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# On separable abelian $p$-groups

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**Abstract**

An $S$-ring (a Schur ring) is said to be *separable* with respect to a class of groups $\mathcal{K}$ if every algebraic isomorphism from the $S$-ring in question to an $S$-ring over a group from $\mathcal{K}$ is induced by a combinatorial isomorphism. A finite group is said to be *separable* with respect to $\mathcal{K}$ if every $S$-ring over this group is separable with respect to $\mathcal{K}$. We provide a complete classification of abelian $p$-groups separable with respect to the class of abelian groups.

*Keywords: Isomorphisms, Schur rings, p-groups.*

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## 1 Introduction

Let $G$ be a finite group. A subring of the group ring $\mathbb{Z}G$ is called an $S$*-ring* (a *Schur ring*) over $G$ if it is determined in a natural way by a special partition of $G$ (the exact definition is given in Section 2). The classes of the partition are called the *basic sets* of the $S$-ring. The concept of the $S$-ring goes back to Schur and Wielandt. They used $S$-rings to study a permutation group containing a regular subgroup [19, 20]. For more details on $S$-rings and their applications we refer the reader to [13].

Let $\mathcal{A}$ and $\mathcal{A}'$ be $S$-rings over groups $G$ and $G'$ respectively. An *algebraic isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ is a ring isomorphism inducing a bijection between the basic sets of $\mathcal{A}$ and the basic sets of $\mathcal{A}'$. Another type of an isomorphism of $S$-rings comes from graph theory. A *combinatorial isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ is defined to be an isomorphism of the corresponding Cayley schemes (see Subsection 2.2). Every combinatorial isomorphism induces the algebraic one. However, the converse statement is not true (the corresponding examples can be found in [6]).

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Let $\mathcal{K}$ be a class of groups. Following [3], we say that an $S$-ring $\mathcal{A}$ is *separable* with respect to $\mathcal{K}$ if every algebraic isomorphism from $\mathcal{A}$ to an $S$-ring over a group from $\mathcal{K}$ is induced by a combinatorial one. We call a finite group *separable* with respect to $\mathcal{K}$ if every $S$-ring over $G$ is separable with respect to $\mathcal{K}$ (see [18]).

The importance of separable $S$-rings comes from the following observation. Suppose that an $S$-ring $\mathcal{A}$ is separable with respect to $\mathcal{K}$. Then $\mathcal{A}$ is determined up to isomorphism in the class of $S$-rings over groups from $\mathcal{K}$ only by the tensor of its structure constants (with respect to the basis of $\mathcal{A}$ corresponding to the partition of the underlying group).

Given a group $G$ denote the class of groups isomorphic to $G$ by $\mathcal{K}_G$. If $G$ is separable with respect to $\mathcal{K}_G$ then the isomorphism of two Cayley graphs over $G$ can be verified efficiently by using the Weisfeiler-Leman algorithm [12]. In the sense of [10] this means that the Weisfeiler-Leman dimension of the class of Cayley graphs over $G$ is at most 3. More information concerned with separability and the graph isomorphism problem is presented in [3, 17].

Denote the classes of cyclic and abelian groups by $\mathcal{K}_C$ and $\mathcal{K}_A$ respectively. The cyclic group of order $n$ is denoted by $C_n$. In the present paper we are interested in abelian groups and especially in abelian $p$-groups which are separable with respect to $\mathcal{K}_A$. The problem of determining of all groups separable with respect to a given class $\mathcal{K}$ seems quite complicated even for $\mathcal{K} = \mathcal{K}_C$. Examples of cyclic groups which are non-separable with respect to $\mathcal{K}_C$ were found in [6]. In [5] it was proved that cyclic $p$-groups are separable with respect to $\mathcal{K}_C$. We prove that a similar statement is also true for $\mathcal{K}_A$.

**Theorem 1.1.** *For every prime $p$ a cyclic $p$-group is separable with respect to $\mathcal{K}_A$.*

The result obtained in [18] implies that an abelian group of order $4p$ is separable with respect to $\mathcal{K}_A$ for every prime $p$. From [9] it follows that for every group $G$ of order at least 4 the group $G \times G$ is non-separable with respect to $\mathcal{K}_{G\times G}$. One can check that a normal subgroup of a group separable with respect to $\mathcal{K}_A$ is separable with respect to $\mathcal{K}_A$ (see also Lemma 2.5). The above discussion shows that a non-cyclic abelian $p$-group separable with respect to $\mathcal{K}_A$ is isomorphic to $C_p \times C_{p^k}$ or $C_p \times C_p \times C_{p^k}$, where $p \in \{2, 3\}$ and $k \geq 1$. The separability of the groups from the first family was proved in [17]. In the present paper we study the question on the separability of the groups from the second family.

**Theorem 1.2.** *The group $C_p \times C_p \times C_{p^k}$, where $p \in \{2, 3\}$ and $k \geq 1$, is separable with respect to $\mathcal{K}_A$ if and only if $k = 1$.*

As an immediate consequence of Theorem 1.1, Theorem 1.2, and the above mentioned results, we obtain a complete classification of abelian $p$-groups separable with respect to $\mathcal{K}_A$.

**Theorem 1.3.** *An abelian $p$-group is separable with respect to $\mathcal{K}_A$ if and only if it is cyclic or isomorphic to one of the following groups:*

$$C_2 \times C_{2^k}, \quad C_3 \times C_{3^k}, \quad C_2^3, \quad C_3^3,$$

*where $k \geq 1$.*

Throughout the paper we write for short "separable" instead of "separable with respect to $\mathcal{K}_A$". The text is organized in the following way. Section 2 contains a background of

$S$-rings. Section 3 is devoted to $S$-rings over cyclic $p$-groups. We finish Section 3 with the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2.

