Complex uniformly resolvable decompositions of K_v

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Dedicated to the good friend and colleague Lorenzo Milazzo
who passed away in March 2019.

Abstract

In this paper we consider complex uniformly resolvable decompositions of the complete graph K_v into subgraphs such that each resolution class contains only blocks isomorphic to the same graph from a given set $\mathcal{H}$ and at least one parallel class is present from each graph of $\mathcal{H}$. We completely determine the spectrum for the cases $\mathcal{H} = \{K_2, P_3, K_3\}$, $\mathcal{H} = \{P_4, C_4\}$, and $\mathcal{H} = \{K_2, P_4, C_4\}$.

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1   Introduction and definitions

Given a set $\mathcal{H}$ of pairwise non-isomorphic graphs, an $\mathcal{H}$-*decomposition* (or $\mathcal{H}$-*design*) of a graph G is a decomposition of the edge set of G into subgraphs (called *blocks*) isomorphic to some element of $\mathcal{H}$. We say that an $\mathcal{H}$-decomposition is *complex*[1] if all $H \in \mathcal{H}$ are present in it.

An $\mathcal{H}$-*factor* of G is a spanning subgraph of G which is a vertex-disjoint union of some copies of graphs belonging to $\mathcal{H}$. If $\mathcal{H} = \{H\}$, we will briefly speak of an H-factor. An $\mathcal{H}$-decomposition of G is *resolvable* if its blocks can be partitioned into $\mathcal{H}$-factors ($\mathcal{H}$-*factorization* or resolution of G). An $\mathcal{H}$-factor in an $\mathcal{H}$-factorization is referred to as a *parallel class*. Note that the parallel classes are mutually edge-disjoint, by definition.

An $\mathcal{H}$-factorization $\mathcal{F}$ of G is called *uniform* if each factor of $\mathcal{F}$ is an H-factor for some graph $H \in \mathcal{H}$. A K_2-factorization of G is known as a 1-*factorization* and its factors are called 1-*factors*; it is well known that a 1-factorization of K_v exists if and only if v is even ([10]). If $\mathcal{H} = \{F_1, \ldots, F_k\}$ and $r_i \geq 0$ for $i = 1, \ldots, k$, we denote by $(F_1, \ldots, F_k)$-URD$(v; r_1, \ldots, r_k)$ a uniformly resolvable decomposition of the complete graph K_v having exactly r_i F_i-factors. A *complex* $(F_1, \ldots, F_k)$-URD$(v; r_1, \ldots, r_k)$ is a uniformly resolvable decomposition of the complete graph K_v into $r_1 + \cdots + r_k$ parallel classes with the requirement that at least one parallel class is present for each $F_i \in \mathcal{H}$, i.e., $r_i > 0$ for $i = 1, \ldots, k$.

Recently, the existence problem for $\mathcal{H}$-factorizations of K_v has been studied and a lot of results have been obtained, especially on the following types of uniformly resolvable $\mathcal{H}$-decompositions: for a set $\mathcal{H}$ consisting of two complete graphs of orders at most five in [3, 13, 14, 15]; for a set $\mathcal{H}$ of two or three paths on two, three, or four vertices in [5, 6, 9]; for $\mathcal{H} = \{P_3, K_3 + e\}$ in [4]; for $\mathcal{H} = \{K_3, K_{1,3}\}$ in [8]; for $\mathcal{H} = \{C_4, P_3\}$ in [11]; for $\mathcal{H} = \{K_3, P_3\}$ in [12]; for 1-factors and n-stars in [7]; and for $\mathcal{H} = \{P_2, P_3, P_4\}$ in [9]. In connection with our current studies the following cases are most relevant:

- perfect matchings and parallel classes of triangles or 4-cycles (that is, $\{K_2, K_3\}$ or $\{K_2, C_4\}$, Rees [13]);

- perfect matchings and parallel classes of 3-paths ($\{K_2, P_3\}$, Bermond et al. [1], Gionfriddo and Milici [5]);

- parallel classes of 3-paths and triangles ($\{K_3, P_3\}$, Milici and Tuza [12]).

