ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 349–368
https://doi.org/10.26493/1855-3974.1684.b0d
(Also available at http://amc-journal.eu)

Girth-regular graphs*

Primož Potočnik
Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia, and
Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia

Janoš Vidali
Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

Received 23 April 2018, accepted 4 March 2019, published online 4 November 2019

Abstract

We introduce a notion of a *girth-regular* graph as a k-regular graph for which there exists a non-descending sequence $(a_1, a_2, \dots, a_k)$ (called the *signature*) giving, for every vertex u of the graph, the number of girth cycles the edges with end-vertex u lie on. Girth-regularity generalises two very different aspects of symmetry in graph theory: that of vertex transitivity and that of distance-regularity. For general girth-regular graphs, we give some results on the extremal cases of signatures. We then focus on the cubic case and provide a characterisation of cubic girth-regular graphs of girth up to 5.

Keywords: Graph, girth-regular, cubic, girth.

Math. Subj. Class.: 05C38

1 Introduction

This paper stems from our research of finite connected vertex-transitive graphs of small girth. The girth (the length of a shortest cycle in the graph) is an important graph theoretical invariant that is often studied in connection with the symmetry properties of graphs. For example, cubic arc-transitive graphs (a graph is called arc-transitive if its automorphism group acts transitively on its *arcs*, where an arc is an ordered pair of adjacent vertices) and cubic semisymmetric (regular, edge-transitive but not vertex-transitive) graphs of girth up

*Supported in part by the Slovenian Research Agency, projects J1-5433, J1-6720, and P1-0294.
E-mail addresses: primoz.potocnik@fmf.uni-lj.si (Primož Potočnik), janos.vidali@fmf.uni-lj.si (Janoš Vidali)

to 9 and 10 have been studied in [5, 11] and [6], respectively, and tetravalent edge-transitive graphs of girths 3 and 4 have been considered in [14]. Recently, a classification of all cubic vertex-transitive graphs of girth up to 5 was obtained in [8].

In our investigation of vertex-transitive graphs of small girth, it became apparent to us that the condition of vertex-transitivity is almost never used in its full strength. What was needed in most of the arguments was only a particular form of uniformity of the distribution of girth cycles throughout the graph. Let us make this more precise.

For an edge e of a graph Γ, let $\epsilon(e)$ denote the number of girth cycles containing the edge e. Let v be a vertex of Γ and let $\{e_1, \dots, e_k\}$ be the set of edges incident to v ordered in such a way that $\epsilon(e_1) \le \epsilon(e_2) \le \dots \le \epsilon(e_k)$. Then the k-tuple $(\epsilon(e_1), \epsilon(e_2), \dots, \epsilon(e_k))$ is called the *signature* of v. A graph Γ is called *girth-regular* provided all of its vertices have the same signature. The signature of a vertex is then called the signature of the graph.

We should like to point out that girth-regular graphs of signature $(a, a, \dots, a)$ for some a have been introduced under the name *edge-girth-regular graphs* in [10], where the authors focused on the families of cubic and tetravalent edge-girth-regular graphs.

By definition, every girth-regular graph is regular (in the sense that all its vertices have the same valence). Further, it is clear that every vertex-transitive as well as every semisymmetric graph is girth-regular. Slightly less obvious is the fact that every distance-regular graph is also girth-regular. The notion of girth-regularity is thus a natural generalisation of all these notions. On the other hand, examples of girth-regular graphs exist that are neither vertex-transitive, nor semisymmetric nor distance-regular (for example, the truncation of a 3-prism is such a graph; see Section 3.2).

The central question we would like to propose and address in this paper is the following:

Question 1.1. Given integers k and g, for which tuples $\sigma = (a_1, a_2, \dots, a_k) \in \mathbb{Z}^k$ does a girth-regular graph of girth g and signature σ exist?

The above question seems to be very difficult if considered in its full generality. We begin by stating three theorems proved in Section 2, which give an upper bound on the entries a_i of the signature in terms of the valence k and the girth g, and consider the case where this upper bound is attained.

Theorem 1.2. *If Γ is a girth-regular graph of valence k, girth g, and signature $(a_1, \dots, a_k)$, then $a_k \le (k-1)^d$, where $d = \lfloor g/2 \rfloor$.*

Theorem 1.3. *If Γ is a connected girth-regular graph of valence k, girth $2d$ for some integer d, and signature $(a_1, \dots, a_k)$ such that $a_k = (k-1)^d$, then $a_1 = a_2 = \dots = a_k$ and Γ is the incidence graph of a generalised d-gon of order $(k-1, k-1)$.*

In particular, if $k = 3$, then $g \in \{4, 6, 8, 12\}$ and Γ is isomorphic to $K_{3,3}$ (if $g = 4$), the Heawood graph (if $g = 6$), the Tutte-Coxeter graph (if $g = 8$) or to the Tutte 12-cage (if $g = 12$).

For a description of the graphs mentioned in the above theorem, see Note 2.2.

Theorem 1.4. *If Γ is a connected 3-valent girth-regular graph of girth $2d+1$ for some integer d and signature (a_1, a_2, a_3) such that $a_3 = 2^d$, then Γ is isomorphic to K_4 or the Petersen graph.*

In the second part of the paper, we focus on 3-valent graphs (also called *cubic* graphs) and obtain a complete classification of cubic girth-regular graphs of girth at most 5 (see

Theorem 1.5 below). Prisms and Möbius ladders are defined in Section 4, the notion of a dihedral scheme and truncation is defined in Section 3.2, and graphs arising from maps are discussed in Section 3.3.

Theorem 1.5. *Let* Γ *be a connected cubic girth-regular of girth* g *with* $g \leq 5$. *Then either the signature of* Γ *is* $(0, 1, 1)$ *and* Γ *is a truncation of a dihedral scheme on some* g*-regular graph (possibly with parallel edges), or one of the following occurs:*

1. $g = 3$ *and* $\Gamma \cong K_4$ *with signature* $(2, 2, 2)$*;*
2. $g = 4$ *and* Γ *is isomorphic to a prism or to a Möbius ladder, with signature* $(4, 4, 4)$ *if* $\Gamma \cong K_{3,3}$*, signature* $(2, 2, 2)$ *if* Γ *is isomorphic to the cube* Q_3*, and signature* $(1, 1, 2)$ *otherwise;*
3. $g = 5$ *and* Γ *is isomorphic to the Petersen graph with signature* $(4, 4, 4)$*, or to the dodecahedron with signature* $(2, 2, 2)$.

Since every vertex-transitive graph is girth-regular, the above result can be viewed as a partial generalisation of the classification [5] of arc-transitive cubic graphs of girth at most 9 and also a recent classification [8] of vertex-transitive cubic graphs of girth at most 5.

Unless explicitly stated otherwise, by a *graph*, we will always mean a finite simple graph, defined as a pair $(V, \sim)$ where V is the *vertex-set* and $\sim$ an irreflexive symmetric *adjacency relation* on V.

However, in Section 3.2 it will be convenient to allow graphs possessing parallel edges; details will be explained there. Finally, in Section 3.3, when considering embeddings of graphs onto surfaces, we will intuitively think of a graph in a topological context as a 1-dimensional CW complex. See that section for details.

2 An upper bound on the signature

This section is devoted to the proof of Theorems 1.2, 1.3 and 1.4 that give an upper bound on the number of girth cycles through an edge in a girth-regular graph and in some cases characterise the graphs attaining this bound.

2.1 Moore graphs and generalised n-gons

We begin by a well-known result that sets a lower bound on the number of vertices for a k-regular graph of finite girth g.

Proposition 2.1 (Tutte [16, 8.39], cf. Brouwer, Cohen & Neumaier [2, §6.7])**.** *Let* Γ *be a* k*-regular graph with* n *vertices and finite girth* $g \geq 2$. *Let* $d = \lfloor g/2 \rfloor$. *Then*

$$n \geq \begin{cases} 1 + k \sum_{j=0}^{(g-3)/2} (k-1)^j = \frac{k(k-1)^d - 2}{k-2} & \text{if } g \text{ is odd,} \\ 2 \sum_{j=0}^{(g-2)/2} (k-1)^j = 2\frac{(k-1)^d - 1}{k-2} & \text{if } g \text{ is even.} \end{cases} \tag{2.1}$$

□

Note 2.2. Let Γ be a k-regular graph of girth g for which equality holds in (2.1). If g is odd, then such an extremal graph is called a *Moore graph*. It is well known (see [7] or [1], for example) that a Moore graph is either a complete graph, an odd cycle, or has girth 5 and valence $k \in \{3, 7, 57\}$. Of the latter, the first two cases uniquely determine the Petersen graph and the Hoffman-Singleton graph, respectively, while no example is known for $k = 57$.

If the girth g is even, then Γ is an incidence graph of a generalised $(g/2)$-gon of order $(k-1, k-1)$ (see [15] or [2, §6.5], for example). For $k = 2$, we have ordinary polygons, and their incidence graphs are even cycles. For $k \geq 3$, such generalised $(g/2)$-gons only exist if $g/2 \in \{2, 3, 4, 6\}$ (see [9, Theorem 1]). In particular, if $k = 3$, then Γ is the incidence graph of a generalised d-gon of order $(2, 2)$, where $d \in \{2, 3, 4, 6\}$. For $d = 2$, this is a geometry with three points incident to three lines, so its incidence graph is $K_{3,3}$. For $d = 3$, we get the Fano plane, and its incidence graph is the Heawood graph, which is the unique cubic arc-transitive graph on 14 vertices. For $d = 4$, there is a unique generalised quadrangle of order $(2, 2)$, cf. Payne & Thas [12, 5.2.3], and its incidence graph is the Tutte-Coxeter graph, also known as the Tutte 8-cage, which is the unique connected cubic arc-transitive graph on 30 vertices. For $d = 6$, there is a unique dual pair of generalised hexagons of order $(2, 2)$, cf. Cohen & Tits [3], and their incidence graph (on 126 vertices), also known as the Tutte 12-cage, is not vertex-transitive. However, the latter graph is edge-transitive, making it *semisymmetric* – in fact, it is the unique cubic semisymmetric graph on 126 vertices (see [4], where this graph is denoted by S126).

