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# 14 Some Thoughts on the Question of Dimensions

A. Kleppe

SACT, Oslo, Norway

**Abstract.** Some issues related to the question of space dimensions are discussed.

**Povzetek.** Obravnavamo nekatere probleme povezane z vprašanjem razsežnosti prostora.

The question of the number of space dimensions is age-old. The naive assumption, that space has three dimensions, seems justified for many different reasons. One argument is the mere experience of space as 3-dimensional, while a stronger argument is for example that the 4-dimensionality of spacetime is a cornerstone in general relativity, where spacetime is modeled as a 4-manifold. The number of dimensions is however not to be taken for granted, as we know e.g. from Kaluza-Klein, and string theory.

Already in the 1780-ies Immanuel Kant [1] reasoned that there must be three dimensions of space, since 3-dimensional space is due to the inverse square law of universal gravitation. His argument was later inverted, as it is the inverse square law that is explained by 3-dimensional space and Gauss' law, since in a D-dimensional space gravitational or electrostatic force varies like $R^{1-D}$, where $R$ is the distance between the two bodies/charges.

Some 140 years later, Paul Ehrenfest [2] argued that in a $d = D+1$-dimensional spacetime with $D > 3$, the orbit of a planet around its sun does not remain stable, and likewise for star's orbit around the center of its galaxy. This is because in an even-dimensional space the different parts of a wave travel at different speeds, while for $D > 3$ and odd, the wave impulses become distorted.

About the same time, in 1922, Hermann Weyl [3] stated that Maxwell's theory of electromagnetism only works for $d = 3 + 1$, and this fact *"...not only leads to a deeper understanding of Maxwell's theory, but also of the fact that the world is four dimensional, which has hitherto always been accepted as merely 'accidental,' become intelligible through it."*

In the end of the 1990-ies Max Tegmark [4] pushed the anthropic principle by arguing that spaces with dimension $D < 3$ are too poor in complexity to allow for intelligent beings like ourselves. Like Ehrenfest, he reasoned that for $D > 3$ neither atoms nor planetary systems can be stable, since for $D > 3$, the two-body problem has no stable orbit solution. Similarly, for $D > 3$ the H-atom has no bound states.

The intuition that four dimensions are 'special' is moreover supported by group theory, as many different families of symmetry groups are distinct in 4-dimensional spaces, while in fewer dimensions they are indistinguishable, and in spaces of higher dimensional spaces they do not exist.

$d = 4$ is also singled out in mathematician Simon Donaldson's work on the classification topological four-manifolds from from the early 1980-ies [5], where it was demonstrated that the most complex geometry and topology occur precisely in four dimensions. Donaldson's work focused on 4-manifolds admitting a differentiable structure, using instantons, i.e. self-dual solutions of the Yang-Mills equations. From gauge theory he derived polynomial invariants, new topological invariants sensitive to the underlying smooth structure of a 4-manifold. This made it possible to deduce the existence of exotic 4-manifolds, differentiable 4-dimensional manifolds which are topologically but not differentiably equivalent to the standard Euclidean $R^4$. This is unique for $d = 4$, in the sense that for $d \neq 4$, there are no exotic smooth structures on $R^d$.

So within the realm of pure mathematics, Donaldson's work singled out 4-dimensionality, echoing Leibniz' claim [6] that our (3+1-dimensional) world is *"the one which is at the same time the simplest in hypothesis and the richest in phenomena."*

A different type of arguments for $d = 3 + 1$, is based on assigning primacy to the Weyl equation and the stability it displays in four dimensions [7]. The dimensions $4 = 3+1$ have the special property that in these dimensions the number of linearly independent matrices that appear in the Weyl equation is exactly equal to the dimension of spacetime. This is relevant for actual, physical spacetime, because the fermions of our world are basically Weyl particles, in the sense that each Dirac particle is to be considered as (composed of) two Weyl particles. The stability of the Weyl equation moreover means that in four dimensions the equation is stable under addition of extra terms. The Weyl equation has the form

$$i\sigma^{\mu} D_{\mu} \psi = 0 \tag{14.1}$$

and that it is stable means that one can add any smooth, Hermitian operator $\mathcal{O}$ to the operator $i\sigma^{\mu} D_{\mu}$,

$$(i\sigma^{\mu} D_{\mu} + \mathcal{O})\psi = 0 \tag{14.2}$$

and again obtain the Weyl equation in the low energy limit of a Taylor expansion.

Since $4 = 3 + 1$ is the "experimental" number of dimensions, and in 4 dimensions the Weyl equation is stable under small modifications, there seems to be genericness.

## 14.1 Manifolds

We regard manifolds as fundamental, the ultimate manifold being "experienced space" itself. A manifold is a topological space that is locally Euclidean, while globally it can have a completely different structure.

