ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 13 (2017) 49–61

On the Erdős-Sós Conjecture for graphs on $n = k + 4$ vertices*

Long-Tu Yuan , Xiao-Dong Zhang

Department of Mathematics, and Ministry of Education, Key Laboratory of Scientific and Engineering Computing, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, P. R. China

Received 1 August 2015, accepted 19 November 2015, published online 21 October 2016

Abstract

The Erdős-Sós Conjecture states that if G is a simple graph of order n with average degree more than $k - 2$, then G contains every tree of order k. In this paper, we prove that Erdős-Sós Conjecture is true for $n = k + 4$.

Keywords: Erdős-Sós Conjecture, tree, maximum degree.

Math. Subj. Class.: 05C05, 05C35

1 Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). Let $G = (V(G), E(G))$ be a graph of order n, where $V(G)$ is the vertex set and $E(G)$ is the edge set with size $e(G)$. The *degree* of $v \in V(G)$, the number of edges incident to v, is denoted $d_G(v)$ and the set of neighbors of v is denoted $N(v)$. If u and v in $V(G)$ are adjacent, we say that u *hits* v or v *hits* u. If u and v are not adjacent, we say that u *misses* v or v *misses* u. If $S \subseteq V(G)$, the induced subgraph of G by S is denoted by $G[S]$. Denote by $D(G)$ the diameter of G. In addition, $\delta(G)$, $\Delta(G)$ and $avedeg(G) = \frac{2e(H)}{|V(H)|}$ are denoted by the minimum, maximum and average degree in $V(G)$, respectively. Let T be a tree on k vertices. If there exists an injection $g : V(T) \to V(G)$ such that $g(u)g(v) \in E(G)$ if $uv \in E(T)$ for $u, v \in V(T)$, we call g an *embedding* of T into G and G contains a copy of T as a subgraph, denoted by $T \subseteq G$. In addition, assume that $T' \subseteq T$ is a subtree of T

*This work is supported by National Natural Science Foundation of China (Nos.11271256 and 11531001), The Joint Israel-China Program (No.11561141001), Innovation Program of Shanghai Municipal Education Commission (No.14ZZ016) and Specialized Research Fund for the Doctoral Program of Higher Education (No.20130073110075).

E-mail addresses: yuanlongtu@sjtu.edu.cn (Long-Tu Yuan), xiaodong@sjtu.edu.cn (Xiao-Dong Zhang)

and g' is an embedding of T' into G. If there exists an embedding $g : V(T) \to V(G)$ such that $g(v) = g'(v)$ for all $v \in V(T')$, we say that g' is $T-$*extensible.*

In 1959, Erdős and Gallai [6] proved the following theorem.

Theorem 1.1. *Let G be a graph with $avedeg(G) > k-2$. Then G contains a path of order k.*

Based on the above result, Later Erdős and Sós proposed the following well known conjecture (for example, see [5]).

Conjecture 1.2. *Let G be a graph with $avedeg(G) > k-2$. Then G contains every tree on k vertices as a subgraph.*

Various specific cases of Conjecture 1.2 have already been proven. For example, Brandt and Dobson [2] proved the conjecture for graphs having girth at least 5. Balasubramanian and Dobson [1] proved this conjecture for graphs without any copy of $K_{2,s}$, $s < \frac{k}{12}+1$. Li, Liu and Wang [15] proved the conjecture for graphs whose complement has girth at least 5. Dobson [3] improved this to graphs whose complements do not contain $K_{2,4}$. More results on this conjecture can be referred to [7, 8, 9] and [11, 12]. On the other hand, in 2003, Mclennan [10] proved the following theorem.

Theorem 1.3. *Let G be a graph with $avedeg(G) > k-2$. Then G contains every tree of order k whose diameter does not excess 4 as a subgraph.*

In 2010, Eaton and Tiner [4] proved the the following two theorems.

Theorem 1.4. *[4] Let G be a graph with $avedeg(G) > k-2$. If $\delta(G) \geq k-4$, then G contains every tree of order k as a subgraph.*

Theorem 1.5. *[4] Let G be a graph with $avedeg(G) > k-2$. If $k \leq 8$, then G contains every tree of order k as a subgraph.*

In 1984, Zhou [17] proved that Conjecture 1.2 holds for $k = n$. Later, Slater, Teo and Yap [13] and Woźniak [16] proved that Conjecture 1.2 holds for $k = n-1$ and $k = n-2$, respectively.

Theorem 1.6. *[16] Let G be a graph of order n with $avedeg(G) > k-2$. If $k = n-2$, then G contains every tree of order k as a subgraph.*

Recently, Tiner [14] proved that Conjecture 1.2 holds for $k = n-3$.

Theorem 1.7. *[14] Let G be a graph of order n with $avedeg(G) > k-2$. If $k \geq n-3$, then G contains every tree of order k as a subgraph.*

In this paper, we establish the following:

Theorem 1.8. *Let G be a graph of order n with $avedeg(G) > k-2$. If $k \geq n-4$, then G contains every tree of order k as a subgraph.*

2 Proof of Theorem 1.8

Let T be any tree of order k. If $k \geq n-3$, or $k \leq 8$ or the diameter of T is at most 4, the assertion holds by Theorems 1.3, 1.5 and 1.7. We only consider $k = n-4 \geq 9$, $D(T) \geq 5$ and prove the assertion by the induction. Clearly the assertion holds for $n = 6$. Hence assume Theorem 1.8 holds for all of the graphs of order fewer than n and let G be a graph of order n. If there exists a vertex v with $d_G(v) < \lfloor \frac{k}{2} \rfloor$, then $avedeg(G-v) > k-2$ and the assertion holds by Theorems 1.7. Furthermore, by Theorem 1.4, without loss of generality, there exists a vertex z in $V(G)$ such that $\lfloor \frac{k}{2} \rfloor \leq d_G(z) = \delta(G) \leq k-5$. Without loss of generality, we can assume that $e(G) = 1 + \lfloor \frac{1}{2}(k-2)(k+4) \rfloor$. Let T be any tree of order k with a longest path $P = a_0a_1 \ldots a_{r-1}a_r$ and $N_T(a_1) \setminus \{a_2\} = \{b_1, \ldots, b_s\}$ and $N_T(a_{r-1}) \setminus \{a_{r-2}\} = \{c_1, \ldots, c_t\}$. Since $avedeg(G) > k-2$, we can consider the following cases: $\Delta(G) = k+3, k+2, k+1, k, k-1$.