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**Notation.**

- The ring of rational integers is denoted by $\mathbb{Z}$.
- Let $X \subseteq G$. The element $\sum_{x \in X} x$ of the group ring $\mathbb{Z}G$ is denoted by $\underline{X}$.
- The order of $g \in G$ is denoted by $|g|$.
- The set $\{x^{-1} : x \in X\}$ is denoted by $X^{-1}$.
- The subgroup of $G$ generated by $X$ is denoted by $\langle X \rangle$; we also set $\mathrm{rad}(X) = \{g \in G : gX = Xg = X\}$.
- If $m \in \mathbb{Z}$ then the set $\{x^m : x \in X\}$ is denoted by $X^{(m)}$.
- Given a set $X \subseteq G$ the set $\{(g, xg) : x \in X, g \in G\}$ of edges of the Cayley graph $\mathrm{Cay}(G, X)$ is denoted by $R(X)$.
- The group of all permutations of a set $\Omega$ is denoted by $\mathrm{Sym}(\Omega)$.
- The subgroup of $\mathrm{Sym}(G)$ induced by right multiplications of $G$ is denoted by $G_{right}$.
- For a set $\Delta \subseteq \mathrm{Sym}(G)$ and a section $S = U/L$ of $G$ we set

$$\Delta^S = \{f^S : f \in \Delta, S^f = S\},$$

  where $S^f = S$ means that $f$ permutes the $L$-cosets in $U$ and $f^S$ denotes the bijection of $S$ induced by $f$.
- If a group $K$ acts on a set $\Omega$ then the set of all orbtis of $K$ on $\Omega$ is denoted by $\mathrm{Orb}(K, \Omega)$.
- If $H \leq G$ then the normalizer of $H$ in $G$ is denoted by $N_G(H)$.
- If $K \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$ then the stabilizer of $\alpha$ in $K$ is denoted by $K_\alpha$.
- The cyclic group of order $n$ is denoted by $C_n$.

## 2 $S$-rings

In this section we give a background of $S$-rings. The most of definitions and statements presented here are taken from [13, 17].

### 2.1 Definitions and basic facts

Let $G$ be a finite group and $\mathbb{Z}G$ the group ring over the integers. The identity element of $G$ is denoted by $e$. A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an *$S$-ring* over $G$ if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of $G$ such that:

(1) $\{e\} \in \mathcal{S}$,

(2) if $X \in \mathcal{S}$ then $X^{-1} \in \mathcal{S}$,

(3) $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathcal{S}\}$.

The elements of $\mathcal{S}$ are called the *basic sets* of $\mathcal{A}$ and the number $|\mathcal{S}|$ is called the *rank* of $\mathcal{A}$. Given $X, Y, Z \in \mathcal{S}$ the number of distinct representations of $z \in Z$ in the form $z = xy$ with $x \in X$ and $y \in Y$ is denoted by $c_{X,Y}^Z$. If $X$ and $Y$ are basic sets of $\mathcal{A}$ then $\underline{X}\,\underline{Y} = \sum_{Z \in \mathcal{S}(\mathcal{A})} c_{X,Y}^Z \underline{Z}$. So the integers $c_{X,Y}^Z$ are structure constants of $\mathcal{A}$ with respect to the basis $\{\underline{X} : X \in \mathcal{S}\}$. It is easy to verify that given basic sets $X$ and $Y$ the set $XY$ is also basic whenever $|X| = 1$ or $|Y| = 1$.

A set $X \subseteq G$ is said to be an $\mathcal{A}$*-set* if $\underline{X} \in \mathcal{A}$. A subgroup $H \leq G$ is said to be an $\mathcal{A}$*-subgroup* if $H$ is an $\mathcal{A}$-set. One can check that for every $\mathcal{A}$-set $X$ the groups $\langle X \rangle$ and $\mathrm{rad}(X)$ are $\mathcal{A}$-subgroups.

A section $U/L$ is said to be an $\mathcal{A}$*-section* if $U$ and $L$ are $\mathcal{A}$-subgroups. If $S = U/L$ is an $\mathcal{A}$-section then the module

$$\mathcal{A}_S = \mathrm{Span}_{\mathbb{Z}} \left\{\underline{X}^{\pi} : X \in \mathcal{S}(\mathcal{A}), X \subseteq U\right\},$$

where $\pi \colon U \to U/L$ is the canonical epimorphism, is an $S$-ring over $S$.

If $K \leq \mathrm{Aut}(G)$ then the set $\mathrm{Orb}(K, G)$ forms a partition of $G$ that defines an $S$-ring $\mathcal{A}$ over $G$. In this case $\mathcal{A}$ is called *cyclotomic* and denoted by $\mathrm{Cyc}(K, G)$.

Let $G$ be abelian. Then from Schur's result [19] it follows that $X^{(m)} \in \mathcal{S}(\mathcal{A})$ for every $X \in \mathcal{S}(\mathcal{A})$ and every $m$ coprime to $|G|$. We say that $X, Y \in \mathcal{S}(\mathcal{A})$ are *rationally conjugate* if $Y = X^{(m)}$ for some $m$ coprime to $|G|$.

### 2.2 Isomorphisms and schurity

Throughout this and the next two subsections $\mathcal{A}$ and $\mathcal{A}'$ are $S$-rings over groups $G$ and $G'$ respectively. A bijection $f \colon G \to G'$ is called *a (combinatorial) isomorphism* from $\mathcal{A}$ over to $\mathcal{A}'$ if

$$\{R(X)^f : X \in \mathcal{S}(\mathcal{A})\} = \{R(X') : X' \in \mathcal{S}(\mathcal{A}')\},$$

where $R(X)^f = \{(g^f, h^f) : (g, h) \in R(X)\}$. If there exists an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ we write $\mathcal{A} \cong \mathcal{A}'$. The group $\mathrm{Iso}(\mathcal{A})$ of all isomorphisms from $\mathcal{A}$ onto itself has a normal subgroup

$$\mathrm{Aut}(\mathcal{A}) = \{f \in \mathrm{Iso}(\mathcal{A}) : R(X)^f = R(X) \text{ for every } X \in \mathcal{S}(\mathcal{A})\}.$$

This subgroup is called the *automorphism group* of $\mathcal{A}$. Note that $\mathrm{Aut}(\mathcal{A}) \geq G_{right}$. If $S$ is an $\mathcal{A}$-section then $\mathrm{Aut}(\mathcal{A})^S \leq \mathrm{Aut}(\mathcal{A}_S)$. An $S$-ring $\mathcal{A}$ over $G$ is said to be *normal* if $G_{right} \trianglelefteq \mathrm{Aut}(\mathcal{A})$. One can check that

$$N_{\mathrm{Aut}(\mathcal{A})}(G_{right})_e = \mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(G). \tag{2.1}$$

Now let $K$ be a subgroup of $\mathrm{Sym}(G)$ containing $G_{right}$. As Schur proved in [19], the $\mathbb{Z}$-submodule

$$V(K, G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathrm{Orb}(K_e, G)\},$$

is an $S$-ring over $G$. An $S$-ring $\mathcal{A}$ over $G$ is called *schurian* if $\mathcal{A} = V(K, G)$ for some $K$ such that $G_{right} \leq K \leq \mathrm{Sym}(G)$. Not every $S$-ring is schurian. The first example of a non-schurian $S$-ring was found by Wielandt in [20, Theorem 25.7]. It is easy to see that $\mathcal{A}$ is schurian if and only if

$$\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(\mathrm{Aut}(\mathcal{A})_e, G). \tag{2.2}$$

Every cyclotomic $S$-ring is schurian. More precisely, if $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq \mathrm{Aut}(G)$ then $\mathcal{A} = V(G_{right} \rtimes K, G)$.

