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On automorphisms of Haar graphs of abelian groups

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Abstract

Let G be a group and $S \subseteq G$. In this paper, a Haar graph of G with connection set S has vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0,g)(1,gs) : g \in G \text{ and } s \in S\}$. Haar graphs are then natural bipartite analogues of Cayley digraphs, and are also called BiCayley graphs. We first examine the relationship between the automorphism group of the Cayley digraph of G with connection set S and the Haar graph of G with connection set S. We establish that the automorphism group of a Haar graph contains a natural subgroup isomorphic to the automorphism group of the corresponding Cayley digraph. In the case where G is abelian, we show there are exactly four situations in which the automorphism group of the Haar graph can be larger than the natural subgroup corresponding to the automorphism group of the Cayley digraph together with a specific involution, and analyze the full automorphism group in each of these cases. As an application, we show that all s-transitive Cayley graphs of generalized dihedral groups have a quasiprimitive automorphism group, can be constructed from digraphs of smaller order, or are Haar graphs of abelian groups whose automorphism groups have a particular permutation group theoretic property.

Keywords: Groups, graphs.

Math. Subj. Class.: 05C15, 05C10

A Haar graph of a group G with connection set S has vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0,g)(1,gs) : g \in G \text{ and } s \in S\}$, where $S \subseteq G$. Haar graphs are natural bipartite analogues of Cayley digraphs, and these graphs have appeared in a variety of contexts and under a variety of names. To the author's knowledge, Haar graphs were introduced in [18], where some of their elementary properties were studied, including some results on isomorphic Haar graphs. Recently there has been a fair amount of work on the isomorphism problem for Haar graphs [4, 5, 21–23, 25, 44], some of it motivated by applications (see

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[7, 27, 41]). Most of this work falls into the two categories of considering the isomorphism problem for graphs of small valency or determining the structure of groups which are BCI-groups. Intuitively, these are groups where isomorphism is determined in the "nicest" possible way.

Our work in this paper was motivated by the isomorphism problem, but in the end we will not consider this problem here. It is well known that the isomorphism problem for a Cayley digraph Γ of a group G depends upon the conjugacy classes of natural subgroups of $\mathrm{Aut}(\Gamma)$ (see [6, Lemma 3.1]). A similar result also holds for Haar graphs [26, Lemma 2.2] or [4, Theorem C]. Thus, information about a Cayley digraph's automorphism group is crucial in determining which other Cayley digraphs of G are isomorphic to it, and similarly, for Haar graphs. Typically, more information is known about the automorphism group of Cayley digraphs of a group G (see for example [9, 11, 13, 17]) than of Haar graphs of G, and for every Haar graph there is a corresponding Cayley digraph with the same connection set. It is thus of natural interest to determine the relationship, if any, between the automorphism group of a Haar graph and the automorphism group of its corresponding Cayley digraph (again, much more is known about the automorphism groups of Cayley digraphs). This is the main focus of the work in this paper.

We will show in Corollary 2.16 that the automorphism group of a Haar graph of an abelian group A falls into four natural families. In two of these families, the automorphism groups are a wreath product and so can be found provided one knows the automorphism groups of the graphs involved in the wreath products (which are always of smaller order and so presumably easier to find). In the third family, the automorphism group of the Haar graph of A is determined, up to conjugacy by $|A|$ natural and explicitly defined permutations, by the automorphism group of the corresponding Cayley digraph. For the fourth and final family the situation is more interesting in that there does not seem to be a natural or obvious relationship between the automorphism group of the Haar graph and its corresponding Cayley digraph. We do, though, give a group theoretic construction for all of these graphs, but unfortunately the group theoretic information needed for the construction does not seem easy to obtain. We should also mention that there is some related work on finding automorphism groups of Haar graphs - see [46].

As an application, we next consider the implications of the above automorphism group results to s-arc-transitive Haar graphs. In particular, we characterize s-arc-transitive Cayley graphs of generalized dihedral groups with abelian subgroup of odd order. In all cases except one (which corresponds to the case in the preceding paragraph where there was no obvious relationship between the automorphism group of the Haar graph and its corresponding Cayley digraph), such s-arc-transitive graphs can be constructed from other highly symmetric graphs and digraphs of smaller order without the use of graph covers.

There is another problem in the literature related to this work. For a graph Γ, its *canonical double cover* is the graph $K_2 \times \Gamma$ and is denoted $B(\Gamma)$. So $V(B(\Gamma)) = \mathbb{Z}_2 \times V(\Gamma)$ and $E(B(\Gamma)) = \{(0,x)(1,y) : xy \in E(\Gamma)\}$. If $\Gamma = \mathrm{Cay}(G,S)$ is a Cayley *graph*, then $B(\Gamma) = \mathrm{Haar}(G,S)$. Automorphism groups of $B(\Gamma)$ for Γ a Cayley graph were first studied in [34] and subsequently by several other authors [28, 35, 39, 43]. The main question is, for a graph Γ (not necessarily a Cayley graph), is $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times \mathrm{Aut}(\Gamma)$? If so, such a graph is *stable*, and if not, *unstable*. Corollary 2.16, for example, refines [28, Theorem 3.2] in the case where Γ is a Cayley graph of an abelian group. Our results also hold for digraphs, so one can consider the Haar graph construction for Cayley digraphs of a group G to be a natural generalization of the canonical double cover of graphs to digraphs.

Some words about notation should be mentioned. Haar graphs are special cases of bi-Cayley graphs of G, which are usually defined as graphs that contain a semiregular subgroup with two orbits that is isomorphic to G [2, 19, 33, 45]. Bi-Cayley graphs need not be bipartite, with the prefix *bi* referring to the two orbits of the semiregular subgroup isomorphic to G, not to the graph being bipartite, while Haar graphs are always bipartite. Additionally, some authors refer to Haar graphs as bi-Cayley graphs with the prefix *bi* presumably referring to the fact that they are bipartite. To make the terminology issues more complex, some authors refer to bi-Cayley graphs as defined here as semi-Cayley graphs. We prefer the term Haar graph to bi-Cayley graph as defined here (this definition will be formally stated below to hopefully eliminate all ambiguity and henceforth we will not use the term bi-Cayley graph), simply because this choice of terminology causes less confusion in that there is only one use of the term "Haar graph" in the literature, at least as far as the author knows!

1 Basic definitions and results on automorphism groups

All groups and graphs are finite. In this section, we define Cayley digraphs and Haar graphs and determine a relationship between their automorphism groups. This section is mainly concerned with Haar graphs of general finite groups G, not necessarily abelian.

Definition 1.1. Let G be a group and $S \subseteq G$. Define a *Cayley digraph of* G, denoted $\mathrm{Cay}(G, S)$, to be the digraph with $V(\mathrm{Cay}(G, S)) = G$ and $A(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call S the *connection set of* $\mathrm{Cay}(G, S)$.

Note that the map $g_L : G \mapsto G$ given by $g_L(x) = gx$ is always an automorphism of $\mathrm{Cay}(G, S)$ for every group G and connection set S. Thus the group $G_L = \{g_L : g \in G\} \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$ for every group G and connection set S. Here, if Γ is a graph or digraph, then $\mathrm{Aut}(\Gamma)$ denotes its automorphism group. Also note that by [6, Proposition 2.1] if $\alpha \in \mathrm{Aut}(G)$, then $\alpha(\mathrm{Cay}(G, S)) = \mathrm{Cay}(G, \alpha(S))$. In Figure 1, we give an example of a Cayley digraph of the cyclic group $\mathbb{Z}_7$.

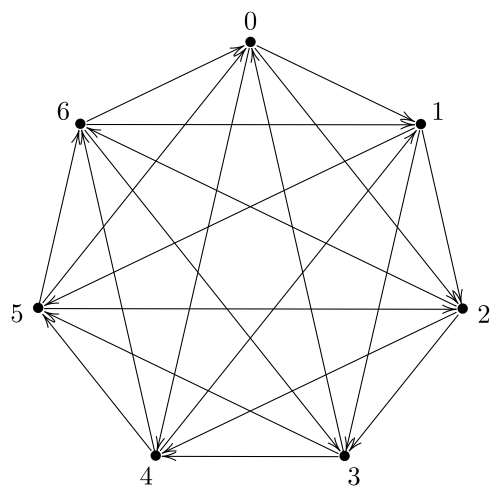

Figure 1: The Cayley digraph $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.

Definition 1.2. Let G be a group and $S \subseteq G$. Define the *Haar graph* $\mathrm{Haar}(G, S)$ with *connection set* S to be the graph with vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0, g)(1, gs) : g \in G \text{ and } s \in S\}$.

Some authors use $H(G, S)$ for $\mathrm{Haar}(G, S)$. We prefer the somewhat longer but more descriptive notation. In Figure 2 we give $\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$. We call $\mathrm{Haar}(G, S)$ the *Haar graph corresponding to* $\mathrm{Cay}(G, S)$. The graph in Figure 2 is the Haar graph corresponding to the Cayley digraph in Figure 1, and is the Heawood graph.

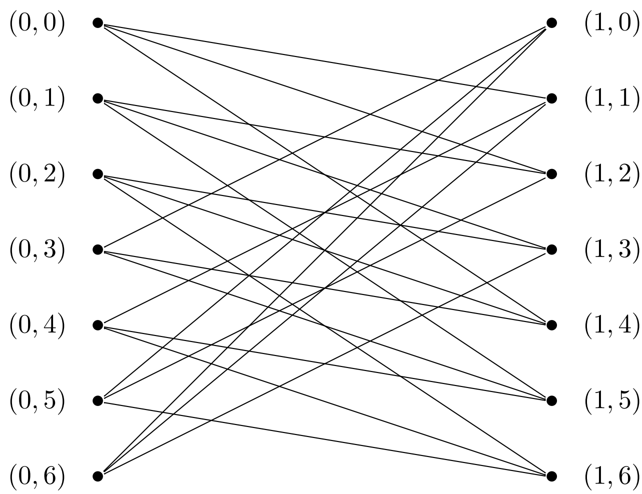

Figure 2: The Heawood graph as $\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$.

