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THE NONLINEAR SHAPING OF THE THERMOMECHANICAL STATUS OF TWO-PHASES BODIES

NELINEARNA DOLOČITEV TERMOMEHANSKEGA STATUSA DVOFAZNIH TELES

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The method and the algorithm for the calculation of the thermomechanical status of bodies are proposed which connect the mechanical behavior of a material at the interchange of heat with the environment. The nonlinear problem of the thermomechanical status of heating of two-phase bodies is solved. The laws of motion of the phase boundary, the temperature field and of the strained state in the rod are given. The outcomes are presented as a relation, of both, temperature and strain upon time and location.

Key words: thermomechanics, heating and mechanical effects, nonlinearity of the first kind, two-phase bodies, mathematical modelling

Predložena sta metoda in algoritem za izračun termomehanskega statusa teles. Povezujeta mehansko vedenje materiala pri izmenjavi toplote z okolico. Rešen je nelinearni problem termomehanskega statusa pri segrevanju dvofaznih teles. Predstavljeni so zakoni, ki opisujejo premikanje fazne meje, temperaturnega polja in deformacijskega stanja v palici. Rešitve so prikazane kot odvisnost temperature in deformacije od časa in kraja.

Ključne besede: termomehanika, segrevanje in mehanske značilnosti, nelinearnost prvega tipa, dvofazna telesa, matematično modeliranje

1 INTRODUCTION

The methods of linear simulation in thermomechanics can not meet the requirements of recent trends in engineering in conventional industrial areas, such as power system, machine industry and especially metallurgy. Therefore, the solution of problems related to the nonstationarity, non-uniformity, nonlinearity and other singularities, and for which the mathematical methods of the classic (linear) theory of heat conduction and mechanics of a deformable solid body are poorly used, is necessary. It is evident, that the achievement of engineering advances is possible only with the application of nonlinear mathematical modeling.

The natural thermomechanical processes are mostly nonlinear, f.i. the radiation heating in metallurgical furnaces, the melting and the solidification of metal and solid phase transitions. Correct mathematical solutions of these nonlinearities of the I, II and III type are necessary for the advance of metallurgical manufacturing and the improvement of economics, ecology, quality and costs. The mathematical difficultness forced for a long time, and often still force, to use a straight linearization of equations of thermomechanics (heat conduction and thermoelasticity). Also proper solutions for the nonlinear boundary value of heat conduction were not found untilly recently. The use of numerical methods predominantly applied to find solution in engineering practice is not always reliable. Therefore it is necessary to develop more simple methods of calculation which would allow to determine analytically solutions with sufficient working accuracy.

In ref. [1,2,3] some problems related to problems of nonlinearity in metallurgical thermomechanics are reviewed, the approaches to their solution are explained and the usefulness of the development of analytical methods for the exploration of nonlinear thermal processes is pointed out.

In the present paper and for the example of a hollow two-phase barrel the method and algorithm for the definition and the solution of the thermomechanical status of a body is presented considering the heat interchange with the environment. Also the problem of solidification of a rod is reviewed.

2 MATHEMATICAL MODELLING AND SOLUTION OF NONLINEAR TASKS OF THE THERMOMECHANICS OF A TWO-PHASE BODY

We shall consider the hollow cylinder of circular cross section with the internal radius b_1 and the external radius b. Let's enter in an undeformed configuration the cylindrical coordinate system (r,ϑ,z), with the axis z coinciding to an axis of the cylinder. Let's further assume that the material of the cylinder can be in two modular statuses - liquid and solid. The behavior of this material in liquid and solid phase is described with equations of growth of an inhomogeneous visco-elastic body. Let's designate through θ^0 the phase transition temperature: at $\theta_2 < \theta^0$ the material is solid, and at $\theta_1 > \theta^0$ - it is in liquid, and assume that up to a strain level the material of the cylinder in solid takes the area of $\omega_2(t) = \{b_0 \le r \le b\}$ and in liquid the area of $\omega_1(t) = \{b_1 \le r \le b_0\}$. The heat interaction of an exterior surface of the cylinder with the environment is characterized with the convection coefficient of heat rejection $\alpha_{\text{кон}}$ and radiation. The interior surface $r = b_1$ remains at the constant temperature f. When at $t = 0$ to the interior surface of the cylinder the pressure $P(t)$ is applied, the reflux of heat begins to the external surface of a body. After that part of the material of the cylinder changes to a solid phase with its boundary moving according the law $r = a = a(t)$.

