More results on r-inflated graphs: Arboricity, thickness, chromatic number and fractional chromatic number

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Abstract

The r-inflation of a graph G is the lexicographic product G with K_r. A graph is said to have thickness t if its edges can be partitioned into t sets, each of which induces a planar graph, and t is smallest possible. In the setting of the r-inflation of planar graphs, we investigate the generalization of Ringel's famous Earth-Moon problem: *What is the largest chromatic number of any thickness-t graph?* In particular, we study classes of planar graphs for which we can determine both the thickness and chromatic number of their 2-inflations, and provide bounds on these parameters for their r-inflations. Moreover, in the same setting, we investigate arboricity and fractional chromatic number as well. We end with a list of open questions.

Keywords: r-inflation, thickness, chromatic number, fractional chromatic number, arboricity.

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1 Introduction

In 1890, P. J. Heawood published his famous article *Map-colour theorem* in the Quarterly Journal of Mathematics [25] that illustrated the flaw in A.B. Kempe's "proof" of the Four Colour Theorem [33]. Heawood's main intention was to investigate generalizations of the

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Four Colour Problem "of which strangely the rigorous proof is much easier [44]." The *Empire Problem* (or M-pire problem, so coined by H. Taylor [21]), was such a problem: one starts with a map G and allows a country to consist of at most M empires. Using Euler's formula, Heawood showed that the largest chromatic number of any M-pire map is at most $6M$ and was further able to prove that the bound was sharp for $M = 2$. In 1981 Herbert Taylor proved that Heawood's bound was sharp for $M = 3$ [21]. The M-pire problem was settled completely by Jackson and Ringel in 1983 [29] with a general construction for an M-pire map with chromatic number $6M$ for any $M \geq 2$. Ringel's interest in the M-pire problem began long before that time: in 1959 he posed a minor variation in one special case. Specifically, Ringel suggested in the case $M = 2$ that the countries should reside on one sphere, while their respective colonies should reside on another, and then wondered what the largest chromatic number of such a graph could be [38]. On the surface, this change might have appeared to make the problem easier to solve. But such was not to be case [28].

The largest chromatic number of a graph arising in Ringel's variation has come to be known [32] as the *Earth-Moon problem*, and such a graph as an *Earth-Moon graph*. The Earth-Moon problem is famous, if not infamous, amongst graph colorers because the state-of-the-art knowledge of the problem has remained remarkably static to this day. In 1959 the largest chromatic number of any Earth-Moon graph was known to lie in $\{8, 9, 10, 11, 12\}$; the 8 is the smallest in the list of possibilities because Ringel achieved an Earth-Moon decomposition of a map whose dual graph is K_8. The 12 is the largest in the list of possibilities because an Earth-Moon map is a 2-pire map and hence Heawood's upper bound applies. In 1973, Thom Sulanke showed that the map whose dual graph is $C_5 \vee K_6$, a 9-chromatic graph, was an Earth-Moon map [41]; his result became publicly known in 1980 when Martin Gardner published Sulanke's decomposition of $C_5 \vee K_6$ in the Scientific American games column [20]. As of 1973 through the writing of this article, the largest chromatic number of any Earth-Moon map is known to lie in $\{9, 10, 11, 12\}$.

Ringel's Earth-Moon problem generalizes to many moons; to this end, a graph G is said to have *thickness t*, written $\Theta(G) = t$, if its edges can partitioned into t sets, each of which induces a planar graph, and t is smallest possible. In that case Ringel's Earth-Moon problem can be rephrased as follows: *What is the largest chromatic number of any thickness-2 graph?* More generally: *What is the largest chromatic number of any thickness-t graph?* With the exception of $t = 1$, where the Four Color Theorem [7, 8, 6, 5, 39] applies, and $t = 2$ as discussed above, the largest chromatic number of any thickness-t graph lies in $\{6t − 2, 6t − 1, 6t\}$ [31].

What makes finding the largest chromatic number of a thickness-t graph difficult? Given an arbitrary graph G and a fixed positive integer $t > 1$, verifying that $\Theta(G) = t$ is an NP-complete problem [35]. Similarly, given an arbitrary graph G and fixed positive integer $k > 2$, verifying that $\chi(G) = k$ is also NP-complete [22, 23]. It appears, then, that the strategy of starting with a graph of known thickness and finding its chromatic number, or vice versa, will not often end in success. Thus it would be useful to have a collection of graph families for which both the thickness and chromatic number are understood.

One of the first families of graphs with known thickness and chromatic number, apart from (most) complete bipartite graphs [10, 9] and complete graphs [3, 43], is a subfamily of Catlin's graphs. Catlin's graphs are the lexicographic product of cycle C_n with the complete graph K_r. Thomassen describes these gaphs as the r-uniform replication of C_n [42]. Plummer, Stiebitz, and Toft use the terminology r-inflation of C_n [37], which is

what we use in this article. In [12] Catlin gave a formula for the chromatic number of the r-inflation of C_n. In [11], Boutin, Gethner, and Sulanke showed that for $n \geq 4$ the 3-inflation of C_n has thickness two.

In [24], the following challenge was offered: "Characterize those planar graphs G for which the thickness of $G[2]$ is precisely two," where $G[2]$ refers to the 2-inflation of G. Much of the work in this paper and [2] was inspired by the challenge and some of its generalizations; a large part of the appeal of the study of the chromatic number, thickness, and other graph parameters of r-inflated graphs is that, in some instances, exact values can be attained and upper bounds can be achieved. In [2] the current authors began the task of finding the thickness of r-inflations and also included some results on chromatic number. Two of the more significant reported results were the determination of the thickness of the r-inflation of the icosahedral graph and the thickness of r-inflated trees. The latter result can be used to obtain a bound on the thickness of $G[r]$ using the arboricity of G. The former achieved an upper bound for both thickness and chromatic number of edge maximal r-inflated planar graphs.

In this article, we develop additional tools for investigating r-inflated graphs. These tools are applied to determine exactly and give bounds for arboricity, thickness, chromatic number, and fractional chromatic number of the r-inflation of the icosahedral graph, dodecahedral graph, maximal planar graphs, trees, cycles, wheels, generalized series parallel graphs, and planar undirected circulant graphs. In Sections 3, 4, and 5 we develop tools and results about (respectively) arboricity, thickness, and fractional chromatic number of r-inflated graphs. In Section 6 we use these tools to determine exact values or bounds for these parameters in a variety of families of inflated planar graphs. A summary of results is then displayed in a table in Section 7. We close with open questions in Section 8.

2   Basics

Definition 2.1. Let G be a graph; the r-*inflation* of G is the lexicographic product $G[K_r]$, and is denoted by $G[r]$. Call $G[2]$ the *clone* of G.

Recall that the lexicographic product $G[H]$ replaces every vertex of G with a copy of H and places edges between all pairs of vertices in copies of H associated with adjacent pairs of vertices of G. That is, the vertices of $G[H]$ are $V(G) \times V(H)$ and there is an edge between (g_1, h_1) and (g_2, h_2) if and only if $g_1 = g_2$ and h_1 is adjacent to h_2 in H (the vertices are in the same copy of H) or g_1 is adjacent to g_2 in G (the vertices are in copies of H associated with adjacent vertices in G).

By definition, we obtain $G[r]$ by replacing each vertex of G by K_r and each edge of G by K_{2r} (which contains a K_r for each vertex of the edge). An r-inflation of G has the following properties, all of which are straightforward to verify.

Observations:

1. If the number of vertices and edges of G are V and E respectively, the number of vertices and edges of $G[r]$ are rV and $\binom{r}{2}V + r^2E$ respectively.

2. $G[sr] = (G[s])\,[r]$. In particular, for any complete graph K_s and any positive integer r, we have $K_s[r] = K_{sr}$.

3. Independence is invariant under inflation. That is, if the independence number of G is α then the independence number of $G[r]$ is α as well.