2.2 Proof of Theorems 1.2, 1.3 and 1.4

Equipped with these facts, we are now ready to prove Theorems 1.2, 1.3 and 1.4. Let us thus assume that Γ is a simple connected girth-regular graph of valence $k \geq 3$, let g be its girth and let $(a_1, a_2, \dots, a_k)$ be its signature. Set $d = \lfloor g/2 \rfloor$.

In order to prove Theorem 1.2, we need to show that $a_k \leq (k-1)^d$, or equivalently, that $\epsilon(e) \leq (k-1)^d$ for every edge e of Γ.

For an integer i and a vertex v of Γ, let $S_i(v)$ denote the set of vertices of Γ that are at distance i from v, and for an edge uv of Γ, let $D^i_j(u, v) = S_i(u) \cap S_j(v)$. If i and j are integers such that $|i - j| \geq 2$, then clearly $D^i_j(u, v) = \emptyset$.

Now let uv be an arbitrary edge of Γ and let $i \in \{2, \dots, d\}$. For simplicity, let $D^i_j = D^i_j(u, v)$. If A and B are two sets of vertices of Γ, let $E(A, B)$ be the set of edges with one end-vertex in A and the other in B. Since $g \geq 2d$, the following facts can be easily deduced:

(1) $D^i_i = \emptyset$ if $i \leq d - 1$;

(2) each of D^{i-1}_i and D^i_{i-1} is an independent set;

(3) each vertex in D^{i-1}_i has precisely one neighbour in D^{i-2}_{i-1}, and if $i \leq d-1$, precisely $k-1$ neighbours in D^i_{i+1};

(3') each vertex in D^i_{i-1} has precisely one neighbour in D^{i-1}_{i-2}, and if $i \leq d-1$, precisely $k-1$ neighbours in D^{i+1}_i;

(4) $|D^i_{i-1}| = |D^{i-1}_i| = (k-1)^{i-1}$;

(5) if g is even, then $\epsilon(uv) = |E(D^{d-1}_d, D^d_{d-1})|$;

(6) if g is odd, then every vertex in D^d_d has precisely one neighbour in each of the sets D^{d-1}_d, D^d_{d-1} and $\epsilon(uv) = |D^d_d|$.

Henceforth, let uv be an arbitrary edge of Γ such that $\epsilon(uv) = a_k$ and let $D^j_i = D^j_i(u, v)$. The structure of Γ with respect to the sets D^j_i is then depicted in Figure 1.

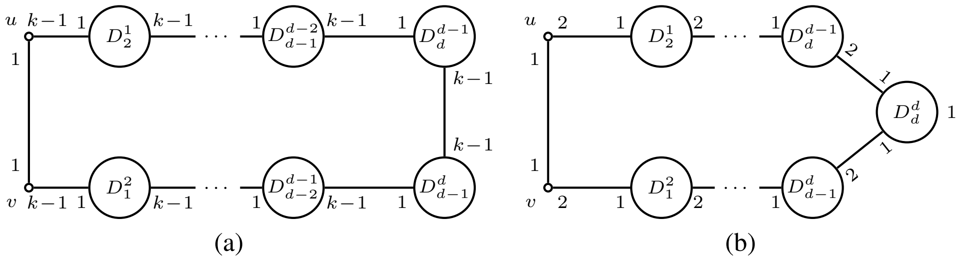


Figure 1: The partitions of the vertices of Γ of girth g corresponding to an edge uv lying on $(k-1)^d$ girth cycles, where $d = \lfloor g/2 \rfloor$. (a) shows the case when g is even, while (b) shows the case when Γ is cubic and g is odd. The sets D^i_j with $i+j < 2d$ are independent sets, while the set D^d_d in the odd case induces a perfect matching.

Suppose first that g is even. Let

$$D = \bigcup_{i=1}^{d} (D^i_{i-1} \cup D^{i-1}_i)$$

and observe that all of the vertices in D, except possibly those in D^{d-1}_d and D^d_{d-1}, have all of their neighbours contained in D. By (4), we see that

$$|D| = 2(1 + (k-1) + \cdots + (k-1)^{d-1}) = 2\frac{(k-1)^d - 1}{k-2}.$$

Moreover, it follows from (1) – (5) that

$$a_k = \epsilon(uv) = |E(D^{d-1}_d, D^d_{d-1})| \leq (k-1)|D^{d-1}_d| = (k-1)^d.$$

This proves Theorem 1.2 in the case when g is even. (The case when g is odd will be considered later.)

To prove Theorem 1.3, assume that $a_k = (k-1)^d$. Then equality holds in the above equation, implying that $|E(D^{d-1}_d, D^d_{d-1})| = (k-1)|D^{d-1}_d|$, which means that each vertex in D^{d-1}_d has $k-1$ neighbours within D^d_{d-1}. This implies that every vertex from the set D has all of its neighbours contained in D, and by connectivity of Γ, we see that $V(\Gamma) = D$. But then by Proposition 2.1 and Note 2.2, the graph Γ is the incidence graph of a generalised $g/2$-gon of order $(k-1, k-1)$. If, in addition, $k = 3$ holds, then Γ is one of the graphs mentioned in the statement of Theorem 1.3. This proves Theorem 1.3.

Let us now move to the case where g is odd, prove Theorem 1.4 and finish the proof of Theorem 1.2. Suppose henceforth that g is odd. Even though Theorem 1.4 is only about cubic graphs, we will try to continue the proof without this assumption for as long as we can. Let

$$D = D^d_d \cup \bigcup_{i=1}^{d} (D^i_{i-1} \cup D^{i-1}_i)$$

and observe that

$$|D| \leq (k-1)^d + 2(1 + (k-1) + \cdots + (k-1)^{d-1}) = \frac{k(k-1)^d - 2}{k-2}.$$

If we prove that every vertex in D has all of its neighbours contained in D, the connectivity of Γ will imply that $V(\Gamma) = D$. But then Proposition 2.1 will imply that Γ is a Moore graph. Since the only cubic Moore graphs are K_4 and the Petersen graph, this will then imply Theorem 1.4.

Note that by (2), (3) and (3'), it follows that the neighbourhoods of all vertices, except possibly those contained in D_d^{d-1}, D_{d-1}^d or D_d^d, are contained in D. By (3), (3') and (6), it follows that $|D_d^d| \leq (k-1)|D_d^{d-1}| = (k-1)^d$, and by (6) we see that

$$a_k = \epsilon(uv) = |D_d^d| \leq (k-1)^d,$$

thus proving Theorem 1.2 also for the case when g is odd.

Assume now that $a_k = (k-1)^d$. Then, by (6), $|D_d^d| = (k-1)^d$, implying that every vertex in D_d^{d-1} (as well as in D_{d-1}^d) has $k-1$ neighbours in D_d^d, and thus none outside the set D. To prove Theorem 1.4, it thus suffices to show that every vertex from D_d^d has all of its neighbours in D.

Since every vertex in D_{d+1}^d or D_d^{d+1} has to have at least one neighbour in D_d^{d-1} or D_{d-1}^d, respectively, and since all of the neighbours of vertices in the latter two sets lie in D_d^d, D_{d-1}^{d-2} and D_{d-2}^{d-1}, it follows that the sets D_{d+1}^d and D_d^{d+1} are empty. By consequence, the sets $D_i^{i+1}(u,v)$ and $D_{i+1}^i(u,v)$ for $i \geq d$ are also empty. Let us summarise that in Lemma 2.3.

Lemma 2.3. *Let Γ be a girth-regular graph of girth $2d+1$ and signature $(a_1, \dots, a_k)$ such that $a_k = (k-1)^d$. If uv is an edge of Γ such that $\epsilon(uv) = a_k$, then for $i \geq d$ the sets $D_i^{i+1}(u,v)$ and $D_{i+1}^i(u,v)$ are empty.*

Suppose now that $V(\Gamma) \neq D$. Then a vertex $y \in D_d^d$ has a neighbour w outside D. Since the girth of Γ is $2d+1$, there exists a unique path of length d from y to u. Let v' be the neighbour of u through which this path passes, and let u' be a neighbour of v' other than u such that $\epsilon(v'u') = \epsilon(uv)$. Let $E_j^i = D_j^i(u', v')$ and observe that by Lemma 2.3, the sets E_d^{d+1} and E_{d+1}^d are empty. Furthermore, since w is not in D but has a neighbour y in D, we see that $d(w,u) = d+1$, implying that $w \in D_{d+1}^{d+1}$.

