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A partial generalization of the Livingstone–Wagner Theorem

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Abstract

For a transitive permutation group G on a finite set Ω, the Livingstone–Wagner Theorem states that if G is k-homogeneous and $2 \le k \le \frac{|\Omega|}{2}$, then G is $(k-1)$-transitive. We conjecture that the number of G-orbits on k-subsets of Ω is greater than or equal to the number of G-orbits on ordered $(k-1)$-tuples of Ω, if $|\Omega|$ is sufficiently large. For the simplest case $k = 3$, we verify this conjecture by establishing a result on edge-colorings of complete digraphs.

Keywords: Groups, transitivity, orbits.

Math. Subj. Class.: 20B20, 20B35

1 Introduction

Let G be a permutation group on a finite set Ω. Denote by Ω/G the set of G-orbits on Ω. Let $\Omega_{(k)}$ and $\Omega_{[l]}$ denote the family of all k-subsets of Ω and the family of all l-tuples of distinct elements of Ω, respectively. The group G is said to be k-homogeneous or l-transitive if G acts transitively on $\Omega_{(k)}$ or $\Omega_{[l]}$ respectively.

Livingstone and Wagner [3, Theorem 1,2] showed that for any group G acting on Ω, and for any k with $2 \le k \le \frac{|\Omega|}{2}$,

(1) the inequality $|\Omega_{(k)}/G| \ge |\Omega_{(k-1)}/G|$ holds. In particular, k-homogeneity implies also $(k-1)$-homogeneity,

(2) if G is k-homogeneous, then G is $(k-1)$-transitive, and

(3) for $k \ge 5$, if G is k-homogeneous, then G is k-transitive.

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Martin and Sagan [4, Theorem 2] generalized (1) by introducing the concept of λ-transitivity as follows. Let S_λ be the set of all partitions of Ω of shape λ. If a permutation group G on Ω acts transitively on S_λ, then G is said to be λ-transitive. Let $\trianglelefteq$ denote the dominance order (see [5, Definition 2.2.2]) on the set of partitions of $|\Omega|$. Martin and Sagan proved that $\lambda \trianglelefteq \mu$ implies $|S_\lambda/G| \geq |S_\mu/G|$. In particular, λ-transitivity implies μ-transitivity. Since $S_{(|\Omega|-k,k)} \cong \Omega_{(k)}$ and $S_{(|\Omega|-k+1,k-1)} \cong \Omega_{(k-1)}$ as G-sets, the Livingstone–Wagner Theorem (1) follows from the Martin–Sagan Theorem. Analogously, (2) will follow if we could prove $|\Omega_{(k)}/G| \geq |\Omega_{[k-1]}/G|$. Since $|\Omega_{(k)}| \geq |\Omega_{[k-1]}|$ holds when $|\Omega| \geq k! + k - 1$, one may expect the inequality $|\Omega_{(k)}/G| \geq |\Omega_{[k-1]}/G|$ to be true for $|\Omega|$ sufficiently large. Since $S_{(|\Omega|-k+1,1^{k-1})} \cong \Omega_{[k-1]}$ as G-sets, and the partitions $(|\Omega| - k, k)$, $(|\Omega| - k + 1, 1^{k-1})$ are incomparable with respect to the dominance order, Theorem 2 of [4] does not apply. For the simplest case $k = 3$, the inequality follows from the following result of Cameron and Saxl [1], provided that $|\Omega_{[2]}/G| \geq 26$.

Proposition 1.1 ([1]). *Let G be a transitive permutation group on a set Ω with $|\Omega| > 2$. Then* $|\Omega_{(3)}/G| \geq \dfrac{|\Omega_{(2)}/G|(|\Omega_{(2)}/G| - 1)}{6}$.

Indeed, since $2|\Omega_{(2)}/G| \geq |\Omega_{[2]}/G|$, Proposition 1.1 implies

$$|\Omega_{(3)}/G| \geq \frac{|\Omega_{[2]}/G|(|\Omega_{[2]}/G| - 2)}{24} \geq |\Omega_{[2]}/G|$$

provided that $|\Omega_{[2]}/G| \geq 26$.

The purpose of this paper is to prove the inequality $|\Omega_{(3)}/G| \geq |\Omega_{[2]}/G|$ provided that $|\Omega| \geq 11$, by counting certain configurations in a regular edge-coloring of a complete digraph (the definition of which is given in the next section). Here we only note that every transitive permutation group G on Ω induces a regular edge-coloring $(\Omega, \mathcal{C}_G, \phi_G, \Psi_G)$ with $\mathcal{C}_G = \Omega_{[2]}/G$, and the number $|\overline{\Omega_3}|$ of equivalence classes is at most $|\Omega_{(3)}/G|$. We prove in Section 5 that if $(\Omega, \mathcal{C}, \phi, \Psi)$ is a regular edge-coloring with $|\Omega| \geq 11$, then $|\overline{\Omega_3}| \geq |\mathcal{C}|$. As a corollary to this result, we obtain the following.

Theorem 1.2. *For any transitive permutation group G on Ω with $|\Omega| \geq 11$, $|\Omega_{(3)}/G| \geq |\Omega_{[2]}/G|$ holds.*

For some permutation groups of degree less than 11, Theorem 1.2 fails to hold. In fact, $C_6, C_3 \rtimes S_2, C_3 \wr S_2$, of degree 6, C_7 of degree 7, $C_4 \wr S_2$ of degree 8, and $C_5 \wr S_2$ of degree 10 are all the counterexamples with degree greater than 5. We refer the reader to [2] for unexplained notation in permutation group theory.