4. If the clique number of G is ω, then the clique number of $G[r]$ is $r\omega$.

5. If the chromatic number of G is χ then the chromatic number of $G[r]$ is at most $r\chi$.

6. Any edge of G (along with its incident vertices) induces a K_{2r} in $G[r]$.

7. If G is vertex transitive then so is $G[r]$. The same cannot be said of edge transitivity.

3   Arboricity and density

Throughout this paper we use the Nash-Williams formula [36] for arboricity of a graph. That is, $\mathrm{arb}(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$ over all subgraphs H of G. Note that over a given subset of vertices of G the maximum $\left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$ occurs on the induced subgraph on that subset. Thus it is sufficient to restrict our search to the induced subgraphs of G.

Definition 3.1. Given a graph G, call $\frac{|E(G)|}{|V(G)|-1}$ the *density* of G, denoted $\mathrm{dens}(G)$. The graph is said to be *uniformly dense* if $\frac{|E(G)|}{|V(G)|-1} \geq \frac{|E(H)|}{|V(H)|-1}$ for all induced subgraphs H of G.

Thus by Nash-Williams, if G is uniformly dense then $\mathrm{arb}(G) = \left\lceil \frac{|E(G)|}{|V(G)|-1} \right\rceil$. Furthermore, using the fact that any planar graph on n vertices has at most $3n - 6$ edges, it is an easy consequence of Nash-Williams that any planar graph has arboricity at most three.

Note that the definition of density (and therefore uniform density) given here agrees with the definition given in [13], but differs slightly from the one given in [27]. However, these definitions agree on connected graphs. The following results from these papers are useful in this work.

Theorem 3.2. [27] If G is a connected vertex transitive or edge transitive graph, then G is uniformly dense.

Theorem 3.3. [13] If G is a tree, a maximal outerplanar graph, or a maximal planar graph, then G is uniformly dense.

The following lemma and theorems are useful in finding the arboricity of r-inflated trees and planar graphs.

Lemma 3.4. Let G be a graph, $v \in V(G)$, and $H = G - \{v\}$. Then

$$\mathrm{dens}(G) < \mathrm{dens}(H) \iff \deg(v) < \mathrm{dens}(G).$$

Proof. Note that $|E(H)| = |E(G)| - \deg(v)$ and $|V(H)| = |V(G)| - 1$. Let $|E(G)| = E, |V(G)| = V, \deg(v) = d$.

$\mathrm{dens}(G) < \mathrm{dens}(H) \iff$
$\frac{E}{V-1} < \frac{E-d}{V-2} \iff$
$EV - 2E < EV - dV - E + d \iff$
$d(V - 1) < E \iff$
$d < \frac{E}{V-1} = \mathrm{dens}(G).$ $\square$

Theorem 3.5. A maximum density subgraph of $G[r]$, which is vertex maximal with that property, is the r-inflation of a subgraph of G.

Proof. The conclusion is immediate if $G[r]$ is uniformly dense. Suppose $G[r] = \mathcal{G}$ is not uniformly dense. Choose any vertex maximal subgraph $\mathcal{H}$ of $\mathcal{G}$ with maximal density. Note that since $\mathcal{H}$ has maximal density it is necessarily an induced subgraph of $\mathcal{G}$. We call two vertices v, u of $\mathcal{G}$ *siblings* if they are in the inflation of the same vertex of G. To prove the theorem it is sufficient to show that if $v \in V(\mathcal{G})$ is not in $V(\mathcal{H})$ then no sibling of v is in $V(\mathcal{H})$.

Let $V(\mathcal{G}) - V(\mathcal{H}) = \{v_1, \ldots, v_k\}$. Define $\mathcal{G}_0 = \mathcal{G}$ and for all $i = 1, \ldots, k$, $\mathcal{G}_i = \mathcal{G}_{i-1} - \{v_i\}$. Since $\mathcal{H}$ is an induced subgraph of $\mathcal{G}$ with the same vertex set as $\mathcal{G}_k$, $\mathcal{G}_k = \mathcal{H}$.

Suppose that there exists j so that v_j has a sibling v'_j that is in $V(\mathcal{H})$. Without loss of generality, we may assume that $j = k$ (or we could reorder the v_i). Notice that siblings have the same degree and the same set of neighbors in $\mathcal{G}$. For each v_i removed in getting to $\mathcal{H}$, v_i is either a neighbor of both v_k and v'_k or of neither. Thus the degrees of v_k, v'_k are the same in each of $\mathcal{G}, \ldots, \mathcal{G}_{k-1}$. In $\mathcal{G}_k$, v_k is removed but v'_k is not. Hence the degree of v_k in $\mathcal{G}_{k-1}$ is one more than the degree of v'_k in $\mathcal{G}_k$.

By the hypothesis on $\mathcal{H}$, $\mathrm{dens}(\mathcal{H}) = \mathrm{dens}(\mathcal{G}_k) > \mathrm{dens}(\mathcal{G}_{k-1})$ (otherwise we would have chosen $\mathcal{G}_{k-1}$ instead of $\mathcal{G}_k = \mathcal{H}$). By Lemma 3.4 this means that $\deg_{\mathcal{G}_{k-1}}(v_k) < \mathrm{dens}(\mathcal{G}_{k-1}) < \mathrm{dens}(\mathcal{G}_k)$. But since we have $\deg_{\mathcal{G}_k}(v'_k) < \deg_{\mathcal{G}_{k-1}}(v_k)$ this means that $\deg_{\mathcal{G}_k}(v'_k) < \mathrm{dens}(\mathcal{G}_k)$.

Thus by Lemma 3.4, $\mathrm{dens}(\mathcal{H} - \{v'_k\}) > \mathrm{dens}(\mathcal{H})$, which is impossible since we chose $\mathcal{H}$ to have maximal density.

Thus if $v \in V(\mathcal{G})$ is not in $V(\mathcal{H})$ then no sibling of v is in $V(\mathcal{H})$. In particular if $v \in V(\mathcal{H})$ then all siblings of v are in $V(\mathcal{H})$. Thus $\mathcal{H}$ is the r-inflation of the induced subgraph of G obtained by deleting from G the vertices corresponding to the K_rs deleted from $\mathcal{G}$ to yield $\mathcal{H}$. $\square$

Theorem 3.6. If a connected graph G has uniform density then so does $G[r]$.

Proof. Assume that G is connected and uniformly dense. Let $|E(G)| = E$ and $|V(G)| = V$. It is not hard to show that if we remove k vertices and ℓ edges from graph G to obtain H, then $\frac{E}{V-1} < \frac{E-\ell}{V-1-k} \iff \frac{\ell}{k} < \frac{E}{V-1}$. Thus since G is uniformly dense, for all ℓ edges and k vertices whose removal from G results in a subgraph, $\frac{E}{V-1} = \mathrm{dens}(G) \leq \frac{\ell}{k}$.

Let H be a subgraph of G and let us compare the density of $H[r]$ with $G[r]$. Note that if we remove k vertices and ℓ edges from G to obtain H, then we remove rk vertices and $\binom{r}{2}k + r^2\ell$ edges from $G[r]$ to obtain $H[r]$. Then $\mathrm{dens}(H[r]) \leq \mathrm{dens}(G[r])$ if and only if

$$\frac{\binom{r}{2}k + r^2\ell}{rk} \geq \frac{\binom{r}{2}V + r^2E}{rV - 1} \iff$$

$$\binom{r}{2}rkV + r^3\ell V - \binom{r}{2}k - r^2\ell \geq \binom{r}{2}rkV + r^3kE \iff$$

$$\ell(r^3V - r^2) \geq k\left(r^3E + \binom{r}{2}\right) \iff$$

$$\frac{\ell}{k} \geq \frac{r^3 E + \binom{r}{2}}{r^3 V - r^2}.$$

Let $A = \frac{r^3 E + \binom{r}{2}}{r^3 V - r^2}$ and let us compare A to $\text{dens}(G) = \frac{E}{V-1}$.

$$A \leq \text{dens}(G) \iff$$

$$\frac{r^3 E + \binom{r}{2}}{r^3 V - r^2} \leq \frac{E}{V - 1} \iff$$

$$r^3 EV + \binom{r}{2} V - r^3 E - \binom{r}{2} \leq r^3 VE - r^2 E \iff$$

$$(V - 1)\binom{r}{2} \leq E(r^3 - r^2) \iff$$

$$\frac{\binom{r}{2}}{r^3 - r^2} = \frac{1}{2r} \leq \frac{E}{V - 1},$$

which is exact since G is connected and therefore contains at least $V - 1$ edges. Thus $A \leq \text{dens}(G)$.