Clearly every Haar graph is a bipartite graph. Notice also that it is very much allowed that $1_G \in S$, while for Cayley digraphs this is usually not allowed. This is because the effect of including 1_G in the connection set of a Cayley digraph is to put a loop at each vertex, and doing this does not usually affect the symmetry properties of Cayley digraphs (e.g. adding a loop at each vertex does not change the automorphism group of a Cayley digraph). In some situations, though, allowing $1_G \in S$ for a Cayley digraph is not only advantageous, but crucial (see for example [3]). In this paper we *allow loops in Cayley digraphs.*

Definition 1.3. Throughout this paper, if $\Gamma = \mathrm{Haar}(G, S)$, then the natural bipartition of $V(\Gamma)$ will be denoted by $\mathcal{B}$, where $\mathcal{B} = \{B_0, B_1\}$, $B_0 = \{(0, g) : g \in G\}$, and $B_1 = \{(1, g) : g \in G\}$.

Notice that the map $\widehat{g_L} \colon \mathbb{Z}_2 \times G \mapsto \mathbb{Z}_2 \times G$ given by $\widehat{g_L}(i, j) = (i, gj)$ is an automorphism of $\mathrm{Haar}(G, S)$ for every group G and connection set S, and corresponds to the subgroup $G_L \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$.

It is easily determined using results in [1] that $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ has automorphism group $\{x \mapsto ax + b : a = 1, 2, 4, b \in \mathbb{Z}_7\}$ which is a metacyclic group of order 21. The Heawood graph is a Haar graph of $\mathbb{Z}_7$, as shown in Figure 2. While the Heawood graph does have a metacyclic subgroup of order 21 corresponding to $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\}))$, the automorphism group of the Heawood graph is actually $\mathbb{Z}_2 \ltimes \mathrm{PGL}(3,2) \cong \mathrm{PGL}(2,7)$ which has order 336 and is an almost simple group. Our first result shows that the automorphism group of a Haar graph always has a natural subgroup isomorphic to the automorphism group of its corresponding Cayley digraph. The Heawood graph example, though, shows the automorphism group of the Haar graph may be much larger.

Lemma 1.4. *Let G be a group, and $\gamma \in S_G$. The map $\hat{\gamma}\colon \mathbb{Z}_2 \times G \mapsto \mathbb{Z}_2 \times G$ given by $\hat{\gamma}(i,j) = (i, \gamma(j))$ is an automorphism of* $\mathrm{Haar}(G,S)$ *if and only if γ is an automorphism of* $\mathrm{Cay}(G,S)$.

Proof. The permutation $\gamma \in S_G$ is in $\mathrm{Aut}(\mathrm{Cay}(G,S))$ if and only if whenever $g \in G$ and $s \in S$, $\gamma(g,gs) = (\gamma(g), \gamma(gs)) \in A(\mathrm{Cay}(G,S))$. This occurs if and only if there exists $s' \in S$ such that $\gamma(gs) = \gamma(g)s'$ which is true if and only if $(0,\gamma(g))(1,\gamma(g)s') = (0,\gamma(g))(1,\gamma(gs)) \in E(\mathrm{Haar}(G,S))$. This last statement is true if and only if $\hat{\gamma} \in \mathrm{Aut}(\mathrm{Haar}(G,S))$. □

We now give a standard notation for the automorphism of $\mathrm{Haar}(G,S)$ induced by an automorphism of $\mathrm{Cay}(G,S)$ which will be used henceforth.

Definition 1.5. Let G be a group and $S \subseteq G$. Let $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$. The automorphism of $\mathrm{Haar}(G,S)$ induced by γ as in Lemma 1.4 will be denoted by $\hat{\gamma}$. That is, if $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$ then the *automorphism of* $\mathrm{Haar}(G,S)$ *corresponding to* γ is $\hat{\gamma}\colon \mathbb{Z}_2 \times G \mapsto \mathbb{Z}_2 \times G$ given by $\hat{\gamma}(i,j) = (i,\gamma(j))$. If $H \leq \mathrm{Aut}(\mathrm{Cay}(G,S))$, then we define $\widehat{H} = \{\hat{h} : h \in H\}$. In particular, the natural semiregular subgroup of $\mathrm{Haar}(G,S)$ isomorphic to G is denoted $\widehat{G_L}$.

In this paper we focus on $\mathrm{Aut}(\mathrm{Haar}(G,S))$ in the special case when G is an abelian group. The reason to restrict our attention to abelian groups is that the automorphism groups of Haar graphs of abelian groups and nonabelian groups are different. As implicitly used in [18], for an abelian group A, $\mathrm{Aut}(\mathrm{Haar}(A,S))$ contains the element $\iota : \mathbb{Z}_2 \times A \mapsto \mathbb{Z}_2 \times A$ given by $\iota(i,j) = (i+1,-j)$. The group $\langle \iota, \widehat{A_L}\rangle$ is transitive and so Haar graphs of abelian groups have transitive automorphism group while Haar graphs of nonabelian groups need not. See for example [15, Proposition 11]. For Haar graphs of abelian groups we may use the automorphism ι defined above to say a little more.

Lemma 1.6. *Let A be an abelian group, and $S \subseteq A$. Then*

$$\mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A,S)) \leq \mathrm{Aut}(\mathrm{Haar}(A,S)).$$

Proof. As A is an abelian group, straightforward computations will show that the map $\iota\colon \mathbb{Z}_2 \times A \mapsto \mathbb{Z}_2 \times A$ given by $\iota(i,j) = (i+1,-j)$ is an automorphism of $\mathrm{Haar}(A,S)$. Set $K = \mathrm{Aut}(\mathrm{Cay}(A,S))$. By Lemma 1.4, $\hat{K} \leq \mathrm{Aut}(\mathrm{Haar}(A,S))$.

Let $a \in A$. Then $(a, a+s) \in A(\mathrm{Cay}(A,S))$ if and only if $(a, a-s) \in A(\mathrm{Cay}(A,-S))$, which occurs if and only if $(a+s, a) \in A(\mathrm{Cay}(A,-S))$. So $\mathrm{Cay}(A,-S)$ can be obtained from $\mathrm{Cay}(A,S)$ by reversing the direction of each arc in $\mathrm{Cay}(A,S)$. Now, $(a,b) \in A(\mathrm{Cay}(A,S))$ if and only if $(\gamma(a),\gamma(b)) \in A(\mathrm{Cay}(A,S))$ for every $\gamma \in \mathrm{Aut}(\mathrm{Cay}(A,S))$. As $(a,b) \in A(\mathrm{Cay}(A,S))$ if and only if $(b,a) \in A(\mathrm{Cay}(A,-S))$, we see $(\gamma(a),\gamma(b)) = \gamma(a,b) \in A(\mathrm{Cay}(A,S))$ if and only if $(\gamma(b),\gamma(a)) = \gamma(b,a) \in A(\mathrm{Cay}(A,-S))$. We conclude $\mathrm{Aut}(\mathrm{Cay}(A,S)) = \mathrm{Aut}(\mathrm{Cay}(A,-S))$.

Define $r\colon A \to A$ by $r(j) = -j$. Then $r \in \mathrm{Aut}(A)$ and $r(\mathrm{Cay}(A,S)) = \mathrm{Cay}(A,-S)$. Also $r^{-1} = r$, and $r(\mathrm{Cay}(A,-S)) = \mathrm{Cay}(A,S)$. Hence

$$\mathrm{Aut}(\mathrm{Cay}(A,S)) = r\mathrm{Aut}(\mathrm{Cay}(A,-S))r.$$

Thus $\gamma \in \mathrm{Aut}(\mathrm{Cay}(A,S))$ if and only if the map $j \mapsto -\gamma(-j)$ is also contained in $\mathrm{Aut}(\mathrm{Cay}(A,S))$. As $\iota\hat{\gamma}\iota(i,j) = (i,-\gamma(-j))$, we see ι normalizes $\hat{K}$. Then the group $\langle \iota, \hat{K}\rangle = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S)) \le \mathrm{Aut}(\mathrm{Haar}(A,S))$ as $|\iota| = 2$ and ι normalizes $\hat{K}$. □

In the case where $S = -S$ and $\mathrm{Cay}(A,S)$ is a graph, we have a slightly nicer result, which is contained in the significantly stronger result [31, Lemma 4.2].

Lemma 1.7. *Let A be an abelian group and $S \subseteq A$ such that $S = -S$. Then $\mathbb{Z}_2 \times \mathrm{Aut}(\mathrm{Cay}(A,S)) \le \mathrm{Aut}(\mathrm{Haar}(A,S))$.*

Proof. Simply observe that if $S = -S$ the map $i \mapsto -i$ is an automorphism of $\mathrm{Cay}(A,S)$ and so the map $(i,j) \mapsto (i+1,j)$ is an automorphism of $\mathrm{Haar}(A,S)$. □

One circumstance in which $\mathrm{Aut}(\mathrm{Haar}(G,S))$ is bigger than $\mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$ is if $\mathrm{Cay}(G,S)$ is connected but $\mathrm{Haar}(G,S)$ is not. For example, for $n \ge 2$, $\mathrm{Cay}(\mathbb{Z}_{2n},\{\pm 1\})$ is a $2n$-cycle with automorphism group D_{2n}, while $\mathrm{Haar}(\mathbb{Z}_{2n},\{\pm 1\})$ is a disjoint union of two $2n$-cycles, with automorphism group isomorphic to $\mathbb{Z}_2 \wr D_{2n}$ (the group wreath product is given in Definition 2.3; here it is enough to observe that this group is bigger). Notice that the vertex sets of these two $2n$-cycles are **not** the sets B_0 and B_1. A perhaps more extreme example is $\mathrm{Cay}(\mathbb{Z}_{2n},S)$, where S is all odd elements of $\mathbb{Z}_{2n}$. The graph $\mathrm{Cay}(\mathbb{Z}_{2n},S) = K_{n,n}$ is connected, but $\mathrm{Haar}(\mathbb{Z}_{2n},S)$ consists of two disjoint copies of $K_{n,n}$. The necessary and sufficient condition for $\mathrm{Haar}(G,S)$ to be connected is $SS^{-1} = \{st^{-1} : s,t \in S\}$ generates G and is given in [14, Lemma 2.3(iii)]. A more appealing formulation for Haar graphs of abelian groups is the following:

Lemma 1.8. *Let A be an abelian group. $\mathrm{Haar}(A,S)$ is disconnected if and only if $S \subseteq a + H$ for some subgroup $H < A$ and $a \in A$.*

Proof. If $S \subseteq a + H$ for some $H < A$ and $a \in A$ then for $s,t \in S$ we have $s - t = a + h_1 - (a + h_2) \in H < A$ for some $h_1, h_2 \in H$, and so $\mathrm{Haar}(A,S)$ is disconnected by [14, Lemma 2.3(iii)]. If $\mathrm{Haar}(A,S)$ is disconnected, then $\langle s - t : s,t \in S\rangle = H < A$ by [14, Lemma 2.3(iii)]. Fix $a \in S$, and let $s \in S$. Then $a - s = -h \in H$ and $s = a + h$. Hence $S \subseteq a + H$. □

Before proceeding, we will need some permutation group theoretic terms.