In this paper we give a complete characterization of the spectrum (the set of all admissible combinations of the parameters) for the following two triplets of graphs and for the pair contained in one of them which is not covered by the cases known so far:

- complex $\{K_2, P_3, K_3\}$-decompositions of K_v (Section 3, Theorem 3.1);

- complex $\{K_2, P_4, C_4\}$-decompositions of K_v (Section 4, Theorem 4.1);

- complex $\{P_4, C_4\}$-decompositions of K_v (Section 5, Theorem 5.1).

We summarize the formulation of those results in the concluding section, where a conjecture related to the method of "metamorphosis" of parallel classes is also raised. We provide the basis for this approach by applying linear algebra in Section 2.

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[1]In the terminology of [2] one may say that in a complex decomposition all $H \in \mathcal{H}$ are *essential* (page 131) or *mandatory* (page 232). In Chapter II.7.9 on uniformly resolvable designs the default is that all block sizes are essential, while in Chapter IV.1 on partially balanced designs no block size is required to be mandatory.

2 Local metamorphosis

In this section we prove three relations between uniform parallel classes of 4-cycles and 4-paths that will be used in the proofs of our main theorems. Before presenting the new statements, we recall the Milici–Tuza–Wilson Lemma from [12].

It will be assumed throughout, without further mention, that all parallel classes are meant on the same vertex set.

Theorem 2.1 ([12]). *The union of two parallel classes of 3-cycles can be decomposed into three parallel classes of P_3.*

The next two results, Theorems 2.2 and 2.3, will directly imply Theorem 2.4 which states a pure metamorphosis from 4-cycles to 4-paths.

Theorem 2.2. *The union of two parallel classes of C_4 is decomposable into two parallel classes of P_4 and one perfect matching.*

Proof. Let the vertices be $v_1, \ldots, v_n$ where n is a multiple of 4. The union of two parallel classes of C_4 forms a 4-regular graph G with $2n$ edges, say $e_1, \ldots, e_{2n}$. We associate a Boolean variable x_i with each edge e_i ($1 \leq i \leq 2n$) and construct a system of linear equations over $GF(2)$, which has $\frac{3n}{2} - 1$ equations over the $2n$ variables. Let us set

$$x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1 \qquad (\text{mod } 2)$$

for each 4-tuple of indices such that $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ are either the edges of a C_4 in a parallel class (call this a C-equation) or are the four edges incident with a vertex v_i (a V-equation). This gives $\frac{3}{2}n$ equations, but the V-equation for v_n can be omitted since the n V-equations sum up to 0 (as each edge is counted twice in the total sum) and therefore the one for v_n follows from the others.

We claim that this system of equations is contradiction-free over $GF(2)$. To show this, we need to prove that if the left sides of a subcollection $\mathcal{E}$ of the equations sum up to 0, then also the right sides have zero sum; that is, the number $|\mathcal{E}|$ of its equations is even.

Observe that each variable is present in precisely three equations: in one C-equation and two V-equations. Hence, to have zero sum on the left side, any x_i should either not appear in any equations of $\mathcal{E}$ or be present in precisely two. This means one of the following two situations.

(T1) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle (in this cyclic order of edges) and its C-equation belongs to $\mathcal{E}$, then precisely two related V-equations must be present in $\mathcal{E}$, namely either those for the vertices $e_{i_1} \cap e_{i_2}$ and $e_{i_3} \cap e_{i_4}$ or those for $e_{i_2} \cap e_{i_3}$ and $e_{i_1} \cap e_{i_4}$.

(T2) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle such that its C-equation does not belong to $\mathcal{E}$ but some x_{i_j} ($1 \leq j \leq 4$) is involved in $\mathcal{E}$, then all the four V-equations for $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ must be present in $\mathcal{E}$.