Since G is uniformly dense, if we remove k vertices and ℓ edges from G to obtain H then $A \leq \text{dens}(G) \leq \frac{\ell}{k}$ which implies that $\text{dens}(H[r]) \leq \text{dens}(G[r])$. Since by Theorem 3.5, one subgraph of $G[r]$ for which maximal density occurs is the r-inflation of a subgraph of G, no other subgraph of $G[r]$ has density larger than that of $G[r]$. Thus $G[r]$ is uniformly dense. $\qquad\square$

4   Thickness

There are two primary tools we use in evaluating the thickness of an inflated graph. The first is a theorem that gives us an upper bound on the thickness of an r-inflation.

Theorem 4.1. [2] If the arboricity of G is k then the thickness of $G[r]$ is at most $k \lceil \frac{r}{2} \rceil$.

To find a lower bound on $\Theta(G[r])$ we can take the ceiling of the number of edges of $G[r]$ divided by the maximum possible number of edges in a planar layer of $G[r]$. We call this the *edge counting bound*. In those instances where the two bounds meet, they provide the thickness of $G[r]$.

As mentioned in [2], it is easy to check that the clone of any cycle contains a K_5 subdivision and thus has thickness two. What about the thickness of other vertex transitive graphs?

Theorem 4.2. Let G be a vertex transitive graph of degree $d \geq 3$ on n vertices. Then

$$\left\lceil \frac{2d + 1}{6} \left(\frac{n}{n - 1} \right) \right\rceil \leq \Theta(G[2]) \leq \left\lceil \frac{d}{2} \left(\frac{n}{n - 1} \right) \right\rceil.$$

Proof. G has n vertices and $\frac{dn}{2}$ edges. Since G is vertex transitive, it is uniformly dense and therefore its arboricity is the ceiling of its density. Thus $\text{arb}(G) = \left\lceil \frac{\frac{dn}{2}}{n-1} \right\rceil = \left\lceil \frac{d}{2} \left(\frac{n}{n-1} \right) \right\rceil$. Thus by Theorem 4.1, the thickness of $G[2]$ is at most $\left\lceil \frac{d}{2} \left(\frac{n}{n-1} \right) \right\rceil$.

By Observation 1 compute that $G[2]$ has $2n$ vertices and $n + 2dn$ edges. Thus using the edge counting bound, a lower bound on the thickness of $G[2]$ is given by $\left\lceil \frac{n+2dn}{6n-6} \right\rceil = \left\lceil \frac{2d+1}{6} \left(\frac{n}{n-1} \right) \right\rceil$.

Note that except for small n, the value of $\frac{n}{n-1}$ is close enough to 1 to not make a difference in the value of the ceiling, except to bump up the value if $\frac{2d+1}{6}$ or $\frac{d}{2}$ is an integer. $\qquad\square$

The difference between these bounds is approximately $\left(\frac{d}{2} - \frac{2d+1}{6} \right) \left(\frac{n}{n-1} \right) = \frac{d-1}{2} \left(\frac{n}{n-1} \right)$ (found by ignoring the ceiling function and simply subtracting). Thus the difference between these bounds grows as the degree grows.

Corollary 4.3. The clone of a vertex transitive graph of degree 3 has thickness two.

5 Chromatic number

Definition 5.1. Given integers k, r and a graph G, a $\frac{k}{r}$-*coloring* of G is an assignment of subsets of size r from $\{1, \ldots, k\}$ so that subsets assigned to adjacent vertices are disjoint. The *fractional chromatic number* of G, denoted $\chi_f(G)$, is $\inf\{ \frac{k}{r} \mid$ there is a $\frac{k}{r}$-coloring of $G\}$.

Since a proper n-coloring is an $\frac{n}{1}$-coloring, it is immediate that $\chi_f(G) \leq \chi(G)$.

Theorem 5.2. [40] For any graph G, $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$ with equality guaranteed when G is vertex transitive.

Thus we have $\frac{r|V(G)|}{\alpha} \leq \chi_f G[r] \leq \chi(G[r]) \leq r\chi(G)$. Note that a $\frac{k}{r}$-coloring of G can be identified with k-coloring of $G[r]$. In particular, we can set up a correspondence between the list of colors assigned to a vertex v in a fractional coloring of G and the colors used in coloring the vertices of $v[r]$ in a proper coloring of $G[r]$. This correspondence shows that G is $\frac{k}{d}$-colorable if and only if the d-inflation of G is k-colorable. Thus an alternate definition for fractional chromatic number is: $\chi_f(G) = \inf\{ \frac{k}{d} \mid G[d]$ is k-colorable$\}$.

6 Results in chosen graph families

In this section we use the tools developed in previous sections to learn about the arboricity, thickness, chromatic number, and fractional chromatic number of the r-inflation of certain graphs or families of graphs.

Note that in many of the computations below we take a complicated-looking bound on a parameter for an r-inflation of a graph on n vertices and write it as a simpler bound that for each r is achieved for all but a finite number of small n (sometimes except for only trivially small n). In the summary table given in the following section we use these simpler bounds, followed by an asterisk *, to indicate that some small differences may occur for small graphs.

6.1 Icosahedral graph

Let I be the icosahedral graph. By Observation 1, since I has 12 vertices and 30 edges, $I[r]$ has $12r$ vertices and $12\binom{r}{2} + 30r^2 = 36r^2 - 6r$ edges.

Arboricity: By Observation 7, since I is vertex transitive so is $I[r]$. By Theorem 3.2 since $I[r]$ is connected and vertex transitive, it has uniform density and therefore $\operatorname{arb}(I[r]) = \left\lceil \frac{36r^2 - 6r}{12r - 1} \right\rceil = \left\lceil 3r - \frac{1}{4} - \frac{1}{48r - 4} \right\rceil = 3r.$

Thickness: In [2] we proved that the thickness of $I[r]$ is precisely r.

Chromatic Number: In [2] we proved that the chromatic number of $I[r]$ is $4r$. Since I is vertex transitive with independence number 3, so is $I[r]$. Thus by Theorem 5.2, $\chi_f(I[r]) = \frac{12r}{3} = 4r$ as well.

6.2 Dodecahedral graph

Let D be the dodecahedral graph. By Observation 1, since D has 20 vertices and 30 edges $D[r]$ has $20r$ vertices and $20\binom{r}{2} + 30r^2 = 40r^2 - 10r$ edges.

Arboricity: By Observation 7, since D is vertex transitive so is $D[r]$. By Theorem 3.2, since $D[r]$ is connected and vertex transitive, it follows that $\operatorname{arb}(D[r]) = \left\lceil \frac{40r^2 - 10r}{20r - 1} \right\rceil = \left\lceil 2r - \frac{2}{5} - \frac{2}{100r - 5} \right\rceil = 2r.$

Thickness: In [24] it was shown that the clone of the dodecahedral graph has thickness two. By Theorem 4.1, since D has arboricity 2 (an easy decomposition into a Hamiltonian path and its complement), the thickness of $D[r]$ is at most $2\left\lceil \frac{r}{2} \right\rceil$. On the other hand, the edge counting bound tells us that the thickness of $D[r]$ is at least $\left\lceil \frac{2}{3}r - \frac{1}{10} - \frac{1}{100r - 10} \right\rceil = \left\lceil \frac{2r}{3} \right\rceil$. Thus $\left\lceil \frac{2r}{3} \right\rceil \leq \Theta(D[r]) \leq 2\left\lceil \frac{r}{2} \right\rceil$.