We shall now partition the set D_d^{d-1} with respect to the distance to the vertices v' and u'. In particular, we will show that D_d^{d-1} is a disjoint union of the sets

$$X = D_d^{d-1} \cap E_{d-2}^{d-3},$$
$$Y = D_d^{d-1} \cap E_{d-2}^{d-1},$$
$$Z = D_d^{d-1} \cap E_d^d.$$

To prove this, note first that a vertex in D_d^{d-1} is at distance $d-1$ from u and thus by (1), it is either at distance $d-2$ or d from v'. Furthermore, those vertices that are at distance $d-2$ from v' are either at distance $d-3$ or $d-1$ from u', and therefore belong to X or Y. Now let x be an element of D_d^{d-1} that is at distance d from v'. Since $E_d^{d+1} = \emptyset$, this implies that x is either in E_d^{d-1} or in E_d^d. If $x \in E_d^{d-1}$, then there exist two distinct paths of length d from x to v', one passing through u and one passing through u', yielding a cycle of length at most $2d$, which is a contradiction. Hence $x \in E_d^d$, and therefore $x \in Z$.

We will now determine the sizes of X, Y and Z. In particular, we will show that:

$$\begin{aligned}|X| &= (k-1)^{d-3},\\ |Y| &= (k-2)(k-1)^{d-3},\\ |Z| &= (k-2)(k-1)^{d-2}.\end{aligned}$$

To prove the first equality, observe that X consists of all the ends of paths of length $d-2$ that start with $v'u'$. The equality for $|X|$ then follows from the fact that there are $(k-1)^{d-3}$ such paths. Further, note that Y consists of all the ends of paths of length $d-2$ that start in v' but do not pass through u' or u. There are $(k-2)(k-1)^{d-3}$ such paths, proving the equality for $|Y|$. Finally, to prove the equality for $|Z|$, observe that Z consists of all the ends of paths of length $d-1$ that start in u but do not pass through v or v'; there are clearly $(k-2)(k-1)^{d-2}$ such paths.

We will now partition the set D^d_d into sets X', Y' and Z' defined as follows. Let x be a vertex of D^d_d and observe that there is a unique path from x to u of length d. If this path passes through X, then we let $x \in X'$, if it passes through Y, then we let $x \in Y'$, and if it passes through Z, we let $x \in Z'$.

Since each vertex in D^{d-1}_d has $k-1$ neighbours in D^d_d and each vertex in D^d_d has precisely one neighbour in D^{d-1}_d, we see that

$$\begin{aligned}|X'| &= (k-1)|X| = (k-1)^{d-2},\\ |Y'| &= (k-1)|Y| = (k-2)(k-1)^{d-2},\\ |Z'| &= (k-1)|Z| = (k-2)(k-1)^{d-1}.\end{aligned}$$

Observe furthermore that a vertex x in X', having a neighbour in X, is at distance at most $d-2$ from u', but since it is at distance d from u, it is at distance exactly $d-2$ from u'. Similarly, $d(x, v') \leq d-1$ and since $d(x, u) = d$, we see that $d(x, v') = d-1$. In particular, $x \in E^{d-2}_{d-1}$ and thus

$$X' = D^d_d \cap E^{d-2}_{d-1} = E^{d-2}_{d-1}.$$

A similar argument shows that

$$Y' = D^d_d \cap E^d_{d-1}.$$

Let us now consider the set Z', and in particular the intersection $A = Z' \cap E^{d-1}_d$. Note that each vertex in Z must have at least one neighbour in A, for otherwise it could not be at distance d from u'. This implies that $|A| \geq |Z| = (k-2)(k-1)^{d-2}$. On the other hand, for a similar reason, each vertex in A must have a neighbour in X'. By comparing the sizes of A and X', we may thus conclude that every vertex in X' has $k-2$ neighbours in A and each vertex in A has precisely one neighbour in X'. In particular, every vertex in X' has all of its neighbours in D, and consequently, the vertex w has no neighbours in X'. Therefore, we have $y \in Y'$. Now recall that $w \in D^{d+1}_{d+1}$, implying that $d(w, v') \geq d$. On the other hand, w has a neighbour in Y', which is a subset of E^d_{d-1}, implying that $d(w, v') = d$. Since $E^{d+1}_d = \emptyset$, it follows that $w \in E^d_d$, and hence there exists a path $wz_1z_2 \ldots z_{d-1}u'$ of length d from w to u'. By considering possibilities for such a path, one can now easily see that $z_1 \in Z'$ and $z_2 \in X'$. But then z_1 has at least four neighbours: z_2, w, a neighbour in Z, and a neighbour in D^d_{d-1}, see Figure 2. This contradicts our assumption that the valence k is 3. This contradiction shows that $V(\Gamma) = D$, and thus completes the proof of Theorem 1.4.

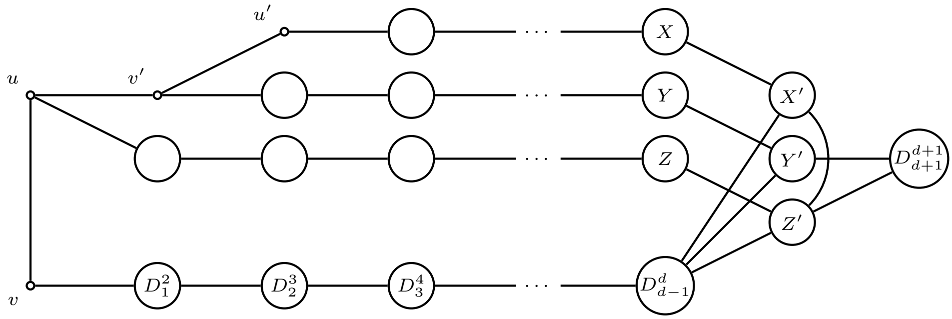


Figure 2: The partitions of the vertices of Γ of girth g, where g is odd, corresponding to the edges uv and $u'v'$, both lying on 2^d girth cycles, where $d = \lfloor g/2 \rfloor$. Assuming there is a vertex $w \in D_{d+1}^{d+1}$, we show that $w \in E_d^d$ has a neighbour in Z', which in turn must have at least four neighbours.

3 Cubic girth-regular graphs

Let us now turn our attention to cubic girth-regular graphs. After proving a few auxiliary lemmas, we will characterise cubic girth-regular graphs of some specific signatures. As an application of our analysis, we provide a characterisation of all cubic girth-regular graphs of girth at most 5 in Sections 4 and 5.

3.1 Auxiliary results

Lemma 3.1. *If (a, b, c) is the signature of a cubic girth-regular graph Γ of girth g, then:*

1. *$a + b + c$ is even,*
2. *$a + b \geq c$, and*
3. *if $a \geq 1$ and $c = a + b$, then g is even.*

Proof. Let u be a vertex of Γ and let and e_1, e_2 and e_3 be the three edges incident to u, lying on a, b and c g-cycles, respectively. Further, let x, y, z be the number of g-cycles the 2-paths e_1e_2, e_2e_3 and e_3e_1 lie on, respectively. Clearly, we have $a = x + z$, $b = x + y$ and $c = y + z$. Then $a + b + c = 2(x + y + z)$, showing that this sum is even.

Further we may express $x = (a + b - c)/2$, $y = (-a + b + c)/2$ and $z = (a - b + c)/2$. Since these numbers are nonnegative, it follows that $a + b \geq c$.

Now suppose that $a \geq 1$ and $c = a + b$. Let us call an edge e with $\epsilon(e) = c$ *saturated* and others *unsaturated*. Note that $c > b$, implying that e_1 and e_2 are unsaturated while e_3 is saturated. Since $y + z = c = a + b = 2x + y + z$, we see that $x = 0$. Since u was an arbitrary vertex of Γ, this shows that a 2-path in Γ consisting of two unsaturated edges belongs to no g-cycles. In particular, when traversing a g-cycle in Γ, saturated and unsaturated edges must alternate, implying that g is even. □

Lemma 3.2. *If the signature of a cubic girth-regular graph is $(0, b, c)$, then $b = c = 1$.*

Proof. Let Γ be a cubic girth-regular graph with signature $(0, b, c)$ and let g be its girth. By part (2) of Lemma 3.1, it follows that $b = c$. Suppose that $b > 1$. Let e be an edge of

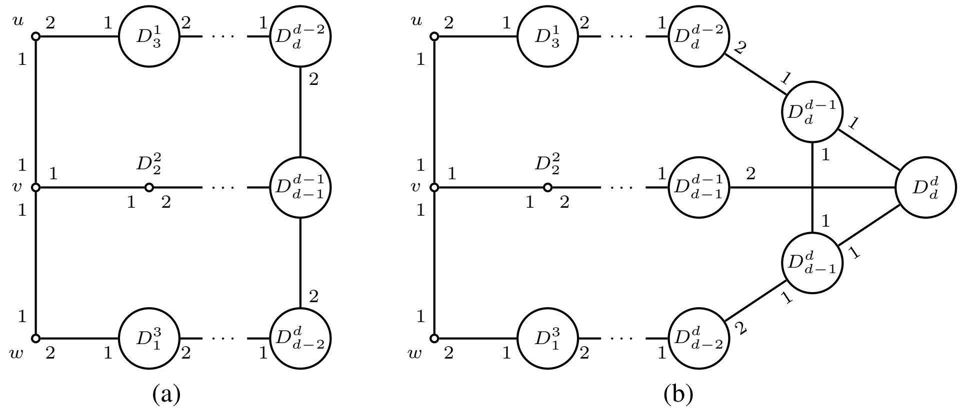


Figure 3: The partitions of the vertices of Γ of girth g corresponding to a 2-path uvw lying on 2^{d-1} girth cycles, where $d = \lfloor g/2 \rfloor$. (a) shows the case when g is even, while (b) shows the case when g is odd. The sets D_j^i with $i + j < 2d$ are independent sets, while the set D_d^d may contain edges. Note that no vertex of D_i^i ($i \in \{d-1, d\}$) with a neighbour in D_{i-1}^{i-1} can have a neighbour in D_i^{i-1} or D_{i-1}^i.