In the first and second parallel class of 4-cycles, respectively, let us denote the number of cycles of type (T1) by a_1 and b_1, and that of type (T2) by a_2 and b_2. Then the number of V-equations in $\mathcal{E}$ is equal to both $2a_1 + 4a_2$ and $2b_1 + 4b_2$, which is the same as the average of these two numbers. Thus, the number $|\mathcal{E}|$ of equations is equal to

$$a_1 + b_1 + \frac{1}{2}((2a_1 + 4a_2) + (2b_1 + 4b_2)) = 2(a_1 + a_2 + b_1 + b_2)$$

that is even, as needed.

Since the system of equations is non-contradictory, it has a solution $\xi \in \{0, 1\}^{2n}$ over $GF(2)$. We observe further that in any C-equation $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ we may switch the values from $\xi(x_{i_j})$ to $1 - \xi(x_{i_j})$ simultaneously for all $1 \leq j \leq 4$, and doing so the modified values remain a solution because the parities of sums in the V-equations do not change either. In this way, we can transform ξ to a basic solution ξ_0 in which every C-equation contains precisely one 1 and three 0s. Since each V-equation contains precisely two or zero variables from each C-equation, it follows that in the basic solution ξ_0 each V-equation, too, contains precisely one 1 and three 0s.

As a consequence, the variables which have $x_i = 1$ in the basic solution define a perfect matching in G (since at most two 1s may occur at each vertex, and then the corresponding V-equation implies that there is precisely one). Moreover, removing those edges from G, each cycle of each parallel class becomes a P_4. In this way we obtain two parallel classes of P_4, and one further class which is a perfect matching. $\qquad\square$

Theorem 2.3. *The edge-disjoint union of a perfect matching and a parallel class of C_4 is decomposable into two parallel classes of P_4.*

Proof. We apply several ideas from the previous proof, but in a somewhat different way. We now introduce Boolean variables $x_1, \ldots, x_n$ for the edges $e_1, \ldots, e_n$ of the 4-cycles only; but still there will be two kinds of linear equations, namely $n/4$ of them for 4-cycles (called C-equations) and $n/2$ of them for the edges of the perfect matching (M-equations). They are of the same form as before:

$$x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1 \qquad (\bmod 2)$$

The C-equations require $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ to be the edges of a C_4 in the parallel class. The M-equations take $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ as the four edges incident with a matching edge. If a matching edge is the diagonal of a 4-cycle, then their equations coincide; and if a matching edge shares just one vertex with a 4-cycle then the M-equation and the C-equation share two variables which correspond to consecutive edges on the cycle. Further, we recall that the matching is edge-disjoint from the cycles, therefore each variable associated with a cycle-edge occurs in precisely two distinct M-equations.

These facts imply that only two types of C-equations can occur in a subcollection $\mathcal{E}$ of equations whose left sides sum up to 0 over $GF(2)$.

(T1) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle and its C-equation belongs to $\mathcal{E}$, then the corresponding C_4 has precisely two (antipodal) vertices for which the M-equations of the incident matching edges are present in $\mathcal{E}$. (At the moment it is unimportant whether those two vertices form a matching edge or not.)

(T2) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle whose C-equation does not belong to $\mathcal{E}$ but some x_{i_j} $(1 \leq j \leq 4)$ is involved in $\mathcal{E}$, then each M-equation belonging to a matching edge incident with some of the four vertices $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ is present in $\mathcal{E}$. (It is again unimportant whether one or both or none of the diagonals of the C_4 in question is a matching edge.)

Let now a_1 and a_2 denote the number of cycles with type (T1) and type (T2), respectively. By what has been said, the number of vertices requiring an M-equation is equal to $2a_1 + 4a_2$. Since each of those equations is now counted at both ends of the

corresponding matching edge, we obtain that $\mathcal{E}$ contains exactly $a_1 + 2a_2$ M-equations; moreover it has a_1 C-equations, by definition. Thus, the number $|\mathcal{E}|$ of equations is equal to $a_1 + (a_1 + 2a_2) = 2(a_1 + a_2)$ which is even. Thus, if the left sides in $\mathcal{E}$ sum up to zero, then also the right sides have sum 0 in $GF(2)$. It proves that the system of the $3n/4$ equations is contradiction-free and has a solution $\xi \in \{0,1\}^n$ over $GF(2)$.