Definition 1.9. Let X be a set and $K \le S_X$ be transitive. A subset $C \subseteq X$ is a *block* of K if whenever $k \in K$, then $k(C) \cap C = \emptyset$ or C. If $C = \{x\}$ for some $x \in X$ or $C = X$, then C is a *trivial block*. Any other block is nontrivial. The set $\mathcal{C} = \{k(C) : k \in K\}$ is a partition of X, called a *block system* of K, and is *nontrivial* if C is nontrivial.

A basic fact about a graph Γ is that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\bar{\Gamma})$, where $\bar{\Gamma}$ is the complement of Γ. But the complement of a Haar graph is not a Haar graph. One could consider “bipartite complements” (defined below) to avoid this problem, but it still need not be the case that the automorphism group of the bipartite complement of a Haar graph Γ is the automorphism group of Γ (although if $\mathcal{B}$ is a block system of the automorphism group of the bipartite complement of Γ we do have equality of automorphism groups under bipartite complements [10, Corollary 4]). Our next result is really an exercise and has certainly appeared as a comment in the literature [38]. We state and prove this result due to its importance to the work in this paper.

Lemma 1.10. *Let Γ be a vertex-transitive bipartite graph with bipartition $\mathcal{B} = \{B_0, B_1\}$. If Γ is connected then $\mathcal{B}$ is a block system of* $\mathrm{Aut}(\Gamma)$.

Proof. We prove the contrapositive, and so suppose there exists $\gamma \in \mathrm{Aut}(\Gamma)$ such that $\gamma(\mathcal{B}) = \mathcal{B}' = \{B_0', B_1'\} \neq \mathcal{B}$. As $\mathcal{B}$ is a bipartition of Γ and $\gamma \in \mathrm{Aut}(\Gamma)$, $\gamma(\mathcal{B}) = \mathcal{B}'$ is also a bipartition of Γ. Let $C_0 = B_0 \cap B_0'$, $C_1 = B_0 \cap B_1'$, $C_2 = B_1 \cap B_0'$, and $C_3 = B_1 \cap B_1'$. As $\mathcal{B} \neq \mathcal{B}'$, none of the sets C_i, $i \in \mathbb{Z}_4$, are empty. If Γ is connected, some vertex v of $C_0 \subset B_0$ is adjacent to some vertex w of B_1. As C_0 is a subset of B_0 and B_0', w must be in both B_1 and B_1', so in C_3. But by a symmetrical argument, any vertex of C_3 can only be adjacent to vertices in C_0, and so there is no path in Γ from v to any vertex of C_2. Hence Γ is disconnected. $\square$

Definition 1.11. Let Γ be a bipartite graph with bipartition $\mathcal{B} = \{B_0, B_1\}$. The *bipartite complement of* Γ is the graph with vertex set $V(\Gamma)$ and two vertices are adjacent if they are in different bipartition classes and are not adjacent in Γ.

Corollary 1.12. *Let G be a group and $S \subseteq G$. If* $\mathrm{Haar}(G, S)$ *and* $\mathrm{Haar}(G, G \setminus S)$ *are both connected then* $\mathrm{Aut}(\mathrm{Haar}(G, S)) = \mathrm{Aut}(\mathrm{Haar}(G, G \setminus S))$.

Proof. By Lemma 1.10, $\mathcal{B}$ is a block system of both $\mathrm{Aut}(\mathrm{Haar}(G, S))$ and $\mathrm{Haar}(G, G \setminus S))$, and so by [10, Corollary 4] $\mathrm{Aut}(\mathrm{Haar}(G, S)) = \mathrm{Aut}(\mathrm{Haar}(G, G \setminus S))$. $\square$

It is not true that if both $\mathrm{Haar}(A, S)$ and $\mathrm{Haar}(A, A - S)$ are disconnected then their automorphism groups are the same. Let $A = \mathbb{Z}_{2n}$, and $S = \langle 2\rangle$ so $A - S = 1 + \langle 2\rangle$. Then both $S + (-S)$ and $(A - S) + [-(A - S)] = \langle 2\rangle$ and so by Lemma 1.8 neither are connected. Both of these graphs are isomorphic to two copies of $K_{n,n}$, but the vertex sets of the $K_{n,n}$'s are different (in one it is even vertices adjacent to even vertices and odd vertices adjacent to odd vertices while in the other it is even vertices adjacent to odd vertices and odd vertices adjacent to even vertices), so their automorphism groups are different, but permutation isomorphic!

2 Characterization of automorphism groups of Haar graphs of abelian groups

We now, with some exceptions, focus on abelian groups. We will use the symbol A when the group under consideration is abelian, and G when a nonabelian group is allowed. We first consider disconnected Haar graphs. We begin with more permutation group terms.

Definition 2.1. Let X be a set and suppose $K \leq S_X$ is a transitive group which has a block system $\mathcal{C}$. Then K has an *induced action on* $\mathcal{C}$, denoted $K/\mathcal{C}$. Namely, for $k \in K$, define $k/\mathcal{C} \colon \mathcal{C} \mapsto \mathcal{C}$ by $k/\mathcal{C}(C) = C'$ if and only if $k(C) = C'$, and set $K/\mathcal{C} = \{k/\mathcal{C} : k \in K\}$. We also define the *fixer of $\mathcal{C}$ in K*, denoted $\mathrm{fix}_K(\mathcal{C})$, to be $\{k \in K : k/\mathcal{C} = 1\}$. That is, $\mathrm{fix}_K(\mathcal{C})$ is the subgroup of K which fixes each block of $\mathcal{C}$ set-wise, and is the kernel of the induced action of K on $\mathcal{C}$.

We observe that for an abelian group A, $H = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A,S))$ has $\mathcal{B}$ as a block system. Here $H/\mathcal{B} \cong \mathbb{Z}_2$ and $\mathrm{fix}_H(\mathcal{B}) = \widehat{K}$, where $K = \mathrm{Aut}(\mathrm{Cay}(A,S))$. We shall also need definitions of wreath products of digraphs and groups.

Definition 2.2. Let Γ_1 and Γ_2 be digraphs. The *wreath product of* Γ_1 *and* Γ_2, denoted $\Gamma_1 \wr \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and arcs $((u,v),(u,v'))$ for $u \in V(\Gamma_1)$ and $(v,v') \in A(\Gamma_2)$ or $((u,v),(u',v'))$ where $(u,u') \in A(\Gamma_1)$ and $v,v' \in V(\Gamma_2)$.

Definition 2.3. Let $G \leq S_X$ and $H \leq S_Y$. Define the *wreath product of* G *and* H, denoted $G \wr H$, to be the set of all permutations of $X \times Y$ of the form $(x,y) \mapsto (g(x), h_x(y))$, where $g \in G$ and each $h_x \in H$.

It is not hard to show that for vertex-transitive digraphs, $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma_1 \wr \Gamma_2)$. See [12] for more information regarding wreath products.

Let $\Gamma = \mathrm{Haar}(A,S)$ be connected, so by Lemma 1.10 $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$. Then $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ has induced actions on B_0 and B_1. Let $f \in F$ with $f(i,j) = (i, \gamma_i(j))$. The induced action of F on B_0 is given by $f \cdot (0,j) = (0, \gamma_0(j))$, and on B_1 by $f * (1,j) = (1, \gamma_1(j))$. We will be considering these induced actions frequently, and will abuse notation by considering the induced action of F on B_0 as an action simply on A, in which case $f \cdot (0,j) = \gamma_0(j)$ (i.e. we just delete the first coordinate if it is clear from context), and similarly for the induced action of F on B_1: $f * (1,j) = \gamma_1(j)$. We will also not write the actions formally, and not use the $\cdot$ and $*$ notation, but instead write F^{B_i}, $i \in \mathbb{Z}_2$ or analogous notation for a subgroup of F. For example, we simply say that A_L is contained in the image of the actions of F on B_0 and B_1 for the induced action of $\widehat{A_L}$ on B_0 and B_1. Or more simply, $\widehat{A_L}^{B_i} = A_L$ as with the above abuse of notation, $\hat{a}_L \cdot (0,j) = a_L(0)$ and $\hat{a}_L(1,j) = a_L(0)$ for every $a \in A$.

Definition 2.4. Let G be a group. We will use the notation $\bar{g}_R$ for the permutations of $\mathbb{Z}_2 \times G$ given by $\bar{g}_R(0,j) = (0,j)$ and $\bar{g}_R(1,j) = (1,jg)$ in what follows.

It is shown in the proof of [32, Lemma 2.2] that for any group G, $S \subseteq G$, and $g \in G$, $\bar{g}_R(\mathrm{Haar}(G,S)) \cong \mathrm{Haar}(G,Sg) \cong \mathrm{Haar}(G,S)$.

Theorem 2.5. *Let A be an abelian group, and $S \subseteq A$. If $\Gamma = \mathrm{Haar}(A,S)$ is disconnected, then there is $a \in A$ and $H < A$ such that $\Gamma = \bar{a}_R^{-1}(\mathrm{Haar}(A, a+S))$ and $\mathrm{Aut}(\Gamma) \cong \bar{a}_R^{-1}(S_{A/H} \wr \mathrm{Aut}(\mathrm{Haar}(H, a+S)))\bar{a}_R$.*

Proof. If Γ is disconnected, then by Lemma 1.8 $S \subseteq -a + H$ for some $a \in A$ and $H < A$. Set $H = \langle S + (-S) \rangle < A$ (this is written additively as A is abelian). Then $\mathrm{Haar}(H, a+S)$ is a connected graph which is a component of $\mathrm{Haar}(A, a+S) = \bar{a}_R(\Gamma)$. Thus $\Gamma = \bar{a}_R^{-1}(\mathrm{Haar}(A, a+S))$. Also, there are $[A:H]$ components of $\mathrm{Haar}(A, a+S)$, and $\mathrm{Haar}(A, a+S) \cong \bar{K}_{A/H} \wr \mathrm{Haar}(H, a+S)$, where $\bar{K}_{A/H}$ is the complement of the complete graph with vertices the cosets of H in A. Then $\mathrm{Aut}(\mathrm{Haar}(A, a+S)) \cong S_{A/H} \wr \mathrm{Aut}(\mathrm{Haar}(H, a+S))$, and $\mathrm{Aut}(\mathrm{Haar}(A,S)) = \bar{a}_R^{-1}\mathrm{Aut}(\mathrm{Haar}(A, a+S))\bar{a}_R$ as $\bar{a}_R^{-1}(\mathrm{Haar}(A, a+S)) = \mathrm{Haar}(A,S)$. □

Another way in which the automorphism group of a Haar graph Γ of an abelian group is bigger than expected is if the graph is connected and the action of $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is not faithful in its action on a block of $\mathcal{B}$. The next result shows that this situation is easy to spot by examining the connection set of Γ.