### 2.3 Algebraic isomorphisms and separability

A bijection $\varphi\colon \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A}')$ is called an *algebraic isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if

$$c_{X,Y}^{Z} = c_{X^{\varphi},Y^{\varphi}}^{Z^{\varphi}}$$

for all $X, Y, Z \in \mathcal{S}(\mathcal{A})$. The mapping $\underline{X} \to \underline{X}^{\varphi}$ is extended by linearity to the ring isomorphism of $\mathcal{A}$ and $\mathcal{A}'$. This ring isomorphism we denote also by $\varphi$. If there exists an algebraic isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ then we write $\mathcal{A} \cong_{\mathrm{Alg}} \mathcal{A}'$. An algebraic isomorphism from $\mathcal{A}$ to itself is called an *algebraic automorphism* of $\mathcal{A}$. The group of all algebraic automorphisms of $\mathcal{A}$ is denoted by $\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$.

Every isomorphism $f$ of $S$-rings preserves the structure constants and hence $f$ induces the algebraic isomorphism $\varphi_f$. However, not every algebraic isomorphism is induced by a combinatorial one (see [6]). Let $\mathcal{K}$ be a class of groups. An $S$-ring $\mathcal{A}$ is defined to be *separable* with respect to $\mathcal{K}$ if every algebraic isomorphism from $\mathcal{A}$ to an $S$-ring over a group from $\mathcal{K}$ is induced by a combinatorial isomorphism.

Put

$$\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})_0 = \{\varphi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}) : \varphi = \varphi_f \text{ for some } f \in \mathrm{Iso}(\mathcal{A})\}.$$

It is easy to see that $\varphi_f = \varphi_g$ for $f, g \in \mathrm{Iso}(\mathcal{A})$ if and only if $gf^{-1} \in \mathrm{Aut}(\mathcal{A})$. Therefore

$$|\,\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})_0| = |\,\mathrm{Iso}(\mathcal{A})|/|\,\mathrm{Aut}(\mathcal{A})|. \tag{2.3}$$

One can verify that for every group $G$ the $S$-ring of rank 2 over $G$ and $\mathbb{Z}G$ are separable with respect to the class of all finite groups. In the former case there exists the unique algebraic isomorphism from the $S$-ring of rank 2 over $G$ to the $S$-ring of rank 2 over a given group of order $|G|$ and this algebraic isomorphism is induced by every bijection. In the latter case every basic set is singleton and hence every algebraic isomorphism is induced by an isomorphism in a natural way.

Let $\varphi\colon \mathcal{A} \to \mathcal{A}'$ be an algebraic isomorphism. One can check that $\varphi$ is extended to a bijection between $\mathcal{A}$- and $\mathcal{A}'$-sets and hence between $\mathcal{A}$- and $\mathcal{A}'$-sections. The images of an $\mathcal{A}$-set $X$ and an $\mathcal{A}$-section $S$ under these extensions are denoted by $X^{\varphi}$ and $S^{\varphi}$ respectively. If $S$ is an $\mathcal{A}$-section then $\varphi$ induces the algebraic isomorphism $\varphi_S\colon \mathcal{A}_S \to \mathcal{A}'_{S'}$, where $S' = S^{\varphi}$. The above bijection between the $\mathcal{A}$- and $\mathcal{A}'$-sets is, in fact, an isomorphism of the corresponding lattices. One can check that

$$\langle X^{\varphi}\rangle = \langle X\rangle^{\varphi} \quad \text{and} \quad \mathrm{rad}(X^{\varphi}) = \mathrm{rad}(X)^{\varphi}$$

for every $\mathcal{A}$-set $X$ (see [4, Equation (10)]). Since $c_{X,Y}^{\{e\}} = \delta_{Y,X^{-1}}|X|$, where $X, Y \in \mathcal{S}(\mathcal{A})$ and $\delta_{Y,X^{-1}}$ is the Kronecker delta, we conclude that $|X| = c_{X,X^{-1}}^{\{e\}}$, $(X^{-1})^{\varphi} = (X^{\varphi})^{-1}$, and $|X| = |X^{\varphi}|$ for every $\mathcal{A}$-set $X$. In particular, $|G| = |G'|$.

### 2.4 Cayley isomorphisms

A group isomorphism $f\colon G \to G'$ is called a *Cayley isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$. If there exists a Cayley isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ we write $\mathcal{A} \cong_{\mathrm{Cay}} \mathcal{A}'$. Every Cayley isomorphism is a (combinatorial) isomorphism, however the converse statement is not true.

### 2.5 Algebraic fusions

Let $\mathcal{A}$ be an $S$-ring over $G$ and $\Phi \leq \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$. Given $X \in \mathcal{S}(\mathcal{A})$ put $X^{\Phi} = \bigcup_{\varphi\in\Phi} X^{\varphi}$. The partition

$$\{X^{\Phi} : X \in \mathcal{S}(\mathcal{A})\}$$

defines an $S$-ring over $G$ called the *algebraic fusion* of $\mathcal{A}$ with respect to $\Phi$ and denoted by $\mathcal{A}^{\Phi}$. Suppose that $\Phi = \{\varphi_f : f \in K\}$ for some $K \leq \mathrm{Iso}(\mathcal{A})$ and $\mathcal{A}$ is schurian. Then one can verify that

$$\mathcal{A}^{\Phi} = V(\mathrm{Aut}(\mathcal{A})K, G).$$

In particular, the following statement holds.