Now, we observe that in any C-equation $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ we may switch the values from $\xi(x_{i_j})$ to $1 - \xi(x_{i_j})$ simultaneously for all $1 \leq j \leq 4$. Doing so, the modified values remain a solution as the parities of sums in the M-equations do not change either. In this way we can transform ξ to a basic solution ξ_0 in which every C-equation contains precisely one 1 and three 0s. Since each M-equation has precisely two or four or zero variables from any C-equation, it follows that in the basic solution ξ_0 each M-equation, too, contains precisely one 1 and three 0s.

As a consequence, the variables (cycle-edges) which have $x_i = 1$ in the basic solution establish a pairing between the edges of the original matching. Hence the set $I = \{e_i : \xi_0(x_i) = 1\}$ together with the edges of the given matching factor determines a P_4-factor. Moreover, removing the edges of I from the 4-cycles, we obtain another parallel class of paths P_4. □

These two types of metamorphosis can be combined to obtain the following third one.

Theorem 2.4. *The union of three parallel classes of C_4 is decomposable into four parallel classes of P_4.*

Proof. Applying Theorem 2.2 we transform the union of the first and the second C_4-class into two P_4-classes and a perfect matching. After that we combine the third C_4-class with the perfect matching just obtained into two further P_4-classes, by Theorem 2.3. □

3 The spectrum for $\mathcal{H} = \{K_2, P_3, K_3\}$

In this section we consider complex uniformly resolvable decompositions of the complete graph K_v into m classes containing only copies of 1-factors (perfect matchings), p classes containing only copies of paths P_3 and t classes containing only copies of triangles K_3. The current problem is to determine the set of feasible triples (m, p, t) such that $m \cdot p \cdot t \neq 0$, for which there exists a complex (K_2, P_3, K_3)-URD$(v; m, p, t)$. A little more than that, for $v = 6$ and $v = 12$ we shall list all the possible (m, p, t) with $m, p, t \geq 0$ such that there exists a uniformly resolvable decomposition (K_2, P_3, K_3)-URD$(v; m, p, t)$.

Theorem 3.1. *The necessary and sufficient conditions for the existence of a complex (K_2, P_3, K_3)-URD$(v; m, p, t)$ are:*

(i) $v \geq 12$ and v is a multiple of 6;

(ii) $3m + 4p + 6t = 3v - 3$.

Moreover, the parameters m, p, t are in the following ranges:

(iii) $1 \leq m \leq v - 7$ and m is odd,
$3 \leq p \leq 3 \cdot \lfloor \frac{v}{4} - 1 \rfloor$,
$1 \leq t \leq \lfloor \frac{v}{2} - 3 \rfloor$.

Proof. We first prove that the conditions are necessary. Divisibility of v by 6 is immediately seen, due to the presence of K_3-classes and 1-factors. We observe further that the number of edges in a parallel class is v for a triangle-class, $2v/3$ for a P_3-class and $v/2$ for a matching. Thus, in any (K_2, P_3, K_3)-URD$(v; m, p, t)$ we must have

$$\frac{mv}{2} + \frac{2pv}{3} + tv = \binom{v}{2}.$$

Dividing it by $v/6$, the assertion of (ii) follows.

As (ii) implies, p is a multiple of 3, say $p = 3x$. Then we obtain

$$m + 4x + 2t = v - 1$$

and also conclude that m is odd. Since $m \geq 1$, $t \geq 1$, and $x \geq 1$, this equation yields

$$m \leq v - 7, \qquad x \leq v/4 - 1, \qquad t \leq v/2 - 3,$$

implying the conditions listed in (iii), and the first one also excludes $v = 6$. This completes the proof that the conditions (i) – (iii) are necessary.