Lemma 2.6. *Let G be a group, and $\Gamma = \mathrm{Haar}(G, S)$ be connected and vertex-transitive. Then $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ in its action on B_0 or B_1 is unfaithful if and only if S is a union of left cosets of some nontrivial subgroup of G.*

Proof. Set $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$. First observe that as Γ is connected, $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$. As $\mathrm{Aut}(\Gamma)$ is transitive, the action of F on B_0 is unfaithful if and only if the action of F on B_1 is unfaithful.

Suppose S is a union of left cosets of some subgroup $H \neq 1$ of A. Clearly for $h \in H$, the map $\bar{h}_R$ is an automorphism of Γ as it fixes left cosets of H. Then $\bar{h}_R$ is contained in the kernel of the action of F on B_0.

Suppose F in its action on B_0 is unfaithful with $K \neq 1$ the kernel of the action. Then $K \triangleleft F$ and so the orbits of K form a block system $\mathcal{C}$ of F^{B_1}. As F^{B_1} contains G_L acting regularly, the blocks of $\mathcal{C}$ are left cosets of some subgroup H of G. As K stabilizes $(0, 0)$, it is clear that if $(0, 0)$ is adjacent to some element $(1, gh)$ for $g \in G$ and $h \in H$, then $(0, 0)$ is adjacent to $(1, gh)$ for every $h \in H$. So S is a union of left cosets of H. As K is nontrivial, $H \neq 1$. □

We next explicitly determine the automorphism groups of such Haar graphs of abelian groups. We will need some additional notation.

Definition 2.7. Let K be a transitive group with block system $\mathcal{C}$. If $\mathcal{D}$ is also a block system of K and each block of $\mathcal{D}$ is a union of blocks of $\mathcal{C}$, then we write $D/\mathcal{C}$ for the set of blocks of $\mathcal{C}$ whose union is D. Let Γ be a vertex-transitive digraph with $K \leq \mathrm{Aut}(\Gamma)$. Define the *block quotient digraph of* Γ *with respect to* $\mathcal{C}$, denoted $\Gamma/\mathcal{C}$, to be the digraph with vertex set $\mathcal{C}$ and arc set $\{(C, C') : C \neq C' \in \mathcal{C} \text{ and } (u, v) \in A(\Gamma) \text{ for some } u \in C \text{ and } v \in C'\}$.

Theorem 2.8. *Let A be an abelian group, $S \subseteq A$, and $\Gamma = \mathrm{Haar}(A, S)$. If Γ is connected, and the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is unfaithful on B_1, then there exists a subgroup $1 < H \leq A$ such that $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_\beta$ where $\beta = |H|$ and S is interpreted as a set of cosets of H in A. Also, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/H, S)) \wr S_\beta$, and denoting the natural bipartition of $\mathrm{Haar}(A/H, S)$ by $\mathcal{D}$, the action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(A/H,S))}(\mathcal{D})$ on $D \in \mathcal{D}$ is faithful.*

Proof. By Lemma 2.6 S is a union of cosets of some subgroup $1 < H \leq A$. We choose H to be maximal with respect to this property - clearly such a maximal subgroup exists. Let $\mathcal{C}$ be the block system of $\langle \iota, \widehat{A_L} \rangle$ formed by the subgroup $\langle \hat{h}_L : h \in H \rangle$. Notice that $\mathcal{C} = \{\{(i, a + h) : h \in H\} : i \in \mathbb{Z}_2, a \in A\}$.

Claim 1: In Γ, for any two distinct blocks $C, C' \in \mathcal{C}$, either every vertex of C is adjacent to every vertex of C' or no vertex of C is adjacent to any vertex of C'.

Clearly if $C, C' \subseteq B_0$ or B_1 then this statement is true as there are no edges with both endpoints inside any B_i, $i = 0, 1$. If, say, $C \subset B_0$ and $C' \subset B_1$ and some vertex $(0, v)$ of C is adjacent to some vertex $(1, v')$ of C', then by definition $v' = v + s$ for some $s \in S$. Then $C = \{(0, w) : w \in v + H\}$ and $C' = \{(1, w') : w' \in v + s + H\}$. Let $(0, w) \in C$ and $(1, w') \in C'$ with $w = v + h$ and $w' = v + s + h'$, $h, h' \in H$. Then

$$(0, w)(1, w') = (0, v + h)(1, v + s + h') = (0, v + h)(1 + v + h + s + (h' - h)) \in E(\Gamma),$$

as $s + (h' - h) \in s + H \subseteq S$, and the claim follows.

It is now straightforward to verify using the definition of the wreath product that $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_\beta$.

Claim 2: $\mathrm{Haar}(A/H, S) \not\cong \Gamma_2 \wr \bar{K}_r$ with $r \geq 2$, where Γ_2 is a connected vertex-transitive graph.

Suppose not, so $\mathrm{Haar}(A/H, S) \cong \Gamma_2 \wr \bar{K}_r$ for some vertex-transitive graph Γ_2 and $r \geq 2$ is maximum. By [12, Theorem 5.7] and the maximality of r, we have $\mathrm{Aut}(\mathrm{Haar}(A, S)) \cong \mathrm{Aut}(\Gamma_2) \wr S_{r\beta}$, so $\mathrm{Aut}(\Gamma)$ has a block system $\mathcal{E}$ with blocks of size $r\beta$ such that $\mathrm{Aut}(\Gamma)/\mathcal{E} = \Gamma_2$. Now, if Γ_2 has an odd cycle, then $\Gamma_2 \wr \bar{K}_r$ has an odd cycle. However, $\Gamma_2 \wr \bar{K}_r$ is a Haar graph and bipartite. So Γ_2 is bipartite, with bipartition $\{F_0, F_1\}$. Observe that both F_0 and F_1 are a set of cosets of H in A, and so $\cup_{a+H\in F_0}(a + H)$ and $\cup_{a+H\in F_1}(a + H)$ are subsets of A.

We now show that $\{\cup_{a+H\in F_0}(a + H), \cup_{a+H\in F_1}(a + H)\} = \mathcal{B}$. It suffices to show that if $E, E' \in \mathcal{E}$ and $E, E' \in F_0$, then $E \cup E' \subseteq B_j$, for some fixed $j = 0, 1$. As Γ is connected, Γ_2 is connected. Thus there is an EE' path $E_0, \ldots, E_t$ in Γ_2. Then $E_i \subseteq F_j$ if i is even and $E_i \subseteq F_{j+1}$ if i is odd. Similarly, the E_i with even subscripts are all contained in B_j while the E_i with odd subscripts are contained in B_{j+1}. As both E and E' are contained in F_0, t is even and $E \cup E' \subseteq B_j$, so $\{\cup_{a+H\in F_0}(a + H), \cup_{a+H\in F_1}(a + H)\} = \mathcal{B}$.

As ι interchanges the blocks of $\mathcal{B}$, ι also interchanges the union of the bipartition sets $\cup F_0$ and $\cup F_1$ of Γ_2. Then $\iota/\mathcal{E}$ is well defined and is not 1 i.e. ι permutes the blocks of $\mathcal{E}$ nontrivially as well. Finally, by [8, Theorem 1.5A] there exists a subgroup $K \leq \langle \iota, \widehat{A_L} \rangle$ such that the block of $\mathcal{E}$ that contains $(0, 0)$ is the orbit of K that contains $(0, 0)$. As $\iota/\mathcal{E} \neq 1$, $K \leq \widehat{A_L}$ is of order $r\beta$. This implies the blocks of $\mathcal{E}$ are orbits of K, and $K = \{\hat{\ell} : \ell \in L \leq A\}$. Also, it should be clear that $H \leq L$ as every block of $\mathcal{C}$ is contained in a block of $\mathcal{E}$. As $r \geq 2$, $H < L$. But this implies that S is a union of cosets of L, contradicting the maximality of H. The claim is established.

By [12, Theorem 5.7] and Claim 2, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/H, S)) \wr \bar{K}_\beta$. That the action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(A/H,S))}(\mathcal{D})$ on $D \in \mathcal{D}$ is faithful follows from Claim 2 and Theorem 2.6. □

We now turn to Haar graphs Γ of abelian groups A that are connected and whose connection set is not a union of cosets of some subgroup of A. These two hypotheses imply that $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$ and that the actions of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_0 and B_1 are faithful. To investigate further, we will need the terminology and some results concerning inequivalent permutation representations.

Definition 2.9. A *permutation representation* of a group K is a homomorphism $\phi\colon K \to S_n$ for some n.

Definition 2.10. Let K be a group, and X and Y sets. Let $\alpha\colon K \mapsto S_X$ and $\omega\colon K \mapsto S_Y$ be permutation representations of K. We say α and ω are *equivalent permutation representations* of K if there exists a bijection $\lambda\colon X \mapsto Y$ such that $\lambda(\alpha(k)(x)) = \omega(k)(\lambda(x))$ for all $x \in X$ and $k \in K$. In this case, we will say that $\alpha(K)$ and $\omega(K)$ are *permutation equivalent.*

The examples of $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ and its corresponding Haar graph, the Heawood graph, provide the next way in which the automorphism group of a Haar graph of an abelian group can be larger than its expected automorphism group. Namely, the full automorphism group of the Heawood graph is $\mathbb{Z}_2 \ltimes \mathrm{PGL}(3, 2)$, and $\mathrm{PGL}(3, 2)$ acts permutation inequivalently on the two blocks B_0 and B_1 of $\mathcal{B}$. It turns out that this is not a coincidence, as we will show. Before turning to that result, we prove a preliminary result which will be crucial.

It does not depend upon A being abelian or even a group, so we will use the symbol X in place of A or G. We keep the notation $B_i = \{(i,x) : x \in X\}$, $i = 0,1$.