2 Regular edge-colorings

Let $\Omega, \mathcal{C}$ be finite sets, and let $\phi : \Omega_{[2]} \to \mathcal{C}$ be a surjective mapping. We call $(\Omega, \mathcal{C}, \phi, \Psi)$ a regular edge-coloring if

(R1) For each $a \in \mathcal{C}$ and each $\alpha \in \Omega$, there exists a positive integer δ_a such that

$$|\{\beta \in \Omega \mid \phi(\alpha, \beta) = a\}| = \delta_a.$$

(R2) There is a bijective mapping $\Psi : \mathcal{C} \to \mathcal{C}$, which maps a color of an edge to that of its opposite:

$$\text{for all } (\alpha, \beta) \in \Omega_{[2]},\ (\Psi \circ \phi)(\alpha, \beta) = \phi(\beta, \alpha).$$

Let G be a transitive permutation group on Ω. We obtain a regular edge-coloring induced by G, denoted $(\Omega, \mathcal{C}_G, \phi_G, \Psi_G)$, as follows. Let $\mathcal{C}_G = \Omega_{[2]}/G$, and define $\phi_G : \Omega_{[2]} \to \mathcal{C}_G$ by $\phi_G(\alpha, \beta) = (\alpha, \beta)^G$ for $(\alpha, \beta) \in \Omega_{[2]}$, where $(\alpha, \beta)^G$ denotes the G-orbit of (α, β). Define Ψ_G by $\Psi_G(\phi_G(\alpha, \beta)) = \phi_G(\beta, \alpha)$. Then (R1) holds by transitivity of G, and clearly (R2) holds. Thus $(\Omega, \mathcal{C}_G, \phi_G, \Psi_G)$ is a regular edge-coloring.

For the remainder of this section, we assume that a regular edge-coloring $(\Omega, \mathcal{C}, \phi, \Psi)$ is given. For $A, B \in \Omega_{(3)}$, we write $A \sim B$ if there exists a bijection π from A to B such that $\phi(\pi(\alpha), \pi(\alpha')) = \phi(\alpha, \alpha')$ for any distinct $\alpha, \alpha' \in A$. Then $\sim$ is an equivalence relation. Let $[A]$ denote the equivalence class of A. For $a, b, c \in \mathcal{C}$ we define

$$[a, b, c] = \big\{\{\alpha, \beta, \gamma\} \in \Omega_{(3)} \mid \phi(\alpha, \beta) = a,\ \phi(\beta, \gamma) = b,\ \phi(\gamma, \alpha) = c\big\}.$$

For each $A = \{\alpha, \beta, \gamma\} \in \Omega_{(3)}$, there exist $a, b, c \in \mathcal{C}$ such that $[A] = [a, b, c]$. Conversely, for any $a, b, c \in \mathcal{C}$, we can see that $[a, b, c]$ is an equivalence class unless it is empty. Let $\overline{\Omega_3}$ denote the set of equivalence classes with respect to $\sim$. For $a, b \in \mathcal{C}$, we define a family of equivalence classes $T_{a,b}$ by

$$T_{a,b} = \{[a, b, x] \mid x \in \mathcal{C},\ [a, b, x] \neq \varnothing\} \subset \overline{\Omega_3}.$$

By (R1), $T_{a,b} = \varnothing$ if and only if $b = \Psi(a)$ and $\delta_a = 1$ for any $a, b \in \mathcal{C}$, $T_{c,c} \neq \varnothing$ for any $c \in \mathcal{C}$ with $c \neq \Psi(c)$, and $T_{\Psi(d),d} \neq \varnothing$ for any $d \in \mathcal{C}$ with $\delta_d \geq 2$. For convenience, we define U_a, V_a for any $a \in \mathcal{C}$ by

$$U_a = T_{\Psi(a),a}, \quad V_a = T_{a,a}. \tag{2.1}$$

3 Some lemmas

Let $(\Omega, \mathcal{C}, \phi, \Psi)$ be a regular edge-coloring. We first prove some useful lemmas.

Lemma 3.1. *Let $a, b, c, d, e \in \mathcal{C}$ with $[a, b, c] \neq \varnothing$. Then $[a, b, c] \in T_{d,e}$ if and only if $\{\Psi(d), e\} \in \big\{\{a, \Psi(c)\}, \{b, \Psi(a)\}, \{c, \Psi(b)\}\big\}$.*

Proof. By the definition, the assertion follows from

$$\begin{aligned}
&[a, b, c] \in T_{d,e} \\
\iff& \text{there exists } x \in \mathcal{C} \text{ such that } [a, b, c] = [d, e, x] \\
\iff& (d, e) \in \{(a, b), (b, c), (c, a), (\Psi(a), \Psi(c)), (\Psi(b), \Psi(a)), (\Psi(c), \Psi(b))\} \\
\iff& (\Psi(d), e) \in \{(\Psi(a), b), (\Psi(b), c), (\Psi(c), a), (a, \Psi(c)), (b, \Psi(a)), (c, \Psi(b))\} \\
\iff& \{\Psi(d), e\} \in \big\{\{a, \Psi(c)\}, \{b, \Psi(a)\}, \{c, \Psi(b)\}\big\}.
\end{aligned}$$

□

Lemma 3.2. *Let a, b, c, d be in $\mathcal{C}$ with $\{a, b\} \neq \{c, d\}$. If $\{\Psi(a), \Psi(b)\} \cap \{c, d\} = \varnothing$, then $T_{\Psi(a),b} \cap T_{\Psi(c),d} = \varnothing$.*