Γ lying on b g-cycles, and let C, C' be two distinct g-cycles containing e. Since $C \neq C'$, there exists a vertex u such that one of the edges incident to u lies on both C and C', while each of the remaining two edges incident to u belongs to exactly one of C and C'. However, this contradicts $a = 0$. □

Corollary 3.3. *If Γ is a cubic girth-regular graph with signature (a, b, c) and girth g, where g is odd, then $a \neq 1$.*

Proof. Suppose that $a = 1$. By part (2) of Lemma 3.1, $c = b$ or $c = b + 1$. If $b = c$, then $a + b + c$ is odd, contradicting part (1) of Lemma 3.1. Hence $c = b + 1 = a + b$, and by part (3) of Lemma 3.1, g is even, contradicting our assumptions. □

Lemma 3.4. *Let Γ be a cubic girth-regular graph of girth g with signature (a, b, c). Let $m = 2^{\lfloor g/2 \rfloor - 1}$. Then $a \geq c - m$ and $b \leq a - c + 2m$.*

Proof. Let us first show that any 2-path in Γ lies on at most m girth cycles. Let uvw be a 2-path in Γ, and let D_j^i be the set of vertices at distance i from u and at distance j from w. Set $d = \lfloor g/2 \rfloor$. Similarly as in the proof of Theorem 1.2, we can see that the number of girth cycles containing the 2-path uvw equals the number of common neighbours of vertices in the sets D_d^{d-2} and D_{d-2}^d if g is even, and the number of edges between the vertices in the sets D_d^{d-1} and D_{d-1}^d if g is odd, see Figure 3. In the even case, $|D_d^{d-2}| = |D_{d-2}^d| = 2^{d-2}$, and each of the vertices from D_d^{d-2} or D_{d-2}^d may have at most two common neighbours with vertices of the other set, so uvw can lie on at most $2^{d-1} = m$ girth cycles. In the odd case, we have $|D_d^{d-1}|, |D_{d-1}^d| \leq 2^{d-1}$, and each vertex from D_d^{d-1} or D_{d-1}^d may have at most one neighbour in the other set, as otherwise we would have a cycle of length $2d < g$. Therefore, uvw can lie on at most m girth cycles also in this case.

As each of a, b, c is the sum of the number of girth cycles two distinct 2-paths sharing the central vertex lie on, the quantity $c - a$ equals the difference between the numbers of girth cycles two such 2-paths lie on, and is therefore at most m, from which $a \geq c - m$ follows. Also, the quantity $-a + b + c$ equals twice the number of girth cycles a 2-path in Γ lies on, and is therefore at most $2m$. From this, $b \leq a - c + 2m$ follows. □

3.2 Dihedral schemes, truncations and signature $(0, 1, 1)$

In this section we will allow graphs to have parallel edges and loops. A graph *with parallel edges and loops* is defined as a triple (V, E, ∂) where V and E are the *vertex-set* and the *edge-set* of the graph and $\partial \colon E \to \{X : X \subseteq V, |X| \leq 2\}$ is a mapping that maps an edge to the set of its end-vertices. If $|\partial(e)| = 1$, then e is a *loop*. Further, we let each edge consist of two mutually inverse *arcs*, each of the two arcs having one of the end-vertices as its *tail*. If the graph has no loops, we may identify an arc with tail v underlying edge e with the pair (v, e). The set of arcs of a graph Γ is denoted by $A(\Gamma)$ and the set of the arcs with their tail being a specific vertex u by $\mathrm{out}_\Gamma(u)$. The valence of a vertex u is defined as the cardinality of $\mathrm{out}_\Gamma(u)$.

A *dihedral scheme* on a graph Γ (possibly with parallel edges and loops) is an irreflexive symmetric relation $\leftrightarrow$ on the arc-set $A(\Gamma)$ such that the simple graph $(A(\Gamma), \leftrightarrow)$ is a 2-regular graph each of whose connected components is the set $\mathrm{out}_\Gamma(u)$ for some $u \in V(\Gamma)$. (Intuitively, we may think of a dihedral scheme as a collection of circles drawn around each vertex u of Γ intersecting each of the arcs in $\mathrm{out}_\Gamma(u)$ once.) Note that, according to this definition, the minimum valence of a graph admitting a dihedral scheme is at least 3.

The group of all automorphisms of Γ that preserve the relation $\leftrightarrow$ will be denoted by $\mathrm{Aut}(\Gamma, \leftrightarrow)$ and the dihedral scheme $\leftrightarrow$ is said to be *arc-transitive* if $\mathrm{Aut}(\Gamma, \leftrightarrow)$ acts transitively on $A(\Gamma)$.

Given a dihedral scheme $\leftrightarrow$ on a graph Γ, let $\mathrm{Tr}(\Gamma, \leftrightarrow)$ be the simple graph whose vertices are the arcs of Γ and two arcs $s, t \in \Gamma$ are adjacent in Γ if either $t \leftrightarrow s$ or t and s are inverse to each other. The graph $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is then called the *truncation of* Γ *with respect to the dihedral scheme* $\leftrightarrow$. Note that $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is a cubic graph which is connected whenever Γ is connected.

As we shall see in Section 3.3, a natural source of arc-transitive dihedral schemes are arc-transitive maps (either orientable or non-orientable). However, not all dihedral schemes arise in this way.

Clearly, the automorphism group $\mathrm{Aut}(\Gamma, \leftrightarrow)$ acts naturally as a group of automorphisms of $\mathrm{Tr}(\Gamma, \leftrightarrow)$, implying that $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is vertex-transitive whenever the dihedral scheme $\leftrightarrow$ is arc-transitive. The following result gives a characterisation of arc-transitive dihedral schemes in group theoretical terms. Here, the symbol $\mathbb{D}_d$ denotes the dihedral group of order $2d$ acting naturally on d points, while $\mathbb{Z}_d$ is the cyclic group acting transitively on d points.

Lemma 3.5. *Let* Γ *be an arc-transitive graph (possibly with parallel edges) of valence* d *for some* $d \geq 3$. *Then* Γ *admits an arc-transitive dihedral scheme if and only if there exists an arc-transitive subgroup* $G \leq \mathrm{Aut}(\Gamma)$ *such that the group* $G_u^{\mathrm{out}_\Gamma(u)}$ *induced by the action of the vertex stabiliser* G_u *on the set* $\mathrm{out}_\Gamma(u)$ *is permutation isomorphic to the transitive action of* $\mathbb{D}_d$, $\mathbb{Z}_d$ *or (when d is even)* $\mathbb{D}_{\frac{d}{2}}$ *on d vertices.*

Proof. Suppose that $\leftrightarrow$ is a dihedral scheme on Γ and that $G = \mathrm{Aut}(\Gamma, \leftrightarrow)$. Then $G_u^{\mathrm{out}_\Gamma(u)}$

preserves the restriction $\leftrightarrow_u$ of the relation $\leftrightarrow$ onto $\mathrm{out}_\Gamma(u)$, and thus acts as a vertex-transitive group of automorphisms on the simple graph $(\mathrm{out}_\Gamma(u), \leftrightarrow_u)$. Since the latter graph is a cycle of length d, we thus see that $G_u^{\mathrm{out}_\Gamma(u)}$ is a transitive subgroup of $\mathbb{D}_d$ and thus permutation isomorphic to one of the transitive actions mentioned in the statement of the lemma.

Conversely, suppose that for some vertex u, the group $G_u^{\mathrm{out}_\Gamma(u)}$ is permutation isomorphic to the transitive action of $\mathbb{D}_d$, $\mathbb{Z}_d$, or (if d is even) $\mathbb{D}_{d/2}$ on d vertices. In all three cases, we may choose an adjacency relation $\leftrightarrow_u$ on $\mathrm{out}_\Gamma(u)$ preserved by $G_u^{\mathrm{out}_\Gamma(u)}$ in such a way that $(\mathrm{out}_\Gamma(u), \leftrightarrow_u)$ is a cycle. For every $v \in V(\Gamma)$, choose an element $g_v \in G$ such that $v^{g_v} = u$, and let $\leftrightarrow_v$ be the relation on $\mathrm{out}_\Gamma(v)$ defined by $s \leftrightarrow_v t$ if and only if $s^{g_v} \leftrightarrow_u t^{g_v}$. Then clearly $(\mathrm{out}_\Gamma(v), \leftrightarrow_v)$ is a cycle, implying that the union $\leftrightarrow$ of all $\leftrightarrow_u$ for $u \in V(\Gamma)$ is a dihedral scheme. Moreover, it is a matter of straightforward computation to show that $\leftrightarrow$ is invariant under G. □

We are now ready to prove the following characterisation of cubic girth-regular graphs of signature $(0, 1, 1)$.

Theorem 3.6. *If Γ is a simple cubic girth-regular graph of girth g with signature $(0, 1, 1)$, then $\Gamma \cong \mathrm{Tr}(\Lambda, \leftrightarrow)$, where $\leftrightarrow$ is a dihedral scheme on a g-regular graph Λ (possibly with parallel edges). Moreover, if Γ is vertex-transitive, then the dihedral scheme is arc-transitive.*

Proof. Let V be the vertex-set of Γ, let $\mathcal{T}$ be the set of girth cycles in Γ, let $\mathcal{M}$ be the set of edges that belong to no girth cycle in Γ, and let $G = \mathrm{Aut}(\Gamma)$. Note that since the signature of Γ is $(0, 1, 1)$, each vertex $v \in V$ is incident to exactly one edge in $\mathcal{M}$ and to exactly one girth cycle in $\mathcal{T}$.