Lemma 2.11. *Let X be a set, and let $F \leq S_{\mathbb{Z}_2 \times X}$ have B_0 and B_1 as orbits. Additionally, assume that the actions of F on B_0 and B_1 are faithful and the action of F on B_0 is permutation equivalent to the action of F on B_1. Then there exists $\bar\epsilon \in S_{\mathbb{Z}_2\times X}$ such that $\bar\epsilon(B_0) = B_0$, $\bar\epsilon^{B_0} = 1$ and every element of $\bar\epsilon^{-1}F\bar\epsilon$ has the form $(i,j) \mapsto (i, g(j))$, where $g \in F^{B_0}$.*

Proof. Let $\alpha\colon F \mapsto F^{B_0}$ and $\omega\colon F \mapsto F^{B_1}$ be permutation representations of F^{B_0} and F^{B_1}, respectively. As F is faithful on B_0 and B_1, both α and ω are faithful permutation representations of F. As the action of F on B_0 is permutation equivalent to the action of F on B_1, there exists a bijection $\lambda\colon B_0 \mapsto B_1$ such that $\lambda(\alpha(g)(0,j)) = \omega(g)(\lambda(0,j))$ for every $j \in X$ and $g \in F$. Let $\beta\colon B_0 \mapsto B_1$ be defined by $\beta(0,j) = (1,j)$ for every $j \in X$, so that β is a bijection. Set $\epsilon = \lambda\beta^{-1}$ so $\epsilon\beta = \lambda$ and $\epsilon \in S_{B_1}$. Substituting $\epsilon\beta$ for λ we have $\epsilon\beta(\alpha(g)(0,j)) = \omega(g)(\epsilon\beta(0,j))$ or equivalently, $\beta(\alpha(g)(0,j)) = \epsilon^{-1}\omega(g)\epsilon(\beta(0,j))$ for every $j \in X$. As ϵ is a bijection and ω a faithful permutation representation of F, it is straightforward to verify that the map $\delta\colon F \mapsto S_{B_1}$ given by $\delta(g) = \epsilon^{-1}\omega(g)\epsilon$ is a faithful permutation representation of F.

Now, as $\alpha(g)(0,j) \in B_0$ and similarly, $\delta(g)(1,j) \in B_1$ for every $j \in X$, $\alpha(g)$ and $\delta(g)$ induce permutations $\bar\alpha(g)$ and $\bar\delta(g)$ in S_X defined by $\alpha(g)(0,j) = (0, \bar\alpha(g)(j))$ and $\delta(g)(1,j) = (1, \bar\delta(g)(j))$ for all $j \in X$. Then

$$\beta(\alpha(g)(0,j)) = \beta(0, \bar\alpha(g)(j)) = (1, \bar\alpha(g)(j))$$

and

$$\epsilon^{-1}\omega(g)\epsilon(\beta(0,j)) = \delta(g)(1,j) = (1, \bar\delta(g)(j))$$

for all $j \in X$. As $\beta(\alpha(g)(0,j)) = \epsilon^{-1}\omega(g)\epsilon(\beta(0,j))$ for every $j \in X$, we see that $\bar\alpha(g)(j) = \bar\delta(g)(j)$ for all $j \in X$, and so $\bar\alpha(g) = \bar\delta(g)$ for all $g \in F$. Define $\bar\epsilon\colon \mathbb{Z}_2 \times X \mapsto \mathbb{Z}_2 \times X$ by $\bar\epsilon(0,j) = (0,j)$ and $\bar\epsilon(1,j) = \epsilon(1,j)$. Let $g \in F$. Then

$$\bar\epsilon^{-1}g\bar\epsilon(0,j) = \alpha(g)(0,j) = (0, \bar\alpha(g)(j))$$

and

$$\bar\epsilon^{-1}g\bar\epsilon(1,j) = \epsilon^{-1}\omega(g)\epsilon(1,j) = (1, \bar\delta(g)(j)) = (1, \bar\alpha(g)(j)),$$

and the result follows. □

Theorem 2.12. *Let G be a group, $S \subseteq G$, $\Gamma = \mathrm{Haar}(G,S)$, and let $F \leq \mathrm{Aut}(\Gamma)$ be the largest subgroup of $\mathrm{Aut}(\Gamma)$ that fixes B_0 and B_1 set-wise. Suppose F satisfies the following conditions:*

(1) *F acts faithfully on both B_0 and B_1, and*

(2) *the action of F on B_0 is permutation equivalent to the action of F on B_1.*

Let L be the group of all elements of $S_{\mathbb{Z}_2\times G}$ of the form $(i,j) \mapsto (i, \ell(j))$ where $\ell \in F^{B_0}$, and $g \in G$ such that $\mathrm{Stab}_F(1,g) = \mathrm{Stab}_F(0,1_G)$. Then $L = \bar g_R^{-1}F\bar g_R$.

Proof. Clearly every element of F can be written as $(i,j)\mapsto(i,\ell_i(j))$, where $\ell_0,\ell_1\in S_G$. By Lemma 2.11 there exists $\bar\epsilon\in S_{\mathbb{Z}_2\times G}$ such that $\bar\epsilon^{B_0}=1$ and every element of $\bar\epsilon^{-1}F\bar\epsilon$ has the form $(i,j)\mapsto(i,\ell(j))$, where $\ell\in F^{B_0}$. Let $\epsilon\in S_G$ such that $\bar\epsilon(1,j)=(1,\epsilon(j))$. Define $g_R\colon G\to G$ by $g_R(x)=xg$, and set $G_R=\{g_R : g\in G\}$. We next show that $\epsilon\in G_R$.

Of course, $\widehat{G_L}\le F$, and $\widehat{G_L}$ has the form $(i,j)\mapsto(i,\ell(j))$, where $\ell\in\widehat{G_L}^{B_0}$. As $\bar\epsilon^{B_0}=1$, we have $\bar\epsilon^{-1}\widehat{g_L}\bar\epsilon=\widehat{g_L}$ for every $g\in G$. Hence $\bar\epsilon$ centralizes $\widehat{G_L}$, and so ϵ centralizes G_L as $\bar\epsilon^{B_0}=1$. As the centralizer of G_L in S_G is G_R (this well known fact can be deduced from [8, Theorem 4.2A(ii)]), we have $\epsilon\in G_R$. As $\epsilon\in G_R$, there exists $g\in G$ such that $L=\bar g_R^{-1}F\bar g_R$. Finally, as $F=\bar g_R L\bar g_R^{-1}$ and $\mathrm{Stab}_L(0,1_G)=\mathrm{Stab}_L(1,1_G)$, $F=\bar g_R L\bar g_R^{-1}$ stabilizes $(1,g)$. □

Corollary 2.13. *Let A be an abelian group with $S\subseteq A$ such that $\Gamma=\mathrm{Haar}(A,S)$ is connected, and consequently $\mathrm{Aut}(\mathrm{Haar}(A,S))$ has $\mathcal{B}$ as a block system. Set $F=\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$. Then one of the following is true:*

(1) *the induced action of F on B_1 is not faithful,*

(2) *the induced action of F on B_0 is permutation inequivalent to the induced action of F on B_1, or*

(3) $\mathrm{Aut}(\mathrm{Haar}(A,S))=\bar a_R^{-1}[\mathbb{Z}_2\ltimes\mathrm{Aut}(\mathrm{Cay}(A,a+S))]\bar a_R$ *for some $a\in A$.*

Proof. We may assume without loss of generality that the action of F on B_1 is faithful. As $\mathrm{Aut}(\Gamma)$ is transitive as A is abelian, we have F is faithful on B_0. With that assumption made, we may then assume without loss of generality that the actions of F on B_0 and B_1 are permutation equivalent. Set $L=\{(i,j)\mapsto(i,g(j)) : g\in F^{B_0}\}$ so $\mathrm{Stab}_L(0,0)=\mathrm{Stab}_L(1,0)$. Applying Theorem 2.12, there is $a\in A$ such that $L=\bar a_R F\bar a_R^{-1}$. By [32, Lemma 2.2], $\bar a_R(\Gamma)=\mathrm{Haar}(A,a+S)$. By Lemma 1.4, $\mathrm{Aut}(\mathrm{Cay}(A,a+S))\cong L$, so $\mathrm{Aut}(\mathrm{Haar}(A,a+S))=\mathbb{Z}_2\ltimes\mathrm{Aut}(\mathrm{Cay}(A,-a+S)$. The result follows as $\mathrm{Aut}(\Gamma)=\bar a_R^{-1}\mathrm{Aut}(\mathrm{Haar}(A,a+S))\bar a_R$. □

Remark 2.14. Suppose $\mathrm{Haar}(A,S)$ is connected and the induced action of F on B_1 is faithful. If $a,b\in A$ with $a\ne b$, then $\bar a_R(\mathrm{Haar}(A,S))\ne\bar b_R(\mathrm{Haar}(A,S))$ as otherwise $\bar b_R^{-1}\bar a_R\in\mathrm{Aut}(\mathrm{Haar}(A,S))$ and the induced action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Cay}(A,S))}(\mathcal{B})$ is not faithful on B_0.

Remark 2.15. While $\mathrm{Haar}(A,S)\cong\mathrm{Haar}(A,a+S)$ for every $a\in A$, $\mathrm{Cay}(A,S)$ is not necessarily isomorphic to $\mathrm{Cay}(A,a+S)$. Indeed, let n be an odd positive integer, $A=\mathbb{Z}_n$, $S=\{\pm1\}$, and $a=1$. Then $\mathrm{Cay}(A,S)$ is a cycle, while $\mathrm{Cay}(A,a+S)=\mathrm{Cay}(\mathbb{Z}_n,\{0,2\})$ is a directed cycle together with a loop at every vertex.

The following result is a combination of Theorems 2.5 and 2.8, and Corollary 2.13, and summarizes the possibilities for $\mathrm{Aut}(\mathrm{Haar}(A,S))$ with A abelian.