2.1 $\Delta(G) = k + 3$

Let $u \in V(G)$ be such vertex that $d_G(u) = k+3$ and let $G' = G - \{u, z\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - (k+3) - (k-5) + 1 > \frac{1}{2}(k+4)(k-2) - (k+3) - (k-5) + 1 = \frac{1}{2}(k^2 - 2k - 2)$. So *avedeg*$(G') > (k^2-2k-2)/(k+2) > k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. Let f' be an embedding of T' into G'. Then let $f = f'$ in T' and $f(a_1) = u$, $X = V(G) \setminus f'(V(T'))$. Since $d_G(u) = k+3$, u hits at least s vertices in X. Hence f can be extended to an embedding of T into G or we can say that f is $T-$extensible.
Remark: For the remainder of this paper we shall always let f' be an embedding of T' into G' and when we do not define the value of f on any vertex of T', we always let $f = f'$ on those vertices.

2.2 $\Delta(G) = k + 2$

Let $u \in V(G)$ be such vertex that $d_G(u) = k+2$. Then there exists only one vertex $x \in V(G) \setminus \{u\}$ not adjacent to u. We consider two subcases: $d_G(x) \leq k-2$ and $d_G(x) \geq k-1$.

2.2.1 $d_G(x) \leq k - 2$

Let $G' = G - \{u, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - (k+2) - (k-2) > \frac{1}{2}(k^2-2k-8)$. So *avedeg*$(G') > (k^2-2k-8)/(k+2) = k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. Then let $f(a_1) = u$ and $X = V(G) \setminus f'(V(T'))$. Since $d_G(u) = k+2$, u hits at least s vertices in X, f is $T-$extensible.

2.2.2 $d_G(x) \geq k - 1$

Since $x \neq z$, we consider the following two cases.

(A). x misses z. Let $G' = G - \{u, z, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_r\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) - (k+1) + 1 > \frac{1}{2}(k^2-4k-2)$. Hence *avedeg*$(G') > (k^2-4k-2)/(k+1) > k-5$ and $\mid V(T') \mid \leq k-3$. By the induction hypothesis, we have $T' \subseteq G'$. Since x misses z, u and $d_G(x) \geq k-1$, x misses at most two vertices of G'. If x hits $f'(a_2)$, let $f(a_1) = x$ and $f(a_r) = u$. Since $d_G(x) \geq k-1$ and u hits all vertices of T', f is $T-$extensible. Hence we assume that x misses $f'(a_2)$. If x hits $f'(a_{r-1})$, let

$f(a_r) = x$ and $f(a_1) = u$. Then f is $T-$extensible. If x misses $f'(a_2)$ and $f'(a_{r-1})$, then x hits all of $V(G') \setminus \{f'(a_2), f'(a_{r-1})\}$, because $D(T) \geq 5$, a_2 and a_{r-1} are not adjacent. Then let $f(a_{r-1}) = x$, $f(a_1) = u$, which implies that f is $T-$extensible.

(B). x hits z. We consider the following two subcases.

(B.1). $d_G(x) > k-1$. Let $G' = G - \{u, z, x\}$, $T' = T - \{a_1, b_1, \ldots, b_s, a_r\}$. Since x misses u and $d_G(x) > k-1$, x misses at most two vertices of G', the assertion can be proven by the similar method of (A).

(B.2). $d_G(x) = k-1$. Then x misses 3 vertices of $V(G) \setminus \{u\}$, says y_1, y_2, y_3.

(a). There exists one vertex y_i with $1 \leq i \leq 3$ such that $d_G(y_i) \geq k+1$. Let $G' = G - \{u, z, y_i, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) - (k+2) - (k-1) + 3 + 1 > \frac{1}{2}(k^2 - 6k + 4)$, which implies *avedeg*$(G') > (k^2 - 6k + 4)/k > k - 6$ and $\mid V(T') \mid \leq k - 4$. Hence by the induction hypothesis, $T' \subseteq G'$. Note that y_i misses at most one vertex of G'. If y_i misses $f'(a_2)$, let $f(a_1) = u, f(a_{r-1}) = y_i$; if y_i misses $f'(a_{r-2})$, let $f(a_{r-1}) = u$,$f(a_1) = y_i$. Thus f is $T-$extensible.

(b). There exists one vertex y_i with $1 \leq i \leq 3$ such that $d_G(y_i) = k$ and y_i misses z. Then the proof is similar to (a) and omitted.

(c). There exists one vertex y_i with $1 \leq i \leq 3$ such that $d_G(y_i) \leq k-2$. Let $G' = G - \{u, y_i, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_r\}$. Then $e(G') \geq e(G) - (k+2) - (k-2) - (k-1) + 1 > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k - 5$ and $\mid V(T') \mid \leq k - 3$. Hence by the induction hypothesis, $T' \subseteq G'$. Similarly as in case (A), there exists an embedding from T into G.

(d). $d_G(y_i) = k$, y_i hits z or $d_G(y_i) = k - 1$ for $i \in \{1, 2, 3\}$.

(d.1). $d_T(a_1) + d_T(a_{r-1}) \geq 5$. Let $G' = G - \{u, z, y_1, y_2, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) - (k-1) - (k-1) - (k-1) + 3 > \frac{1}{2}(k^2 - 8k + 10)$ which implies *avedeg*$(G') > (k^2 - 8k + 10)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k - 5$. Hence by the induction hypothesis, $T' \subseteq G'$. Moreover, x misses only one vertex of G'. If x misses $f'(a_2)$, let $f(a_1) = u, f(a_{r-1}) = x$; if x misses $f'(a_{r-2})$, let $f(a_{r-1}) = u, f(a_1) = x$. In both situations, f is $T-$extensible.

(d.2). $d_T(a_1) = d_T(a_{r-1}) = 2$. Let $G' = G - \{u, z\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) + 1 > \frac{1}{2}(k^2 - 2k)$, which implies *avedeg*$(G') > (k^2 - 2k)/(k+2) > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Moreover, u hits all vertices of $V(G) \setminus \{x\}$ and z hits x. Let $f(a_1) = u$ or z and $f(a_0) = z$ or u. Then f is $T-$extensible.

2.3 $\Delta(G) = k + 1$

Let $u \in V(G)$ be such vertex that $d_G(u) = k+1$ with u missing vertices x_1 and x_2. Without loss of the generality, we can assume $d_G(x_1) \geq d_G(x_2)$ and $d_T(a_1) \geq d_T(a_{r-1})$.

2.3.1 $d_T(a_1) + d_T(a_{r-1}) \geq 5$

We consider the following two cases: (A) and (B).

(A). x_1 misses x_2.

(A.1) $d_G(x_1) + d_G(x_2) \leq 2k - 3$. Let $G' = G - \{u, x_1, x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - (k+1) - (2k-3) > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k - 5$ and $\mid V(T') \mid \leq k - 3$. Hence by the induction hypothesis, $T' \subseteq G'$. Let $f(a_1) = u$. It is easy to see that f is $T-$extensible.

(A.2). $d_G(x_1) + d_G(x_2) \geq 2k - 2$.