Chromatic Number: By Theorem 5.2, since $D[r]$ is vertex transitive $\chi_f(D[r]) = \frac{20r}{8} = \frac{5}{2}r$ and thus $\frac{5r}{2}$ is a lower bound for $\chi(D[r])$. In particular, $5 = \chi_f(D[2]) \leq \chi(D[2])$. However, $\chi_f(D) = \frac{5}{2}$ (again by Theorem 5.2). By our observation in Section 5, a $\frac{5}{2}$-fractional coloring of D corresponds to a proper 5-coloring of $D[2]$. Thus 5 is also an upper bound on the chromatic number and hence $\chi(D[2]) = 5$. Suppose that r is even. By Observation 2 if we let $H = D[2]$ then $H[\frac{r}{2}] = D[r]$. By Observation 5, $\chi(D[r]) = \chi(H[\frac{r}{2}]) \leq \frac{r}{2}\chi(H) = \frac{r}{2}\chi(D[2]) = \frac{5r}{2}$. Thus when r is even, $\frac{5r}{2}$ also provides an upper bound and therefore $\chi(D[r]) = \frac{5r}{2}$.

6.3 Trees

By Observation 1, since a tree T on n vertices has $n - 1$ edges, $T[r]$ has rn vertices and $\binom{r}{2}n + r^2(n-1) = \frac{3}{2}r^2n - \frac{1}{2}rn - r^2$ edges.

Arboricity: By Theorem 3.3, trees are uniformly dense. By Theorem 3.6, since a tree T is connected, $T[r]$ is uniformly dense. Thus $\operatorname{arb}(T[r]) = \left\lceil \frac{\frac{3}{2}r^2n - \frac{1}{2}rn - r^2}{rn - 1} \right\rceil = \left\lceil \frac{3}{2}r - \frac{1}{2} - \frac{2r^2 - 3r + 1}{2rn - 2} \right\rceil$. For a given r, for all but finitely many small n, this quantity is equal to $\left\lceil \frac{3r - 1}{2} \right\rceil$.

Thickness: We showed in [2] that the thickness of $T[r]$ is at most $\left\lceil \frac{r}{2} \right\rceil$ and for all but finitely many trees, equality holds. For completeness, we show that computation here. The edge counting bound for $\Theta(T[r])$ yields $\Theta(T[r]) \geq \left\lceil \frac{1}{2}r - \frac{1}{6} - \frac{r^2-3r+1}{3rn-6} \right\rceil$. For a given value of r, for all but a finite number of small n, this is $\left\lceil \frac{r}{2} \right\rceil$. Thus for each r we have $\Theta(T[r]) = \left\lceil \frac{r}{2} \right\rceil$ for all but a finite number of small n.

Chromatic Number: By Observation 4, since $\omega(T) = 2, \omega(T[r]) = 2r$. This implies that $\chi(T[r]) \geq 2r$. However, by Observation 5, since $\chi(T) = 2$, $\chi(T[r]) \leq 2r$ and thus $\chi(T[r]) = 2r$.

6.4 Maximal planar graphs

Let G be a maximal planar graph on n vertices. By Observation 1, since a maximal planar graph on n vertices has $3n - 6$ edges, $G[r]$ has rn vertices and $\binom{r}{2}n + r^2(3n - 6) = \frac{7}{2}r^2n - \frac{1}{2}rn - 6r^2$ edges.

Arboricity: By Theorem 3.3, a maximal planar graph G is uniformly dense. By Theorem 3.6 since a maximal planar graph is connected, this tells us that $G[r]$ is uniformly dense. Thus $\mathrm{arb}(G[r]) = \left\lceil \frac{\frac{7}{2}r^2n - \frac{1}{2}rn - 6r^2}{rn-1} \right\rceil = \left\lceil \frac{7}{2}r - \frac{1}{2} - \frac{6r^2 - \frac{7}{2}r + \frac{1}{2}}{rn-1} \right\rceil$. For each r, for all but a finite number of small n, this quantity is equal to $\left\lceil \frac{7r-1}{2} \right\rceil$.

Thickness: By Theorem 4.1, since the arboricity of G is at most 3, $\Theta(G[r]) \leq 3 \left\lceil \frac{r}{2} \right\rceil$. On the other hand, using the edge counting bound we get

$$\Theta(G[r]) \geq \left\lceil \frac{7}{6}r - \frac{1}{6} - \frac{6r^2 - 7r + 1}{3rn - 6} \right\rceil.$$

For each r, for all but a finite number of small n, this is $\left\lceil \frac{7r-1}{6} \right\rceil$. Thus, for each r, for all but a finite number of small n, $\left\lceil \frac{7r-1}{6} \right\rceil \leq \Theta(G[r]) \leq 3 \left\lceil \frac{r}{2} \right\rceil$.

Chromatic Number: By the Four Color Theorem, $\chi(G) \leq 4$. By Observation 5, $\chi(G[r]) \leq 4r$.

6.5 Cycles

By Observation 1, since C_n has n vertices and n edges, $C_n[r]$ has rn vertices and $\binom{r}{2}n + r^2n = \frac{3}{2}r^2n - \frac{1}{2}rn$ edges.

Arboricity: By Observation 7, since C_n is vertex transitive, so is $C_n[r]$. By Theorem 3.2, since $C_n[r]$ is connected and vertex transitive, it is uniformly dense. Thus $\mathrm{arb}(C_n[r]) = \left\lceil \frac{\frac{3}{2}r^2n - \frac{1}{2}rn}{rn-1} \right\rceil = \left\lceil \left(\frac{3}{2}r - \frac{1}{2}\right)\left(\frac{rn}{rn-1}\right) \right\rceil$. By studying the cases when r is even and when r is odd, we can see that, for all but trivially small n, this is $\left\lfloor \frac{3r+1}{2} \right\rfloor$.

Thickness: A cycle C_n is the union of a path on n vertices, P_n, and an edge e. Since P_n is a tree, by the results in Section 6.3 (and by Proposition 4 in [2]), we have $\Theta(P_n[r]) = \left\lceil \frac{r}{2} \right\rceil$ for all but finitely many small values of n. The edge e, when r-inflated, gives rise to a $K_{r,r}$ which by [10], has thickness $\left\lfloor \frac{r+5}{4} \right\rfloor$. Since $C_n[r]$ is the disjoint union of $P_n[r]$ and $K_{r,r}$ we have $\Theta(C_n[r]) \leq \left\lceil \frac{r}{2} \right\rceil + \left\lfloor \frac{r+5}{4} \right\rfloor$. When r is even, this quantity reduces to $\left\lfloor \frac{3r+5}{4} \right\rfloor$; when r is odd, it reduces to $\left\lfloor \frac{3r+7}{4} \right\rfloor$.

On the other hand, the edge counting bound yields $\Theta(C_n[r]) \geq \left\lceil \frac{\frac{3}{2}r^2n - \frac{1}{2}rn}{3rn-6} \right\rceil =$ $\left\lceil \frac{1}{2}r - \frac{1}{6} + \frac{3r-1}{3rn-6} \right\rceil$. For all but a few small values of n this is equal to $\left\lceil \frac{r}{2} \right\rceil$. Thus $\frac{r}{2} \leq \Theta(C_n[r]) \leq \lfloor \frac{3r+5}{4} \rfloor$ when r is even and $\left\lceil \frac{r}{2} \right\rceil \leq \Theta(C_n[r]) \leq \lfloor \frac{3r+7}{4} \rfloor$ when r is odd. As previously mentioned, for all n, $\Theta(C_n[2]) = 2$. Thus when $r = 2$, the upper bound is achieved.