Proof. Suppose $T_{\Psi(a),b} \cap T_{\Psi(c),d} \neq \varnothing$. Then there exists $x \in \mathcal{C}$ such that $[\Psi(a), b, x] \in T_{\Psi(a),b}$, $[\Psi(a), b, x] \neq \varnothing$ and $[\Psi(a), b, x] \in T_{\Psi(c),d}$. Lemma 3.1 implies that

$$\{c, d\} \in \big\{\{\Psi(a), \Psi(x)\}, \{b, a\}, \{x, \Psi(b)\}\big\}.$$

Thus $\{\Psi(a), \Psi(b)\} \cap \{c, d\} \neq \varnothing$, as $\{a, b\} \neq \{c, d\}$. □

Lemma 3.3. *Let* $a, b, c \in \mathcal{C}$ *with* $\Psi(a) = a$ *and* $b \notin \{c, \Psi(c)\}$. *Then* $T_{a,b} \cap T_{a,c} = \{[a, b, \Psi(c)]\}$ *if* $[a, b, \Psi(c)] \neq \varnothing$, *and* $T_{a,b} \cap T_{a,c} = \varnothing$ *otherwise.*

Proof. If $[a, b, \Psi(c)] \neq \varnothing$, then $[a, b, \Psi(c)] \in T_{a,b} \cap T_{a,c}$, as $[a, b, \Psi(c)] = [a, c, \Psi(b)]$. Suppose $[a, b, x] \in T_{a,b} \cap T_{a,c}$, where $x \in \mathcal{C}$. Lemma 3.1 implies that $\{a, c\} \in \big\{\{a, \Psi(x)\}, \{b, a\}, \{x, \Psi(b)\}\big\}$. Since $b \notin \{c, \Psi(c)\}$, either $\Psi(x) = c$ or $(a, c) = (\Psi(b), x)$ holds. In the former case, we have $[a, b, x] = [a, b, \Psi(c)]$. In the latter case, we also have $[a, b, x] = [\Psi(b), \Psi(a), \Psi(x)] = [a, b, \Psi(c)]$. Thus $T_{a,b} \cap T_{a,c} = \{[a, b, \Psi(c)]\}$ provided that $[a, b, \Psi(c)] \neq \varnothing$, and $T_{a,b} \cap T_{a,c} = \varnothing$ if $[a, b, \Psi(c)] = \varnothing$. □

Lemma 3.4. *Let* $\{a, b\}, \{c, d\}$ *be distinct 2-sets of* $\mathcal{C}$ *and suppose* $\Psi(w) = w$ *for each* $w \in \{a, b, c, d\}$. *Then* $T_{e,e} \cap T_{a,b} \cap T_{c,d} = T_{e,\Psi(e)} \cap T_{a,b} \cap T_{c,d} = \varnothing$ *holds for any* $e \in \mathcal{C}$.

Proof. If $e \neq \Psi(e)$, then we have $\{e, \Psi(e)\} \cap \{a, b\} = \varnothing$, and the assertion follows from Lemma 3.2. Hence we suppose that $e = \Psi(e)$.

Assume $T_{e,e} \cap T_{a,b} \cap T_{c,d} \neq \varnothing$. As $T_{a,b} \cap T_{c,d} \neq \varnothing$, Lemma 3.2 implies that $\{a, b\} \cap \{c, d\} \neq \varnothing$. Thus we may assume $a = c$, and so $b \neq d$, as $\{a, b\} \neq \{c, d\}$. Lemma 3.3 implies $T_{a,b} \cap T_{a,d} = \{[a, b, d]\}$.

By Lemma 3.2, we can see that $T_{e,e} \cap T_{a,b} \neq \varnothing$ implies $\{e\} \cap \{a, b\} \neq \varnothing$, and that $T_{e,e} \cap T_{a,d} \neq \varnothing$ implies $\{e\} \cap \{a, d\} \neq \varnothing$. Since $b \neq d$, we have $a = e$, and $[a, b, d] \in T_{a,a}$. Lemma 3.1 implies $\{a\} \in \big\{\{a, d\}, \{b, a\}, \{d, b\}\big\}$. Hence, either $d = a$ or $b = a$, a contradiction. □

Lemma 3.5. *Suppose* $a \in \mathcal{C}$, $a \neq \Psi(a)$, *and* $b, c, d, e \in \mathcal{C} \setminus \{a, \Psi(a)\}$. *If* $[a, b, c] = [a, d, e] \neq \varnothing$, *then* $b = d$ *and* $c = e$.

Proof. Since $[a, b, c] = [a, d, e] \in T_{a,d}$, Lemma 3.1 implies that

$$\{\Psi(a), d\} \in \big\{\{a, \Psi(c)\}, \{b, \Psi(a)\}, \{c, \Psi(b)\}\big\}.$$

By assumption, we have $\{\Psi(a), d\} = \{b, \Psi(a)\}$, and $b = d$. Similarly, $c = e$ is obtained. □

For $\mathcal{D} \subset \mathcal{C}$, we define $f(\mathcal{D})$ by $f(\mathcal{D}) = 2 + 2\sum_{d\in\mathcal{D}} \delta_d$, where δ_d is defined as in (R1). If $\mathcal{D} = \Psi(\mathcal{D})$, then we define $\Delta(\mathcal{D})$ by $\Delta(\mathcal{D}) = \{[x, y, z] \mid x, y, z \in \mathcal{C}\setminus\mathcal{D},\ [x, y, z] \neq \varnothing\}$.