Corollary 2.16. *Let A be an abelian group, $S\subseteq A$, and $\Gamma=\mathrm{Haar}(A,S)$. Then one of the following is true:*

(1) *Γ is disconnected, then there is $a\in A$ and $H<A$ such that $\Gamma=\bar a_R^{-1}(\mathrm{Haar}(A,a+S))$ and $\mathrm{Aut}(\Gamma)\cong\bar a_R^{-1}(S_{A/H}\wr\mathrm{Aut}(\mathrm{Haar}(H,a+S)))\bar a_R$,*

(2) *Γ is connected, and the action of* $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ *is unfaithful on B_1. There exists a subgroup $1 < H \leq A$ such that $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_\beta$ where $\beta = |H|$ and S is interpreted as a set of cosets of H in A, and $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/H, S)) \wr S_\beta$. Additionally, denoting the natural bipartition of $\mathrm{Haar}(A/H, S)$ by $\mathcal{D}$, the action of $\mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(A/H,S))}(\mathcal{D})$ on $D \in \mathcal{D}$ is faithful,*

(3) $\mathrm{Aut}(\Gamma) \cong \bar{a}_R^{-1}\mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A, a + S))\bar{a}_R$ *for some $a \in A$, or*

(4) *the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_1 is faithful but the actions on B_0 and B_1 are not equivalent permutation groups.*

We now give a group theoretic description of the graphs in (4) of Corollary 2.16.

Theorem 2.17. *Let A be an abelian group and $S \subseteq A$ such that $\Gamma = \mathrm{Haar}(A, S)$ is connected and S is not a union of cosets of some subgroup of A. Let $F = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$, $H = F^{B_0}$, and $L = \mathrm{Stab}_H(b)$, where $b \in B_0$. If the actions of F on B_0 and B_1 are inequivalent, then there exists $\sigma \in \mathrm{Aut}(H)$ which is an involution and maps L to a subgroup R of H which is not conjugate in H to L.*

Proof. The hypothesis implies that the action of F on B_0 and B_1 is faithful by Lemma 2.6. As $\iota \in \mathrm{Aut}(\Gamma)$, conjugation of F by ι induces automorphisms σ and δ which map F^{B_0} to F^{B_1}, and F^{B_1} to F^{B_0}, so we may view F^{B_1} as being contained in H as the image of δ. As ι has order 2, so does σ. Also, as the action of F on B_0 and B_1 are not equivalent, by [8, Lemma 1.6B] $\sigma(L) = R$ is not the stabilizer of a point in B_0, so R is not conjugate in H to L. □

We next give a partial converse of the previous result. We begin with a more general construction of bipartite graphs than that of the Haar graphs.

Definition 2.18. Let G be a group, let $L, R \leq G$, and let D be a union of double cosets of R and L in G, that is $D = \cup_i R d_i L$. Define a bipartite graph $\Gamma = B(G, L, R; D)$ with bipartition $V(\Gamma) = G/L \cup G/R$ (here G/L and G/R are the sets of left cosets of L and R in G) and edge set $E(\Gamma) = \{\{gL, gdR\} : g \in G, d \in D\}$. We call Γ the *bi-coset graph* of G with respect to L, R and D.

The bi-coset graphs $B(G, L, R; D)$ were introduced in [14], where they were shown to be well-defined bipartite graphs whose automorphism group contain a natural subgroup isomorphic to G. Haar graphs are bi-coset graphs with $L = R = 1$. In [14, Lemma 2.6] a sufficient condition to ensure that a bi-coset graph is vertex-transitive is given, and then several more specific circumstances were given where this condition was satisfied. We will need another such more specific circumstance.

Lemma 2.19. *Let G be a group with $L \leq G$ and $\sigma \in \mathrm{Aut}(G)$ an involution such that $\sigma(L) = R$. The bi-coset graph $\Gamma = B(G, L, R; S)$ is vertex-transitive with $\sigma \in \mathrm{Aut}(\Gamma)$ provided $S = LDR$ with $\sigma(D) = D^{-1} = D$.*

Proof. By [14, Lemma 2.6] we need only show that $S^{-1} = \sigma(S)$. Then

$$\sigma(S) = \sigma(LDR) = \sigma(L)\sigma(D)\sigma(R) = RD^{-1}L = R^{-1}D^{-1}L^{-1} = (LDR)^{-1} = S^{-1}.$$

□

Definition 2.20. Let K be a group and $H \leq K$. The largest normal subgroup of K contained in H is called the *core* of H in K. If the core of H in K is trivial, then we say H is *core-free* in K.

It is well known that the left action of a group K on a subgroup $H \leq K$ is faithful if and only if H is core-free in K.

Theorem 2.21. *Let A be an abelian group. Suppose $A_L \leq K \leq S_A$ with $L = \mathrm{Stab}_K(a)$, $a \in A$, $\sigma \in \mathrm{Aut}(K)$ an involution, and $R = \sigma(L) \leq K$ which is not conjugate in K to L and satisfies $R \cap A_L = 1$. Then the bi-coset graph $\Gamma = B(K, L, R; LR) \cong \mathrm{Haar}(A, S)$ for some $S \subseteq A$, and there is a subgroup $H \leq \mathrm{Aut}(\Gamma)$ such that H fixes $\mathcal{B}$, $H \cong K$, the action of H on B_0 is faithful, and the action of H on $B_0 = K/L$ is inequivalent to the action of H on $B_1 = K/R$.*

Proof. By [14, Lemma 2.3] the left multiplication action of K on $V(\Gamma)$ is contained in $\mathrm{Aut}(\Gamma)$, fixes $\mathcal{B}$, and is transitive on B_0 and B_1. Denote by H the corresponding subgroup of $\mathrm{Aut}(\Gamma)$. Then H^{B_0} is permutation equivalent to $K \leq S_A$ as their stabilizers are the same, namely L. It is clear that the left multiplication action of K on B_0 is faithful as L is core-free in K as L is the stabilizer of a point in $K \leq S_A$. Similarly, as $\sigma \in \mathrm{Aut}(K)$ and $\sigma(L) = R$, R is core-free in K (as it is the isomorphic image of a core-free subgroup of K) so the left multiplication action of K on K/R is faithful as well. Note that as left multiplication of R by an element of K fixes R if and only if it is contained in R, $\mathrm{Stab}_{H^{B_1}}(R) = R$. We see that the action of H on B_1 is faithful, and the action of H on B_0 is inequivalent to the action of H on B_1. Setting $D = \{1_K\}$ we see by Lemma 2.19 that $\sigma \in \mathrm{Aut}(\Gamma)$ as $D^{-1} = D$, and so Γ is vertex-transitive.

To see Γ is isomorphic to a Haar graph of A, let $\tilde{A} \leq \mathrm{Aut}(\Gamma)$ be the subgroup of $\mathrm{Aut}(\Gamma)$ induced by left multiplication by elements of A_L in K. Of course, $\tilde{A}$ is transitive on B_0 as $A_L L = K$. Suppose $\tilde{a}_1 R = \tilde{a}_2 R$ for some $\tilde{a}_1, \tilde{a}_2 \in \tilde{A}$. This occurs if and only if there are $a_1, a_2 \in A$ such that $(a_1)_L R = (a_2)_L R$, which occurs if and only if $(a_2^{-1} a_1)_L R = R$. As $A_L \cap R = 1$, we see $(a_1)_L = (a_2)_L$. Then $\tilde{A}$ is faithful on B_1. As $\tilde{A} \cap R = 1$, $\tilde{A}^{B_1}$ is semiregular. By [42, Proposition 4.1] $\tilde{A}$ has one orbit of size $|A|$, and so is transitive. The result follows by [14, Lemma 2.5]. □

Theorem 2.21 is only a partial converse to Theorem 2.17 as it is possible that the graph constructed in the result (or any graph constructed in a similar way) will have an automorphism group larger than the group K in the statement. For example, if $LR = K$ is transitive, the graph $B(K, L, R; LR)$ constructed will be a complete bipartite graph with automorphism group $K_2 \wr S_n$ with $n = |A|$. The next example shows that this can occur.

Example 2.22. Let $K = S_6$, $L = \mathrm{Stab}_{S_6}(x) \cong S_5$ and σ be the outer automorphism of S_6 of order 2. Set $R = \sigma(L)$. Then $B(K, L, R; LR) \cong K_{6,6}$.

Proof. We show that L is transitive on R. That is, we will show that $\sigma(S_5)$ is transitive, where we view S_5 as the stabilizer of the point 0 in the set $\mathbb{Z}_6$ on which we view S_6 permuting. Now, the three cycle $(3, 4, 5)$ is mapped by σ to a product of two disjoint 3-cycles (see for example [20]). So $\sigma(3, 4, 5)$ has two orbits of size 3. This implies that $\sigma(S_5)$ is transitive, or $\sigma(S_5)$ has two orbits of size 3, in which case its maximum order is $|S_3| \cdot |S_3| = 36$. But S_5 has order 120, so $\sigma(S_5)$ is transitive and the result follows. □

Note that the graph $B(K, L, R; LR)$ as constructed in the previous example does not satisfy the hypothesis of Theorem 2.21 as $R \cap A_L \neq 1$.

Definition 2.23. Let $n \geq 3$ be an integer, q a prime-power, and set

$$L = \mathrm{PG}(n-1, q) = \{r\vec{v} : r \in \mathbb{F}_q \text{ and } \vec{v} \text{ is in the vector space } \mathbb{F}_q^n\},$$

the set of lines (or projective points) in $\mathbb{F}_q^n$. Let H be the set of hyperplanes in $\mathbb{F}_q^n$. Define a graph $B(\mathrm{PG}(n-1, q))$ to have vertex set $L \cup H$ and edges $\{L, H\}$ where $L \subseteq H$.

Note that the 'B' used in defining the above graphs has an entirely different meaning from the 'B' used in the previous example. No confusion can arise though as the parameters of the families are completely different. Also, it is known that $B(2, 2)$ is isomorphic to the Heawood graph. Our next example, which is also fairly well-known, shows it is a member of a much larger family of graphs with many of the same properties.

Example 2.24. The graph $B(\mathrm{PG}(n-1, q))$ is isomorphic to a Haar graph of the cyclic group of order $(q^n - 1)/(q - 1)$. Additionally, if $n \geq 3$, $F = \mathrm{fix}_{\mathrm{Aut}(B(\mathrm{PG}(n-1,q)))}(\mathcal{B})$ in its induced actions on B_0 and B_1 are inequivalent representations of $\mathrm{PGL}(n, q)$ with the actions being on points and hyperplanes.