**Lemma 2.1.** *Let $\mathcal{A}$ be a schurian $S$-ring over $G$ and $K \leq \mathrm{Iso}(\mathcal{A})$. Then $\mathcal{A}^{\Phi}$, where $\Phi = \{\varphi_f : f \in K\}$, is also schurian.*

### 2.6 Wreath and tensor products

Let $\mathcal{A}$ be an $S$-ring over a group $G$ and $S = U/L$ an $\mathcal{A}$-section. The $S$-ring $\mathcal{A}$ is called the *$S$-wreath product* if $L \trianglelefteq G$ and $L \leq \mathrm{rad}(X)$ for all basic sets $X$ outside $U$. In this case we write

$$\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}.$$

The $S$-wreath product is called *non-trivial* or *proper* if $e \neq L$ and $U \neq G$. If $U = L$ we say that $\mathcal{A}$ is the *wreath product* of $\mathcal{A}_L$ and $\mathcal{A}_{G/L}$ and write $\mathcal{A} = \mathcal{A}_L \wr \mathcal{A}_{G/L}$.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be $S$-rings over groups $G_1$ and $G_2$ respectively. Then the set

$$\mathcal{S} = \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) = \{X_1 \times X_2 : X_1 \in \mathcal{S}(\mathcal{A}_1), X_2 \in \mathcal{S}(\mathcal{A}_2)\}$$

forms a partition of $G = G_1 \times G_2$ that defines an $S$-ring over $G$. This $S$-ring is called the *tensor product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ and denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$.

**Lemma 2.2.** *The tensor product of two separable $S$-rings is separable.*

*Proof.* As noted in [18, Lemma 2.6], the statement of the lemma follows from [1, Theorem 1.20]. □

**Lemma 2.3** ([17, Lemma 4.4])**.** *Let $\mathcal{A}$ be the $S$-wreath product over an abelian group $G$ for some $\mathcal{A}$-section $S = U/L$. Suppose that $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are separable and* $\mathrm{Aut}(\mathcal{A}_U)^S = \mathrm{Aut}(\mathcal{A}_S)$. *Then $\mathcal{A}$ is separable. In particular, the wreath product of two separable $S$-rings is separable.*

Let $\Omega$ be a finite set. Permutation groups $K,\ K' \leq \mathrm{Sym}(\Omega)$ are called $2$-*equivalent* if $\mathrm{Orb}(K, \Omega^2) = \mathrm{Orb}(K', \Omega^2)$. A permutation group $K \leq \mathrm{Sym}(\Omega)$ is called $2$-*isolated* if it is the only group which is 2-equivalent to $K$.

**Lemma 2.4.** *Let $\mathcal{A}$ be the $S$-wreath product over an abelian group $G$ for some $\mathcal{A}$-section $S = U/L$. Suppose that $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are separable, $\mathcal{A}_U$ is schurian, and the group* $\mathrm{Aut}(\mathcal{A}_S)$ *is* $2$*-isolated. Then $\mathcal{A}$ is separable.*

*Proof.* Since $\mathcal{A}_U$ is schurian, the groups $\mathrm{Aut}(\mathcal{A}_U)^S$ and $\mathrm{Aut}(\mathcal{A}_S)$ are 2-equivalent. Indeed,

$$\mathrm{Orb}(\mathrm{Aut}(\mathcal{A}_U)^S, S^2) = \mathrm{Orb}(\mathrm{Aut}(\mathcal{A}_S), S^2) = \{R(X) : X \in \mathcal{S}(\mathcal{A}_S)\}.$$

This implies that $\mathrm{Aut}(\mathcal{A}_U)^S = \mathrm{Aut}(\mathcal{A}_S)$ because $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated. Therefore the conditions of Lemma 2.3 hold and $\mathcal{A}$ is separable. □

**Lemma 2.5.** *Let $H$ be a normal subgroup of a group $G$, $\mathcal{B}$ an $S$-ring over $H$, $\varphi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{B}) \setminus \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{B})_0$. Then there exists $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}) \setminus \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})_0$, where $\mathcal{A} = \mathcal{B} \wr \mathbb{Z}(G/H)$, such that $\psi^H = \varphi$.*

*Proof.* Define $\psi$ as follows: $X^\psi = X^\varphi$ for $X \in \mathcal{S}(\mathcal{A}_H)$ and $X^\psi = X$ for $X \in \mathcal{S}(\mathcal{A}) \setminus \mathcal{S}(\mathcal{A}_H)$. Let us prove that $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$. To do this it suffices to check that $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z$ for all $X, Y, Z \in \mathcal{S}(\mathcal{A})$. Suppose that $X, Y \in \mathcal{S}(\mathcal{A}_H)$. If $Z \in \mathcal{S}(\mathcal{A}_H)$ then $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X^\varphi,Y^\varphi}^{Z^\varphi} = c_{X,Y}^Z$. If $Z \notin \mathcal{S}(\mathcal{A}_H)$ then $Z^\psi \notin \mathcal{S}(\mathcal{A}_H)$ and hence $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z = 0$.

Now suppose that exactly one of the sets $X, Y$, say $X$, lies inside $H$. Then $Y^\psi = Y$ and $X \cup X^\psi \subseteq H \leq \mathrm{rad}(Y)$. So $\underline{X}\,\underline{Y} = \underline{X^\psi}\,\underline{Y} = |X|\underline{Y}$. This implies that $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z = |X|$ whenever $Z = Y$ and $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z = 0$ otherwise.

Finally, suppose that $X, Y \notin \mathcal{S}(\mathcal{A}_H)$. In this case $X^\psi = X$ and $Y^\psi = Y$. If $Z \notin \mathcal{S}(\mathcal{A}_H)$ then $Z^\psi = Z$ and hence $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z$. If $Z \in \mathcal{S}(\mathcal{A}_H)$ then $Z$ and $Z^\psi$ enter the element $\underline{XY}$ with the same coefficients because $H = \mathrm{rad}(X) \cap \mathrm{rad}(Y)$. Therefore $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z$. Thus, $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$.