Lemma 3.6. *Let* $\mathcal{D} \subset \mathcal{C}$. *If* $f(\mathcal{D}) < |\Omega|$, *then for any* $a \in \mathcal{C}$ *there exist* $b, c \in \mathcal{C} \setminus \mathcal{D}$ *such that* $[a, b, \Psi(c)] \neq \varnothing$. *Moreover, if* $\mathcal{D} = \Psi(\mathcal{D})$, *then* $\Delta(\mathcal{D}) \neq \varnothing$.

Proof. Let $\alpha, \beta \in \Omega$ with $\phi(\alpha, \beta) = a$. Then

$$\begin{aligned}
&\big|\{\omega \in \Omega \setminus \{\alpha, \beta\} \mid \phi(\alpha, \omega) \in \mathcal{D} \text{ or } \phi(\beta, \omega) \in \mathcal{D}\}\big| \\
&\leq \big|\{\omega \in \Omega \setminus \{\alpha, \beta\} \mid \phi(\alpha, \omega) \in \mathcal{D}\}\big| + \big|\{\omega \in \Omega \setminus \{\alpha, \beta\} \mid \phi(\beta, \omega) \in \mathcal{D}\}\big| \\
&\leq 2\sum_{d\in\mathcal{D}} \delta_d \\
&< |\Omega| - 2 = |\Omega \setminus \{\alpha, \beta\}|.
\end{aligned}$$

Thus there exists $\gamma \in \Omega \setminus \{\alpha, \beta\}$ such that $\phi(\alpha, \gamma) \notin \mathcal{D}$ and $\phi(\beta, \gamma) \notin \mathcal{D}$. Setting $b = \phi(\beta, \gamma)$ and $c = \phi(\alpha, \gamma)$, we obtain $\{\alpha, \beta, \gamma\} \in [a, b, \Psi(c)]$ with $b, c \in \mathcal{C} \setminus \mathcal{D}$.

Suppose $\mathcal{D} = \Psi(\mathcal{D})$, and let $x \in \mathcal{C} \setminus \mathcal{D}$. By the first part of the lemma, there exist $y, z \in \mathcal{C} \setminus \mathcal{D}$ such that $[x, y, \Psi(z)] \neq \varnothing$. Since $\Psi(\mathcal{D}) = \mathcal{D}$, we have $\Psi(z) \notin \mathcal{D}$, and hence $[x, y, \Psi(z)] \in \Delta(\mathcal{D})$. □

4 Lower bounds of $|\overline{\Omega_3}|$

This section is devoted to establishing some lower bounds for $|\overline{\Omega_3}|$, which will be needed later. As in the previous section, suppose that $(\Omega, \mathcal{C}, \phi, \Psi)$ is a regular edge-coloring. We define subsets $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{K}_1, \mathcal{K}_2, \mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{C}$ by

$$\begin{aligned} &\mathcal{K} = \{a \in \mathcal{C} \mid \Psi(a) \neq a\},\ \mathcal{L} = \{a \in \mathcal{C} \mid \Psi(a) = a\},\ \mathcal{M} = \{a \in \mathcal{C} \mid \delta_a \geq 2\}, \\ &\mathcal{K}_1 = \mathcal{K} \setminus \mathcal{M},\ \mathcal{K}_2 = \mathcal{K} \cap \mathcal{M},\ \mathcal{L}_1 = \mathcal{L} \setminus \mathcal{M},\ \mathcal{L}_2 = \mathcal{L} \cap \mathcal{M}, \end{aligned} \tag{4.1}$$

and we define integers k, l, m by

$$k = \tfrac{|\mathcal{K}|}{2},\ l = |\mathcal{L}|,\ m = |\mathcal{M}|. \tag{4.2}$$

Lemma 4.1. *Let $\{a_1, b_1\}, \dots, \{a_s, b_s\}$ be distinct subsets of $\mathcal{C}$ with $T_{\Psi(a_i), b_i} \neq \varnothing$, for all i. Let $X = \bigcup_{1 \leq i \leq s} T_{\Psi(a_i), b_i}$. Then $|X| \geq \lceil \frac{s}{3} \rceil$. In particular*

$$|\overline{\Omega_3}| \geq \left\lceil \frac{2m + |\mathcal{C}|(|\mathcal{C}| - 1)}{6} \right\rceil.$$

Proof. Since $T_{\Psi(a_i), b_i} \neq \varnothing$, we have

$$\begin{aligned} s &\leq \left|\{([x,y,z], j) \mid x,y,z \in \mathcal{C},\ 1 \leq j \leq s,\ [x,y,z] \in T_{\Psi(a_j), b_j}\}\right| \\ &= \sum_{[x,y,z] \in \overline{\Omega_3}} \left|\{j \mid 1 \leq j \leq s,\ [x,y,z] \in T_{\Psi(a_j), b_j}\}\right| \\ &= \sum_{[x,y,z] \in X} \left|\{j \mid 1 \leq j \leq s,\ [x,y,z] \in T_{\Psi(a_j), b_j}\}\right| \\ &\leq \sum_{[x,y,z] \in X} 3 \qquad \text{(by Lemma 3.1)} \\ &= 3|X|. \end{aligned}$$

Since we can take $m + |\mathcal{C}_{(2)}|$ subsets $\{a_i, b_i\}$ with $T_{\Psi(a_i), b_i} \neq \varnothing$, the second part follows. □

Lemma 4.2. *For any $a \in \mathcal{K}_2$, the inequality $|U_a \cup V_a| \geq 2$ holds.*