For the change of an external load we need to define the law of the motion of the boundary of phases $a = a(t)$, of temperature $\theta_i = \theta_i\,(t,r)$ (i = 1,2) and of the strained state of the barrel assuming a plane stress mode and a linear creep law.

The mathematical definition of the nonlinear task of thermomechanics for the cylinder in occurrence of a phase transition due to the heat interchange with the environment requires the determination of unknowns characteristics in the following equations:

$$\varepsilon_{r,i}(t,r) = \frac{\partial u_{r,i}(t,r)}{\partial r},\ \varepsilon_{\vartheta,i}(t,r) = \frac{u_{\vartheta,i}(t,r)}{r},$$

$$\sigma_{r,1}(t,r) - \sigma_{\vartheta,1}(t,r) = 2G(t-\tau^*(r))(\varepsilon_{r,1}(t,r) - \varepsilon_{\vartheta,1}(t,r)) -$$

$$- \int_{\tau^*(r)}^{t} R_i(t-\tau^*(r),\ \tau-\tau^*(r))(\varepsilon_{r,i}(\tau,r) - \varepsilon_{\vartheta,i}(\tau,r))d\tau,$$

$$\frac{\partial \sigma_{r,i}(t,r)}{\partial r} = \frac{\sigma_{r,i}(t,r) - \sigma_{\vartheta,i}(t,r)}{r},\ \varepsilon_{r,i}(t,r) = \varepsilon_{\vartheta,i}(t,r)$$

$$\rho_i c_i \frac{\partial \theta_i(t,r)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda_i r \frac{\partial \theta_i(t,r)}{\partial r}\right) \quad (i = 1,2). \qquad (1)$$

In this equation and further the index "1" is related to the arguments of the liquid and the index "2" - the solid material; and are $G(t-\tau^*(r))$ - the elastic and instantaneous shear modulus; $R(t-\tau^*(x),\tau-\tau^*(x))$ - the core of the relaxation of the visco-elastic material; $\tau^*(x)$ - the moment of transition of an element of the material to the solid modular status; $\rho_i = \rho_i(\theta_i)$, $r_i = r_i(\theta_i)$, $\lambda_i = \lambda_i(\theta_i)$ - the density, the thermal capacity and the coefficient of heat conduction of a material and $\theta_i(t,x)$ - the allocation of temperature in the phase.

The boundary and the initial conditions of the relations are:

$$\sigma_{r,1}(t,b_1) = -P(t),\ \sigma_{r,2}(t,b) = 0,$$

$$\sigma_{r,1}(t,a(t)) = \sigma_{r,2}(t,a(t)),$$

$$u_{r,1}(t,a(t)) = u_{r,2}(t,a(t)),\ \theta_1(t,b_1) = f,$$

$$-\lambda_2 \left.\frac{\partial \theta_2(t,r)}{\partial r}\right|_{r=b} = \alpha_{\text{кон}}\left[\theta_2(t,b) - \theta_{\text{ср}}\right] + \varepsilon_{12}\sigma_0\theta_2^4(t,b),$$

$$\theta_2(t,a(t)) = \theta_1(t,a(t)) - \Delta) = \theta^0,$$

$$-\lambda_1 \left.\frac{\partial \theta_1(t,r)}{\partial r}\right|_{r=a(t)} = -\lambda_2 \left.\frac{\partial \theta_2(t,r)}{\partial r}\right|_{r=a(t)} + \rho_1 \mu \frac{da(t)}{dt},$$