(a). $d_G(x_1) = k-1$ Then $d_G(x_2) = k-1$ and x_1 misses $\{u, x_2, y_1, y_2\}$. If $y_1, y_2 \neq z$, let $G' = G - \{u, z, x_1, x_2, y_1\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) - (2k-2) - (k+1) + 3 > \frac{1}{2}(k^2 - 8k + 8)$, which implies *avedeg*$(G') > (k^2 - 8k + 8)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k - 5$. Hence by the induction hypothesis, $T' \subseteq G'$. Note that x_1 misses only one vertex of G'. If x_1 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_1$; if x_1 misses $f'(a_{r-2})$, let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. In both situations, f is $T-$extensible. Now assume that $y_1 = z$ or $y_2 = z$. Let $G' = G - \{u, x_1, x_2, y_1, y_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) - (2k-2) - (k+1) + 2 + 1 > \frac{1}{2}(k^2 - 8k + 8)$, which implies *avedeg*$(G') > (k^2 - 8k + 8)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k - 5$. Let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. Then f is $T-$extensible.

(b). $d_G(x_1) \geq k$. Let $G' = G - \{u, z, x_1, x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) - (2k+2) + 1 + 2 > \frac{1}{2}(k^2 - 6k + 2)$, which implies *avedeg*$(G') > (k^2 - 6k + 2)/k > k - 6$ and $\mid V(T') \mid \leq k - 4$. Hence by the induction hypothesis, $T' \subseteq G'$. Note that x_1 misses at most one vertex of G'. If x_1 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_1$; if x_1 misses $f'(a_{r-2})$, let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. In both situations, f is $T-$extensible.

(B). x_1 hits x_2.

(B.1). $d_G(x_1) + d_G(x_2) \leq 2k - 2$. The proof is similar to (A.1) and omitted.

(B.2). $d_G(x_1) + d_G(x_2) \geq 2k - 1$. The proof is similar to (A.2) with (a)$d_G(x_1) = k$,$d_G(x_2) = k - 1$ or k, (b)$d_G(x_1) = k + 1$.

2.3.2 $d_T(a_1) = d_T(a_{r-1}) = 2$.

We consider the following four cases.

(A). There exists a vertex $v \neq u$ of degree at most k such that it hits both x_1 and x_2. Let $G' = G - \{u, v\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+1) - k + 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $\mid V(T') \mid \leq k - 2$. Hence by the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, let $f(a_1) = u$. If $f'(a_2)$ misses u, then $f'(a_2) = x_1$ or x_2 and let $f(a_1) = v, f(a_0) = u$. Thus f is $T-$extensible.

(B). There exists a vertex $v \neq u$ of degree at least $k+1$ such that it hits both x_1 and x_2. Then $d_G(v) = k + 1$ and v misses y_1 and y_2. Since the case $z \in \{x_1, x_2, y_1, y_2\}$ is much easier, we may suppose $z \neq x_1, x_2, y_1, y_2$. Let $G' = G - \{u, v, z\} - x_1x_2 - y_1y_2$ and $T' = T - \{a_0, a_1, a_r\}$. Then $e(G') \geq e(G) - 2(k+1) - (k-5) + 1 - 2 > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k - 5$ and $\mid V(T') \mid \leq k - 3$. Hence by the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_1$ or x_2, and $f'(a_{r-1}) = y_1$ or y_2, then let $f(a_1) = v$ and $f(a_r) = u$. If $f'(a_2) = x_1$ and $f'(a_{r-1}) = x_2$, then let $f(a_1) = v, f(a_{r-1}) = u$, because u hits all the neighbours of $f'(a_{r-1})$. If $f'(a_2) = y_1$,$f'(a_{r-1}) = y_2$, then let $f(a_1) = u$ and $f(a_{r-1}) = v$. For the rest situations, it is easy to find an embedding from T into G.

(C). There is no vertex in $V(G) \setminus \{u\}$ hitting both x_1 and x_2, and x_1 misses x_2. Then $d_G(x_1) + d_G(x_2) \leq k + 1$. Let $G' = G - \{u, x_1, x_2\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+1) - (k+1) > \frac{1}{2}(k^2 - 2k - 12)$, Since $k \geq 9$, *avedeg*$(G') > (k^2 - 2k - 12)/(k+1) > k - 4$ and $\mid V(T') \mid \leq k - 2$. By theorem 1.7, $T' \subseteq G'$. Let $f(a_1) = u$. Then f is $T-$extensible.

(D). There is no vertex in $V(G) \setminus \{u\}$ hitting both x_1 and x_2, and x_1 hits x_2. Then $d_G(x_1) + d_G(x_2) \leq k+3$. If $d_G(x_1) + d_G(x_2) \leq k+2$, the assertion follows from (C). Hence assume that $d_G(x_1) + d_G(x_2) = k+3$. If $z \neq x_1, x_2$, then z has to hit x_1 or x_2, say that z hits x_1. Let $G' = G - \{u, z\} - x_1x_2$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) + 1 - 1 > \frac{1}{2}(k^2 - 2k)$, which implies *avedeg*$(G') > (k^2 - 2k)/(k+2) > k-4$ and $\mid V(T') \mid \leq k-2$. Hence by the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, let $f(a_1) = u$; if $f'(a_2) = x_1$, let $f(a_1) = z$ and $f(a_0) = u$. If $f'(a_2) = x_2$ and if there is a vertex w in T' such that $f'(w) = x_1$, let $f(w) = u$, $f(a_1) = x_1$ and $f(a_0) = z$, because u hits all neighbours of $f'(w)$ in G'; if $f'(a_2) = x_2$ and there does not exist any vertex w in T' such that $f'(w) = x_1$, let $f(a_1) = x_1$, and $f(a_0) = z$. In all situations, f is $T-$extensible. If $z = x_1$ or x_2, then let $G' = G - \{u, z\}$ and $T' = T - \{a_0, a_1\}$. Similarly, we can find an embedding from T into G.

2.4 $\Delta(G) = k$

Let $u \in V(G)$ be a vertex of degree $d_G(u) = k$ and misses three vertices x_1, x_2, x_3. Denote by $S = \{x_1, x_2, x_3\}$.

2.4.1 $G[S]$ contains no edges.

Let $G' = G - \{u\}$ and $T' = T - \{a_0\}$. Then $e(G') \geq e(G) - k > \frac{1}{2}(k^2 - 8)$, which implies *avedeg*$(G') > (k^2 - 8)/(k+3) > k-3$ and $\mid V(T') \mid \leq k-1$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_1)$ hits u, let $f(a_0) = u$; if $f'(a_1) = x_i$, $1 \leq i \leq 3$, let $f(a_1) = u$. Since u hits all neighbours of $f'(a_1)$ in G', f is $T-$extensible.

2.4.2 $G[S]$ contains exactly one edge.

Without loss of the generality, x_1 hits x_2, $d_G(x_1) \geq d_G(x_2)$, and $d_T(a_1) \geq d_T(a_{r-1})$. We consider two cases.