To prove the sufficiency of (i) – (ii), we consider $v \geq 18$ first. Since v is a multiple of 6 according to (i), there exists a Nearly Kirkman Triple System of order v, which means $m = 1$ perfect matching and $t = \frac{v}{2} - 1$ parallel classes of triangles. More generally, for every odd m in the range $1 \leq m \leq v - 7$, there exists a collection of m perfect matchings and $t = \frac{v-1-m}{2}$ parallel classes of triangles, which together decompose K_v; this was proved in [13]. From such a system, for every $0 < x < \frac{v-1-m}{4}$, we can take $2x$ parallel classes of triangles. Applying Theorem 2.1 [12], also proved independently by Wilson (unpublished), we obtain $3x$ parallel classes of paths P_3. This gives a complex (K_2, P_3, K_3)-URD$(v; m, 3x, \frac{v-1-m}{2} - 2x)$. For $v = 12$, the statement follows by Proposition 3.3 below. □

3.1   Small cases

Obviously, the proofs of the necessary conditions that v is a multiple of 6, and that the equality $3m + 4p + 6t = 3v - 3$ must be satisfied by every (K_2, P_3, K_3)-URD$(v; m, p, t)$, do not use the assumption $m \cdot p \cdot t \neq 0$.

Proposition 3.2. *There exists a (K_2, P_3, K_3)-URD$(6; m, p, t)$ if and only if $(m, p, t) \in \{(5, 0, 0), (3, 0, 1), (1, 3, 0)\}$.*

Proof. Putting $v = 6$, the equation $3m + 4p + 6t = 15$ has exactly four solutions (m, p, t) over the nonnegative integers. The case $(1, 0, 2)$ would correspond to an NKTS(6) which is known not to exist [13]. The case $(5, 0, 0)$ corresponds to a 1-factorization of the complete graph K_6 which is known to exist [10]. The case of $(1, 3, 0)$ is just the same as a (K_2, P_3)-URD$(6; 1, 3)$ that is known to exist [6]. To see the existence for $(3, 0, 1)$, consider $V(K_6) = \mathbb{Z}_6$ and the following classes: $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$, $\{\{1, 5\}, \{2, 6\}, \{3, 4\}\}$, $\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}$, $\{(1, 2, 3), (4, 5, 6)\}$. □

Proposition 3.3. *There exists a (K_2, P_3, K_3)-URD$(12; m, p, t)$ if and only if $(m, p, t) \in \{(11, 0, 0), (9, 0, 1), (7, 3, 0), (7, 0, 2), (5, 0, 3), (5, 3, 1), (3, 6, 0), (3, 3, 2), (3, 0, 2), (1, 6, 1), (1, 3, 3)\}$.*

Proof. Checking the nonnegative integer solutions of $3m + 4p + 6t = 33$, the case of $(1, 0, 5)$ would correspond an NKTS(12) which is known not to exist [13]. The case of $(11, 0, 0)$ corresponds to a 1-factorization of the complete graph K_{12} that is known to exist [2]. The result for the cases $(9, 0, 1)$, $(7, 0, 2)$, $(5, 0, 3)$, $(3, 0, 4)$ follows by [13]. Applying Theorem 2.1 to $(7, 0, 2)$, $(5, 0, 3)$, $(3, 0, 4)$, we obtain the existence for $(7, 3, 0)$, $(5, 3, 1)$, $(3, 3, 2)$, and $(3, 6, 0)$. The existence for the case $(1, 3, 3)$ is shown by the following construction. Let $V(K_{12}) = \mathbb{Z}_{12}$, and consider the following parallel classes:

- matching: $\{\{1, 6\}, \{2, 4\}, \{3, 0\}, \{5, 11\}, \{7, 9\}, \{8, 10\}\}$;

- paths: $\{\{5, 0, 11\}, \{8, 1, 7\}, \{9, 2, 3\}, \{6, 4, 10\}\}$, $\{\{1, 3, 8\}, \{7, 5, 10\}, \{4, 9, 0\}, \{2, 11, 6\}\}$, $\{\{0, 6, 5\}, \{2, 7, 4\}, \{9, 8, 11\}, \{3, 10, 1\}\}$;

- triangles: $\{\{0, 1, 2), \{3, 4, 5\}, \{6, 7, 8\}, \{9, 10, 11\}\}$, $\{\{0, 4, 8\}, \{3, 7, 11\}, \{2, 6, 10\}, \{1, 5, 9\}\}$, $\{\{0, 7, 10\}, \{3, 6, 9\}, \{2, 5, 8\}, \{1, 4, 11\}\}$.

