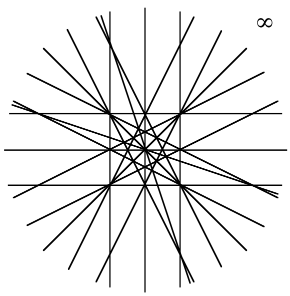
$\mathcal{A}(19,1)$



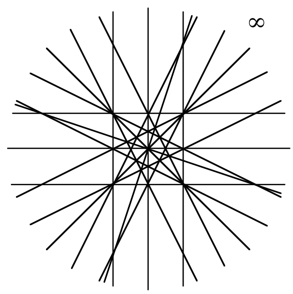
$\mathcal{A}(19,2)$



$\mathcal{A}(19,3)$

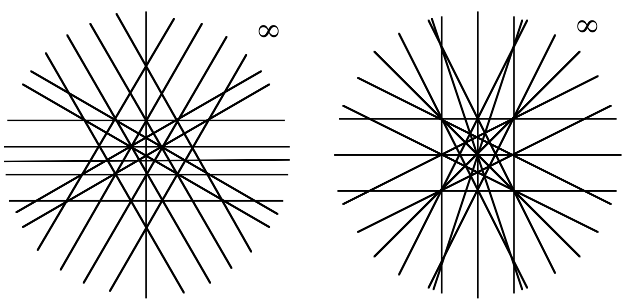


$\mathcal{A}(19,4)$

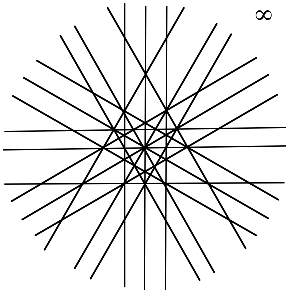


$\mathcal{A}(19,5)$

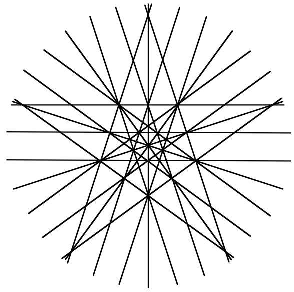
$\mathcal{A}(19,4)$ and $\mathcal{A}(19,5)$ differ by the order of the points at-infinity of different multiplicities.