If $\psi$ is induced by an isomorphism then [4, Lemma 3.4] implies that $\psi^H = \varphi$ is also induced by an isomorphism. We obtain a contradiction with the assumption of the lemma and the lemma is proved. □

## 3 $S$-rings over cyclic $p$-groups

In this section we prove Theorem 1.1. Before the proof we recall some results on $S$-rings over cyclic $p$-groups. The most of them can be found in [7, 8]. Throughout the section $p$ is an odd prime, $G$ is a cyclic $p$-group and $\mathcal{A}$ is an $S$-ring over $G$. We say that $X \in \mathcal{S}(\mathcal{A})$ is *highest* if $X$ contains a generator of $G$. Put $\mathrm{rad}(\mathcal{A}) = \mathrm{rad}(X)$, where $X$ is highest. Note that $\mathrm{rad}(\mathcal{A})$ does not depend on the choice of $X$ because every two basic sets are rationally conjugate and hence have the same radicals.

**Lemma 3.1.** *The $S$-ring $\mathcal{A}$ is schurian and one of the following statements holds for $\mathcal{A}$:*

*(1)* $|\mathrm{rad}(\mathcal{A})| = 1$ *and* $\mathrm{rk}(\mathcal{A}) = 2$*;*

*(2)* $|\mathrm{rad}(\mathcal{A})| = 1$*,* $\mathcal{A}$ *is normal, and* $\mathcal{A} = \mathrm{Cyc}(K, G)$ *for some* $K \leq K_0$*, where* $K_0$ *is the subgroup of* $\mathrm{Aut}(G)$ *of order* $p - 1$*;*

*(3)* $|\mathrm{rad}(\mathcal{A})| > 1$ *and* $\mathcal{A}$ *is the proper generalized wreath product.*

*Proof.* The $S$-ring $\mathcal{A}$ is schurian by the main result of [16]. The other statements of the lemma follow from [8, Theorem 4.1, Theorem 4.2 (1)] and [7, Lemma 5.1, Equation (1)]. □

**Lemma 3.2.** *Let $S$ be an $\mathcal{A}$-section with $|S| \geq p^2$. The following statements hold:*

*(1) If Statement (2) of Lemma 3.1 holds for $\mathcal{A}$ then Statement (2) of Lemma 3.1 holds for $\mathcal{A}_S$;*

*(2) If* $\mathrm{rk}(\mathcal{A}_S) = 2$ *then* $\mathrm{Aut}(\mathcal{A})^S = \mathrm{Sym}(S)$.

*Proof.* Statement (1) of the lemma follows from [8, Corollary 4.4] and Statement (2) of the lemma follows from [8, Theorem 4.6 (1)]. □

**Lemma 3.3.** *Suppose that Statement (2) of Lemma 3.1 holds for* $\mathcal{A}$*. Then* $\mathrm{Aut}(\mathcal{A})$ *is* 2*-isolated.*

*Proof.* By [15, Lemma 8.2], it suffices to prove that $\mathrm{Aut}(\mathcal{A})_e$ has a faithful regular orbit. The $S$-ring $\mathcal{A}$ is normal. So (2.1) implies that $\mathrm{Aut}(\mathcal{A})_e \leq \mathrm{Aut}(G)$. Let $X \in \mathcal{S}(\mathcal{A})$ be highest. Since $\mathcal{A}$ is cyclotomic, each element of $X$ is a generator of $G$. If $f \in \mathrm{Aut}(\mathcal{A})_e$ fixes some $x \in X$ then $f$ is trivial because $f \in \mathrm{Aut}(G)$ and $x$ is a generator of $G$. Besides, $\mathcal{A}$ is schurian and hence $X \in \mathrm{Orb}(\mathrm{Aut}(\mathcal{A})_e, G)$ by (2.2). Therefore $X$ is a regular orbit of $\mathrm{Aut}(\mathcal{A})_e$. The group $\mathrm{Aut}(G)$ is cyclic because $p$ is odd. So both of the groups $\mathrm{Aut}(\mathcal{A})_e$ and $\mathrm{Aut}(\mathcal{A})_e^X$ are cyclic groups of order $|X|$. Thus, $X$ is a faithful regular orbit of $\mathrm{Aut}(\mathcal{A})_e$ and the lemma is proved. □

**Lemma 3.4.** *Suppose that Statement (2) of Lemma 3.1 holds for* $\mathcal{A}$ *and* $\varphi$ *is an algebraic isomorphism from* $\mathcal{A}$ *to an* $S$*-ring* $\mathcal{A}'$ *over an abelian group* $G'$*. Then* $G'$ *is cyclic.*

*Proof.* By the hypothesis,

$$\mathcal{A} = \mathrm{Cyc}(K, G) \text{ for some } K \leq \mathrm{Aut}(G) \text{ with } |K| \leq p - 1.$$

The group $E = \{g \in G : |g| = p\}$ is an $\mathcal{A}$-subgroup of order $p$ because $\mathcal{A}$ is cyclotomic. The group $E' = E^{\varphi}$ is an $\mathcal{A}'$-subgroup of order $p$ by the properties of an algebraic isomorphism. Assume that $G'$ is non-cyclic. Then there exists $X' \in \mathcal{S}(\mathcal{A}')$ containing an element of order $p$ outside $E'$. Let $X \in \mathcal{S}(\mathcal{A})$ such that $X^{\varphi} = X'$. The set $X$ consists of elements of order greater than $p$ because $G$ is cyclic and all elements of order $p$ from $G$ lie inside $E$. The identity element $e$ of $G$ enters the element $\underline{X}^p$ with a coefficient dividing by $p$ because $x^p \neq e$ for each $x \in X$. The identity element $e'$ of $G'$ enters the element $(\underline{X}')^p$ with a coefficient which is not divided by $p$ because $(x')^p = e'$ for some $x' \in X'$ and $|X'| \leq p - 1$. Since $\varphi$ is an algebraic isomorphism, we have

$$(\underline{X}^p)^{\varphi} = (\underline{X}')^p \quad \text{and} \quad \{e\}^{\varphi} = \{e'\}.$$

This implies that $e$ and $e'$ must enter $\underline{X}^p$ and $(\underline{X}')^p$ respectively with the same coefficients, a contradiction. Therefore $G'$ is cyclic and the lemma is proved. □

**Lemma 3.5.** *Suppose that* $|\mathrm{rad}(\mathcal{A})| > 1$*. Then there exists an* $\mathcal{A}$*-section* $S = U/L$ *such that* $\mathcal{A}$ *is the proper* $S$*-wreath product,* $|\mathrm{rad}(\mathcal{A}_U)| = 1$*, and* $|L| = p$*.*