(A). $d_T(a_1) + d_T(a_{r-1}) \geq 5$.

(A.1). $d_G(x_2) \geq k-1$. If $x_3 \neq z$, let $G' = G - \{u, z, x_3\} - x_1x_2$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - k - (k-5) - k - 1 > \frac{1}{2}(k^2 - 4k)$, which implies *avedeg*$(G') > (k^2 - 4k)/(k+1) > k-5$ and $\mid V(T') \mid \leq k-3$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, then let $f(a_1) = u$; if $f'(a_2) = x_1$ and $x_2 \notin f'(V(T'))$, then let $f(a_1) = x_2$; if $f'(a_2) = x_1$ and $x_2 \in f'(V(T'))$ and $f'(w) = x_2$, then let $f(w) = u$, $f(a_2) = x_1$, and $f(a_1) = x_2$. Hence f is $T-$extensible. On the other hand, if $x_3 = z$, let $G' = G - \{u, z\} - \{x_1x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Similarly, we can prove that the assertion holds.

(A.2). $d_G(x_3) \geq k-1$. By (A.1), we can assume that $d_G(x_2) \leq k-2$. If $z \neq x_1, x_2$, let $G' = G - \{u, z, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - k - (k-5) - (k-2) - k - k + 2 + 1 > \frac{1}{2}(k^2 - 8k + 12)$, which implies *avedeg*$(G') > (k^2 - 8k + 12)/(k-1) > k-7$ and $\mid V(T') \mid \leq k-5$. Hence by the induction hypothesis, $T' \subseteq G'$. Moreover, x_3 misses at most one vertex of $V(G')$. If x_3 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_3$; if x_3 hits $f'(a_2)$, let $f(a_{r-1}) = u$ and $f(a_1) = x_3$. then f is $T-$extensible. On the other hand, if $x_1 = z$ or $x_2 = z$, let $G' = G - \{u, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Using the same above argument, we can prove the assertion.

(A.3). $d_G(x_1) = k$,$d_G(x_2) \le k-2$ and $d_G(x_3) \le k-2$. If $z \ne x_2, x_3$, let $G' = G - \{u, z, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Hence $e(G') \ge e(G)-k-(k-5)-(k-2)-k-(k-2)+2 > \frac{1}{2}(k^2-8k+10)$, which implies *avedeg*$(G') > (k^2-8k+10)/(k-1) > k-7$ and $\mid V(T') \mid \le k-5$. By the induction hypothesis, $T' \subseteq G'$. Note that x_1 misses at most one vertex in $V(G')$. If x_1 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_1$; if x_1 hits $f'(a_2)$, let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. Hence f is $T-$extensible. On the other hand, if $x_2 = z$ or $x_3 = z$, let $G' = G - \{u, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. By the same above argument, we can prove the assertion.

(A.4). $d_G(x_1) \le k-1$,$d_G(x_2) \le k-2$ and $d_G(x_3) \le k-2$. Then there exists a vertex u' in $V(G) \setminus \{x_1, x_2, x_3, u\}$ with degree at least $k-1$. Otherwise, by $\delta(G) \le k-5$, we have *avedeg*$(G) \le \frac{k+(k-1)(k-2)+(k-1)+2(k-2)+(k-5)}{k+4} \le k-2$, which is a contradiction. Let $G' = G - \{u, u'\} - \{x_1x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \ge e(G) - k - k + 1 - 1 > \frac{1}{2}(k^2-2k-8)$, which implies *avedeg*$(G') > (k^2-2k-8)/(k+2) = k-4$ and $\mid V(T') \mid \le k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, let $f(a_1) = u$; if $f'(a_2)$ misses u, let $f(a_2) = u$ and $f(a_1) = u'$. Then f is $T-$extensible.

(B). $d_T(a_1) = 2$ and $d_T(a_{r-1}) = 2$. If there exists a vertex w that hits both x_1 and x_3, let $G' = G - \{u, w\} - x_1x_2$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - 2k + 1 - 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k + 8)/(k + 2) = k - 4$ and $\mid V(T') \mid \le k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_1$ or x_3, let $f(a_1) = w$ and $f(a_0) = u$; if $f'(a_2) = x_2$ and $x_1 \notin f'(V(T'))$, let $f(a_1) = x_1$ and $f(a_0) = w$; if $f'(a_2) = x_2$ and $x_1 \in f'(V(T'))$,$f'(v) = x_1$, let $f(v) = u, f(a_1) = x_1$ and $f(a_0) = w$. In the above situations, f is $T-$extensible. On the other hand, if there is no vertex hits both x_1 and x_3, or x_2 and x_3. then $d_G(x_1)+d_G(x_3) \le k$, $d_G(x_2)+d_G(x_3) \le k$. Since $d_G(x_i) \ge \lfloor \frac{k}{2} \rfloor$ and $k \ge 9$, $d_G(x_i) \le k-2$. Hence, similarly as in (A.4), there exists a vertex u' in $V(G) \setminus \{x_1, x_2, x_3, u\}$ with degree at least $k-1$, and an embedding of T into G.

2.4.3 $G[S]$ contains exactly two edges

Without loss of the generality, assume that x_1 hits both x_2 and x_3. We consider the following two cases.

(A). $d_T(a_1) = 2$. Let $G' = G - \{u, x_1\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - 2k > \frac{1}{2}(k^2-2k-8)$, which implies *avedeg*$(G') > (k^2-2k-8)/(k+2) > k-4$ and $\mid V(T') \mid \le k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_1) = x_1$; Moreover, if $x_3 \notin f'(V(T'))$, let $f(a_0) = x_3$; if $x_3 \in f'(V(T'))$ and $f'(v) = x_3$, let $f(v) = u, f(a_1) = x_1$, and $f(a_0) = x_3$. Hence, f is T-extensible. If $f'(a_2) \ne x_2, x_3$, then it is easy to find an embedding from T to G.

(B). $d_T(a_1) \ge 3$.

(a). $d_G(x_1) \ge k-1$. If $z \ne x_2, x_3$, let $G' = G - \{u, z, x_1\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \ge e(G) - k - (k-5) - k + 1 > \frac{1}{2}(k^2 - 4k + 4)$, which implies *avedeg*$(G') > (k^2 - 4k + 4)/(k+1) > k-5$ and $\mid V(T') \mid \le k-3$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_1) = x_1$. Moreover, if $x_3 \notin f'(V(T'))$, let $f(b_1) = x_3$; if $x_3 \in f'(V(T'))$ and $f'(v) = x_3$, let $f(v) = u, f(a_1) = x_1$ and $f(b_1) = x_3$. Because u hits all neighbours of $f'(v)$ and $d_G(x_1) \ge k-1$, f is $T-$extensible. If $f'(a_2) \ne x_2, x_3$, it is easy to find an embedding from T to G. On the other hand, if $z = x_2$ or x_3 (say x_2), let $G' = G - \{u, x_1, x_2\}$, by the same argument above, the assertion holds.