Proof. Let $\alpha, \beta \in \Omega$ be such that $\phi(\alpha, \beta) = a$. There exists $\gamma \in \Omega \setminus \{\alpha, \beta\}$ such that $\phi(\beta, \gamma) = a$ and $\phi(\alpha, \gamma) \neq a$, as

$$|\{\omega \in \Omega \setminus \{\beta\} \mid \phi(\alpha, \omega) = a\}| = \delta_a - 1 < \delta_a = |\{\omega \in \Omega \mid \phi(\beta, \omega) = a\}|.$$

We have $[\{\alpha, \beta, \gamma\}] = [a, a, \phi(\gamma, \alpha)] \in V_a$. If $|U_a \cup V_a| = 1$, then $U_a = V_a$, and $[a, a, \phi(\gamma, \alpha)] \in U_a$. Lemma 3.1 implies

$$\{a\} \in \bigl\{\{a, (\Psi \circ \phi)(\gamma, \alpha)\}, \{a, \Psi(a)\}, \{\phi(\gamma, \alpha), \Psi(a)\}\bigr\},$$

which is a contradiction. □

Lemma 4.3. $|\overline{\Omega_3}| \geq \max \left\{ m, \left\lceil \frac{4m + l(l-1)}{6} \right\rceil \right\}$.

Proof. Let $\mathcal{K}'$ denote a subset of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}' \cup \Psi(\mathcal{K}')$ and $\mathcal{K}' \cap \Psi(\mathcal{K}') = \varnothing$. Let $\Gamma = (\bigcup_{a\in\mathcal{K}'\cap\mathcal{M}} U_a \cup V_a) \cup (\bigcup_{b\in\mathcal{L}_2} U_b)$. Lemma 3.2 implies that $|\Gamma| = \sum_{a\in\mathcal{K}'\cap\mathcal{M}} |U_a \cup V_a| + \sum_{b\in\mathcal{L}_2} |U_b|$. By Lemma 4.2, we have that $|\Gamma| \geq 2|\mathcal{K}'\cap\mathcal{M}| + |\mathcal{L}_2| = m$. In particular, $|\overline{\Omega_3}| \geq m$.

We define $X = \{\{c,d\} \in \mathcal{L}_{(2)} \mid T_{c,d} \cap \Gamma \neq \varnothing\}$. We claim $|X| \leq |\Gamma|$. Indeed, if $|X| > |\Gamma|$, then there exist $\{c,d\}, \{c',d'\} \in X$ such that $\Gamma \cap T_{c,d} \cap T_{c',d'} \neq \varnothing$. By the definition of Γ, this contradicts Lemma 3.4, and the claim holds. Equivalently, $|\mathcal{L}_{(2)} \setminus X| \geq \binom{l}{2} - |\Gamma|$. Now, Lemma 4.1 implies that

$$|\overline{\Omega_3}| \geq \Big|\Gamma \cup \bigcup_{\{c,d\}\in\mathcal{L}_{(2)}\setminus X} T_{c,d}\Big| = |\Gamma| + \Big|\bigcup_{\{c,d\}\in\mathcal{L}_{(2)}\setminus X} T_{c,d}\Big| \geq \left\lceil \frac{4m + l(l-1)}{6} \right\rceil. \qquad \square$$

Lemma 4.4. *Let $\mathcal{D} \subset \mathcal{C}$. If $\mathcal{D} = \Psi(\mathcal{D})$, then*

$$|\overline{\Omega_3}| \geq |\Delta(\mathcal{D})| + \Big|\Big\{[a,x,y] \mid a \in \mathcal{D},\ x,y \in \mathcal{C}\setminus\mathcal{D},\ [a,x,y] \neq \varnothing\Big\}\Big| + \Big|\bigcup_{b,c\in\mathcal{D}} T_{b,c}\Big|.$$

Proof. Lemma 3.1 implies that the sets corresponding to the terms of the right-hand side are disjoint. Hence the claim holds. □

Lemma 4.5. *Suppose $\mathcal{D} \subset \mathcal{C}$, $\mathcal{D} = \Psi(\mathcal{D})$ and $f(\mathcal{D}) < |\Omega|$. Then $|\overline{\Omega_3}| \geq 1 + |\mathcal{D}| + |\bigcup_{b,c\in\mathcal{D}} T_{b,c}|$.*

Proof. By assumption, Lemma 3.6 implies that $\Delta(\mathcal{D}) \neq \varnothing$. Also, for each $a \in \mathcal{D}$, Lemma 3.6 implies that there exist $x_a, y_a \in \mathcal{C}\setminus\mathcal{D}$ such that $[a, x_a, \Psi(y_a)] \neq \varnothing$. We have $x_a \in \mathcal{C}\setminus\mathcal{D}$ and $\Psi(y_a) \in \mathcal{C}\setminus\mathcal{D}$, as $\Psi(\mathcal{D}) = \mathcal{D}$. Lemma 4.4 implies that

$$|\overline{\Omega_3}| \geq |\Delta(\mathcal{D})| + \sum_{a\in\mathcal{D}} \Big|\Big\{[a, x_a, \Psi(y_a)]\Big\}\Big| + \Big|\bigcup_{b,c\in\mathcal{D}} T_{b,c}\Big| \geq 1 + |\mathcal{D}| + \Big|\bigcup_{b,c\in\mathcal{D}} T_{b,c}\Big|. \qquad \square$$

5 Proof of the main result

In this section we prove Theorem 1.2. Let a regular edge-coloring $(\Omega, \mathcal{C}, \phi, \Psi)$ be given. Let U_c, V_c, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, k, l, m be as in (2.1), (4.1) and (4.2).