$$\left.\theta_i(t,r)\right|_{t=0} = \theta_i(0,r) \qquad (2)$$

The relations (1) and (2) are the nonlinear equations of thermomechanics for the cylinder in presence of a phase transition due to the heat interaction with the environment. The problem is reduced to the solution of connected nonlinear integro-differential equations. For the solution the approximated analytical [1] and the a step-by-step method [4] were used. In the solution the nonlinear equations of heat conduction are treated as a combination of solutions of linear equations with different boundary conditions for each time interval. The temperature and the mechanical properties of the material vary between the time intervals in a stepwise way. The connection between intervals is obtained with the initial conditions. The selection of the time intervals depends on the intensity of the thermal loading. For the definition of the temperature field of an elements exposed to thermal loading of major intensity the most rational is the use of approximate relations for the performances of materials and mediums constant within the limits of the temperature range.

The interval of time $[0,t_{\text{кон}}]$, in which the strained state in the cylinder is examined, is divided to sub-intervals $t_k = k\Delta$, $\Delta = t_{\text{кон}}/N$, k = 0,1,...,N. In this way the process of continuous growth or crystallization is substituted with a digital process. For each period of time the equation of heat conduction and the elastic contact equations for two bodies $\omega_1(t)$, $\omega_2(t)$ are established. Using a variation of the Gibbs principle, the law of motion of a phase boundary is than determined, which takes into account the relation position of temperature and mechanical fields and their change in time.

According to [4] the solution of the equation of heat conduction for a system of two cylindrical bodies $\omega_1(t)$ and $\omega_2(t)$ for the iteration k is:

$$\theta_{1,k}(t_k,r) = f + \frac{\lambda_{2,k}}{\lambda_{1,k}}\left\{\frac{\alpha_k\left[\theta^0 - \theta_{\text{ср}}\right]}{\frac{\lambda_{2,k}}{b} + \alpha_k \ln\frac{b}{a_k(t_k)}} + \right.$$

$$\left. + \frac{\mu\rho_{1,k}}{\lambda_{2,k}} a_k(t_k)\left[a_k(t_k) - a_k(t_{k+1})\right]\right\}\ln\frac{b_1}{r}$$

$$\theta_{2,\mathrm{k}}(t_{\mathrm{k}}, r) = \theta^0 \frac{\alpha_{\mathrm{k}}\left[\theta^0 - \theta_{\mathrm{cp}}\right]}{\dfrac{\lambda_{2,\mathrm{k}}}{b} + \alpha_{\mathrm{k}} \ln \dfrac{b}{a_{\mathrm{k}}(t_{\mathrm{k}})}} \ln \frac{r}{a_{\mathrm{k}}(t_{\mathrm{k}})} \tag{3}$$

In this equation $\alpha_k = \alpha_k(\theta_{2,k})$ is the reduced coefficient of heat rejection [5] which describes the intensity of convection and radiative heat interchange with the environment.

The strained state in the barrel in the instant t_k of the k iteration [6] is determined with a Lame treatment and the equation:

$$\sigma^{\mathrm{k}}_{\substack{r,1\\ \vartheta,1}}(t_{\mathrm{k}}, r) = \frac{P(t_{\mathrm{k}})b_1^2 - P_{\mathrm{k}}(t_{\mathrm{k}})a_{\mathrm{k}}^2(t_{\mathrm{k}})}{a_{\mathrm{k}}^2(t_{\mathrm{k}}) - b_1^2} \mp \frac{b_1^2 a_{\mathrm{k}}^2(t_{\mathrm{k}})}{r^2} \frac{P(t_{\mathrm{k}}) - P_{\mathrm{k}}(t_{\mathrm{k}})}{a_{\mathrm{k}}^2(t_{\mathrm{k}}) - b_1^2}$$

$$u^{\mathrm{k}}_{r,1}(t_{\mathrm{k}}, r) = \frac{1 - v_1^{\mathrm{k}}}{E_1^{\mathrm{k}}} \frac{P(t_{\mathrm{k}})b_1^2 - P_{\mathrm{k}}(t_{\mathrm{k}})a_{\mathrm{k}}^2(t_{\mathrm{k}})}{a_{\mathrm{k}}^2(t_{\mathrm{k}}) - b_1^2} r + \frac{1 + v_1^{\mathrm{k}}}{E_1^{\mathrm{k}}} \frac{a_{\mathrm{k}}^2(t_{\mathrm{k}}) b_1^2}{r} \frac{P(t_{\mathrm{k}}) - P_{\mathrm{k}}(t_{\mathrm{k}})}{a_{\mathrm{k}}^2(t_{\mathrm{k}}) - b_1^2}$$