(b). $d_G(x_1) \leq k-2$, $d_G(x_2) = k$ or $d_G(x_3) = k$ (say $d_G(x_2) = k$). Then there exists a vertex $y \in V(G) \setminus \{u, x_1, x_2, x_3\}$ such that x_2 misses y. So x_2 misses u, x_3 and y and u misses x_3. By Case 2.4.2, we can assume y hits x_3. Further, by (a), we can assume $d_G(y) \leq k-2$. If $z \neq x_1, y$, let $G' = G - \{u, z, x_2, x_3, y\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - k - (k-5) - k - k - (k-2) + 3 > \frac{1}{2}(k^2 - 8k + 12)$, which implies *avedeg*$(G') > (k^2 - 8k + 12)/(k-1) > k-7$ and $\mid V(T') \mid \leq k-5$. By the induction hypothesis, $T' \subseteq G'$. Further, if $f'(a_2) = x_1$, let $f(a_1) = x_2$ and $f(a_{r-1}) = u$; if $f'(a_{r-2}) = x_1$, let $f(a_{r-1}) = x_2$ and $f(a_1) = u$. Hence f is $T-$extensible. On the other hand, if $z = y$, let $G' = G - \{u, x_2, x_3, y\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$; if $z = x_1$, let $G' = G - \{u, z, x_2, x_3, y\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then by the same argument, it is easy to prove that the assertion holds.

(c). $d_G(x_1) \leq k-2$, $d_G(x_2) = k-1$ and $d_G(x_3) = k-1$. Let $G' = G - \{u, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - k - (k-1) - (k-1) > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k-5$ and $\mid V(T') \mid \leq k-3$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_1$, let $f(a_1) = x_2$, which f is $T-$extensible. If $f'(a_2) \neq x_1$, it is easy to find an embedding from T to G.

(d). $d_G(x_1) \leq k-2$, and $d_G(x_2) \leq k-2$ or $d_G(x_3) \leq k-2$ (say $d_G(x_2) \leq k-2$), hence $d_G(x_3) \leq k-1$ by (b). Then there exists a vertex $u' \in V(G) \setminus \{x_1, x_2, x_3, u\}$ of degree at least $k-1$, otherwise $2e(G) \leq (k-1)(k-2) + (k-5) + k + 2(k-2) + (k-1) \leq (k+4)(k-2)$ which is impossible. Let $G' = G - \{u, u', x_1\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - 2k - (k-2) + 1 > \frac{1}{2}(k^2 - 4k - 2)$, which implies *avedeg*$(G') > (k^2 - 4k - 2)/(k+1) > k-5$ and $\mid V(T') \mid \leq k-3$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2)$ hits u, let $f(a_1) = u$; if $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_2) = u$ and $f(a_1) = u'$ since u hits all the neighbours of $f'(a_2)$. Then f is $T-$extensible.

2.4.4 $G[S]$ contains exactly three edges

The following two cases are considered.

(A). $d_T(a_1) = 2$. If there exists an $1 \leq i \leq 3$ (say $i = 1$) such that $d_G(x_1) \leq k-1$, let $G' = G - \{u, x_1\} - x_2x_3$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - k - (k-1) - 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_1) = x_1$. Moreover, if $x_3 \notin f'(V(T'))$, let $f(a_0) = x_3$; and if $x_3 \in f'(V(T'))$ and $f'(v) = x_3$, let $f(v) = u, f(a_1) = x_1, f(a_0) = x_3$. Hence f is $T-$extensible. On the other hand, if $d_G(x_1) = d_G(x_2) = d_G(x_3) = k$, let $G' = G - \{u, x_1\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2k > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis,$T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3, let $f(a_1) = x_1$; if $f'(a_2) \neq x_2, x_3$, let $f(a_1) = u$. Hence f is T-extensible.

(B). $d_T(a_1) \geq 3$. If there exists an $1 \leq i \leq 3$ (say $i = 1$) such that $d_G(x_1) \geq k-1$, Let $G' = G - \{u, z, x_1\} - x_2x_3$. By the same argument as Case 2.4.3.(B).(a), the assertion holds. The rest is similar as Case 2.4.3.(B).(d).

2.5 $\Delta(G) = k - 1$

Since $\Delta(G) = k - 1$ and $\delta(G) \leq k - 5$, there exist at least four vertices of degree $k - 1$. Otherwise $2e(G) \leq 3(k-1)+k(k-2)+(k-5) = (k-2)(k+4)$, which is a contradiction. Let u_i be vertex of $d_G(u_i) = k - 1$ missing four vertices of $S_i = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\}$ for $i = 1, 2, 3, 4$. If there exists a vertex u_i with $1 \leq i \leq 4$ such that $G[S_i]$ contains at most one edge. Let $G' = G - \{u_i\} - E(G[S_i])$ and $T' = T - \{a_0\}$. Then $e(G') \geq e(G) - (k-1) - 1 > \frac{1}{2}(k^2 - 8)$, which implies *avedeg*$(G') > (k^2 - 8)/(k + 3) > k - 3$ and $\mid V(T') \mid\leq k - 1$. By the induction hypothesis, $T' \subseteq G'$. If u_i hits $f'(a_1)$, let $f(a_0) = u_i$, and if u_i misses $f'(a_1)$, let $f(a_1) = u_i$. Then f is $T-$extensible. Hence we assume that $G[S_i]$ contains at least two edges for $i = 1, 2, 3, 4$.

2.5.1 $d_T(a_1) \geq 3, d_T(a_{r-1}) \geq 2$

We consider the number of the edges in $G[u_1, u_2, u_3, u_4]$.

(A). $G[u_1, u_2, u_3, u_4]$ contains at least one edge, say u_1 hits u_2. If $z \notin S_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$, let $G' = G - \{u_1, u_2, z\} - E(G[S_1])$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - 2(k-1) - (k-5) + 1 - 6 > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k + 1) > k - 5$ and $\mid V(T') \mid\leq k - 3$. By the induction hypothesis, $T' \subseteq G'$. Hence if u_1 hits $f'(a_2)$, let $f(a_1) = u_1$; and if u_1 misses $f'(a_2)$, let $f(a_2) = u_1$ and $f(a_1) = u_2$. Since u_1 hits all the neighbours of $f'(a_2)$ in G', f is $T-$extensible. On the other hand, if $z \in S_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$, say $z = x_{11}$. Let $G' = G - \{u_1, u_2, z\} - E(G[x_{12}, x_{13}, x_{13}])$. By the same argument, the assertion holds.

(B). $G[u_1, u_2, u_3, u_4]$ contains no edges.