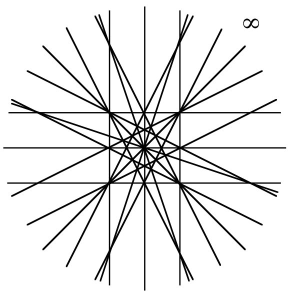
$\mathcal{A}(19,6)$



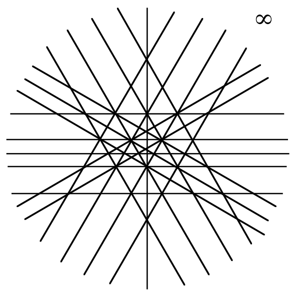
$\mathcal{A}(19,7)$



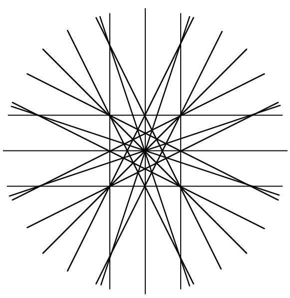
$\mathcal{A}(20,2)$



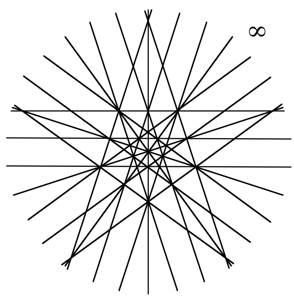
$\mathcal{A}(20,3)$



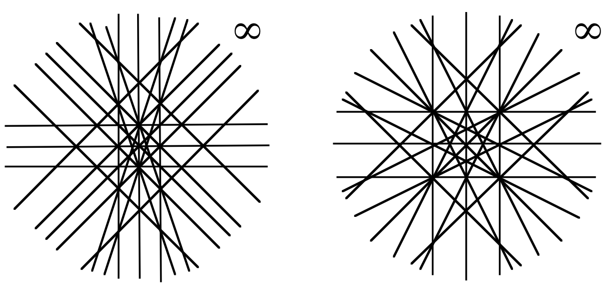
$\mathcal{A}(20,4)$



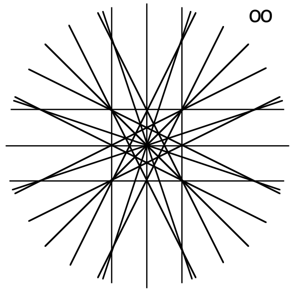
$\mathcal{A}(20,5)$



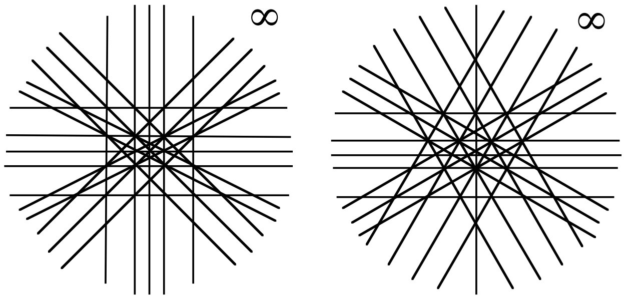
$\mathcal{A}(21,2)$



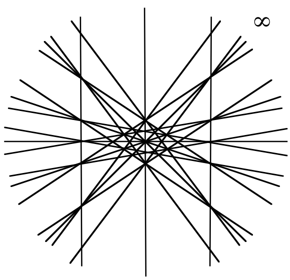
$\mathcal{A}(21,3)$



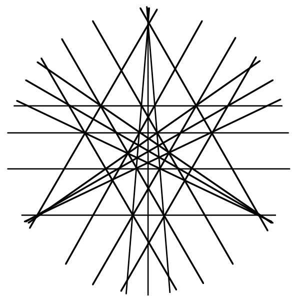
$\mathcal{A}(21,4)$



$\mathcal{A}(21,5)$



$\mathcal{A}(21,6)$



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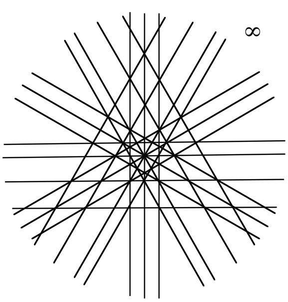
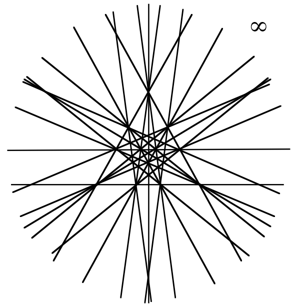
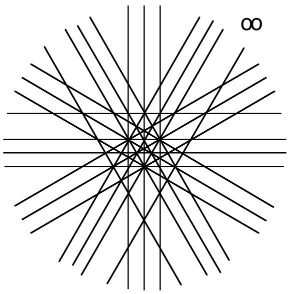