Lemma 5.1. *Assume $|\Omega| \geq 8$. If $m = 0$ or $|\mathcal{C}| \geq 6$, then $|\overline{\Omega_3}| \geq |\mathcal{C}|$ holds.*

Proof. If $m = 0$, then $|\mathcal{C}| = |\Omega| - 1 > 6$. If $|\mathcal{C}| > 6$, then Lemma 4.1 implies $|\overline{\Omega_3}| \geq |\mathcal{C}|$. If $|\mathcal{C}| = 6$, then $m \geq 1$, as $|\Omega| \geq 8$. Therefore the result follows from Lemma 4.1. □

Lemma 5.2. *If $|\Omega| \geq 11$ and $\mathcal{K}_1 \neq \varnothing$, then $|\overline{\Omega_3}| \geq |\mathcal{C}|$.*

Proof. Let $c \in \mathcal{K}_1$. The condition (R1) implies that there exist $\alpha_1, \alpha_2, \alpha_3 \in \Omega$ such that $\phi(\alpha_1, \alpha_2) = \phi(\alpha_2, \alpha_3) = c$. We define $d = \phi(\alpha_3, \alpha_1)$, $\mathcal{D} = \{c, \Psi(c)\}$, and $\mathcal{E} = \mathcal{C}\setminus\mathcal{D}$. Since $|\Omega| \geq 11$ and $\delta_c = 1$, we get that $\mathcal{E} \neq \varnothing$, and that

$$\text{for all } \omega \in \Omega\setminus\{\alpha_1, \alpha_3\},\ \phi(\omega, \alpha_2) \in \mathcal{E}. \tag{5.1}$$

By (R1), for any $e \in \mathcal{E}\setminus\{d\}$, there exists $\beta_e \in \Omega\setminus\{\alpha_1\}$ such that $\phi(\alpha_3, \beta_e) = e$. We have $\phi(\beta_e, \alpha_2) \in \mathcal{E}$ by (5.1). It follows that

$$|\{(x,y) \mid x \in \mathcal{E}\setminus\{d\},\ y \in \mathcal{E},\ [c,x,y] \neq \varnothing\}| \geq |\mathcal{E}\setminus\{d\}|. \tag{5.2}$$

As $f(\mathcal{D}) = 6 < |\Omega|$, Lemma 3.6 implies $\Delta(\mathcal{D}) \neq \varnothing$. Lemma 4.4 implies that

$$\begin{aligned}|\overline{\Omega_3}| &\geq |\Delta(\mathcal{D})| + \left|\{[a,x,y] \mid a \in \mathcal{D},\ x,y \in \mathcal{E},\ [a,x,y] \neq \varnothing\}\right| + \left|\bigcup_{b,b' \in \mathcal{D}} T_{b,b'}\right| \\ &= |\Delta(\mathcal{D})| + |\{[c,x,y] \mid x,y \in \mathcal{E},\ [c,x,y] \neq \varnothing\}| + |V_c| \\ &= |\Delta(\mathcal{D})| + |\{(x,y) \mid x,y \in \mathcal{E},\ [c,x,y] \neq \varnothing\}| + |V_c| \\ &= |\Delta(\mathcal{D})| + |\{(x,y) \mid x \in \mathcal{E} \setminus \{d\},\ y \in \mathcal{E},\ [c,x,y] \neq \varnothing\}| \\ &\qquad + |\{y \mid y \in \mathcal{E},\ [c,d,y] \neq \varnothing\}| + |V_c|.\end{aligned}$$

Combining the inequalities $|\Delta(\mathcal{D})| \geq 1$, $|V_c| \geq 1$, and (5.2), we obtain

$$|\overline{\Omega_3}| \geq 1 + |\mathcal{C}| - |\{c, \Psi(c), d\}| + 0 + 1 \geq |\mathcal{C}| - 1.$$

Suppose $|\overline{\Omega_3}| < |\mathcal{C}|$. Then equality is forced in each of the above inequalities. In particular, $|\Delta(\mathcal{D})| = 1$, $d \notin \{c, \Psi(c)\}$, and $[c,d,y] = \varnothing$ for any $y \in \mathcal{E}$.

Now, if there exists $\gamma \in \Omega \setminus \{\alpha_1\}$ such that $\phi(\alpha_3, \gamma) = d$, then (5.1) implies $\phi(\gamma, \alpha_2) \in \mathcal{E}$. This would imply $[c,d,y] \neq \varnothing$, where $y = \phi(\gamma, \alpha_2) \in \mathcal{E}$, which is a contradiction. Thus there is no such γ. That is, $\delta_d = 1$. Hence $f(\mathcal{D} \cup \{d, \Psi(d)\}) \leq 10 < |\Omega|$, and Lemma 3.6 implies $\Delta(\mathcal{D} \cup \{d, \Psi(d)\}) \neq \varnothing$. We have $\Delta(\mathcal{D}) = \Delta(\mathcal{D} \cup \{d, \Psi(d)\})$, as $|\Delta(\mathcal{D})| = 1$. However, since $f(\{c, \Psi(c)\}) = 6 < |\Omega|$, Lemma 3.6 implies that there exist $z, w \in \mathcal{E}$ such that $[d,z,w] \neq \varnothing$. Since $[d,z,w] \in \Delta(\mathcal{D})$ and $[d,z,w] \notin \Delta(\mathcal{D} \cup \{d, \Psi(d)\})$, we have $\Delta(\mathcal{D}) \neq \Delta(\mathcal{D} \cup \{d, \Psi(d)\})$, which is a contradiction. Therefore $|\overline{\Omega_3}| \geq |\mathcal{C}|$. □