(B.1). If there exist two vertices, say u_1 and u_2, in $\{u_1, u_2, u_3, u_4\}$ such that u_1 misses y_1 and u_2 misses y_2, where $y_1 \neq y_2$ and $y_1, y_2 \notin \{u_1, \ldots, u_4\}$. Let $G' = G - \{u_1, u_2, u_3, u_4\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - 4(k-1) > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > (k^2 - 6k)/k = k - 6$ and $\mid V(T') \mid\leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = y_1$, let $f(a_1) = u_2$ and $f(a_{r-1}) = u_1$; if $f'(a_2) = y_2$, let $f(a_1) = u_1$ and $f(a_{r-1}) = u_2$. Moreover, if $f'(a_{r-2}) = y_1$, let $f(a_1) = u_1$ and $f(a_{r-1}) = u_2$; and if $f'(a_{r-2}) = y_2$, let $f(a_1) = u_2$ and $f(a_{r-1}) = u_1$. Therefore, f is $T-$extensible.

(B.2). There exist a vertex $y \notin \{u_1, \ldots, u_4\}$ such that y misses $u_1, \ldots, u_4$. Then $G[u_1, \ldots, u_4, y]$ contains no edges.

(a). $d_T(a_{r-1}) = 2$. Then there exists a vertex w hits $\{u_1, u_2, u_3, u_4\}$ and y. Let $G' = G - \{u_1, w\}$ and $T' = T - \{a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 2(k-1) + 1 > \frac{1}{2}(k^2 - 2k - 2)$, which implies *avedeg*$(G') > (k^2 - 2k - 2)/(k + 2) > k - 4$ and $\mid V(T') \mid\leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_{r-2}) = u_2, u_3, u_4$ or y, let $f(a_{r-1}) = w$ and $f(a_r) = u_1$; and if $f'(a_{r-2}) \neq u_2, u_3, u_4, y$, let $f(a_{r-1}) = u_1$ and $f(a_r) = w$. Therefore f is $T-$extensible.

(b). $d_T(a_{r-1}) \geq 3$. If $z \neq y$, let $G' = G - \{u_1, u_2, u_3, u_4, y, z\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - 4(k-1) - (k-1) - (k-5) + 4 > \frac{1}{2}(k^2 - 10k + 20)$, which implies *avedeg*$(G') > (k^2 - 10k + 20)/(k - 2) > k - 8$ and $\mid V(T') \mid\leq k - 6$. By the induction hypothesis, $T' \subseteq G'$. Let $f(a_1) = u_1$ and $f(a_{r-1}) = u_2$. Then f is $T-$extensible. On the other hand, if $z = y$, let $G' = G - \{u_1, u_2, u_3, u_4, z\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. By the same argument, the assertion holds.

2.5.2 $d_T(a_1) = 2, d_T(a_{r-1}) = 2$.

We will discuss the following four cases: $(A), (B), (C)$ and (D).

(A). There exists a $1 \le i \le 4$, say $i = 1$, such that $G[S_1]$ contains two or three edges. If u_1 hits one vertex, say u_2, of three vertices u_2, u_3, u_4. Let $G' = G - \{u_1, u_2\} - E(G[S_1])$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $\mid V(T') \mid \le k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if u_1 hits $f'(a_2)$, let $f(a_1) = u_1$; and if u_1 misses $f'(a_2)$, let $f(a_2) = u_1$ and $f(a_1) = u_2$. Since u_1 hits all the neighbours of $f'(a_2)$ in G', f is $T-$extensible. Therefore, we assume that u_1 misses u_j for $j = 2, 3, 4$. Then u_1 misses $x_{11} = u_2, x_{12} = u_3, x_{13} = u_4, x_{14}$ and $G[u_2, u_3, u_4, x_{14}]$ contains two or three edges.

(A.1). x_{14} hits one vertex, say u_2, of three vertices u_2, u_3, u_4. Let $G' = G - \{u_1, u_2, u_3, u_4\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 4(k-1) > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > (k^2 - 6k)/k = k - 6$ and $\mid V(T') \mid \le k - 4$. By the induction hypothesis, $T' \subseteq G'$. Since $G[u_2, u_3, u_4, x_{14}]$ contains two or three edges, there exists a vertex, say u_3, of two vertices u_3,u_4 misses at most one vertex, say y_1, in $V(G) \setminus \{u_1, u_2, u_4, x_{14}\}$. Hence if $f'(a_2) = x_{14}$ or y_1, and $f'(a_{r-2}) = y_1$ or x_{14}, let $f(a_1) = u_2$ or u_1 and $f(a_{r-1}) = u_1$ or u_2, then f is $T-$extensible. For the rest cases, it is easy to find an embedding from T to G.

(A.2). x_{14} misses three vertices u_2, u_3, u_4. Then $G[u_2, u_3, u_4]$ contains two or three edges. We can assume that u_2 hits u_3 and u_4. If u_3 misses u_4, u_3 misses at most one vertex, says y_1, in $V(G) \setminus \{u_1, u_2, u_4, x_{14}\}$. Then let $G' = G - \{u_1, x_{14}, u_3, u_4\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. By the similar argument as Case (A.1), the assertion holds. Hence we can assume that u_3 hits u_4 and u_3 misses z_1, z_2, u_1, x_{14}. Let $G' = G - \{u_1, x_{14}, u_3, u_4\} - \{z_1 z_2\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 4(k-1) + 1 - 1 > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > (k^2 - 6k)/k = k - 6$ and $\mid V(T') \mid \le k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = z_1$ or z_2, and $f'(a_{r-2}) = z_2$ or z_1, let $f(a_2) = u_3$, $f(a_1) = u_4$, $f(a_{r-1}) = u_1$. Therefore f is $T-$ extensible. If $f'(a_2) = z_1$ or z_2, and $f'(a_{r-2}) = u_2$, let $f(a_1) = u_1$, $f(a_{r-1}) = u_4$. Therefore f is $T-$ extensible. For the rest cases, it is easy to find an embedding from T to G.

(B). There exists a $1 \le i \le 4$, say $i = 1$, such that $G[S_1]$ contains exactly four edges.

(B.1). There exists a vertex, say x_{11}, of degree 3 in $G[S_1]$ and $\mid E(G[S_1]) \mid \le 5$. Then x_{11} hits x_{12}, x_{13} and x_{14}. Let $G' = G - \{u_1, x_{11}\} - E(G[x_{12}, x_{13}, x_{14}])$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - 2(k-1) - 2 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $\mid V(T') \mid \le k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if u_1 hits $f'(a_2)$, let $f(a_1) = u_1$, which implies that f is $T-$extensible. If u_1 misses $f'(a_2)$ and $f'(a_2) = x_{12}$, let $f(a_1) = x_{11}$. Moreover, if x_{13} or $x_{14} \notin f'(V(T'))$, then let $f(a_0) = x_{13}$ or x_{14}. Then f is $T-$extensible. If x_{13} and $x_{14} \in f'(V(T'))$, $f'(w) = x_{13}$ or x_{14}, let $f(w) = u_1, f(a_0) = x_{13}$ or x_{14}. Then f is $T-$extensible. For the rest cases, it is easy to find an embedding from T to G.