*Proof.* From [17, Lemma 5.2] it follows that there exists an $\mathcal{A}$-section $U/L_1$ such that $\mathcal{A}$ is the proper $U/L_1$-wreath product and $|\mathrm{rad}(\mathcal{A}_U)| = 1$. Let $L$ be a subgroup of $L_1$ of order $p$. Then the lemma holds for $S = U/L$. □

**Lemma 3.6** ([5, Theorem 1.3])**.** *Every* $S$*-ring over a cyclic* $p$*-group is separable with respect to* $\mathcal{K}_C$*.*

*Proof of the Theorem 1.1.* The statement of the theorem for $p \in \{2, 3\}$ was proved in [17, Lemma 5.5]. Further we assume that $p \geq 5$. Let $\mathcal{A}$ be an $S$-ring over a cyclic $p$-group $G$ of order $p^k$, where $k \geq 1$. Prove that $\mathcal{A}$ is separable. We proceed by induction on $k$. If $k = 1$ then $G$ is the unique up to isomorphism group of order $p$ and the statement of the theorem follows from Lemma 3.6.

Let $k \geq 2$. One of the statements of Lemma 3.1 holds for $\mathcal{A}$. If Statement (1) of Lemma 3.1 holds for $\mathcal{A}$ then $\mathrm{rk}(\mathcal{A}) = 2$ and hence $\mathcal{A}$ is separable. Suppose that Statement (2) of Lemma 3.1 holds for $\mathcal{A}$. Let $\varphi$ be an algebraic isomorphism from $\mathcal{A}$ to an $S$-ring $\mathcal{A}'$ over an abelian group $G'$. Due to Lemma 3.4, the group $G'$ is cyclic. So $\varphi$ is induced by an isomorphism by Lemma 3.6. Therefore $\mathcal{A}$ is separable.

Now suppose that Statement (3) of Lemma 3.1 holds for $\mathcal{A}$. Then $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ for some $\mathcal{A}$-section $S = U/L$ with $L > e$ and $U < G$. The $S$-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are separable by the induction hypothesis. Due to Lemma 3.5 we may assume that $\mathrm{rad}(\mathcal{A}_U) = e$ and $|L| = p$. In this case $\mathrm{rk}(\mathcal{A}_U) = 2$ or Statement (2) of Lemma 3.1 holds for $\mathcal{A}_U$. If $\mathrm{rk}(\mathcal{A}_U) = 2$ or $|S| = 1$ then $U = L$ and $\mathcal{A}$ is separable by Lemma 2.3.

Assume that Statement (2) of Lemma 3.1 holds for $\mathcal{A}_U$. If $|S| \geq p^2$ then Statement (2) of Lemma 3.1 holds for $\mathcal{A}_S$ by Statement (1) of Lemma 3.2. Lemma 3.3 implies that $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated. The $S$-ring $\mathcal{A}_U$ is cyclotomic and hence it is schurian. Therefore $\mathcal{A}$ is separable by Lemma 2.4.

It remains to consider the case when $|S| = p$. In this case $|U| = p^2$. If $\mathrm{rad}(X) > L$ for every $X \in \mathcal{S}(\mathcal{A})$ outside $U$ then $\mathrm{rad}(X) \geq U$ for every $X \in \mathcal{S}(\mathcal{A})$ outside $U$ because $G$ is cyclic. This yields that $\mathcal{A} = \mathcal{A}_U \wr \mathcal{A}_{G/U}$ and hence $\mathcal{A}$ is separable by Lemma 2.3.

Suppose that there exists $X \in \mathcal{S}(\mathcal{A})$ outside $U$ with $\mathrm{rad}(X) = L$. The remaining part of the proof is divided into two cases.

**Case 1:** $\langle X \rangle < G$. In this case put $S_1 = \langle X \rangle / L$. The $S$-ring $\mathcal{A}$ is the $S_1$-wreath product and $|S_1| \geq p^2$. Note that $|\mathrm{rad}(\mathcal{A}_{S_1})| = 1$ because $\mathrm{rad}(X) = L$. So Statement (1) or Statement (2) of Lemma 3.1 holds for $\mathcal{A}_{S_1}$. In the former case $\mathrm{Aut}(\mathcal{A}_{\langle X \rangle})^{S_1} = \mathrm{Aut}(\mathcal{A}_{S_1}) = \mathrm{Sym}(S_1)$ by Statement (2) of Lemma 3.2 and $\mathcal{A}$ is separable by Lemma 2.3. In the latter case $\mathrm{Aut}(\mathcal{A}_{S_1})$ is 2-isolated by Lemma 3.3. Since $\mathcal{A}_{\langle X \rangle}$ is schurian, the conditions of Lemma 2.4 hold for $S_1$ and $\mathcal{A}$ is separable by Lemma 2.4.

**Case 2:** $\langle X \rangle = G$. In this case $|\mathrm{rad}(\mathcal{A}_{G/L})| = 1$ because $\mathrm{rad}(X) = L$. Let $\pi \colon G \to G/L$ be the canonical epimorphism. Clearly, $\pi(U)$ is an $\mathcal{A}_{G/L}$-subgroup and $\pi(X)$ lies outside $\pi(U)$. So $\mathrm{rk}(\mathcal{A}_{G/L}) > 2$ and hence Statement (2) of Lemma 3.1 holds for $\mathcal{A}_{G/L}$.

Let $\varphi$ be an algebraic isomorphism from $\mathcal{A}$ to an $S$-ring $\mathcal{A}'$ over an abelian group $G'$. Put $U' = U^{\varphi}$ and $L' = L^{\varphi}$. Clearly,

$$L' \leq U'. \tag{3.1}$$

The algebraic isomorphism $\varphi$ induces the algebraic isomorphism $\varphi_U$ from $\mathcal{A}_U$ to $\mathcal{A}_{U'}$, where $U' = U^{\varphi}$. From Lemma 3.4 it follows that

$$U' \cong C_{p^2}. \tag{3.2}$$

Also $\varphi$ induces the algebraic isomorphism $\varphi_{G/L}$ from $\mathcal{A}_{G/L}$ to $\mathcal{A}_{G'/L'}$. Lemma 3.4 implies that $G'/L'$ is cyclic. Since $|L'| = |L| = p$, we conclude that

$$G' \cong C_{p^k} \quad \text{or} \quad G' \cong C_p \times C_{p^{k-1}}.$$

However, in the latter case $L'$ is not contained in a cyclic group of order $p^2$ because $G'/L'$ is cyclic. This contradicts to (3.1) and (3.2). So $G'$ is cyclic and $\varphi$ is induced by an isomorphism by Lemma 3.6. Therefore $\mathcal{A}$ is separable and the theorem is proved. □

# 4 Proof of Theorem 1.2

**Proposition 4.1.** *The group $C_p^3$ is separable for $p \in \{2, 3\}$.*

Before we prove Propostion 4.1 we give the lemma providing a description of $S$-rings over these groups.