Lemma 5.3. *If $|\Omega| \geq 11$ and $\mathcal{L} = \varnothing$, then $|\overline{\Omega_3}| \geq |\mathcal{C}|$.*

Proof. By assumption, we have $\mathcal{C} = \mathcal{K}_1 \cup \mathcal{M}$. If $\mathcal{K}_1 = \varnothing$, then the result follows from Lemma 4.3. Otherwise the result follows from Lemma 5.2. □

Lemma 5.4. *If $|\Omega| \geq 11$ and $\mathcal{K} = \varnothing$, then $|\overline{\Omega_3}| \geq |\mathcal{C}|$.*

Proof. By Lemma 5.1, we may assume $l \leq 5$ and $m \geq 1$. If $m \geq 2$, then Lemma 4.3 yields the result. Assume $m = 1$. Set $\mathcal{C} = \{1, \ldots, l\}$. Then we may assume without loss of generality that $\delta_1 \geq 2$. Since $f(\mathcal{C} \setminus \{1\}) = 2l < |\Omega|$, Lemma 3.6 yields $[c,1,1] \neq \varnothing$ for each $c \in \mathcal{C}$. Therefore $|\overline{\Omega_3}| \geq l$. □

Lemma 5.5. *Let $(\Omega, \mathcal{C}, \phi, \Psi)$ be a regular edge-coloring with $|\Omega| \geq 11$ and $|\mathcal{C}| = 3$. Then $|\overline{\Omega_3}| \geq 3$ holds.*

Proof. We argue by contradiction. Assume $|\overline{\Omega_3}| < 3$. Lemmas 5.3 and 5.4 imply $\mathcal{C} = \{1, \overrightarrow{1}, \overleftarrow{1}\}$, where $\Psi(1) = 1$ and $\Psi(\overrightarrow{1}) = \overleftarrow{1}$. Also, $|\overline{\Omega_3}| < |\mathcal{C}|$ implies $\delta_{\overrightarrow{1}} \geq 2$ by Lemma 5.2. We have $\delta_1 = 1$ by Lemma 4.3. Since $|U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}}| \geq 2$ by Lemma 4.2, it follows from $|\overline{\Omega_3}| \leq 2$ that

$$\overline{\Omega_3} = U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}}. \tag{5.3}$$

Now, since $[1, \overrightarrow{1}, \overleftarrow{1}] \notin U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}}$ by Lemma 3.1, we have $[1, \overrightarrow{1}, \overleftarrow{1}] = \varnothing$, which implies $T_{1,\overrightarrow{1}} = \{[1, \overrightarrow{1}, \overrightarrow{1}]\}$ by (5.3). Since $f(\{1\}) = 4 < |\Omega|$, Lemma 3.6 implies $\Delta(\{1\}) \neq \varnothing$. Hence

$$\overline{\Omega_3} = \{[1, \overrightarrow{1}, \overrightarrow{1}]\} \cup \Delta(\{1\}). \tag{5.4}$$

We compare (5.3) with (5.4). Since $[1, \overrightarrow{1}, \overrightarrow{1}] \notin U_{\overrightarrow{1}}$ by Lemma 3.2, we have $U_{\overrightarrow{1}} = \Delta(\{1\})$, which implies $U_{\overrightarrow{1}} = \{[\overrightarrow{1}, \overrightarrow{1}, \overleftarrow{1}]\}$. Hence $\overline{\Omega_3} = \{[1, \overrightarrow{1}, \overrightarrow{1}], [\overrightarrow{1}, \overrightarrow{1}, \overleftarrow{1}]\}$.

Let $\alpha_1 \in \Omega$. Pick distinct $\beta_1, \alpha_2, \beta_2, \alpha_3 \in \Omega$ in such a way that $\phi(\alpha_1, \beta_1) = 1$, $\phi(\alpha_1, \alpha_2) = \overrightarrow{1}$, $\phi(\alpha_2, \beta_2) = 1$, and $\phi(\alpha_2, \alpha_3) = \overrightarrow{1}$. Notice that $\delta_1 = 1$. Since $\phi(\alpha_1, \alpha_2) = \phi(\alpha_2, \alpha_3) = \overrightarrow{1}$, we have $\phi(\alpha_1, \alpha_3) = \overrightarrow{1}$. Since $\phi(\alpha_1, \alpha_2) = \overrightarrow{1}$ and $\phi(\alpha_2, \beta_2) = 1$, we have $\phi(\beta_2, \alpha_1) = \overrightarrow{1}$. Similarly, we have $\phi(\alpha_3, \beta_2) = \overrightarrow{1}$. Thus, $[\{\alpha_1, \alpha_3, \beta_2\}] = [\overrightarrow{1}, \overrightarrow{1}, \overrightarrow{1}]$, a contradiction. □

We are now ready to prove our main result.