(B.2). The degree of every vertex in $G[S_1]$ is two. We assume that x_{11} hits x_{12}, x_{12} hits x_{13}, x_{13} hits x_{14}, x_{14} hits x_{11}.

(a). u_1 hits all vertices of $\{u_2, u_3, u_4\}$.

(a.1). There exists a vertex u_i, say u_2, in $\{u_2, u_3, u_4\}$ which misses x_{11}, x_{12}, x_{13} and x_{14}. Let $G' = G - \{u_1, u_2, x_{11}, x_{12}\} - x_{13}x_{14}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 4(k-1) + 1 > \frac{1}{2}(k^2 - 6k + 2)$, which implies *avedeg*$(G') > (k^2 - 6k + 2)/k > k - 6$ and $\mid V(T') \mid \le k - 4$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_{13}, f'(a_{r-2}) = x_{14}$, let $f(a_1) = x_{12}, f(a_0) = x_{11}, f(a_{r-2}) = u_1, f(a_{r-1}) =$

u_2. Since u_1 hits all the neighbours of $f'(a_{r-2})$ in G', f is $T-$extensible. For the rest cases, similarly, it is easy to find an embedding from T to G.

(a.2). There exists a vertex, say u_2, in $\{u_2, u_3, u_4\}$ such that it hits at least two vertices of $\{x_{11}, x_{12}, x_{13}, x_{14}\}$, say u_2 hits x_{11} and x_{13}, or u_2 hits x_{11} and x_{12}.

If u_2 hits x_{11} and x_{13}, let $G' = G - \{u_1, u_2\} - x_{11}x_{12} - x_{12}x_{13} - x_{13}x_{14}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ or x_{13}, let $f(a_1) = u_2$; if $f'(a_2) = x_{12}$, let $f(a_2) = u_1$ and $f(a_1) = u_2$; if $f'(a_2) = x_{14}$ and $x_{13} \notin f'(V(T'))$, let $f(a_1) = x_{13}$ and $f(a_0) = u_2$; if $f'(a_2) = x_{14}$ and $x_{13} \in f'(V(T'))$, let $f(v) = u_1, f(a_1) = x_{13}, f(a_0) = u_2$, because there is a vertex v, $f'(v) = x_{13}$ and u_1 hits all the neighbours of $f'(v)$ in G'. Therefore f is $T-$extensible.

If u_2 hits x_{11} and x_{12}, let $G' = G - \{u_1, u_2\} - x_{12}x_{13} - x_{13}x_{14} - x_{11}x_{14}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ or x_{12}, let $f(a_1) = u_2$; if $f'(a_2) = x_{13}$ or x_{14}, let $f(a_2) = u_1, f(a_1) = u_2$, because u_1 hits all the neighbours of $f'(a_2)$ in G'. Therefore f is $T-$extensible.

(a.3). u_i hits exactly one vertex of $\{x_{11}, x_{12}, x_{13}, x_{14}\}$ for $i = 2, 3, 4$.

(i). There exist two vertices of $\{u_2, u_3, u_4\}$ such that they hit the same vertex in $\{x_{11}, x_{12}, x_{13}, x_{14}\}$, says both u_2 and u_3 hit x_{14}.

If u_2 and u_3 misses the same vertices, say, $\{x_{11}, x_{12}, x_{13}, y\}$, then u_2 hits u_3. Further, if $G[x_{11}, x_{12}, x_{13}, y]$ contains at most three edges or has a vertex of degree 3, the assertion follows from Case 2.5.2.(A) or Case 2.5.2.(B.1). Therefore we can assume that y hits both x_{11} and x_{13}. Let $G' = G - \{u_2, u_3, x_{11}, x_{12}\} - x_{13}y$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. The assertion follows from Case 2.5.2. (B.2).(a.1).

If u_2 misses $\{x_{11}, x_{12}, x_{13}, y_1\}$ and u_3 misses $\{x_{11}, x_{12}, x_{13}, y_2\}$ with $y_1 \neq y_2$, let $G' = G - \{u_1, u_2, u_3, x_{14}\} - x_{11}x_{12} - x_{12}x_{13}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 4(k-1) + 4 - 2 > \frac{1}{2}(k^2 - 6k + 4)$, which implies *avedeg*$(G') > k - 6$ and $\mid V(T') \mid \leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ or x_{13}, let $f(a_1) = x_{14}$,$f(a_0) = u_3$ or u_2 or let $f(a_2) = u_1$,$f(a_1) = u_3$ or u_2. If $f'(a_2) = x_{12}$, let $f(a_2) = u_1$,$f(a_1) = u_3$ or u_2. If $f'(a_2) = y_1$ or y_2, let $f(a_1) = u_3$ or u_2. Since there is a choice which uses distinct vertices of $\{u_1, u_2, u_3, x_{14}\}$ for any two vertices of $\{x_{11}, x_{12}, x_{13}, y_1, y_2\}$, we can find an embedding from T to G. (For example, if $f'(a_2) = x_{11}$,$f'(a_{r-2}) = x_{13}$, let $f(a_1) = x_{14}$,$f(a_0) = u_3$,$f(a_{r-2}) = u_1$,$f(a_{r-1}) = u_2$.)

(ii). $\{u_2, u_3, u_4\}$ hits the different vertices of $\{x_{11}, x_{12}, x_{13}, x_{14}\}$. Without loss of generality, we assume that u_2 hits x_{11} and u_3 hits x_{13}, u_2 misses y_1 and u_3 misses y_2. Let $G' = G - \{u_1, u_2, u_3, x_{13}\} - x_{11}x_{12} - x_{11}x_{14}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 4(k-1) + 3 + 0 - 2 > \frac{1}{2}(k^2 - 6k + 2)$, which implies *avedeg*$(G') > k - 6$ and $\mid V(T') \mid \leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{12}$ or x_{14}, let $f(a_1) = x_{13}$ and $f(a_0) = u_3$, or let $f(a_2) = u_1$ and $f(a_1) = u_2$, if $f'(a_2) = y_1$ or y_2, let $f(a_1) = u_1$, if $f'(a_2) = x_{11}$, let $f(a_1) = u_2$, Therefore f is $T-$extensible. For the rest cases, by the same argument, it is easy to find an embedding from T to G.

(b). u_1 hits one or two vertices of $\{u_2, u_3, u_4\}$. Without loss of the generality, we assume that u_1 hits u_2 and u_1 misses u_4. Then $u_4 \in \{x_{11}, x_{12}, x_{13}, x_{14}\}$, say $u_4 = x_{14}$, u_4 misses u_1,x_{12},z_1,z_2.