**Lemma 4.2.** *Let $\mathcal{A}$ be an $S$-ring over $C_p^3$, where $p \in \{2, 3\}$. Then $\mathcal{A}$ is schurian and one of the following statements holds:*

*(1)* $\mathrm{rk}(\mathcal{A}) = 2$*;*

*(2) $\mathcal{A}$ is the tensor product of smaller $S$-rings;*

*(3) $\mathcal{A}$ is the proper $S$-wreath product of two $S$-rings with $|S| \le p$;*

*(4) $p = 3$ and $\mathcal{A} \cong_{\mathrm{Cay}} \mathcal{A}_i$, where $\mathcal{A}_i$ is one of the 14 exceptional $S$-rings whose parameters are listed in Table 1.*

**Remark 4.3.** In Table 1 the notation $k^m$ means that an $S$-ring have exactly $m$ basic sets of size $k$.

Table 1: Parameters of the 14 exceptional $S$-rings $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{14}$.

| $S$-ring | rank | sizes of basic sets |
|---|---|---|
| $\mathcal{A}_1$ | 3 | $1, 13^2$ |
| $\mathcal{A}_2$ | 4 | 1, 6, 8, 12 |
| $\mathcal{A}_3$ | 4 | $1, 2, 12^2$ |
| $\mathcal{A}_4$ | 5 | $1, 4^2, 6, 12$ |
| $\mathcal{A}_5$ | 5 | $1, 2, 8^3$ |
| $\mathcal{A}_6$ | 6 | $1, 2, 6^4$ |
| $\mathcal{A}_7$ | 7 | $1, 2, 4^4, 8$ |
| $\mathcal{A}_8$ | 7 | $1, 2, 3^2, 6^3$ |
| $\mathcal{A}_9$ | 8 | $1, 2, 4^6$ |
| $\mathcal{A}_{10}$ | 9 | $1, 2^3, 4^5$ |
| $\mathcal{A}_{11}$ | 10 | $1, 2^5, 4^4$ |
| $\mathcal{A}_{12}$ | 10 | $1^3, 3^6, 6$ |
| $\mathcal{A}_{13}$ | 11 | $1^3, 3^8$ |
| $\mathcal{A}_{14}$ | 14 | $1, 2^{13}$ |

*Proof.* The statement of the lemma can be checked with the help of the GAP package COCO2P [11]. □

*Proof of the Proposition 4.1.* From [17, Theorem 1, Lemma 5.5] it follows that the group $C_p^k$ is separable for $p \in \{2,3\}$ and $k \leq 2$. Let $\mathcal{A}$ be an $S$-ring over $G \cong C_p^3$, where $p \in \{2,3\}$. Then one of the statements of Lemma 4.2 holds for $\mathcal{A}$. If Statement (1) of Lemma 4.2 holds for $\mathcal{A}$ then, obviously, $\mathcal{A}$ is separable. If Statement (2) of Lemma 4.2 holds for $\mathcal{A}$ then $\mathcal{A}$ is separable by Lemma 2.2. Suppose that Statement (3) of Lemma 4.2 holds for $\mathcal{A}$. Then $\mathcal{A}$ is the proper schurian $S$-wreath product for some $\mathcal{A}$-section $S = U/L$ with $|S| \leq 3$. Since $\mathcal{A}$ is schurian, $\mathcal{A}_U$ is also schurian. Note that $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated becasue $|S| \leq 3$. Therefore $\mathcal{A}$ is separable by Lemma 2.4.

Suppose that Statement (4) of Lemma 4.2 holds for $\mathcal{A}$ and $\varphi$ is an algebraic isomorphism from $\mathcal{A}$ to an $S$-ring $\mathcal{A}'$ over an abelian group $G'$. Clearly, if $\mathcal{A}'$ is separable then $\varphi^{-1}$ is induced by an isomorphism and hence $\varphi$ is also induced by an isomorphism. If $G' \cong C_{p^3}$ then $\mathcal{A}'$ is separable by Theorem 1.1; if $G' \cong C_p \times C_{p^2}$ then $\mathcal{A}'$ is separable by [17, Theorem 1]; if $G' \cong C_p^3$ and one of the Statements (1)–(3) of Lemma 4.2 holds for $\mathcal{A}'$ then $\mathcal{A}'$ is separable by the previous paragraph. So in the above cases $\varphi$ is induced by an isomorphism. Thus, we may assume that $G' \cong C_p^3$ and Statement (4) of Lemma 4.2 holds for $\mathcal{A}'$.

Two algebraically isomorphic $S$-rings have the same rank and sizes of basic sets. So information from Table 1 implies that $\mathcal{A}_i \ncong_{\mathrm{Alg}} \mathcal{A}_j$ whenever $i \neq j$. Therefore we may assume that

$$\mathcal{A} = \mathcal{A}' = \mathcal{A}_i$$

for some $i \in \{1,\ldots,14\}$. Using the package COCO2P again, one can find that

$$|\operatorname{Aut}_{\mathrm{Alg}}(\mathcal{A}_j)| = |\operatorname{Iso}(\mathcal{A}_j)|/|\operatorname{Aut}(\mathcal{A}_j)|$$

for every $j \in \{1,\ldots,14\}$. In view of (2.3) this yields that $\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_j) = \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_j)_0$ for every $j \in \{1,\ldots,14\}$. So $\varphi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_i)_0$ and hence $\varphi$ is induced by an isomorphism. Thus, $\mathcal{A}$ is separable and the proposition is proved. □

**Proposition 4.4.** *The group $C_p \times C_p \times C_{p^k}$ is non-separable for $p \in \{2,3\}$ and $k \geq 2$.*

*Proof.* In view of Lemma 2.5 to prove that the group $C_p \times C_p \times C_{p^k}$ is non-separable for $p \in \{2,3\}$ and $k \geq 2$ it is sufficient to construct an $S$-ring $\mathcal{A}$ over $C_p \times C_p \times C_{p^2}$, $p \in \{2,3\}$, and an algebraic isomorphism $\varphi$ from $\mathcal{A}$ to itself which is not induced by an isomorphism.