Chromatic Number: By [12, 19], $\chi(C_{2k+1}[r]) = 2r + \left\lceil \frac{r}{k} \right\rceil$. Thus, for a given r, for all but a finite number of small k, $\chi(C_{2k+1}[r]) = 2r + 1$. By Theorem 5.2, since $C_n[r]$ is vertex transitive, $\chi_f(C_{2k+1}[r]) = \frac{|V(C_{2k+1}[r])|}{\alpha(C_{2k+1}[r])} = \frac{r(2k+1)}{k} = 2r + \frac{r}{k}$. Analogous reasoning shows that $\chi(C_{2k}[r]) = 2r$ and $\chi_f(C_{2k}[r]) = 2r$.

6.6 Wheels

The n-wheel W_n is the join of C_n and a single vertex. By Observation 1, since W_n has $n+1$ vertices and $2n$ edges $W_n[r]$ has $rn + r$ vertices and $\binom{r}{2}(n+1) + 2r^2n = \frac{5}{2}r^2n - \frac{1}{2}rn + \frac{1}{2}r^2 - \frac{1}{2}r$ edges.

Arboricity: By Theorem 3.6, since W_n is connected and uniformly dense so is $W_n[r]$. Thus $\mathrm{arb}(W_n[r]) = \left\lceil \frac{\frac{5}{2}r^2n - \frac{1}{2}rn + \frac{1}{2}r^2 - \frac{1}{2}r}{rn+r-1} \right\rceil = \left\lceil \frac{5}{2}r - \frac{1}{2} - \frac{2r^2 - \frac{5}{2}r + \frac{1}{2}}{rn+r-1} \right\rceil$. For a given r, for all but finitely many small n, this is $\left\lceil \frac{5r-1}{2} \right\rceil$.

Thickness: By Theorem 4.1, since W_n has arboricity two $\Theta(W_n[r]) \leq 2\left\lceil \frac{r}{2} \right\rceil$. On the other hand, the edge counting bound yields $\Theta(W_n[r]) \geq \left\lceil \frac{5}{6}r - \frac{1}{6} - \frac{2r^2 - 5r + 1}{3rn+3r-6} \right\rceil$. For a given r for all but a finitely many small n this is equal to $\left\lceil \frac{5r-1}{6} \right\rceil$. Thus for a given r, for all but finitely many small n, $\left\lceil \frac{5r-1}{6} \right\rceil \leq \Theta(W_n[r]) \leq 2\left\lceil \frac{r}{2} \right\rceil$.

Chromatic Number: Since W_n is the join of C_n and K_1, $W_n[r]$ is the join of $C_n[r]$ and $K_1[r] = K_r$. Thus $\chi(W_n[r]) = \chi(C_n[r]) + \chi(K_r)$. That is,

$$\chi(W_n[r]) = \left\{ \begin{array}{ll} 3r + \left\lceil \frac{r}{k} \right\rceil & \text{if } n = 2k+1 \\ 3r & \text{if } n = 2k \end{array} \right. .$$

By Theorem 5.2, $\chi_f(W_{2k+1}[r]) \geq 2r + \frac{2r}{k}$ and $\chi_f(W_{2k}[r]) \geq 2r + \frac{r}{k}$.

6.7 Generalized series parallel graphs

In [2] it was shown that the arboricity of an outerplanar graph G is at most two, and hence the thickness of its clone, $G[2]$, is at most two as well. Toward the goal of understanding which planar graphs G satisfy $\Theta(G[2]) = 2$, we move one step away from outerplanar graphs to the *generalized series parallel graphs* or *GSP graphs*. Series parallel graphs (or *SP graphs*) have their roots in electrical networks of the same name; see, for example, [15]. The advantage of studying *GSP* graphs over *SP* graphs is that the former class is broader and contains the outerplanar graphs [34].

Definition 6.1. A graph G is a *GSP graph* if it can be reduced to K_2 through any sequence of the operations below. A graph G is an *SP graph* if it can be reduced to K_2 using only the first two operations.

1. Remove a parallel edge.

2. Replace an edge subdivided by a vertex with a single edge (i.e., suppress a vertex of degree 2)

3. Delete a vertex of degree 1.

The 2-connected SP graphs are precisely the 2-connected graphs with no K_4 minor [14]. The SP graphs are precisely those graphs with treewidth at most two [17, 16]. Some consequences are as follows. Let G be a simple GSP graph on n vertices. Then G is connected and planar with at most $2n - 3$ edges. By definition, G must have a vertex of degree 2 or less. The latter implies that the maximum chromatic number of any GSP graph is 3. Finally, $K_{2,n}$ for all $n \geq 3$ are GSP graphs, none of which are outerplanar. Thus GSP graphs may be viewed as a class of planar graphs that lie somewhere between outerplanar and arbitrary planar graphs. We show that the clone of any simple GSP graph has thickness at most two, and in general, that the r-inflation of any simple GSP graph has thickness at most $2 \left\lceil \frac{r}{2} \right\rceil$. We do so by first computing the arboricity of any GSP graph.

Arboricity:

Lemma 6.2. Let G be a simple GSP graph. Then $\mathrm{arb}(G) \leq 2$.

Proof. Assume by induction that all simple *GSP* graphs on k or fewer vertices have arboricity at most two and suppose G is a simple *GSP* graph on $k + 1$ vertices. Without loss of generality, assume G is not a tree; thus it has arboricity at least two. It remains to show that G has arboricity exactly two. One can show directly from the definition that every *GSP* graph has at least one vertex of degree two or less. Choose such a vertex v. If v has degree 2, then replace the edge that v subdivides with a single edge e (operation 2 in Definition 6.1) and if e is a parallel edge, remove it also (operation 1 in Definition 6.1). Call the new graph G'. By construction, G' is a simple *GSP* graph on k vertices, and hence by induction has arboricity at most two. If $\mathrm{arb}(G') = 2$, it is straightforward to return v and its two incident edges to the forest decomposition while maintaining the arboricity of G' by adding one pendant edge to each of the two forests. On the other hand, if $\mathrm{arb}(G') = 1$, add v and one pendant edge to the forest G', and consider v and the remaining pendant edge as a second forest. Thus in either case $\mathrm{arb}(G) = \mathrm{arb}(G') = 2$. Finally, if G has no vertex of degree two, it must have some vertex of degree 1; let v be such a vertex. Deleting v from G yields a simple *GSP* graph G' (operation 3 in Definition 6.1) that, by induction, has arboricity exactly 2 (since we have assumed that G is not a tree). As before, return the pendant edge with v as an endpoint to either forest in the arboricity two decomposition of G'; thus $\mathrm{arb}(G') = \mathrm{arb}(G) = 2$. By induction, then, simple *GSP* graphs have arboricity at most two. $\square$

Because any GSP graph on n vertices has at most $2n - 3$ edges, and $\left\lceil \frac{2n-3}{n-1} \right\rceil = \left\lceil 2 - \frac{1}{n-1} \right\rceil = 2$, Nash-Williams [36] together with Lemma 6.2 imply that any GSP graph G is uniformly dense. In that case, by Theorem 3.6, $G[r]$ is uniformly dense as well. Since the number of edges in $G[r]$ is bounded above by $n\binom{r}{2} + r^2(2n - 3)$ (Observation 1) we have that the arboricity of $G[r]$ is bounded above by $\left\lceil \frac{5r^2n - rn - 6r^2}{2rn - 2} \right\rceil$. For a fixed r, and for all but finitely many n, this quantity is exactly $\left\lceil \frac{5r-1}{2} \right\rceil$.

Thickness: Using Lemma 6.2 we bound the thickness of the clone and of the r-inflation of a GSP graph.

Corollary 6.3. Let G be a simple *GSP* graph. Then the thickness of $G[2]$ is at most two.