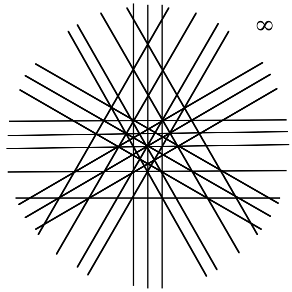
$\mathcal{A}(22,2)$



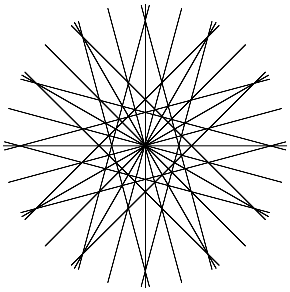
$\mathcal{A}(22,3)$



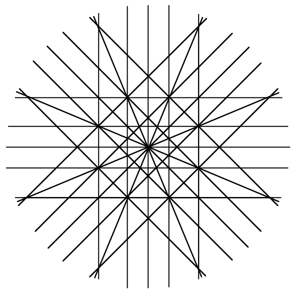
$\mathcal{A}(22, 4)$



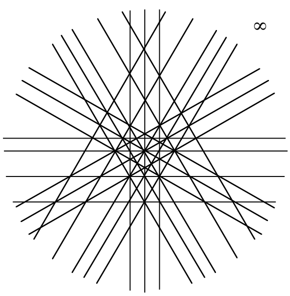
$\mathcal{A}(23, 1)$



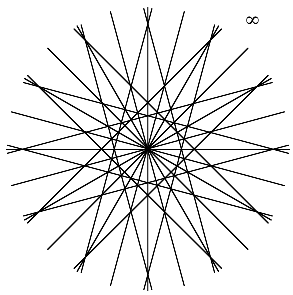
$\mathcal{A}(24, 1)$



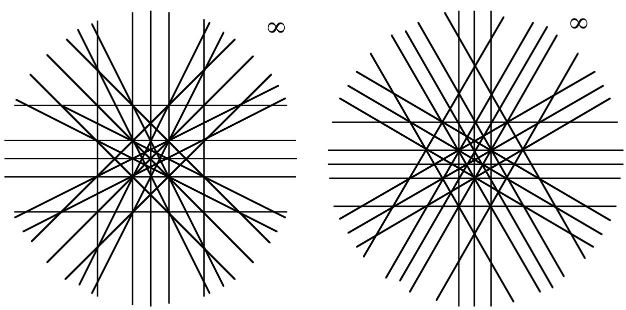
$\mathcal{A}(24, 2)$



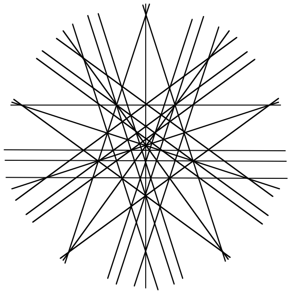
$\mathcal{A}(24, 3)$



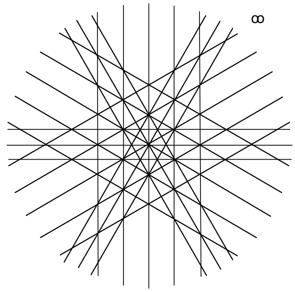
$\mathcal{A}(25, 1)$



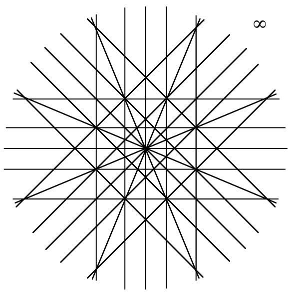
$\mathcal{A}(25, 2)$



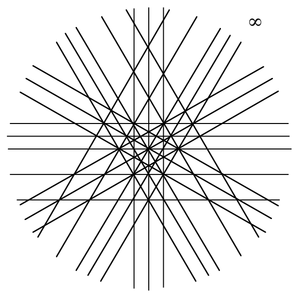
$\mathcal{A}(25, 3)$



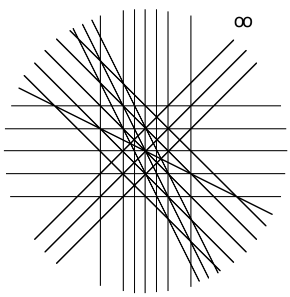
$\mathcal{A}(25, 4)$



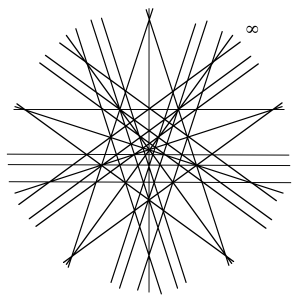
$\mathcal{A}(25, 5)$



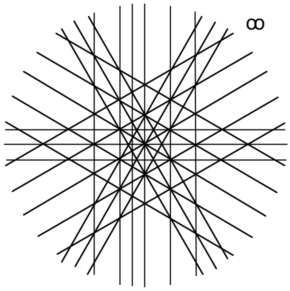