A differentiable manifold of dimension D locally looks like $R^D$, so it has open sets, continuous functions and differentiable functions. The structure of a manifold $M$ is studied by mapping $M$ into $R^n$, and then back to $M$. So if $f_1$ and $f_2$ are two different overlapping mappings from $M$ to $R^n$, $f_j : M \to R^n, j = 1, 2$, and a transition map is a mapping relating these maps, $T(p) = f_1[f_2^{-1}(p)]$.

The manifold structure is defined by the properties of the transition functions, e.g. if each transition function is a smooth map, the manifold itself is smooth.

The transition map is not well-defined unless both charts are restricted to the intersection of their domains of definition. With differentiable transition functions T, the manifold is differentiable, and functions on the manifold can be differentiated.

In this way a manifold's structure is established by relating it to something well-known, namely real numbers. They at least seem to be well-known, but in a certain sense the reals are just as abstract as any other abstract notions; mathematician Gregory Chaitin [8] goes so far as to claim that *"most individual real numbers are like mythical beasts, like Pegasus or unicorns"*.

His reasoning is that a real number is a number which is measured with arbitrary - infinite - precision. Each point on the number line is a real number, which from a geometrical point of view is seemingly uncomplicated, but arithmetically it is more problematic.

If you want to compute a number, you use an algorithm that you run on a computer, say. The number of computers and algorithms is however countable. Thereby, the number of computable reals is countable.

- The number of computable reals = The number of computer programs = The number of natural numbers = $\aleph_0$
- The number of uncomputable reals = The number of all reals = $\aleph_1$

Since $\aleph_1 \gg \aleph_0$, most reals are not computable. We can refer to them, but not compute them, and in this sense most reals do not 'exist'.

In spite of the fact that we in reality *handle* only a small subset of the real numbers, the real numbers are still well-known, and it is natural to study the structure of some abstract space by mapping it into the realm of reals.

### 14.1.1 Classification of manifolds

Up to four dimensions, manifolds are classified by geometric structure, while manifolds of higher dimensions are classified algebraically.

Generically, manifolds are classified by their invariants, the most familiar being the manifold *orientability* and the manifold *genus* or the related Euler characteristic $\chi$. That a manifold is orientable basically establishes that it is not a Möbius band, ensuring that helicity or rotation can be unambiguously defined on the manifold.

The intuitively compelling genus $g$ corresponds, loosely speaking, to the number of handles or holes in a manifold, and a compact 2-dimensional manifold is completely characterized by its genus and orientability.

In dimensions $2 < d \leq 4$, the characterizing invariants are the manifold's orientability and its Euler characteristic, closely related to the genus. In 3 dimensions, the Euler characteristic $\chi$ of a (convex) polyhedron is the relation

$$\chi = V - E + F = 2, \tag{14.3}$$

where $V, E, F$ are the vertices, edges and faces of the polyhedron. Since a (closed) convex polyhedron is homeomorphic to the sphere $S^2$, the sphere also has the

Euler characteristic 2, and for a compact, orientable surface the Euler characteristic $\chi$ and the genus $g$ are related by

$$\chi = 2 - 2g. \tag{14.4}$$

So while $S^2$ has Euler characteristic 2, the Euler characteristic of the torus, which has genus $g = 1$, is $\chi = 2 - 2 = 0$.

The classification of manifolds basically concerns whether two manifolds are homeomorphic or not: if the manifold X is homeomorphic to the manifold Y, then $\chi(X) = \chi(Y)$. The classification of smooth (differentiable) closed manifolds is well understood, even though the methods of classification differ for $d < 4$ and $d > 4$. In four dimensions, $d = 4$, the classification is however not possible, except for simply connected manifolds, the reason being that in 4-dimensions manifolds display a much higher degree of complexity than in other dimensions. What Simon Donaldson showed in 1982 was that there is a large class of 4-manifolds which admit no smooth structure at all, and even if there exists a smooth structure it need not be unique. The exotic manifolds, which are homeomorphic but not diffeomorphic to Euclidean $R^4$, thus single out 4 dimensions as encompassing a maximum of complexity.

## 14.2 Four dimensions

Locally a manifold resembles 'experienced space', and is naturally easy to intuit. But there are subtleties to keep in mind, for example that a manifold's dimension is actually locally defined. This can be illustrated by a sheet of paper, where the surface of the sheet is 2-dimensional, while the border of the sheet is 1-dimensional. Each connected part of the manifold however has a fixed dimension, i.e. all the points in a connected manifold have the same dimension.

Manifolds moreover display different properties in different dimensions. One way of 'probing' the different dimensions is to study the hypercube in N dimensions.