Theorem 5.6. *Let $(\Omega, \mathcal{C}, \phi, \Psi)$ be a regular edge-coloring. If $|\Omega| \geq 11$, then $|\overline{\Omega_3}| \geq |\mathcal{C}|$.*

Proof. Let $\mathcal{K} = \{\overrightarrow{1}, \ldots, \overrightarrow{k}, \overleftarrow{1}, \ldots, \overleftarrow{k}\}$, $\mathcal{L} = \{1, \ldots, l\}$, and $\Psi(\overrightarrow{i}) = \overleftarrow{i}$. By Lemmas 5.1, 5.3, 5.4, and 5.5, we only need to consider the cases where $(k, l, |\mathcal{C}|) = (1, 2, 4)$, $(1, 3, 5)$, and $(2, 1, 5)$. In each case, we may assume $\delta_{\overrightarrow{i}} \geq 2$ for $1 \leq i \leq k$, by Lemma 5.2. Lemma 4.2 implies that

$$|U_{\overrightarrow{i}} \cup V_{\overrightarrow{i}}| \geq 2. \quad (1 \leq i \leq k) \tag{5.5}$$

Also, Lemma 3.2 implies that

$$(U_{\overrightarrow{i}} \cup V_{\overrightarrow{i}}) \cap T_{j,j'} = \varnothing. \quad (1 \leq i \leq k,\ 1 \leq j, j' \leq l) \tag{5.6}$$

Notice that $T_{a,b} \neq \varnothing$ unless $b = \Psi(a)$ and $\delta_a = 1$, for any $a, b \in \mathcal{C}$.

Suppose $(k, l) = (1, 2)$. If $\delta_1 = \delta_2 = 1$, then Lemma 4.5 implies $|\overline{\Omega_3}| \geq 4$, so we may assume without loss of generality that $\delta_1 \geq 2$. If $|U_1 \cup T_{1,2}| \geq 2$, then $|\overline{\Omega_3}| \geq |U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}} \cup U_1 \cup T_{1,2}| = |U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}}| + |U_1 \cup T_{1,2}| \geq 4$ by (5.5) and (5.6). If $|U_1 \cup T_{1,2}| = 1$, then $U_1 = T_{1,2}$, and $U_1 \cup T_{1,2} = \{[1, 1, 2]\}$ by Lemma 3.3. In particular, we get $[1, 1, 2] \neq \varnothing$. Lemmas 3.1 and 3.2 imply that $\{[1, 1, 2]\}$, $U_{\overrightarrow{1}}$, $T_{1,\overrightarrow{1}}$, and $T_{2,\overrightarrow{1}}$ are pairwise disjoint. Hence we obtain $|\overline{\Omega_3}| \geq 4$ by collecting these families.

Next suppose $(k, l) = (1, 3)$. If $[1, 2, 3] = \varnothing$, then Lemma 3.3 implies that $T_{1,2}$, $T_{1,3}$, and $T_{2,3}$ are pairwise disjoint. Hence, $|\overline{\Omega_3}| \geq |U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}} \cup T_{1,2} \cup T_{1,3} \cup T_{2,3}| \geq 5$ by (5.5) and (5.6). We may assume that $[1, 2, 3] \neq \varnothing$. Lemmas 3.1 and 3.2 imply that $\{[1, 2, 3]\}$, $U_{\overrightarrow{1}}$, $T_{1,\overrightarrow{1}}$, $T_{2,\overrightarrow{1}}$, and $T_{3,\overrightarrow{1}}$ are pairwise disjoint. Collecting these families, we have $|\overline{\Omega_3}| \geq 5$.

Finally suppose $(k, l) = (2, 1)$. Assume $|T_{1,\overrightarrow{1}} \cup T_{1,\overrightarrow{2}}| \geq 2$. Lemma 3.2 implies that $(T_{1,\overrightarrow{1}} \cup T_{1,\overrightarrow{2}})$, $T_{\Psi(\overrightarrow{1}),\overrightarrow{2}}$, $U_{\overrightarrow{1}}$, and $U_{\overrightarrow{2}}$ are pairwise disjoint. Hence $|\overline{\Omega_3}| \geq |T_{1,\overrightarrow{1}} \cup T_{1,\overrightarrow{2}} \cup T_{\Psi(\overrightarrow{1}),\overrightarrow{2}} \cup U_{\overrightarrow{1}} \cup U_{\overrightarrow{2}}| \geq 5$. We may assume that $|T_{1,\overrightarrow{1}} \cup T_{1,\overrightarrow{2}}| = 1$. Then $T_{1,\overrightarrow{1}} = T_{1,\overrightarrow{2}}$, and $T_{1,\overrightarrow{1}} \cup T_{1,\overrightarrow{2}} = \{[1, \overrightarrow{1}, \overleftarrow{2}]\}$ by Lemma 3.3. In particular, we get $[1, \overrightarrow{1}, \overleftarrow{2}] \neq \varnothing$. Lemma 3.2 implies that $\{[1, \overrightarrow{1}, \overleftarrow{2}]\}$, $U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}}$, and $U_{\overrightarrow{2}} \cup V_{\overrightarrow{2}}$ are pairwise disjoint. Therefore $|\overline{\Omega_3}| \geq |\{[1, \overrightarrow{1}, \overleftarrow{2}]\} \cup U_{\overrightarrow{1}} \cup V_{\overrightarrow{1}} \cup U_{\overrightarrow{2}} \cup V_{\overrightarrow{2}}| \geq 5$ by (5.5). □

Proof of Theorem 1.2. Let $(\Omega, \mathcal{C}_G, \phi_G, \Psi_G)$ be the regular edge-coloring induced by G. By the definition of induced regular edge-coloring, $|\mathcal{C}_G| = |\Omega_{[2]}/G|$ holds. For $A, B \in \Omega_{(3)}$, if $A^g = B$ for some $g \in G$, then $[A] = [B]$, hence $|\overline{\Omega_3}| \leq |\Omega_{(3)}/G|$ holds. The result follows from Theorem 5.6. □

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