For an edge $v'v \in \mathcal{M}$, let C and C' be the girth cycles that pass through v and v', respectively, and let $\partial(v'v) = \{C, C'\}$. This allows us to define a graph $\Lambda = (\mathcal{T}, \mathcal{M}, \partial)$.

Note that since $C, C' \in V(\Lambda)$ are girth cycles of Γ, we have $C \neq C'$, and so Λ has no loops. This allows us to view an arc of Λ as a pair (C, e) where $e \in \mathcal{M}$ and C is a girth cycle of Γ passing through one of the two end-vertices of e. For two such pairs (C_1, e_1) and (C_2, e_2) we write $(C_1, e_1) \leftrightarrow (C_2, e_2)$ if and only if $C_1 = C_2$ and the end-vertices of e_1 and e_2 that belong to C_1 are two consecutive vertices of C_1. Then $\leftrightarrow$ is a dihedral scheme on Λ. Let $\Gamma' = \mathrm{Tr}(\Lambda, \leftrightarrow)$.

We will now show that $\Gamma' \cong \Gamma$. By the definition of truncation, the vertex-set of Γ' equals the arc-set of Λ. For an arc (C, e) of Λ let $\varphi(C, e)$ be the unique end-vertex of e that belongs to C. Since each vertex of Γ is incident to exactly one edge in $\mathcal{M}$ and exactly one cycle in $\mathcal{T}$, it follows that φ is a bijection between $V(\Gamma')$ and $V(\Gamma)$. If (C_1, e_1) and (C_2, e_2) are adjacent in Γ', then either $(C_1, e_1) \leftrightarrow (C_2, e_2)$ or (C_1, e_1) and (C_2, e_2) are inverse arcs in Γ'. In the first case, $C_1 = C_2$ and the vertices $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are adjacent on C_1. In the second case, $e_1 = e_2$ and the vertices $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are the two end-vertices of e_1. In both cases $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are adjacent in Γ. By a similar argument we see that whenever $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are adjacent in Γ, (C_1, e_1) and (C_2, e_2) are adjacent in Γ'. Since both Γ and Γ' are simple graphs (one by assumption, the other by definition), this shows that φ is a graph isomorphism.

Suppose now that G is transitive on the vertices of Γ. Since both sets $\mathcal{T}$ and $\mathcal{M}$ are invariant under the action of G, there exists a natural action of G on Λ that preserves the dihedral scheme $\leftrightarrow$; that is, $G \leq \mathrm{Aut}(\Lambda, \leftrightarrow)$. Now let (C_1, e_1) and (C_2, e_2) be two arcs of Λ, and for $i \in \{1,2\}$, let v_i be the unique end-vertex of e_i that lies on C_i. Since G is vertex-transitive on Γ, there exists $g \in G$ mapping v_1 to v_2. Since C_i is the unique girth-cycle through v_i for $i \in \{1,2\}$, it follows that $C_1^g = C_2$. Similarly, since e_i is the unique edge in $\mathcal{M}$ incident with v_i for $i \in \{1,2\}$, it follows that $e_1^g = e_2$. This shows that G acts transitively on the arcs of Λ. □

Note 3.7. Parallel edges occur in the graph Λ as in Theorem 3.6 whenever there exist two girth cycles in Γ such that there are at least two edges with an end-vertex in each of the two girth cycles. In fact, it can be easily seen that in a girth-regular graph Γ with signature $(0,1,1)$, there are at most two such edges between any two girth cycles, leading to at most two parallel edges between each two vertices, with the exception of the case when Γ is the 3-prism (see Section 4) and Λ is the graph with two vertices and three parallel edges between them.

Note 3.8. No nontrivial bound on the girth of the graph Λ as in Theorem 3.6 can be given. In fact, we can construct a family of graphs of constant girth such that their truncations with respect to appropriate dihedral schemes are cubic girth-regular graphs with signature $(0,1,1)$ and unbounded girth. Let Λ be a graph obtained by doubling all edges in a k-regular graph of girth at least $\frac{k+1}{2}$ – the girth of Λ is then 2. Equip Λ with a dihedral scheme $\leftrightarrow$ such that each two arcs with a common tail belonging to two parallel edges are antipodal in the connected component of the graph defined by $\leftrightarrow$ they belong to. Then $\mathrm{Tr}(\Lambda, \leftrightarrow)$ is a cubic girth-regular graph of girth k and signature $(0,1,1)$.

3.3 Maps and signatures $(2,2,2)$ and $(1,1,2)$

In this section, it will be convenient to think of a graph (possibly with parallel edges) as a topological space having the structure of a regular 1-dimensional CW complex with the vertices of the graph corresponding to the 0-cells of the complex and the edges corresponding to the 1-cells. A simple closed walk (that is, a closed walk that traverses each edge at most once) in the graph then corresponds to a closed curve in the corresponding topological space which may intersect itself only in the points that correspond to the vertices of the graph.

Given a graph Γ (viewed as a CW complex) and a set of simple closed walks $\mathcal{T}$ in Γ, one can construct a 2-dimensional CW complex in the following way. First, take a collection $\mathcal{D}$ of topological disks, one for each walk in $\mathcal{T}$. Then choose a surjective continuous mapping from the boundary of each disk to the closed curve in Γ representing the corresponding walk in $\mathcal{T}$, such that the preimage of each point that is not a vertex of the graph is a singleton. Finally, identify each point of the boundary of the disk with its image under that continuous mapping. Note that the resulting topological space is independent of the choice of the homeomorphisms $\mathcal{D}$ and thus depends only on the choice of the graph and the set of closed walks $\mathcal{T}$.

When Γ is connected and the resulting topological space is a closed surface (either orientable or non-orientable), the CW complex is also called a *map*. Its open 2-cells are then called the *faces* of the map, the closed walks in $\mathcal{T}$ are called the *face-cycles* and the graph Γ is the *skeleton* of the map. A map whose skeleton is a k-regular graph and all of whose face cycles are of length m is called an $\{m,k\}$-map. The following lemma provides a sufficient condition on the set of cycles $\mathcal{T}$ under which the resulting 2-dimensional CW complex is indeed a map.

Lemma 3.9. *Let* Γ *be a graph and* $\mathcal{T}$ *a set of simple closed walks in* Γ *such that every edge of* Γ *belongs to precisely two walks in* $\mathcal{T}$*. For two arcs* s *and* t *with a common tail, write* $s \leftrightarrow t$ *if and only if the underlying edges of* s *and* t *are two consecutive edges on a walk in* $\mathcal{T}$*. If* $\leftrightarrow$ *is a dihedral scheme, then* Γ *is the skeleton of a map whose face cycles are precisely the walks in* $\mathcal{T}$*.*

Proof. Let us think of Γ as a 1-dimensional CW complex and let us turn it into a 2-dimensional CW complex by adding to it one 2-cell for each walk in $\mathcal{T}$ as described above.

Let us now prove that the resulting topological space $\mathcal{M}$ is a closed surface. It is clear that the internal vertices of the 2-cells have a regular neighbourhood. Further, since each edge of Γ lies on precisely two walks in $\mathcal{T}$, the internal points of edges also have a regular neighbourhood, made up from two half-disks, each contained in the 2-cell glued to one of the walks in $\mathcal{T}$ passing through that edge. Finally, let u be a vertex of Γ, let k be the valence of u, and let $\{s_i : i \in \mathbb{Z}_k\}$ be the set of arcs with the initial vertex u such that $s_0 \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_{k-1} \leftrightarrow s_0$. By the definition of $\leftrightarrow$, each pair of arcs (s_i, s_{i+1}) $(i \in \mathbb{Z}_k)$ lies on a unique walk C_i in $\mathcal{T}$. Note that $C_i \neq C_{i+1}$, for otherwise the edge underlying s_{i+1} would lie on only one walk in $\mathcal{T}$. This implies that a regular neighbourhood of u in $\mathcal{M}$ can be built by taking appropriate half-disks from the 2-cells corresponding to the cycles C_i $(i \in \mathbb{Z}_k)$, and gluing them together in the order suggested by the relation $\leftrightarrow$. This shows that $\mathcal{M}$ is a 2-manifold without a boundary. Finally, since Γ is finite, $\mathcal{M}$ is compact, and thus a closed surface. Hence, $\mathcal{M}$ is a map with Γ as its skeleton. □

Each face of a map can be decomposed further into *flags*, that is, triangles with one vertex in the centre of a face, one vertex in the centre of an edge on the boundary of that face and one in a vertex incident with that edge. In most cases, a flag can be viewed as a triple consisting of a vertex, an edge incident to that vertex, and a face incident to both the vertex and the edge.

An automorphism of a map is then defined as a permutation of the flags induced by a homeomorphism of the surface that preserves the embedded graph. A map is said to be *vertex-transitive* or *arc-transitive* provided that its automorphism group induces a vertex-transitive or arc-transitive group on the skeleton of the map, respectively.

Note 3.10. If a map is built from a graph Γ and a set of simple closed walks $\mathcal{T}$ as in Lemma 3.9, then each automorphism of Γ that preserves the set of walks $\mathcal{T}$ clearly extends to an automorphism of the map.