Proof. It is well-known that $\mathrm{PGL}(n, q)$ contains a regular cyclic subgroup of order $(q^n - 1)/(q - 1)$ - see for example [24, Theorem 3] or [29, Theorem 1.1]. It is easy to see that $F^{B_0} \cong F^{B_1}$ contains $\mathrm{PGL}(n, q)$ as a point contained in a hyperplane is mapped to a point contained in a hyperplane by an element of $\mathrm{PGL}(n, q)$. That F is not larger than $\mathrm{PGL}(n, q)$ basically follows from the Fundamental Theorem of Finite Geometry. By [18, Lemma 4.2] we see $B(\mathrm{PG}(n-1, q))$ is isomorphic to a Haar graph, and, as $n \geq 3$, points and hyperplanes have different dimensions. It is then not hard to see that the stabilizer in $\mathrm{PGL}(n, q)$ of a line does not stabilize any hyperplane. Thus the induced actions of F on B_0 and B_1 are inequivalent by [8, Lemma 1.6B]. □

3 Applications to arc-transitive graphs

The study of s-arc-transitive graphs was initiated in a celebrated paper by Tutte [40]. There has been strong and consistent interest in s-arc-transitive graphs for several decades. Perhaps the most important tool in this area is Praeger's Normal Quotient Lemma [36]. This lemma shows how to reduce an s-arc-transitive graph Γ to an s-arc-transitive quotient of Γ provided one can find $N \triangleleft \mathrm{Aut}(\Gamma)$ that has at least three orbits. If $\mathrm{Aut}(\Gamma)$ is quasiprimitive, then one can study such groups and graphs using the O'Nan-Scott Theorem [8, Theorem 4.1A] and Praeger's quasiprimitive counterpart [37]. So other techniques are necessary to deal with the case when Γ only has normal subgroups with exactly two orbits. In this case, if Γ is disconnected, then there is an obvious reduction to s-arc-transitive graphs of smaller order. If Γ is connected, then there are edges between the two orbits and so no edges inside the orbits. Hence Γ is bipartite, and if the subgroup N of $\mathrm{Aut}(\Gamma)$ fixing the parts of the bipartition set-wise contains a semiregular subgroup isomorphic to G, then Γ is a Haar graph of G by [14, Lemma 2.4]. Thus the study of Haar graphs is essential to the study of s-arc-transitive graphs.

Definition 3.1. Let $s \geq 1$, and Γ a digraph. An *s-arc* of Γ is a sequence $x_0, x_1, \ldots, x_s$ of vertices of Γ such that $(x_i, x_{i+1}) \in A(\Gamma)$, $0 \leq i \leq s-1$, and $x_i \neq x_{i+2}$, $0 \leq i \leq s-2$. The digraph Γ is *s-arc-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs of Γ.

As an arc-transitive graph without isolated vertices is vertex-transitive, we restrict our attention to vertex-transitive Haar graphs.

Definition 3.2. Let Γ be a digraph, and $s \geq 1$. A sequence of arcs $a_1, \ldots, a_s \in A(\Gamma)$ is an *alternating s-arc* if there exists vertices $x_0, \ldots, x_s \in V(\Gamma)$, $x_i \neq x_{i+2}$, and for $1 \leq m \leq s$ the arc $a_m = (x_{m-1}, x_m)$ if m is odd while $a_m = (x_m, x_{m-1})$ if m is even. An *alternating s-arc-transitive digraph* is a digraph whose automorphism group is transitive on the set of alternating s-arcs. The vertices $x_0, \ldots, x_s$ are the *vertex-sequence* of the alternating s-arc $a_1, \ldots, a_s$.

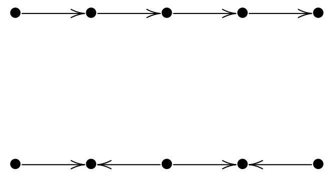

Figure 3: A 4-arc (top) versus an alternating 4-arc (bottom).

Clearly if $s = 1$, then an alternating s-arc is simply an s-arc. If $s \geq 2$ then an alternating s-arc can be obtained from an s-arc by reversing the direction of every other arc - see Figure 3. Now choose an s-arc, and fix an orientation of this s-arc. An s-arc-transitive *graph* is transitive on the set of all s-arcs in Γ with this fixed reorientation, so an s-arc-transitive graph is trivially alternating s-arc-transitive. The next result gives a relationship between alternating s-arc-transitive Cayley digraphs and s-arc-transitive Haar graphs.

Theorem 3.3. *Let G be a group, $S \subseteq G$, and $s \geq 1$. If $\mathrm{Cay}(G, S)$ is an alternating s-arc-transitive digraph and $\mathrm{Haar}(G, S)$ is vertex-transitive, then $\mathrm{Haar}(G, S)$ is s-arc-transitive. Conversely, if $\mathrm{Aut}(\mathrm{Haar}(G, S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G, S))$ and $\mathrm{Haar}(G, S)$ is s-arc-transitive, then $\mathrm{Cay}(G, S)$ is alternating s-arc-transitive.*

Proof. Suppose $\mathrm{Cay}(G, S)$ is alternating s-arc-transitive and $\mathrm{Haar}(G, S)$ is vertex-transitive. Let $P_1 = (i, x_0), (i+1, x_1), \ldots, (i+s, x_s)$ and $P_2 = (j, y_0), (j+1, y_1), \ldots, (j+s, y_s)$ be two s-arcs in $\mathrm{Haar}(G, S)$. As $\mathrm{Aut}(\mathrm{Haar}(G, S))$ is transitive, replacing P_1 or P_2 by $\iota(P_1)$ or $\iota(P_2)$, for appropriate $\iota \in \mathrm{Aut}(\mathrm{Haar}(G, S))$, if necessary, we may assume without loss of generality that $i = j = 0$. Then there exist $t_1, \ldots, t_s \in S$ such that for $1 \leq m \leq s$

$$x_m = \begin{cases} x_0 t_1 t_2^{-1} \ldots t_{m-1}^{-1} t_m, & \text{if } m \text{ is odd,} \\ x_0 t_1 t_2^{-1} \ldots t_{m-1} t_m^{-1}, & \text{if } m \text{ is even.} \end{cases}$$

Similarly, there exist $u_1, \ldots, u_m \in S$ such that for $1 \leq m \leq s$

$$y_m = \begin{cases} y_0 u_1 u_2^{-1} \ldots u_{m-1}^{-1} u_m, & \text{if } m \text{ is odd,} \\ y_0 u_1 u_2^{-1} \ldots u_{m-1} u_m^{-1}, & \text{if } m \text{ is even.} \end{cases}$$

The arcs $a_m = (x_m, x_m t_m)$ if m is even and $a_m = (x_{m-1} t_m^{-1}, x_{m-1})$ if m is odd, $0 \leq m \leq s - 1$, are then contained in $A(\mathrm{Cay}(G, S))$, and $Q_1 = a_0, \ldots, a_{s-1}$ is an alternating s-arc in $\mathrm{Cay}(G, S)$. Similarly, the arcs $b_m = (y_m, x_y u_m)$ if m is even and $b_m = (y_{m-1} u_m^{-1}, y_{m-1})$ if m is odd, $0 \leq m \leq s-1$, are then contained in $A(\mathrm{Cay}(G, S))$, and $Q_2 = b_0, \ldots, b_{s-1}$ is an alternating s-arc in $\mathrm{Cay}(G, S)$. As $\mathrm{Cay}(G, S)$ is alternating s-arc-transitive, there exists $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G, S))$ such that $\gamma(Q_1) = Q_2$. Then $\hat{\gamma} \in \mathrm{Aut}(\mathrm{Haar}(G, S))$, and $\hat{\gamma}(P_1) = P_2$. The result follows.

Conversely, suppose $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$ and $\mathrm{Haar}(G,S)$ is s-arc-transitive. Let $P_1 = a_1, \ldots, a_s$ and $P_2 = b_1, \ldots, b_s$ be alternating s-arcs in $\mathrm{Cay}(G,S)$ with vertex-sequences $x_0, x_1, \ldots, x_s$ and $y_0, y_1, \ldots, y_s$, respectively. Then there exist $t_1, \ldots, t_s \in S$ such that for $1 \le m \le s$ we have

$$x_m = \begin{cases} x_0 t_1 t_2^{-1} \ldots t_{m-1}^{-1} t_m, & \text{if } m \text{ is odd,} \\ x_0 t_1 t_2^{-1} \ldots t_{m-1} t_m^{-1}, & \text{if } m \text{ is even.} \end{cases}$$

Similarly, there exist $u_1, \ldots, u_m \in S$ such that for $1 \le m \le s$

$$y_m = \begin{cases} y_0 u_1 u_2^{-1} \ldots u_{m-1}^{-1} u_m, & \text{if } m \text{ is odd,} \\ y_0 u_1 u_2^{-1} \ldots u_{m-1} u_m^{-1}, & \text{if } m \text{ is even.} \end{cases}$$

Corresponding to these two alternating s-arcs, there are two s-arcs $(0, x_0), (1, x_1), \ldots, (s, x_s)$ and $(0, y_0), (1, y_1), \ldots, (s, y_s)$ in $\mathrm{Haar}(G,S)$. As $\mathrm{Haar}(G,S)$ is s-arc-transitive, there exists $\delta \in \mathrm{Aut}(\mathrm{Haar}(G,S))$ such that

$$\delta((0, x_0), (1, x_1), \ldots, (s, x_s)) = (0, y_0), (1, y_1), \ldots, (s, y_s).$$

Then $\delta(0, x_0) = (0, y_0)$ so $\delta \in \mathrm{fix}_{\mathrm{Aut}(\mathrm{Haar}(G,S))}(\mathcal{B})$. As $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$, $\delta = \hat{\gamma}$ for some $\gamma \in \mathrm{Aut}(\mathrm{Cay}(G,S))$. Then $\gamma(x_i) = y_i$ and so $\gamma(P_1) = P_2$. So $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive. □

We next consider alternating s-arc-transitivity when $\mathrm{Cay}(G,S)$ contains arcs and edges. We will need a preliminary lemma.