Proof. The result follows directly from Theorem 4.1 that the thickness of the clone of a graph is at most its arboricity.   □

Corollary 6.4. Let G be a simple *GSP* graph. Then the thickness of $G[r]$ is at most $2\left\lceil\frac{r}{2}\right\rceil$.

Proof. The result follows directly from Theorem 4.1 and Lemma 6.2.   □

Chromatic Number: As previously mentioned, a *GSP* graph G always has a vertex of degree two or less and thus by induction, $\chi(G) \leq 3$. It then follows from Observation 5 that $\chi(G[r]) \leq 3r$.

6.8   Planar undirected circulant graphs

Finally we turn our attention to the clones and r-inflations of planar undirected circulant graphs. The motivation for considering these particular graphs is that a large class of them have arboricity three but their clones have thickness two. We begin with the definition of circulant graph and use the notation and some of the results in [26].

Definition 6.5. Let $\{a_1, a_2, \ldots, a_m\}$ a set of m positive integers, with $m \geq 1$. Suppose G is a graph on n vertices $V(G) = \{0, 1, \ldots, n-1\}$, with edges given by $E(G) = \{(i, i + a_j) : 0 \leq i \leq n-1, \ 1 \leq j \leq m\}$. Then G is called a *circulant graph* and is denoted by $C_n(a_1, a_2, \ldots, a_m)$.

The study of circulant graphs, those graphs whose adjacency matrices are circulants, began in the late 1960s: see, for example [1, 18, 4]. In 2003, C. Heuberger completely characterized planar undirected circulant graphs; we use this characterization to prove that the clone of any planar circulant graph has thickness two; in addition, we provide upper and lower bounds on the the thickness of the r-inflation of these graphs. Note that when $m = 1$ it is easy to show that $C_n(a_1)$ is either a cycle or else a disjoint union of cycles; either way, when $G = C_n(a)$ for some positive integer a, the clone of G has thickness two. Thus the interesting cases occur when $m \geq 2$. Before stating the results in [26], we mention a few preliminary well known facts about circulant graphs.

Remarks:

Let $G = C_n(a_1, a_2, \ldots, a_m)$ be a circulant graph.

1. Then G is connected if and only if $gcd(a_1, a_2, \ldots, a_m) = 1$.
2. If G is disconnected, then its components are mutually isomorphic.
3. G is regular of degree δ, where

$$\delta = \begin{cases} 2m, & \text{if } a_j \not\equiv \frac{n}{2} \pmod{n} \text{ for all } 1 \leq j \leq m \\ 2m - 1, & \text{otherwise} \end{cases} \tag{6.1}$$

4. G is vertex transitive.

One more definition is needed before stating the required result from [26].

Definition 6.6. Let p be prime and suppose $k \in \mathbb{Z}$. Then the *p-adic valuation of k*, written $v_p(k)$, is the largest integer ℓ such that $p^\ell | k$.

Theorem 6.7. [26] Let $G = C_n(a_1, a_2, \ldots, a_m)$ be a circulant graph.

1. If $m \geq 3$ then G is non-planar.

2. Suppose $m = 2$ and consider $G = C_n(a_1, a_2)$. Then G is planar if and only if one of the following two conditions is satisfied:

   (a) $a_i \equiv \pm 2a_j \pmod{n}$ and $v_2(a_j) < v_2(n)$, where $(i, j) = (1, 2)$ or $(2, 1)$
   (b) $a_i = \frac{n}{2}$ and $1 \leq v_2(n) \leq v_2(a_j)$, where $(i, j) = (1, 2)$ or $(2, 1)$

We now have most of the tools needed to do an analysis of planar undirected circulant graphs on an even number of vertices.

Arboricity: Let $G = C_{2n}(1, 2)$, $n \geq 2$; then G is 4-regular with $4n$ edges. Since G is vertex transitive, it is uniformly dense and hence $\mathrm{arb}(G) = \left\lceil \frac{4n}{2n-1} \right\rceil = 3$. By Observation 7, $G[r]$ is vertex transitive and by Observation 1 has $2nr$ vertices and $nr^2 - nr + 4r^2 n$ edges. Hence for a fixed value of r, we have $\mathrm{arb}(G[r]) = \left\lceil \frac{5r^2 n - nr}{2nr-1} \right\rceil = \left\lceil \frac{5r-1}{2} \right\rceil$ for all but finitely many values of n.

Thickness: Like the icosahedral graph whose arboricity is three and whose clone has thickness two, we find that the even planar 4-regular circulant graphs provide an infinite family of graphs whose arboricity does not give the best bound for the thickness of their clones.

Theorem 6.8. Let $G = C_{2n}(a_1, a_2)$ be a planar undirected circulant graph on an even number of vertices. Then the thickness of $G[2]$ is two.

Proof. As mentioned earlier, when $m = 1$, for all $a, n \in \mathbb{Z}^+$, the clone of $C_n(a)$ has thickness two. As shown in [26], when $m \geq 3$, the undirected circulant $C_n(a_1, a_2, \ldots, a_m)$ is not planar. Furthermore, by Remarks 1 and 2, since a disconnected circulant graph has mutually isomorphic components, it suffices to consider only planar circulants $C_{2n}(a_1, a_2)$ with $\gcd(a_1, a_2) = 1$. In case (b), by Remarks 3 and 4, since $G = C_{2n}(a_1, a_2)$ is vertex transitive of degree three, Corollary 4.3 implies that $G[2]$ has thickness two regardless of whether or not the graph $C_{2n}(\frac{n}{2}, a_2)$ is planar.

Thus it remains to show that a planar circulant graph G satisfying condition (a) of Theorem 6.7 (as proved in [26]), when cloned, has thickness two. To this end, as noted in [26], we use the following straightforward observation: If $G = C_{2n}(a_1, a_2)$ with $\gcd(a_1, 2n) = 1$, then G is isomorphic to $C_{2n}(1, a_1^{-1} a_2 \pmod{2n})$, where the roles of a_1 and a_2 can be interchanged.

In particular, observe that since $\gcd(a_1, a_2, 2n) = 1$ and $a_1 \equiv \pm 2a_2 \pmod{2n}$, we have $\gcd(\pm 2a_2 + 2kn, a_2, 2n) = 1$ for some $k \in \mathbb{Z}$. Thus $\gcd(a_2, 2n) = 1$, in which case $C_{2n}(\pm 2a_2 \pmod{2n}, a_2)$ is isomorphic to $C_{2n}(1, a_2^{-1}(\pm 2a_2 \pmod{2n})) = C_{2n}(1, \pm 2 \pmod{2n})$. Notice that the 2-adic valuation criteria in Theorem 6.7(a) is trivially satisfied since $2n$ is even and a_2 is odd. The planar circulant graph families that we must consider are $C_{2n}(1, 2)$ and $C_{2n}(1, 2n-2)$, which are identical for each fixed value of n. Thus the problem is reduced to showing that for any $n \in \mathbb{Z}^+$ the clone of the planar circulant graph

$C_{2n}(1, 2)$ has thickness two. In this case, since $C_{2n}(1, 2)$ has $4n$ edges, the arboricity is larger than two, and hence we must use an alternative method to show that its clone has thickness two.