Finally, we apply Theorem 2.1 to the case $(1, 3, 3)$ and infer that a (K_2, P_3, K_3)-URD$(12; 1, 6, 1)$ exists, too. $\square$

4   The spectrum for $\mathcal{F} = \{K_2, P_4, C_4\}$

In this section we consider complex uniformly resolvable decompositions of the complete graph K_v into m parallel 1-factors, p parallel classes of 4-paths, and c parallel classes of 4-cycles. The current problem is to determine the set of feasible triples (m, p, c) such that $m \cdot p \cdot c \neq 0$, for which there exists a complex (K_2, P_3, C_4)-URD$(v; m, p, c)$. The case $\{P_4, C_4\}$ will be discussed in Section 5.

Theorem 4.1. *The necessary and sufficient conditions for the existence of a complex (K_2, P_4, C_4)-URD$(v; m, p, c)$ are:*

(i) $v \geq 8$ *and v is a multiple of* 4*;*

(ii) $2m + 3p + 4c = 2v - 2$.

Moreover, the parameters m, p, c are in the following ranges:

(iii) $1 \leq c \leq \frac{v}{2} - 3$,
   $1 \leq m \leq v - 6$,
   $2 \leq p \leq 2\lfloor \frac{v-4}{3} \rfloor$;

(iv) p *is even; and if $p \equiv 2 \pmod 4$, then also m is even.*

Proof. We first show that the conditions are necessary. Since K_v has a C_4-factor, v must be a multiple of 4. Further, as a C_4-, K_2-, and P_4-factor respectively cover exactly v, $v/2$, and $3v/4$ edges, in a (K_2, P_4, C_4)-URD$(v; m, p, c)$ we have

$$\frac{mv}{2} + \frac{3pv}{4} + cv = \binom{v}{2}.$$

This equality directly implies (ii) and we may also conclude that p is even and, further, if $p \equiv 2 \pmod 4$, then m must be even as well. Putting $p = 2x$ we obtain

$$m + 3x + 2c = v - 1.$$

By our condition, all the three types of parallel classes are present in the decomposition, i.e. we have $c \geq 1$, $m \geq 1$, and $x \geq 1$. These, together with the equality above, imply the necessity of (iii).

Next we prove the sufficiency of (i) – (ii). We first take a 1-factorization of $K_{v/2}$ into $v/2 - 1$ perfect matchings, which exists because v is a multiple of 4. Now, replace each vertex of $K_{v/2}$ with two non-adjacent vertices, and each edge with a copy of the complete bipartite graph $K_{2,2}$, i.e. a 4-cycle whose missing diagonals are the non-adjacent vertex pairs.[2] This blow-up results in $v/2 - 1$ parallel classes of C_4 inside K_v, and the missing edges can be taken as a perfect matching. Let C be the set of the parallel classes of C_4 and x be a nonnegative integer such that $0 < x \leq \lfloor \frac{v-4}{3} \rfloor$. The construction splits into two cases depending on the parity of x.

If x is even, take $\frac{3x}{2}$ parallel classes from C. Applying Theorem 2.4, we transform the $\frac{3x}{2}$ parallel classes of C_4 into $2x = p$ parallel classes of paths P_4. For any given $y = c$ in the range $0 < y \leq \lfloor \frac{v-2}{2} - \frac{3x}{2} \rfloor$, keep y classes of C_4 and transform the remaining $\frac{v-2}{2} - \frac{3x}{2} - y$ classes of C_4 into $2(\frac{v-2}{2} - \frac{3x}{2} - y) = m - 1$ classes of 1-factors. In this way we obtain a complex (K_2, P_4, C_4)-URD$(v; v - 3x - 2y - 1, 2x, y)$.