Let $G = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, where $|a| = |b| = p$ and $|c| = p^2$. Put $A = \langle a\rangle$, $B = \langle b\rangle$, $C = \langle c\rangle$, $c_1 = c^p$, and $C_1 = \langle c_1\rangle$. Firstly consider the case $p = 2$. Let $f \in \mathrm{Aut}(G)$ such that

$$f\colon (a,b,c) \to (a, bac_1, ca)$$

and $\mathcal{A} = \mathrm{Cyc}(\langle f\rangle, G)$. It easy to see that $|f| = 2$ and the basic sets of $\mathcal{A}$ are the following

$$T_0 = \{e\},\quad T_1 = \{a\},\quad T_2 = \{c_1\},\quad T_3 = \{ac_1\},$$
$$X_1 = cA,\quad X_2 = c^3A,$$
$$Y_1 = b\langle ac_1\rangle,\quad Y_2 = ba\langle ac_1\rangle,$$
$$Z_1 = bcC_1,\quad Z_2 = bcaC_1.$$

Define a permutation $\varphi$ on the set $\mathcal{S}(\mathcal{A})$ as follows:

$$T_0^{\varphi} = T_0,\quad T_1^{\varphi} = T_1,\quad T_2^{\varphi} = T_3,\quad T_3^{\varphi} = T_2,$$
$$X_1^{\varphi} = X_1,\quad X_2^{\varphi} = X_2,$$

$$Y_1^\varphi = Z_1, \quad Y_2^\varphi = Z_2, \quad Z_1^\varphi = Y_1, \quad Z_2^\varphi = Y_2.$$

It easy to see that $|\varphi| = 2$. The straightforward check implies that $\varphi$ is an algebraic isomorphism from $\mathcal{A}$ to itself. Let us check, for example, that $c_{Y_1,Y_2}^{T_2} = c_{Y_1^\varphi,Y_2^\varphi}^{T_2^\varphi}$. We have $\underline{Y_1 Y_2} = 2a + 2c_1$ and $\underline{Y_1}^\varphi \underline{Y_2}^\varphi = \underline{Z_1 Z_2} = 2a + 2ac_1$. So $c_{Y_1,Y_2}^{T_2} = c_{Y_1^\varphi,Y_2^\varphi}^{T_2^\varphi} = 2$.

Note that $\mathcal{A}$ corresponds to a Kleinian quasi-thin scheme of index 4 in the sense of [14]. The $S$-ring $\mathcal{A}$ is cyclotomic and hence it is schurian. Assume that $\varphi$ is induced by an isomorphism. Then the algebraic fusion $\mathcal{A}^{\langle\varphi\rangle}$ is schurian by Lemma 2.1. However, computer calculations made by using the package COCO2P [11] (see also [21]) imply that $\mathcal{A}^{\langle\varphi\rangle}$ is non-schurian, a contradiction. Therefore, $\varphi$ is not induced by an isomorphism and hence $\mathcal{A}$ is non-separable.

Now let $p = 3$. Let $f_1, f_2, f_3 \in \mathrm{Aut}(G)$ such that

$$f_1\colon (a,b,c) \to (a^{-1}, b^{-1}, c^{-1}),\ f_2\colon (a,b,c) \to (a,b,cc_1),\ f_3\colon (a,b,c) \to (a,ba,c).$$

The direct check implies that $|f_1| = 2$, $|f_2| = |f_3| = 3$, and $f_1, f_2, f_3$ pairwise commute. Put $K = \langle f_1\rangle \times \langle f_2\rangle \times \langle f_3\rangle$ and $\mathcal{A} = \mathrm{Cyc}(K, G)$. The basic sets of $\mathcal{A}$ are the following:

$$T_0 = \{e\},\ T_1 = \{a, a^{-1}\},\ T_2 = \{c_1, c_1^{-1}\},\ T_3 = \{ac_1, a^{-1}c_1^{-1}\},\ T_4 = \{a^{-1}c_1, ac_1^{-1}\},$$
$$X_1 = cC_1 \cup c^{-1}C_1, \quad X_2 = caC_1 \cup c^{-1}a^{-1}C_1, \quad X_3 = ca^{-1}C_1 \cup c^{-1}aC_1,$$
$$Y_1 = bA \cup b^{-1}A, \quad Y_2 = bc_1A \cup b^{-1}c_1^{-1}A, \quad Y_3 = b^{-1}c_1A \cup bc_1^{-1}A,$$
$$Z_1 = \{bc, b^{-1}c^{-1}\}(A \times C_1), \quad Z_2 = \{b^{-1}c, bc^{-1}\}(A \times C_1).$$

Let $\varphi$ be a permutation on the set $\mathcal{S}(\mathcal{A})$ such that $T_3^\varphi = T_4$, $T_4^\varphi = T_3$, and $X^\varphi = X$ for every $X \in \mathcal{S}(\mathcal{A}) \setminus \{T_3, T_4\}$. Clearly, $|\varphi| = 2$. Note that for every $X, Y \in \mathcal{S}(\mathcal{A}) \setminus \{T_3, T_4\}$ the elements $\underline{T_3}$ and $\underline{T_4}$ enter with non-zero coefficients the element $\underline{XY}$ only in the following cases: $X = Y = Z_i$; $X = X_i, Y = X_j,\ i \neq j$; $X = Y_i, Y = Y_j,\ i \neq j$. The straightforward check using this observation implies that $\varphi$ is an algebraic isomorphism from $\mathcal{A}$ to itself.

If $\varphi$ is induced by an isomorphism then $\mathcal{A}^{\langle\varphi\rangle}$ is schurian by Lemma 2.1. However, $\mathcal{A}^{\langle\varphi\rangle}$ coincides with the non-schurian $S$-ring constructed in [2, pp. 8–10] in case of $G \cong C_3 \times C_3 \times C_9$, a contradiction. Thus, $\varphi$ is not induced by an isomorphism and hence $\mathcal{A}$ is non-separable. The proposition is proved. □

Theorem 1.2 is an immediate consequence of Proposition 4.1 and Proposition 4.4.

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