To complete the last part of the proof, note that a plane drawing of $C_{2n}(1, 2)$ can be accomplished by embedding two plane drawings of C_n, one inside the bounded region of the other, and then drawing in a Hamiltonian cycle by alternating vertices from the two copies of C_n. We alter the standard vertex labeling of $C_{2n}(1, 2)$ to better suit the given plane embedding. Specifically, if the vertices of the inner C_n are labeled, in order, $0, 1, \ldots, n-1$ and the vertices of the outer C_n are labeled, in order, $n, n+1, \ldots, 2n-1$, then we use the Hamiltonian cycle given by $0, n, 1, n+1, 2, n+2, \ldots, i, n+i, \ldots, n-1, 2n-1, 0$. Figure 1 shows such a plane drawing of $C_{10}(1, 2)$, where the bold edges display a perfect matching of $C_{10}(1, 2)$. In general, the edges

$$\{m_i = (i, i+n) : 0 \leq i \leq n-1\} \tag{6.2}$$

yield a perfect matching of $C_{2n}(1, 2)$, the fact of which is needed in the remainder of the proof. The vertex labeling scheme for cloning a graph G on $2n$ vertices is as follows:
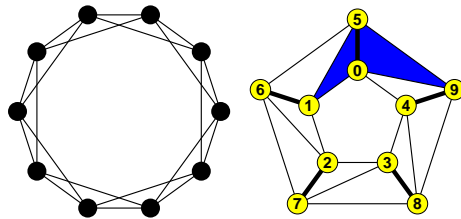

Figure 1: The graph $G = C_{10}(1, 2)$ and a plane embedding; the vertices on the right are labeled to facilitate the proof of Theorem 6.8. The bold edges constitute a perfect matching of G. The highlighted region illustrates the unique quadrilateral associated with matching edge $(0, 5)$.

a vertex labeled i in G is replaced by a pair of vertices i and $i + 2n$ in $G[2]$ for $i = 0, \ldots, 2n-1$. Specifically, vertices i and $i + 2n$ induce a K_2 in $G[2]$ and are called *siblings* in keeping with the terminology used in Theorem 3.5. Moreover, by Observation 6, an edge of G induces a K_4 in $G[2]$; a K_4 is an edge-disjoint union of two paths of length three. Let us focus on the perfect matching edges m_i of $C_{2n}(1, 2)$ as given in (6.2). First, each matching edge $m_i = (i, i+n)$ in G uniquely determines a quadrilateral, namely, the one given by $i, i+1 \pmod{n}, i+n$, and $i-1+n$ for $i = 0, \ldots, n-1$. Each matching edge m_i in G induces a K_4 in $G[2]$ by way of the vertices $i, i+2n, i+n$, and $i+3n$. Decompose each such K_4 into two paths of length three: $P_1(i) = i, i+2n, i+3n, i+n$ and $P_2(i) = i+2n, i+n, i, i+3n$. The paths are edge disjoint, and $P_1(i) \cup P_2(i)$ induces a K_4. See Figure 2.

Now subdivide every m_i by two vertices; each such subdivided edge induces a path of length three. Note that subdividing an edge does not alter the genus of a graph. Next add edges from vertices $i+1$ and $i-1+n$ to the two subdivision vertices for $i = 0, \ldots, n-1$; call the new graph G'. By construction G' is planar. Make two copies of G' and call them $G_1[2]$ and $G_2[2]$. Label the vertices in $G_1[2]$ by maintaining the labeling of G and then labeling the subdivision vertices of matching edge m_i to yield the path $P_1(i)$ for
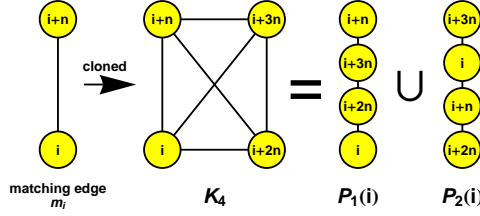
Figure 2: Decomposition of cloned matching edge m_i into paths $P_1(i)$ and $P_2(i)$.

$i = 0, \ldots, n - 1$. Label the vertices in $G_2[2]$ by replacing the vertices from the original labeling of G with their siblings. Then effect $P_2(i)$ on the subdivision vertices of each matching edge m_i. The placement of the subdivision labels in each of $G_1[2]$ and $G_2[2]$ is unambiguous since the labels of the endpoints of the subdivided edges are given initially. See Figure 3.
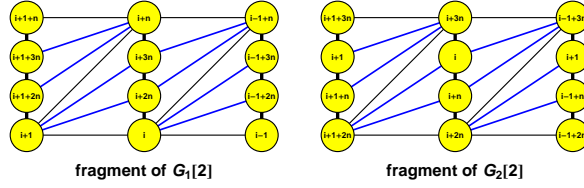

Figure 3: Fragments of the graphs $G_1[2]$ and $G_2[2]$.

It suffices to show that $G[2] \subset G_1[2] \cup G_2[2]$. Alternatively, if $N_H(x)$ denotes the neighborhood of vertex x in graph H, then it is enough to show that it holds $N_{G[2]}(i) \subset N_{G_1[2] \cup G_2[2]}(i)$. To this end, suppose i is an arbitrary vertex in G and let $0 \le i \le n - 1$. Recall that G is 4-regular and observe that $N_G(i) = \{i-1, i+1, i+n, i-1+n\}$. Therefore, $N_{G[2]}(i) = \{i-1, i+1, i+n, i-1+n, i+2n, i-1+2n, i+1+2n, i+3n, i-1+3n\}$. By construction, $N_{G_1[2]}(i) = \{i+1, i+2n, i-1, i-1+2n, i-1+3n, i-1+n\}$ and $N_{G_2[2]}(i) = \{i-1+3n, i+3n, i+n, i+1+2n\}$ and thus $N_{G[2]}(i) \subset N_{G_1[2] \cup G_2[2]}(i)$, as desired. Notice that edge $(i, i-1+3n)$ appears in both of $N_{G_1[2]}(i)$ and $N_{G_2[2]}(i)$, which means that $G_1[2] \cup G_2[2]$ is not simple, but this fact is not relevant to the discussion at hand: in a thickness-two decomposition of $G = C_{2n}(1, 2)$ one could simply remove the edges $\{(i, i-1+3n) : i = 0, \ldots, n-1\}$ from $G_1[2]$. Finally, the same method can be used to show that any vertex x in $G[2]$ satisfies $N_{G[2]}(x) \subset N_{G_1[2] \cup G_2[2]}(x)$ and this completes the proof.   □

Example. An edge decomposition of $G = C_{10}(1, 2)$ by way of $G_1[2]$ and $G_2[2]$ is given in Figure 4. Note that the multiple edges have been retained.

The same technique, together with the lemma below as detailed in [2] can be modified to prove the following theorem:

Theorem 6.9. The thickness of the r-inflation of $C_{2n}(1, 2)$ is at most r.
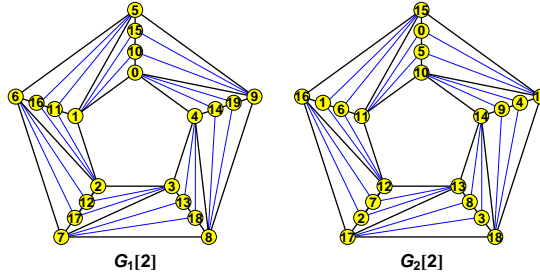
Figure 4: $G = C_{10}(1, 2)$ is contained in $G_1[2] \cup G_2[2]$.

We omit the lengthy proof here, but the key ideas are contained in the proof of Theorem 6.8 and Section 4 of [2]. In particular, any edge from a graph G becomes a K_{2r} in $G[r]$. Further K_{2r} is an edge disjoint union of r Hamiltonian paths of length $2r - 1$. For example, in the proof of Theorem 6.8, matching edges became paths of length three in each of the two planar layers, and the union of corresponding paths of length three induced a K_4. That idea, together with the lemma below, is at the core of the construction for the general case.

Lemma 6.10. Let K_{2r} be a complete graph on $2r$ vertices with vertices labeled $v_1, \ldots, v_{2r}$. The edges of K_{2r} can be partitioned into r Hamiltonian paths, $P_1, P_2, \ldots, P_r$, each with one endpoint in $\{v_1, v_2, \ldots, v_r\}$ and the other in $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$.

Finally, since the number of edges in $G[r]$ is $5r^2n - rn$ and the number of vertices is $2rn$, the edge counting bound tells us that for a given fixed r, we have $\Theta(G[r]) \geq \lceil \frac{5r^2n - rn}{6rn - 6} \rceil = \lceil \frac{5r-1}{6} \rceil$ for all but finitely many values of n.