$\mathcal{A}(25, 6)$



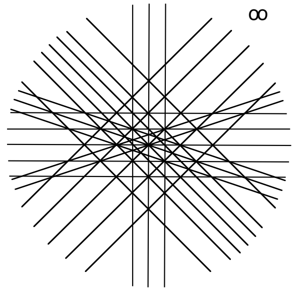
$\mathcal{A}(25, 7)$



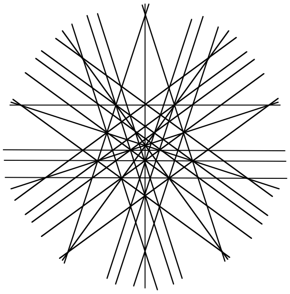
$\mathcal{A}(26, 2)$



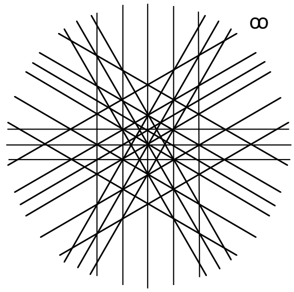
$\mathcal{A}(26, 3)$



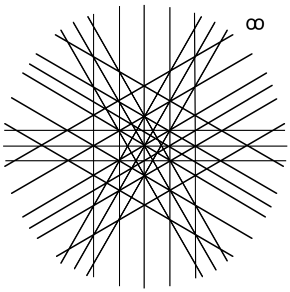
$\mathcal{A}(26, 4)$



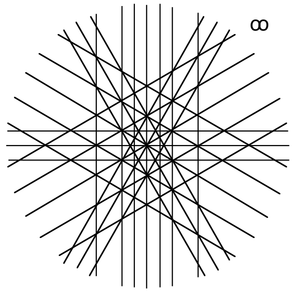
$\mathcal{A}(27,1)$



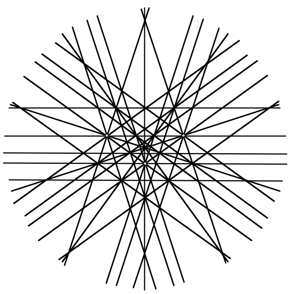
$\mathcal{A}(27,2)$



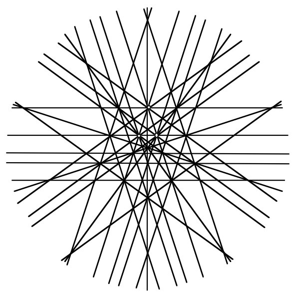
$\mathcal{A}(27,3)$



$\mathcal{A}(27,4)$

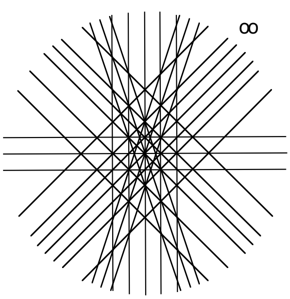


$\mathcal{A}(28,2)$

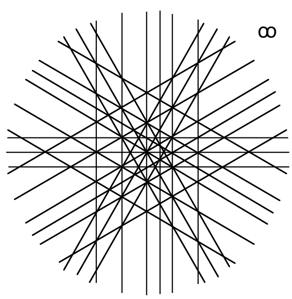


$\mathcal{A}(28,3)$

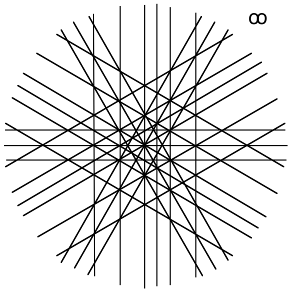
In $\mathcal{A}(28,3)$ one of the triangles determined by the 3 sextuple points contains no quintuple point. In $\mathcal{A}(28,2)$ there is no such triangle.