The 3D cube is characterized by its Euler characteristic $\chi = V - E + F = 2$, where $V, F$ and $E$ are the vertices, edges and faces, thus the 0-dimensional, 1-dimensional and 2-dimensional simplices constituting the cube.

In N dimensions the Euler characteristic of the N-dimensional cube can be expressed as $\chi_N = \sum_{k=0}^{N-1} (-1)^k S_k$, where $S_k$ is the number of k-dimensional simplices, which constitute the N-dimensional cube, $k = 0,..., N - 1$.

In order to calculate the number of simplices $S_k$ for each dimension k, we start by considering the 3D cube. The 3-cube has $2^3$ corners (vertices) V. At each corner (vertex) of the cube there is one vertex and 3 convening edges. Each edge "shares" two vertices with other edges, so the total number of egdes on the cube is $E = 3V/2$.

At each corner of the cube there are moreover 3 convening faces, each face "sharing" four vertices. Thus the total number of faces is $F = 3V/4$. Therefore the Euler characteristic is

$$\chi_{3D} = V[1 - 3/2 + 3/4] = V/4 = 2, \tag{14.5}$$

since V = $2^3$ = 8.

We repeat the reasoning for the 4-dimensional cube, the tesseract. At each corner there are 4 orthogonal convening edges, and since each 2-dimensional face is spanned by two orthogonal egdes, the 4 edges span $4x3/2! = 6$ faces, so there are 6 faces at each corner. Each edge "shares" 2 vertices with other edges, and each face "shares" 4 vertices with other faces, so the total number of edges and faces of the tesseract is $4V/2$ and $6V/4$, respectively. In addition, there are the 3D cubes, and each 3-simplex is spanned by 3 orthogonal egdes. From the 4 orthogonal edges that convene at each of the tesseract's corners, we thus get $4x3x2/3! = 4$ three-dimensional cubes, and since each cube "shares" 8 vertices with other 3-cubes, the total number of 3-cubes on the tesseract is 4V/8. The Euler characteristic is then

$$\chi_{4D} = V[1 - 4/2 + 6/4 - 4/8] = 0, \tag{14.6}$$

in agreement with the Euler characteristic being 0 for *any odd* dimensional manifold, since the manifold we are considering is the *surface* of the 4D tesseract, which is 3-dimensional, i.e. odd, just like the surface of the 3D cube is 2-dimensional, and thus has Euler characteristic 2.

Generally speaking, at each corner, or vertex, of the $N$-cube, there are $N$ convening edges. A $k$-dimensional simplex is spanned by $k$ of these edges, just like a (2-dimensional) face on the 3D cube is spanned by 2 of the 3 edges that meet at each of the cube's vertices.

Each $k$-dimensional simplex of the cube moreover has $2^k$ vertices (to be "shared"), so at each corner of the hypercube, there are

$$\frac{N(N-1)...(N-(k-1))}{k!2^k} = \frac{N!}{(N-k)!k!2^k} \tag{14.7}$$

$k$-dimensional simplices, which gives the Euler characteristic of a N-cube

$$\chi = 2^N \sum_{k=0}^{N-1} (-1)^k \frac{N!}{(N-k)!k!2^k} \tag{14.8}$$

oscillating between 0 and 2, corresponding to surfaces of odd and even dimension, respectively.

That there are generally valid formulae like the one for the Euler characteristic does however not mean that we can transcribe everything from one dimension to an other. It is for example hard to carry the concept of volume between different dimensions.

One way of looking at it, is to consider the longest straight line that we can draw within a N-dimensional hypercube with edge length $l$, namely the line which connects two opposite corners and has the length $L = \sqrt{l^2 + l^2 + l^2 + ...} = \sqrt{N}l$.

With the normalization $L = 1$ we get $l = 1/\sqrt{N}$, whereby the volume of the hypercube is $l^N = N^{-N/2}$, which shrinks with the dimension. There is however nothing magical with the higher $N$, this result is valid in any dimensions. Just compare the diagonals of a square and a 3D-cube:

In 2D we would get that $l_2 = 1/\sqrt{2}$, and in 3D that $l_3 = 1/\sqrt{3}$, the crux is that the normalizing "**1**" is not the same in the two expressions. If we were to carry the normalization in 2D over to 3D, we would actually get: $l_3^{normalized} = \sqrt{3} l_2 = \sqrt{\frac{3}{2}}$.

In conclusion, it is difficult to visualize, or "derive" a picture of one dimension by analogy of another. To go from three to four dimensions is not like going from two to three dimensions, both because we lack phenomenological experience of four dimensions, and also because the degree of complexity is so much higher in four than in three dimensions.

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