Chromatic Number: The planar circulant graphs of degree four are the most interesting of the family of planar circulant graphs. It turns out that the chromatic number of the clone of any such graph can be determined up to a point. By construction, the independence number of $C_{2n}(1, 2)$ is roughly $\frac{1}{3}$ of the number of vertices in $C_{2n}(1, 2)$ and in particular $\alpha(C_{2n}(1, 2)) = \lfloor \frac{2n}{3} \rfloor$ When $3|n$, $\chi(C_{2n}(1, 2)) = 3$ in which case, $\chi(C_{2n}(1, 2)[2]) \leq 6$ by Observation 5. On the other hand, since the clique number of $C_{2n}(1, 2)$ is 3, by Observation 4, the clique number of $C_{2n}(1, 2)[2]$ is six, and hence $\chi(C_{2n}(1, 2)[2]) \geq 6$. Altogether $\chi(C_{2n}(1, 2)[2]) = 6$. When 3 does not divide n, $\chi(C_{2n}(1, 2)[2]) = 4$ [26], which means, by Observation 5, $\chi(C_{2n}(1, 2)[2]) \leq 8$. On the other hand, $\chi(C_{2n}(1, 2)[2]) \geq \frac{4n}{\lfloor \frac{2n}{3} \rfloor} > \frac{4n}{\frac{2n}{3}} = 6$ and thus $\chi(C_{2n}(1, 2)) \in \{7, 8\}$. Experimental evidence shows that $\chi = 8$ does occur, so the upper bound is achieved. Similarly, when $3|n$, we have $\chi(C_{2n}(1, 2)[r]) = 3r$ and otherwise $\chi(C_{2n}[r]) \in (3r, 4r]$. Finally, by Observation 3, since $C_{2n}(1, 2)$ has independence number $\alpha = \lfloor \frac{2n}{3} \rfloor$, so does $C_{2n}(1, 2)[r]$. Thus $\chi_f(C_{2n}(1, 2)[r]) = \frac{2nr}{\lfloor \frac{2n}{3} \rfloor}$.

7 Summary table

See Table 1 for a summary of the results found in Section 6. In this table, * indicates that for each r the value is exact for all but a finite number of small graphs.

	Arboricity	Thickness	χ	χ_f
Icos. Graph	3	1	4	4
Cloned	6	2	8	8
r-Inflated	3r	r	4r	4r
Dodec. Graph	2	1	3	$\frac{5}{2}$
cloned	4	2	5	5
r-Inflated	2r	$\left\lceil\frac{2r}{3}\right\rceil \leq \Theta \leq 2\left\lceil\frac{r}{2}\right\rceil$	$\frac{5r}{2}$ for even r	$\frac{5}{2}r$
Tree	1	1	2	
cloned	3	1	4	
r-inflated	$\left\lceil\frac{3r-1}{2}\right\rceil^*$	$\left\lceil\frac{r}{2}\right\rceil^*$	2r	
Maximal Planar	3^*	1	≤ 4	
cloned	7^*	3^*	≤ 8	
r-inflated	$\left\lceil\frac{7r-1}{2}\right\rceil^*$	$\left\lceil\frac{7r-1}{6}\right\rceil^* \leq \Theta \leq 3\left\lceil\frac{r}{2}\right\rceil$	$\leq 4r$	
Even Cycle C_{2k}	2	1	2	2
cloned	3^*	2	4	4
r-Inflated	$\left\lfloor\frac{3r+1}{2}\right\rfloor^*$	$\left\lceil\frac{r}{2}\right\rceil^* \leq \Theta \leq \left\lceil\frac{r}{2}\right\rceil + \left\lfloor\frac{r+5}{4}\right\rfloor$	2r	2r
Odd Cycle C_{2k+1}	2	1	3	$2+\frac{1}{k}$
cloned	3^*	2	5^*	$4+\frac{2}{k}$
r-Inflated	$\left\lfloor\frac{3r+1}{2}\right\rfloor^*$	$\left\lceil\frac{r}{2}\right\rceil^* \leq \Theta \leq \left\lceil\frac{r}{2}\right\rceil + \left\lfloor\frac{r+5}{4}\right\rfloor$	$2r+1^*$	$2r+\frac{r}{k}$
Even Wheel W_{2k}	2	1	3	$\geq 2+\frac{1}{k}$
cloned	5^*	2	6	$\geq 4+\frac{2}{k}$
r-Inflated	$\left\lceil\frac{5r-1}{2}\right\rceil^*$	$\left\lceil\frac{5r-1}{6}\right\rceil^* \leq \Theta \leq 2\left\lceil\frac{r}{2}\right\rceil$	3r	$\geq 2r+\frac{r}{k}$
Odd Wheel W_{2k+1}	2	1	4	$\geq 2+\frac{2}{k}$
cloned	5^*	2	7^*	$\geq 4+\frac{4}{k}$
r-Inflated	$\left\lceil\frac{5r-1}{2}\right\rceil^*$	$\left\lceil\frac{5r-1}{6}\right\rceil^* \leq \Theta \leq 2\left\lceil\frac{r}{2}\right\rceil$	$3r+1^*$	$\geq 2r+\frac{2r}{k}$
Gen. Series Parallel	2	1	≤ 3	
cloned	$\leq 5^*$	2	≤ 6	
r-Inflated	$\leq \left\lceil\frac{5r-1}{2}\right\rceil^*$	$\leq 2\left\lceil\frac{r}{2}\right\rceil$	$\leq 3r$	
Circulant $C_{2n}(1,2)$	3	1	≤ 4	$\frac{2n}{\left\lfloor\frac{2n}{3}\right\rfloor}$
cloned	5^*	2	≤ 8	$\frac{4n}{\left\lfloor\frac{2n}{3}\right\rfloor}$
r-Inflated	$\left\lceil\frac{5r-1}{2}\right\rceil^*$	$\left\lceil\frac{5r-1}{6}\right\rceil^* \leq \Theta \leq r$	$\leq 4r$	$\frac{2nr}{\left\lfloor\frac{2n}{3}\right\rfloor}$

Table 1: Summary.

8   Open questions

1. The Petersen graph P has thickness two and arboricity two, and hence $P[2]$ has thickness two. Similarly, the degree three circulants satisfying $a_1 = \frac{n}{2}$ but for which the 2-adic valuation condition is not satisfied in Theorem 6.7 are thickness two and so are their clones. Characterize those thickness two graphs H for which the thickness of $H[2]$ is precisely two.

2. More generally, if G has thickness r, what can be said of $\Theta(G[r])$ for any $r \geq 2$?

3. Are cloned planar graphs that have thickness two always permuted layer graphs? In particular, is it always the case that a cloned planar graph can be decomposed into two layers, one of which is isomorphic to a subgraph of the other?

4. The technique used in Section 6.5 can be slightly modified to give an upper bound on the thickness of the r-inflation of any unicyclic graph G. That is, removing any edge from the cycle in G leaves a forest. The r-inflation of a forest has thickness at most $\left\lceil \frac{r}{2} \right\rceil$. The removed edge, once r-inflated, induces a $K_{r,r}$, which has thickness $\left\lfloor \frac{r+5}{4} \right\rfloor$. Thus $\Theta(G)$ is bounded above by $\left\lceil \frac{r}{2} \right\rceil + \left\lfloor \frac{r+5}{4} \right\rfloor$. Can this method or a modification be applied to more general classes of graphs?

5. Jackson and Ringel wrote about variations on Ringel's Earth-Moon problem in [30]. In the spirit of their article, given a graph G, instead of asking for a t-layer decomposition of G on t spheres, allow any set of t surfaces, both orientable and nonorientable. What can be reasonably asked and answered in this setting?

9   Acknowledgement

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