If $u_2 \neq z_1, z_2$, then u_2 hits u_4. Let $G' = G - \{u_1, u_2, u_4, x_{12}\} - z_1z_2$ and $T' =$

$T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 4(k-1) + 2 - 1 > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > k - 6$ and $\mid V(T') \mid \leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ and $f'(a_{r-2}) = x_{13}$, let $f(a_1) = u_4, f(a_{r-2}) = u_1$ and $f(a_{r-1}) = u_2$, if $f'(a_2) = z_1$ and $f'(a_{r-2}) = z_2$, let $f(a_1) = u_1, f(a_{r-2}) = u_4$ and $f(a_{r-1}) = u_2$. Therefore f is T-extensible. For the rest cases, it is easy to find an embedding from T to G. If $u_2 = z_1$ or z_2, say $u_2 = z_1$, let $G' = G - \{u_1, u_2, u_4, x_{12}\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. This situation is much easier than the above case.

(c). u_1 misses all the vertices of $\{u_2, u_3, u_4\}$. Without loss of generality, we assume $u_2 = x_{11}$, $u_3 = x_{12}$, $u_4 = x_{13}$. Let u_2 miss $\{u_1, x_{13}, y_1, y_2\}$. If $G[u_1, x_{13}, y_1, y_2]$ contains two, or three edges, or a vertex of degree 3, the assertion follows from Case 2.5.2 (A). and Case 2.5.2 (B.1). Hence we assume that u_1 hits y_1, y_1 hits $u_4 = x_{13}$, u_4 hits y_2 and y_2 hits u_1. Hence the assertion follows from Case 2.5.2. (B.2). (a) and Case 2.5.2. (B.2).(b).

(C). There exists a $1 \leq i \leq 4$, say $i = 1$, such that $G[x_{11}, x_{12}, x_{13}, x_{14}]$ contains five edges. Then we assume that x_{11} hits x_{12}, x_{13} and x_{14}. Let $G' = G - \{u_1, x_{11}\} - E(G[x_{12}, x_{13}, x_{14}])$ and $T' = T - \{a_0, a_1\}$. The assertion follows from the proof of Case 2.5.2 (B.1).

(D). There exists a $1 \leq i \leq 4$, say $i = 1$, such that $G[x_{11}, x_{12}, x_{13}, x_{14}]$ contains six edges. If $d_G(x_{11}) \leq k - 2$, similar as Case 2.5.2 (B.1), we can prove the assertion. So we can assume $d_G(x_{11}) = d_G(x_{12}) = d_G(x_{13}) = d_G(x_{14}) = k - 1$, we can also assume if $d_G(x) = k - 1$, and x misses y then $d_G(y) = k - 1$, furthermore we can assume x hits all of the vertices whose degree is less than $k - 1$. Let $G' = G - \{u_1, z\}$, z hits all of $\{x_1, x_2, x_3, x_4\}$, $T' = T - \{a_0, a_1\}$. So $e(G') \geq e(G) - (k-1) - (k-5) + 1 > \frac{1}{2}(k^2 - 2k + 6)$. *avedeg*$(G') > (k^2 - 2k + 6)/(k+2) > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction assumption, $T' \subseteq G'$. If $f'(a_2)$ hits u_1, then $f(a_1) = u_1, f(a_0) = z$. If $f'(a_2)$ misses u_1, then $f(a_0) = u_1, f(a_1) = z$. f is T-extensible.

References

[1] S. Balasubramanian and E. Dobson, Constructing trees in graphs with no $k_{2,s}$, *Journal of Graph Theory* **56** (2007), 301–310, http://dx.doi.org/10.1002/jgt.20261.

[2] S. Brandt and E. Dobson, The Erdős-Sós conjecture for graphs of girth 5, *Discrete Math.* **150** (1996), 411–414, selected papers in honour of Paul Erdős on the occasion of his 80th birthday (Keszthely, 1993), http://dx.doi.org/10.1016/0012-365X(95)00207-D.

[3] E. Dobson, Constructing trees in graphs whose complement has no $K_{2,s}$, *Combin. Probab. Comput.* **11** (2002), 343–347, http://dx.doi.org/10.1017/S0963548302005102.

[4] N. Eaton and G. Tiner, On the Erdős-Sós conjecture and graphs with large minimum degree, *Ars Combin.* **95** (2010), 373–382.

[5] P. Erdős, Some problems in graph theory, *Theory of Graphs and Its Applications* (1965), 29–36.

[6] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar* **10** (1959), 337–356 (unbound insert).

[7] G. Fan, The Erdős-Sós conjecture for spiders of large size, *Discrete Math.* **313** (2013), 2513–2517, http://dx.doi.org/10.1016/j.disc.2013.07.021.

[8] G. Fan and L. Sun, The Erdős-Sós conjecture for spiders, *Discrete Math.* **307** (2007), 3055–3062, http://dx.doi.org/10.1016/j.disc.2007.03.018.

[9] P. E. Haxell, Tree embeddings, *J. Graph Theory* **36** (2001), 121–130, http://dx.doi.org/10.1002/1097-0118(200103)36:3<121::AID-JGT1000>3.0.CO;2-U.

[10] A. McLennan, The Erdős-Sós conjecture for trees of diameter four, *J. Graph Theory* **49** (2005), 291–301, http://dx.doi.org/10.1002/jgt.20083.

[11] J.-F. Saclé and M. Woźniak, The Erdős-Sós conjecture for graphs without C_4, *J. Combin. Theory Ser. B* **70** (1997), 367–372, http://dx.doi.org/10.1006/jctb.1997.1758.

[12] A. F. Sidorenko, Asymptotic solution for a new class of forbidden r-graphs, *Combinatorica* **9** (1989), 207–215, http://dx.doi.org/10.1007/BF02124681.

[13] P. J. Slater, S. K. Teo and H. P. Yap, Packing a tree with a graph of the same size, *J. Graph Theory* **9** (1985), 213–216, http://dx.doi.org/10.1002/jgt.3190090203.

[14] G. Tiner, On the Erdős-Sós conjecture for graphs on $n = k + 3$ vertices, *Ars Combin.* **95** (2010), 143–150.

[15] M. Wang, G.-j. Li and A.-d. Liu, A result of Erdős-Sós conjecture, *Ars Combin.* **55** (2000), 123–127.

[16] M. Woźniak, On the Erdős-Sós conjecture, *J. Graph Theory* **21** (1996), 229–234, http://dx.doi.org/10.1002/(SICI)1097-0118(199602)21:2<229::AID-JGT13>3.3.CO;2-2.

[17] B. Zhou, A note on the Erdős-Sós conjecture, *Acta Math. Sci. (English Ed.)* **4** (1984), 287–289.