If $\mathcal{M}$ is a map on a surface $\mathcal{S}$, then the sets V, E and F of the vertices, edges and faces, respectively, satisfy the *Euler formula*

$$|V| - |E| + |F| = \chi(\mathcal{S})$$

where $\chi(\mathcal{S})$ is the *Euler characteristic* of the surface $\mathcal{S}$. It is well known that $\chi(\mathcal{S}) \leq 2$ with equality holding if and only if $\mathcal{S}$ is homeomorphic to a sphere. Moreover, if $\chi(\mathcal{S})$ is odd, then $\mathcal{S}$ is non-orientable.

As the following two results show, skeletons of maps arise naturally when analysing cubic vertex-transitive graphs of signature $(2, 2, 2)$ or $(1, 1, 2)$.

Theorem 3.11. *Let* Γ *be a simple connected cubic girth-regular graph of girth* g *and order* n *with signature* $(2, 2, 2)$*. Then* g *divides* $3n$ *and* Γ *is the skeleton of a* $\{g, 3\}$*-map* *embedded on a surface with Euler characteristic*

$$\chi = n\left(\frac{3}{g} - \frac{1}{2}\right).$$

Moreover, every automorphism of Γ *extends to an automorphism of the map. In particular, if* Γ *is vertex-transitive, so is the map.*

Proof. Let $\mathcal{T}$ be the set of girth cycles of Γ. Since the valence of Γ is 3, it follows easily that the relation $\leftrightarrow$ from Lemma 3.9 satisfies the conditions stated in the lemma; that is, $\leftrightarrow$ is a dihedral scheme. Lemma 3.9 thus yields a map $\mathcal{M}$ whose skeleton is Γ and whose face-cycles are precisely the walks in $\mathcal{T}$; in particular, $\mathcal{M}$ is a $\{g, 3\}$-map, as claimed.

Since Γ is a cubic graph with n vertices, it has $3n/2$ edges, and since each vertex lies on three face-cycles and since each face-cycle contains g vertices, the map $\mathcal{M}$ has $3n/g$ faces (showing that g must divide $3n$). The Euler characteristic of $\mathcal{M}$ thus equals $n - \frac{3n}{2} + \frac{3n}{g} = n(\frac{3}{g} - \frac{1}{2})$.

Since every automorphism of Γ preserves $\mathcal{T}$, it extends to an automorphism of $\mathcal{M}$ (see Note 3.10). □

Theorem 3.11 has the following interesting consequence.

Corollary 3.12. *There exists only finitely many connected cubic girth-regular graphs with signature* $(2, 2, 2)$ *of girth at most* 5.

Proof. Suppose that Γ is a connected cubic girth-regular graph with signature $(2, 2, 2)$ of girth g and order n. By Theorem 3.11, Γ is a skeleton of a map on a surface of Euler characteristic $\chi = n(3/g - 1/2)$. Hence, if $g \leq 5$, then $\chi \geq n/10$, and since $\chi \leq 2$, it follows that $n \leq 20$. □

Note 3.13. For each $g \geq 6$, there are infinitely many girth-regular graphs of girth g with signature $(2, 2, 2)$.

If $\mathcal{M}$ is a map and Γ is its skeleton, then one can define a dihedral scheme $\leftrightarrow$ on Γ by letting $s \leftrightarrow t$ whenever the arcs s and t have a common tail and the underlying edges of s and t are two consecutive edges on some face-cycle of $\mathcal{M}$. The truncation $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is then simply referred to as the *truncation of the map* $\mathcal{M}$ and denoted $\mathrm{Tr}(\mathcal{M})$. Note that this construction in some sense complements Lemma 3.9. We are now equipped for a characterisation of cubic girth-regular graphs with signature $(1, 1, 2)$.

Theorem 3.14. *Let* Γ *be a simple connected cubic girth-regular graph of girth* g *with* n *vertices and signature* $(1, 1, 2)$. *Then* g *is even and* Γ *is the truncation of some map* $\mathcal{M}$ *with face cycles of length* $g/2$. *In particular,* $g/2$ *divides* n. *Moreover, if* Γ *is vertex-transitive,* $\mathcal{M}$ *is an arc-transitive* $\{g/2, \ell\}$*-map for some* $\ell > g$.

Proof. By part (3) of Lemma 3.1 we know that g is even and in particular, $g \geq 4$. Let $\mathcal{X}$ be the set of edges of Γ that belong to exactly one girth cycle and let $\mathcal{Y}$ be the set of edges that belong to two girth cycles. Since the signature of Γ is $(1, 1, 2)$, every vertex of Γ is incident to two edges in $\mathcal{X}$ and one edge in $\mathcal{Y}$. Consequently, the edges in $\mathcal{Y}$ form a perfect matching of Γ and the subgraph induced by the edges in $\mathcal{X}$ is a union of vertex-disjoint cycles of Γ that cover all the vertices of Γ. Let us denote the set of these cycles by $\mathcal{C}$.

Observe also that two edges in $\mathcal{X}$ sharing a common end-vertex, say v, cannot be two consecutive edges on the same girth cycle, for otherwise that would be a unique girth cycle through v, contradicting the fact that the third edge incident with v belongs to two girth cycles. Since the edges in $\mathcal{Y}$ form a complete matching of Γ, the same holds for the edges in $\mathcal{Y}$, implying that the edges on any girth cycle alternate between the sets $\mathcal{X}$ and $\mathcal{Y}$.

For an edge e in $\mathcal{Y}$ with end-vertices u and v, let C_u and C_v be the unique cycles in $\mathcal{C}$ that pass through u and v, respectively, and define $\partial(e)$ to be the pair $\{C_u, C_v\}$. Let $\Lambda = (\mathcal{C}, \mathcal{Y}, \partial)$. Note that since the edges of Λ are precisely those edges of Γ that belong to $\mathcal{Y}$, we may think of the arc-set $A(\Lambda)$ as being the set of arcs of Γ that underlie edges in $\mathcal{Y}$. Note also that it may happen that for some $e \in \mathcal{Y}$, we may have $C_u = C_v$ and then the graph Λ has loops. If D is a girth cycle of Γ, then the edges of D that belong to $\mathcal{Y}$ induce a simple closed walk in the graph Λ of length $g/2$, which we denote $\hat{D}$.

Let $\mathcal{T}$ be the set of walks $\hat{D}$ where D runs through the set of girth cycles of Γ. Since edges of Λ correspond to the edges of Γ that pass through two girth cycles of Γ, each edge of Λ belongs to two walks in $\mathcal{T}$. As $|\mathcal{Y}| = n/2$, it follows that $g/2$ divides n.

Let $\leftrightarrow$ be the relation on the arcs of Λ defined by $\mathcal{T}$ as explained in Lemma 3.9. It is easy to see that $\leftrightarrow$ is a dihedral scheme. Indeed, let $C \in \mathcal{C}$ be a vertex of Λ viewed as a cycle in Γ and let $v_0, v_1, \ldots, v_{k-1} \in V(\Gamma)$ be its vertices listed in a cyclical order as they appear on C. Further, for each $i \in \mathbb{Z}_k$, let s_i be the arc of Γ with tail v_i that underlies an edge contained in $\mathcal{Y}$. The arc s_i can thus also be viewed as an arc of Λ. Observe that $\mathrm{out}_\Lambda(C) = \{s_0, s_1, \ldots, s_{k-1}\}$ and that $s_0 \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_{k-1} \leftrightarrow s_0$. In particular, $\leftrightarrow$ is a dihedral scheme.

By Lemma 3.9, there exists a map $\mathcal{M}$ with skeleton Λ in which $\mathcal{T}$ is the set of face-cycles. Moreover, $\leftrightarrow$ equals the dihedral scheme arising from that map.

Let $\Gamma' = \mathrm{Tr}(\mathcal{M})$ and let s be a vertex of Γ'. Then s is an arc of Λ and thus also an arc of Γ underlying an edge in $\mathcal{Y}$. By letting $\varphi(s)$ be the tail of s (viewed as an arc of Γ), we define a mapping $\varphi\colon V(\Gamma') \to V(\Gamma)$. Note that the mapping which assigns to a vertex $v \in V(\Gamma)$ the unique arc of Γ with tail v that underlies an edge in $\mathcal{Y}$ is the inverse of φ, showing that φ is a bijection. Furthermore, note that two vertices s and t of Γ' are adjacent in Γ' if and only if one of the following happens: (1) they are inverse to each other as arcs of Λ; or (2) they have a common tail and $s \leftrightarrow t$. In case (1), $\varphi(s)$ and $\varphi(t)$ are adjacent in Γ via an edge in $\mathcal{Y}$, while in case (2), $\varphi(s)$ and $\varphi(t)$ are adjacent in Γ via an edge in $\mathcal{X}$. Conversely, if for some $s, t \in V(\Gamma')$, the images $\varphi(s)$ and $\varphi(t)$ are adjacent in Γ, then either s and t are inverse to each other as arcs of Λ (this happens if $\varphi(s)$ and $\varphi(t)$ form an edge in $\mathcal{Y}$), or s and t have a common tail and $s \leftrightarrow t$ (this happens if $\varphi(s)$ and $\varphi(t)$ form an edge in $\mathcal{X}$). In both cases, s and t are adjacent in Γ'. This implies that φ is an isomorphism of graphs and thus $\Gamma \cong \mathrm{Tr}(\mathcal{M})$, as claimed.