If x is odd, take $\frac{3(x-1)}{2} + 2$ parallel classes from C. By Theorems 2.2 and 2.4, we can transform the $\frac{3(x-1)}{2} + 2$ parallel classes of C_4 into $2x = p$ parallel classes of paths P_4 and a 1-factor. For any given $y = c$ in the range $0 < y \leq \frac{v-2}{2} - \frac{3(x-1)}{2} - 2$, keep y classes of C_4 and transform the remaining $\frac{v-2}{2} - \frac{3(x-1)}{2} - 2 - y$ classes of C_4 into $2(\frac{v-2}{2} - \frac{3(x-1)}{2} - 2 - y) = m - 2$ classes of 1-factors. In this way, we obtain a complex (K_2, P_4, C_4)-URD$(v; v - 3x - 2y - 1, 2x, y)$.

The result, for every $v \equiv 0 \pmod 4$, $0 < x \leq \lfloor \frac{v-4}{3} \rfloor$, and $0 < y \leq \lfloor \frac{v-2}{2} - \frac{3x}{2} \rfloor$, is a uniformly resolvable decomposition of K_v into into $v - 1 - 3x - 2y = m$ classes containing only copies of 1-factors, $2x = p$ classes containing only copies of paths P_4, and $y = c$ classes containing only copies of 4-cycles C_4. This finishes the proof of the theorem. □

5   The spectrum for $\mathcal{H} = \{P_4, C_4\}$

Finally, we consider complex uniformly resolvable decompositions of the complete graph K_v into p classes containing only copies of paths P_4 and c classes containing only copies of 4-cycles C_4.

Theorem 5.1. *The necessary and sufficient conditions for the existence of a complex* (P_4, C_4)-*URD$(v; p, c)$ are:*

 (i) $v \geq 8$ *and v is a multiple of 4;*

 (ii) $3p + 4c = 2v - 2$.

Proof. Necessity is a consequence of Theorem 4.1, since we did not need to assume $m > 0$ in that part of its proof. Turning to sufficiency, the condition $3p + 4c = 2v - 2$ implies that $3p \equiv 2v - 2 \pmod 4$. This gives $p = 2 + 4x$ and $c = \frac{v-4}{2} - 3x$. For a construction, we start with a decomposition of K_v into a perfect matching $\tilde{F}$ and $\frac{v-2}{2}$ parallel classes of C_4 as in the proof of Theorem 4.1. By Theorem 2.3, we can transform one class of C_4 and F

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[2]With another terminology, this "blow-up" is the lexicographic product $K_{v/2}[2K_1]$.

into two classes of paths P_4. Then, by Theorem 2.4, we transform $3x$ parallel classes of C_4 into $4x$ parallel classes of P_4. The result, for every x such that $0 \leq x \leq \lfloor \frac{v-6}{6} \rfloor$, is a uniformly resolvable decomposition of K_v into $2 + 4x$ classes containing only copies of paths P_4 and $\frac{v-4}{2} - 3x$ classes containing only copies of 4-cycles C_4. This completes the proof. □

6 Conclusion

Combining Theorems 3.1, 4.1, and 5.1, we obtain the main result of this paper.

Theorem 6.1.

(i) *A complex (K_2, P_3, K_3)-URD$(v; m, p, t)$ exists if and only if $v \geq 12$, v is a multiple of 6, and $3m + 4p + 6t = 3v - 3$.*

(ii) *A complex (K_2, P_4, C_4)-URD$(v; m, p, t)$ exists if and only if $v \geq 8$, v is a multiple of 4, and $2m + 3p + 4t = 2v - 2$.*

(iii) *A complex (P_4, C_4)-URD$(v; p, t)$ exists if and only if $v \geq 8$, v is a multiple of 4, and $3p + 4t = 2v - 2$.*

Concerning the local metamorphosis studied in Section 2, we pose the following conjecture as a common generalization of Theorems 2.1 and 2.4.

Conjecture 6.2. *The union of $k-1$ parallel classes of C_k is decomposable into k parallel classes of P_k.*

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