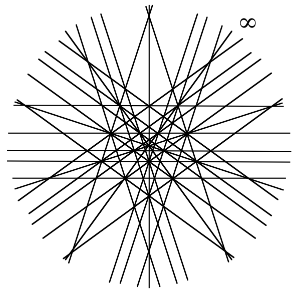
$\mathcal{A}(28,4)$



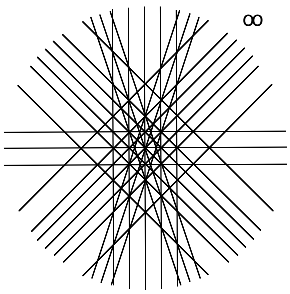
$\mathcal{A}(28,5)$



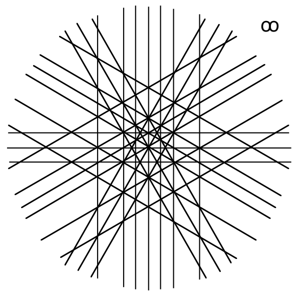
$\mathcal{A}(28,6)$



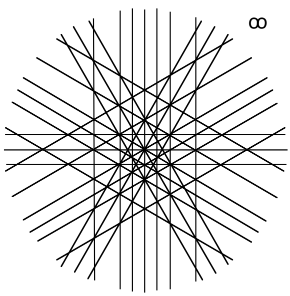
$\mathcal{A}(29,2)$



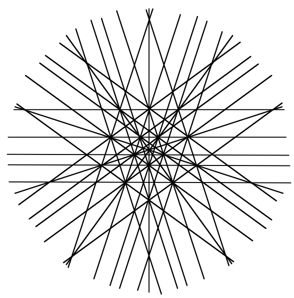
$\mathcal{A}(29,3)$



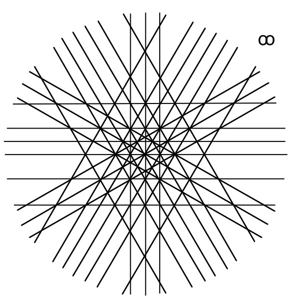
$\mathcal{A}(29,4)$



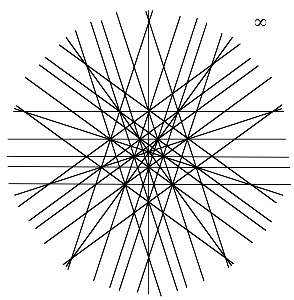
$\mathcal{A}(29,5)$



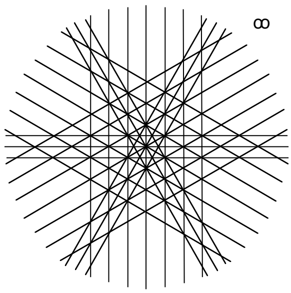
$\mathcal{A}(30,2)$



$\mathcal{A}(30,3)$



$\mathcal{A}(31,1)$



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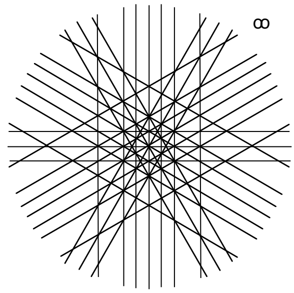
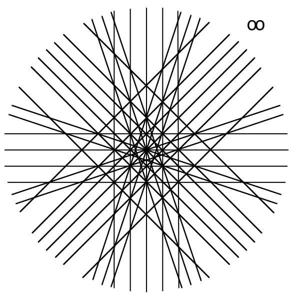
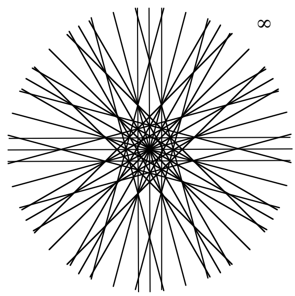


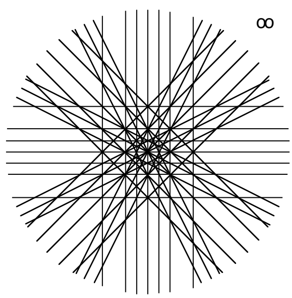
$\mathcal{A}(31,3)$



$\mathcal{A}(34,2)$



$\mathcal{A}(37,2)$



$\mathcal{A}(37,3)$

References

[1] G. Barthel, F. Hirzebruch and T. Höfer, *Geradenkonfigurationen und Algebraische Flächen*, Vieweg, Braunschweig, 1987.

[2] P. Erdös and G. Purdy, Extremal problems in combinatorial geometry, in: R. L. Graham, M. Grötschel and L. Lovász (eds.), *Handbook of Combinatorics*, Elsevier, New York, 1995, 809–874.

[3] S. Felsner, *Geometric Graphs and Arrangements*. Vieweg, Wiesbaden, 2004.

[4] J. E. Goodman, Pseudoline arrangements, in: J. E. Goodman and J. O'Rourke (eds.), *Handbook of Discrete and Computational Geometry*, 2nd ed., Chapman & Hall/CRC, New York, 2004, 97–128.

[5] B. Grünbaum, Arrangements of hyperplanes, in: R. C. Mullin *et al.* (eds.), *Proc. Second Louisiana Conf. on Combinatorics, Graph Theory and Computing*, Louisiana State University, Baton Rouge, 41–106.

[6] B. Grünbaum, *Arrangements and Spreads*, CBMS Regional Conference Series in Mathematics, No. 10., AMS, 1972, reprinted 1980.

[7] F. Hirzebruch, Singularities of algebraic surfaces and characteristic numbers, in: *Proc. Lefschetz Centenial Conf.*, Mexico City 1984, Part I, Contemporary Mathematics 58, AMS, 1986, 141–155.

[8] E. Melchior, Über Vielseite der projektiven Ebene, *Deutsche Mathematik* **5** (1941), 461–475.