Since every automorphism of Γ preserves each of the sets $\mathcal{Y}$ and $\mathcal{X}$ (and thus also $\mathcal{C}$), it clearly induces an automorphism of the graph Λ which preserves the set $\mathcal{T}$. In particular, every automorphism of Γ induces an automorphism of the map $\mathcal{M}$.

Finally, suppose that Γ is vertex-transitive. Then all cycles of $\mathcal{C}$ have the same length $\ell > g$. As each vertex of a cycle of $\mathcal{C}$ is incident to precisely one edge of $\mathcal{Y}$, it follows that Λ is an ℓ-regular graph, and $\mathcal{M}$ is then a $\{g/2, \ell\}$-map. Let G be a group of automorphisms of Γ acting transitively on $V(\Gamma)$. Note that every vertex of Γ is the tail of precisely one arc of Γ that underlies an edge of $\mathcal{Y}$. In view of our identification of the arcs of Γ' with the arcs of Γ that underlie an edge in $\mathcal{Y}$, we thus see that the transitivity of the action of G on $V(\Gamma)$ implies the transitivity of the action of G on the arcs of $\mathcal{M}$. □

4 Cubic girth-regular graphs of girths 3 and 4

Before stating the theorem about girth-regular cubic graphs of girth 3, let us point out that every cubic graph admits a unique dihedral scheme, which is preserved by every automorphism of the graph. This allows us to talk about truncations of cubic graphs without specifying the dihedral scheme.

Theorem 4.1. *Let* Γ *be a connected cubic girth-regular graph of girth* 3*. Then one of the following holds:*

(a) Γ *is isomorphic to the complete graph* K_4*;*

(b) Γ *has signature* $(0, 1, 1)$ *and is isomorphic to the truncation of a cubic graph.*

Proof. Let (a, b, c) be the signature of Γ. By Theorem 1.2 it follows that $c \leq 2$. If $c = 2$, then Theorem 1.4 implies that Γ is isomorphic to K_4. On the other hand, if $c = 1$, then Lemmas 3.1 and 3.2 imply that the signature of Γ is $(0, 1, 1)$, and by Theorem 3.6, it follows that Γ is the truncation of a cubic graph. □

Let us now move our attention to graphs of girth 4. Before stating the classification theorem, let us define two families of cubic vertex-transitive graphs.

For $n \geq 3$, let the *n-Möbius ladder* M_n be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_{2n}, \{-1, 1, n\})$. Note that such a graph has girth 4. The graph M_n has signature $(4, 4, 4)$ if $n = 3$ (in this case it is isomorphic to the complete bipartite graph $K_{3,3}$), and $(1, 1, 2)$ if $n \geq 4$. An n-Möbius ladder can also be seen as the skeleton of the truncation of the $\{2, 2n\}$-map with a single vertex embedded on a projective plane.

For $n \geq 3$, the *n-prism* Y_n is defined as the Cartesian product $C_n \,\square\, K_2$ or, alternatively, as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(-1, 0), (1, 0), (0, 1)\})$. The girth of Y_3 is 3, while the girth of Y_n for $n \geq 4$ is 4. The graph Y_n has signature $(2, 2, 2)$ if $n = 4$ (in this case it is isomorphic to the cube Q_3), and $(1, 1, 2)$ if $n \geq 5$. An n-prism can also be seen as the skeleton of the truncation of the $\{2, n\}$-map with two vertices embedded on a sphere, i.e., an n-gonal hosohedron.

Theorem 4.2. *Let* Γ *be a connected cubic girth-regular graph of girth* 4*. Then* Γ *is isomorphic to one of the following graphs:*

(a) the n-Möbius ladder M_n for some $n \geq 3$;

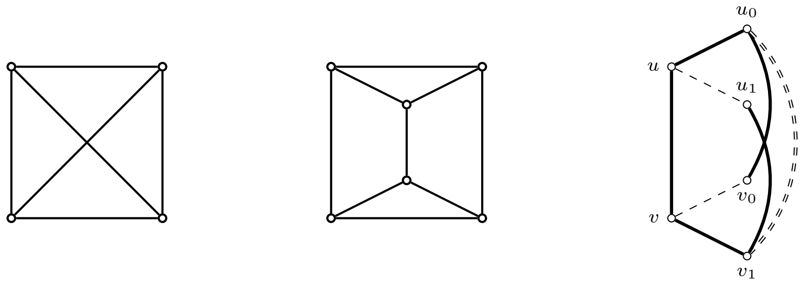
(b) the n-prism Y_n for some $n \geq 4$;

(c) $\mathrm{Tr}(\Lambda, \leftrightarrow)$ *for some tetravalent graph* Λ *and a dihedral scheme* $\leftrightarrow$ *on* Λ*.*

Proof. Let (a, b, c) be the signature of Γ. By Theorem 1.2, we see that $c \leq 4$, and by Theorem 1.3, if $c = 4$, then the signature of Γ is $(4, 4, 4)$ and $\Gamma \cong K_{3,3} \cong M_3$.

Suppose now that $c = 3$. Then, by Lemma 3.1, $a + b$ is odd, and by Lemma 3.2, $a \geq 1$. Hence either $a = 1$ and then $b = 2$, or $a = 2$ and then $b = 3$. The possible signatures in this case are thus $(1, 2, 3)$ and $(2, 3, 3)$. Let us show that neither can occur.

Let uv be an edge of Γ lying on three 4-cycles, and u_0, u_1 and v_0, v_1 be the remaining neighbours of u and v, respectively. There must be three edges with one end-vertex in $\{u_0, u_1\}$ and the other in $\{v_0, v_1\}$; without loss of generality, these edges are u_0v_0, u_0v_1 and u_1v_1 (see Figure 4(c)). Then the edges uu_0 and vv_1 already lie on three 4-cycles,



(a) (b) (c)

Figure 4: (a) The graph K_4 of girth 3 with signature $(2, 2, 2)$. (b) The graph Y_3 of girth 3 with signature $(1, 1, 2)$. (c) Constructing a graph of girth 4 with $c = 3$. The dashed edges should lie on two 4-cycles, however the doubled edge already lies on three 4-cycles.

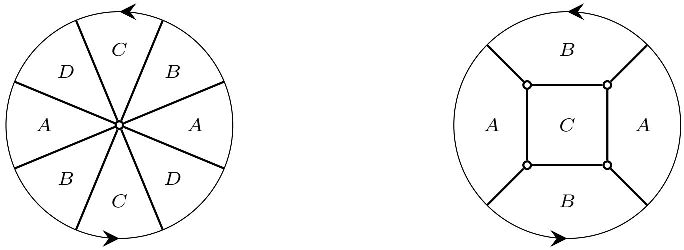
so we have $b = 3$, and thus $a = 2$. In particular, $\epsilon(uu_1) = \epsilon(vv_0) = 2$. Then the edge u_1v_1, being incident to both u_1 and v_1, belongs to precisely three 4-cycles, that is $\epsilon(u_1v_1) = 3$. Similarly, $\epsilon(u_0v_0) = 3$. It follows that the edge u_0v_1 lies on precisely two 4-cycles. However, we have already determined three 4-cycles on which u_0v_1 lies; these are uu_0v_1v, $v_0u_0v_1v$, and $uu_0v_1u_1$. This contradiction shows that the case $c = 3$ is not possible.

Suppose now that $c = 2$. By Lemma 3.1, $a + b$ is even, and by Lemma 3.2, $a \geq 1$. Hence the signature of Γ is either $(1, 1, 2)$ or $(2, 2, 2)$.

If $(a, b, c) = (1, 1, 2)$, then, by Theorem 3.14, Γ is the skeleton of the truncation of a connected map $\mathcal{M}$ with face cycles of length 2. Since every edge belongs to two faces and every face is surrounded by two edges, the number of faces equals the number of edges. The Euler characteristics $\chi(\mathcal{S})$ of the underlying surface $\mathcal{S}$ thus equals $|V(\mathcal{M})|$. Since $\chi(\mathcal{S}) \leq 2$, it follows that $\mathcal{M}$ has one or two vertices, depending on whether $\mathcal{S}$ is the projective plane or the sphere – in particular, the skeleton of $\mathcal{M}$ is an ℓ-regular graph for some $\ell > 4$. If $\mathcal{M}$ has one vertex only, then it consists of $\ell/2$ loops embedded onto the projective plane in such a way that its truncation is the Möbius ladder M_n with $n = \ell/2 \geq 4$, see Figure 5(a) (note that $M_3 \cong K_{3,3}$ has signature $(4, 4, 4)$). On the other hand, if $\mathcal{M}$ has two vertices, then $\mathcal{M}$ is the map with two vertices and ℓ parallel edges embedded onto the sphere. The graph Γ is then isomorphic to the n-prism Y_n with $n = \ell \geq 5$.

If $(a, b, c) = (2, 2, 2)$, then, by Theorem 3.11, Γ is the skeleton of a $\{4, 3\}$-map embedded on a surface of Euler characteristic $\chi = n/4 > 0$. As above, $\chi \leq 2$ and thus $\chi = 1$ or 2. For $\chi = 1$, we get the hemicube on the projective plane (see Figure 5(b)), and its skeleton is isomorphic to K_4 of girth 3. For $\chi = 2$, we get the cube on a sphere, and its skeleton is isomorphic to Y_4 with signature $(2, 2, 2)$. This completes the case $c = 2$.