Lemma 3.4. *Let Γ be a digraph such that every vertex has in- and out-degree at least two, and $s \ge 2$. If Γ is alternating s-arc-transitive, then Γ is alternating $(s-1)$-arc-transitive.*

Proof. Let $a_1, \ldots, a_{s-1}$ and $b_1, \ldots, b_{s-1}$ be alternating $(s-1)$-arcs in Γ with vertex-sequences $x_0, \ldots, x_{s-1}$ and $y_0, \ldots, y_{s-1}$. As x_{s-1} and y_{s-1} have in- and out-degree at least two, $a_1, \ldots, a_{s-1}$ and $b_1 \ldots, b_{s-1}$ can be extended to alternating s-arcs $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$. As Γ is alternating s-arc-transitive, there is $\gamma \in \mathrm{Aut}(\Gamma)$ such that

$$\gamma(a_1, \ldots, a_s) = b_1, \ldots, b_s.$$

So $\gamma(a_1, \ldots, a_{s-1}) = b_1, \ldots, b_{s-1}$ and Γ is alternating $(s-1)$-arc-transitive. □

Corollary 3.5. *Let G be a group, $S \subseteq G$ such that $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$. $\mathrm{Haar}(G,S)$ is s-arc-transitive if and only if*

(1) *$S = S^{-1}$ and $\mathrm{Cay}(G,S)$ is s-arc-transitive, or*

(2) *$S \cap S^{-1} = \emptyset$ and $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive.*

Proof. Suppose $\mathrm{Haar}(G,S)$ is s-arc-transitive. By Theorem 3.3, $\mathrm{Cay}(G,S)$ is alternating s-arc-transitive. Also, $|S| \ge 2$ as $\mathrm{Aut}(\mathrm{Haar}(G,S)) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(G,S))$, so by Lemma 3.4 we have that $\mathrm{Cay}(G,S)$ is arc-transitive. This implies $\mathrm{Cay}(G,S)$ cannot contain both edges and arcs, so $S = S^{-1}$ or $S \cap S^{-1} = \emptyset$. If $S \cap S^{-1} = \emptyset$, then (2) follows. Otherwise, $S = S^{-1}$ so $\mathrm{Cay}(G,S)$ is a graph, and is alternating s-arc-transitive if and only if it is s-arc-transitive and (1) follows.

For the converse, we have already observed that an s-arc-transitive graph is alternating s-arc-transitive, and so the result holds if $S = S^{-1}$. If $S \cap S^{-1} = \emptyset$, then the result follows by Theorem 3.3. □

Definition 3.6. Let Γ be a digraph. we say that Γ is a *strict digraph* if for every arc $(a, b) \in A(\Gamma)$, the arc $(b, a) \notin A(\Gamma)$.

Note that if G is a group and $S \subseteq G$ such that $S \cap S^{-1} = \emptyset$, then $\mathrm{Cay}(G, S)$ is a strict digraph.

Definition 3.7. Let A be an abelian group. The group $\mathbb{Z}_2 \ltimes A \cong \langle \iota, \widehat{A_L} \rangle$ is a *generalized dihedral group*.

Definition 3.8. A transitive permutation group $G \leq S_n$ is *quasiprimitive* if every nontrivial normal subgroup of G is transitive.

We now characterize s-arc-transitive Cayley graphs of generalized dihedral groups with abelian normal subgroup of odd order.

Theorem 3.9. *Let $s \geq 2$ and Γ be an s-arc-transitive Cayley graph of a generalized dihedral group G with a normal abelian subgroup A of odd order n and index 2 in G. Then one of the following is true:*

(1) Γ *is disconnected,*

(2) $\mathrm{Aut}(\Gamma)$ *is quasiprimitive or primitive,*

(3) $\Gamma \cong K_{n,n}$,

(4) Γ *is isomorphic to a Haar graph corresponding to an s-arc-transitive Cayley graph of A,*

(5) Γ *is isomorphic to a Haar graph corresponding to an alternating s-arc-transitive Cayley strict digraph of A, or*

(6) Γ *is isomorphic to a Haar graph of A and its corresponding Cayley digraph need not be s-arc-transitive. In this case, $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$ and the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_1 is faithful and inequivalent to the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_0.*

Proof. We assume (1) - (5) do not hold, and show (6) holds. As $\mathrm{Aut}(\Gamma)$ is neither quasiprimitive or primitive, there exists $1 < N \triangleleft \mathrm{Aut}(\Gamma)$ that is not transitive. Let $\mathcal{C}$ be the nontrivial block system of $\mathrm{Aut}(\Gamma)$ formed by the orbits of N. As Γ is connected, there is some edge in Γ between C_1 and $C_2 \in \mathcal{C}$ with $C_1 \neq C_2$. As Γ is edge-transitive, there can be no edges inside any block of $\mathcal{C}$.

Case 1: N has two orbits. As there are no edges inside any block of $\mathcal{C}$, Γ is a connected bipartite graph with bipartition $\mathcal{C}$. So $\mathrm{Aut}(\Gamma)/\mathcal{C} \cong \mathbb{Z}_2$ has order 2. As G_L has order $2n = |G|$, we see $\mathrm{fix}_{G_L}(\mathcal{C})$ has order n. As the only proper normal subgroups of G of odd order are contained in A as A is of odd order, $\mathrm{fix}_{G_L}(\mathcal{C})$ contains a semiregular subgroup with two orbits isomorphic to A. Then Γ is isomorphic to a Haar graph of A by [14, Lemma 2.5]. So we may assume without loss of generality that $\Gamma = \mathrm{Haar}(A, S)$ (and $\mathcal{C} = \mathcal{B}$).

If the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is not faithful, then by Theorem 2.16 we have $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_{|H|}$ for some maximal subgroup $1 < H \leq A$. If $|A/H| = 1$, then $\Gamma \cong K_{n,n} \cong \mathrm{Haar}(G, A)$, contradicting our assumption that (3) does not hold. If $|A/H| \geq 2$, then it is straightforward to show that no such wreath product is 2-arc-transitive as $|V(\mathrm{Haar}(A/H, S))| \geq 4$. To see this, first observe that $\mathbb{Z}_2 \times A/H$ is a block system of

$\mathrm{Aut}(\Gamma)$ as $\Gamma \cong \mathrm{Haar}(A/H, S) \wr \bar{K}_{|H|}$ and H was chosen to be maximal. Then some 2-arcs in Γ are of the form $((0, ah), (1, ah), (0, ah'))$ for some $a \in G$ and $h, h' \in H$ with $h' \neq h$, and some 2-arcs in Γ are of the form $(0, ah), (1, ah), (0, bh)$ where $h \in H$, and $a, b \in G$ with $aH \neq bH$. As $\mathbb{Z}_2 \times A/H$ is a block system of $\mathrm{Aut}(\Gamma)$, no automorphism of Γ can map a 2-arc of the first kind, whose vertices are in two blocks of $\mathbb{Z}_2 \times A/H$, to a 2-arc of the second kind, whose vertices are in three blocks of $\mathbb{Z}_2 \times A/H$. Thus the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ is faithful. If the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_1 is equivalent to the action of $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ on B_0, then by Theorem 2.16 we may assume $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \ltimes \mathrm{Aut}(\mathrm{Cay}(A, S))$, in which case (4) or (5) would occur by Corollary 3.5. Then (6) follows from Theorem 2.16.

Case 2: If N has at least three orbits, then, as Γ is connected, Γ is a cover of some s-arc-transitive graph by the Praeger Normal Cover Lemma [36], so N is semiregular. Let M be the largest subgroup of N that is normal in $\mathrm{Aut}(\Gamma)$ and is contained in $\widehat{A_L}$. Let $\mathcal{D}$ be the block system of $\mathrm{Aut}(\Gamma)$ formed by the orbits of M.

Suppose $M = 1$. As $\mathcal{C}$ is the set of left cosets of some subgroup of G, and as the square of every element of G is contained in A, $\mathcal{C}$ consists of blocks of size 2. So N is a semiregular group of order 2. As A has odd order, $\widehat{A_L}/\mathcal{D} \cong A$ is a semiregular subgroup of order $n = |A|$ permutating n blocks, and so is regular. Then $\langle \widehat{A_L}, M \rangle$ is transitive, $\widehat{A_L} \cap M = 1$, and as M has order 2, $\langle \widehat{A_L}, M \rangle$ is abelian. We conclude that $\langle \widehat{A_L}, M \rangle \cong \widehat{A_L} \times M$ is abelian. Thus Γ is an s-arc-transitive Cayley graph of an abelian group, and as n is odd, we have Γ is isomorphic to a Cayley graph of $\mathbb{Z}_{2n}$. By [30, Theorem 1.1], $\Gamma = K_{2n}$, $\Gamma = K_{n,n}$ or Γ is $K_{n,n}$ with a 1-factor deleted. If $\Gamma = K_n$, then $\mathrm{Aut}(\Gamma)$ is primitive, a contradiction. By hypothesis, $\Gamma \neq K_{n,n}$. Finally, $K_{n,n}$ with a 1-factor deleted is 2-arc-transitive but not 3-arc-transitive and is isomorphic to $\mathrm{Haar}(A, A - \{0\})$. So $K_{n,n}$ with a 1-factor deleted is the Haar graph corresponding to the 2-arc-transitive Cayley graph $K_n = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n - \{0\})$, also a contradiction.

Suppose $M \neq 1$. Then M has at least as many orbits as N. If G/M is abelian, then the commutator subgroup G' of G is contained in M, and as $G' \cong A$, $[G : G'] = 2$, M has at most two orbits, a contradiction. Thus G/M is nonabelian. Then A/M is semiregular in G/M with two orbits, so $\Gamma/\mathcal{D}$ is isomorphic to a Haar graph of A/M by [14, Lemma 2.5]. Then $\Gamma/\mathcal{D}$ is connected, and so $\mathcal{B}/\mathcal{D}$ is a block system of $\mathrm{Aut}(\Gamma/\mathcal{D})$, and so of $\mathrm{Aut}(\Gamma)/\mathcal{D}$. Then $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$, and this case reduces to the one above with $N = \mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ having two orbits. □

Note that $K_{n,n}$ does not satisfy 3.9(4) as $K_{n,n} = \mathrm{Haar}(A, A)$ but $\mathrm{Cay}(A, A)$ has loops, and even if the loops were deleted, K_n is 2-arc-transitive but not 3-arc-transitive while $K_{n,n}$ is 3-arc-transitive. So including $K_{n,n}$ in the conclusion of the above theorem is not superfluous.

Also, the Heawood graph is 4-arc-transitive by the Foster Census [16], while its corresponding Cayley digraph $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ is arc-transitive but not 2-arc-transitive. This follows as by [1, Theorem 2] $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\}))$ has automorphism group $G = \{x \mapsto ax + b : a \in \{1, 2, 4\}, b \in \mathbb{Z}_7\}$, and it is arc-transitive. However, as G has order 21, $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$ cannot be 2-arc-transitive as there are 42 2-arcs in $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$.

Finally, while we have shown that there are graphs satisfying Theorem 3.9(6) holds, and some of the other possibilities obviously hold, it is not clear that for each of the six possibilities, there are corresponding graphs.

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