If $c = 1$, then since $a + b + c$ is even, we see that $a = 0$ and $b = 1$, and then by Theorem 3.6, Γ is the truncation of a 4-regular graph with respect to some dihedral scheme. □



(a) (b)

Figure 5: (a) A $\{2,8\}$-map with a single vertex, four edges and four labelled faces embedded on the projective plane. Its truncation has the graph M_4 with signature $(1,1,2)$ as its skeleton. (b) The hemicube on the projective plane with labelled faces. Its skeleton is the graph K_4.

5 Cubic girth-regular graphs of girth 5

Theorem 5.1. *Let* Γ *be a connected cubic girth-regular graph of girth* 5*. Then either the signature of* Γ *is* $(0,1,1)$ *and* Γ *is the truncation of a* 5*-regular graph with respect to some dihedral scheme, or* Γ *is isomorphic to the Petersen graph or to the dodecahedron graph.*

Proof. Let (a,b,c) be the signature of Γ. By Theorem 1.2, we see that $c \le 4$. If $c = 4$, then by Theorem 1.3 and Theorem 1.4, the signature of Γ is $(4,4,4)$ and Γ is isomorphic to the Petersen graph. We may thus assume that $c \le 3$.

If $a = 0$, then by Lemma 3.2 the signature of Γ is $(0,1,1)$, and then by Theorem 3.6, Γ is the truncation of a 5-regular graph with respect to some dihedral scheme. Moreover, by Corollary 3.3, $a \ne 1$. We may thus assume that $a \ge 2$.

If $c = 2$, then the signature of Γ is $(2,2,2)$ and by Theorem 3.11, Γ is the skeleton of a $\{5,3\}$-map embedded on a surface of Euler characteristic $\chi = n/10$, where n is the order of the graph Γ. In particular, $\chi \in \{1,2\}$. If $\chi = 1$, then $n = 10$ and since the girth of Γ is 5, Proposition 2.1 and Note 2.2 imply that Γ is the Petersen graph (whose signature is in fact $(4,4,4)$). If $\chi = 2$, then $n = 20$ and Γ is the skeleton of a $\{5,3\}$-map on the sphere. It is well known that there is only one such map, namely the dodecahedron.

Finally, suppose that $c = 3$. Then, by Lemma 3.1, $a + b$ is odd, and since $a \ge 2$, the signature of Γ is $(2,3,3)$. We will now show that this possibility does not occur.

Let uv be an edge of Γ lying on three 5-cycles, and u_0, u_1, v_0, v_1 be vertices of Γ with adjacencies $u_0 \sim u \sim u_1$ and $v_0 \sim v \sim v_1$. Then there should be three vertices adjacent to one of u_0, u_1 and one of v_0, v_1. Without loss of generality, let w_{00}, w_{10}, w_{11} be vertices such that $u_0 \sim w_{00} \sim v_0 \sim w_{10} \sim u_1 \sim w_{11} \sim v_1$. Further, let x be the neighbour of u_0 other than u and w_{00}, and let y be the neighbour of v_1 other than v and w_{11}, see Figure 6(a). Observe that $x \ne y$, for otherwise the edge uv would belong to four 5-cycles. Note also that x is not adjacent to any of the three neighbours of v, for otherwise the girth of Γ would be at most 4.

The signature implies that for each vertex, two edges incident to it lie on three 5-cycles. Suppose that uu_0 lies on three 5-cycles. As x and v have no common neighbours, w_{00} and x must have a common neighbour with u_1, so we have $w_{00} \sim w_{11}$ and $x \sim w_{10}$, see Figure 6(b). But then the edge uu_1 lies on four 5-cycles, contradiction. Therefore, the edge

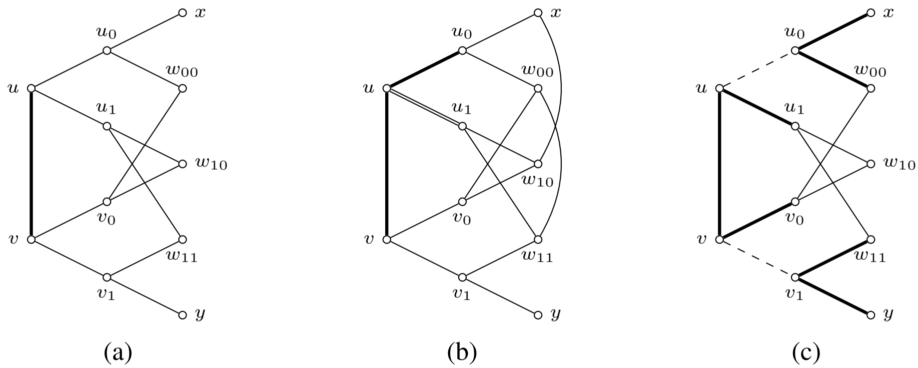


Figure 6: Constructing a graph of girth 5 with $c = 3$. The bold edges lie on three 5-cycles, and the dashed edges lie on two 5-cycles. The general setting is shown in (a). In (b), the arc (u, u_0) is assumed to lie on three 5-cycles, but the doubled edge then lies in four 5-cycles. In (c), the obtained distribution of edges among cycles is shown, which, however, cannot be completed.

uu_0 must lie on two 5-cycles, and a similar argument shows the same for vv_1. Thus, the arcs uu_1, vv_0, u_0x, u_0w_{00}, v_1w_{11} and v_1y must lie on three 5-cycles, see Figure 6(c).

Since the edge u_0w_{00} lies on three 5-cycles, there should be three vertices adjacent to one of u, x and one of v_0 and the remaining neighbour of w_{00}. Similarly, v_1w_{11} lying on three 5-cycles implies that there should be three vertices adjacent to one of v, y and one of u_1 and the remaining neighbour of w_{11}. As w_{10} is the only potential common neighbour for x, v_0 and for y, u_1, it follows that at least one of these pairs does not have a common neighbour. Without loss of generality, we may assume that x and v_0 do not have a common neighbour. The vertex u already has a common neighbour with v_0, and it must also have a common neighbour with the remaining neighbour of w_{00}. Then the remaining neighbour of w_{00} must be w_{11}, which however has no common neighbour with x, contradiction. Therefore, $(a, b, c) = (2, 3, 3)$ is not possible. □

6 Concluding remarks

Theorem 1.5 gives a complete classification of simple connected cubic girth-regular graphs of girths up to 5. While extending the classification to non-simple graphs (i.e., girths 1 and 2) is straightforward, increasing the girth leads to exponentially many more possible signatures. For example, the census of connected cubic vertex-transitive graphs on at most 1280 vertices by Potočnik, Spiga and Verret [13] shows that 9 distinct signatures appear among graphs of girth 6, while many more signatures are allowed by the results in Sections 1, 2 and Subsection 3.1. A classification of connected cubic vertex-transitive graphs of girth 6 will thus be given in a follow-up paper.

References

[1] E. Bannai and T. Ito, On finite Moore graphs, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20** (1973), 191–208, doi:10.15083/00039786.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, volume 18 of

Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1989, doi: 10.1007/978-3-642-74341-2.

[3] A. M. Cohen and J. Tits, On generalized hexagons and a near octagon whose lines have three points, *European J. Combin.* **6** (1985), 13–27, doi:10.1016/s0195-6698(85)80017-2.

[4] M. Conder, A. Malnič, D. Marušič and P. Potočnik, A census of semisymmetric cubic graphs on up to 768 vertices, *J. Algebraic Combin.* **23** (2006), 255–294, doi:10.1007/s10801-006-7397-3.

[5] M. Conder and R. Nedela, Symmetric cubic graphs of small girth, *J. Comb. Theory Ser. B* **97** (2007), 757–768, doi:10.1016/j.jctb.2007.01.001.

[6] M. Conder and S. S. Zemljič, private communication with M. Conder, 2017.

[7] R. M. Damerell, On Moore graphs, *Proc. Cambridge Philos. Soc.* **74** (1973), 227–236, doi: 10.1017/s0305004100048015.

[8] E. Eiben, R. Jajcay and P. Šparl, Symmetry properties of generalized graph truncations, *J. Comb. Theory Ser. B* **137** (2019), 291–315, doi:10.1016/j.jctb.2019.01.002.

[9] W. Feit and G. Higman, The nonexistence of certain generalized polygons, *J. Algebra* **1** (1964), 114–131, doi:10.1016/0021-8693(64)90028-6.

[10] R. Jajcay, G. Kiss and Š. Miklavič, Edge-girth-regular graphs, *European J. Combin.* **72** (2018), 70–82, doi:10.1016/j.ejc.2018.04.006.

[11] K. Kutnar and D. Marušič, A complete classification of cubic symmetric graphs of girth 6, *J. Comb. Theory Ser. B* **99** (2009), 162–184, doi:10.1016/j.jctb.2008.06.001.

[12] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2nd edition, 2009, doi:10.4171/066.

[13] P. Potočnik, P. Spiga and G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *J. Symbolic Comput.* **50** (2013), 465–477, doi:10.1016/j.jsc.2012.09.002.

[14] P. Potočnik and S. Wilson, Tetravalent edge-transitive graphs of girth at most 4, *J. Comb. Theory Ser. B* **97** (2007), 217–236, doi:10.1016/j.jctb.2006.03.007.

[15] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13–60, `http://www.numdam.org/item?id=PMIHES_1959__2__13_0`.

[16] W. T. Tutte, *Connectivity in Graphs*, number 15 in Mathematical Expositions, University of Toronto Press, Toronto, 1966, `http://www.jstor.org/stable/10.3138/j.ctvfrxhc8`.