$$\sigma^{\mathrm{k}}_{\substack{r,2\\ \vartheta,2}}(t_{\mathrm{k}}, r) = \frac{P(t_{\mathrm{k}})a_{\mathrm{k}}^2(t_{\mathrm{k}})}{b^2 - a_{\mathrm{k}}^2(t_{\mathrm{k}})}\left(1 \mp \frac{b^2}{r^2}\right)$$

$$u^{\mathrm{k}}_{r,2}(t_{\mathrm{k}}, r) = \frac{P(t_{\mathrm{k}})a_{\mathrm{k}}^2(t_{\mathrm{k}})}{E_{\mathrm{k}}^2(b^2 - a_{\mathrm{k}}^2(t_{\mathrm{k}}))}\left[(1 - v_2^{\mathrm{k}})r + (1 + v_2^{\mathrm{k}})\frac{b^2}{r}\right] \tag{4}$$

where $v_{\mathrm{i}}^{\mathrm{k}} = v_{\mathrm{i}}^{\mathrm{k}}(\theta_{\mathrm{i,k}})$, $E_{\mathrm{i}}^{\mathrm{k}} = E_{\mathrm{i}}^{\mathrm{k}}(\theta_{\mathrm{i,k}})$ - are the Poisson constant and Young modulus for each phase.

Following a method explained in [1] for the condition for an extreme of an entropy function (variation principle of Gibbs)

$$S^{(\mathrm{k})} = \frac{V^{(\mathrm{k})} - A^{(\mathrm{k})} + \mu\rho_1 a(t_{\mathrm{k}}) + \alpha_1\left[\theta_2(t_{\mathrm{k}}, b) - \theta_{\mathrm{cp}}\right]}{\theta^0}$$

with $A^{(k)}$ - activity of external forces and $V^{(k)}$ - internal energy. After some transformations, the following expression is derived for the definition of the law of motion of a phase boundary in the k interval:

$$\rho_{1,\mathrm{k}} c_{1,\mathrm{k}} \Delta\theta_{1,\mathrm{k}} + \mu\rho_{1,\mathrm{k}} + \frac{\alpha_{\mathrm{k}}^2 \lambda_{2,\mathrm{k}}(\theta^0 - \theta_{\mathrm{cp}})}{b a_{\mathrm{k}}\left(\dfrac{\lambda_{2,\mathrm{k}}}{b} + \alpha_{\mathrm{k}} \ln \dfrac{b}{a_{\mathrm{k}}}\right)^2} + \Psi = 0 \tag{5}$$

where Ψ is a function of time, temperature, heating, mechanical properties, and the geometrical arguments of the system. This function is developed in ref. [1,2].

The algorithm for the definition of the thermomechanical status for the Ω phase transition of a body and the j period of time is constructed in the following way:

1) we fit the phase $a_j(t_k)$ boundary and from the solution of the equation for heat conduction the temperatures $\theta_i^j(x)$ (i = 1,2) in the body Ω is deduced;
2) we solve a contact task for two bodies with the constant area $\omega_1^j(t_k)$ and $\omega_2^j(t_k)$, and define the field of movements $\boldsymbol{u}_j^{(k)}$;
3) for available values ($\boldsymbol{u}_j^{(k)}$, $\theta_j^{(k)}$, $a_j^{(k)}$) we institute the internal energy $V_j^{(k)}$ and entropy $S_j^{(k)}$ of the body Ω;
4) from the first law of thermodynamics and for the maximum of entropy $S_j^{(k)}$ for the body Ω the true position the phase boundar $a_{*j}^{(k)}$ is determined;
5) at the found position of the phase $a_{*j}^{(k)}$ the true boundary temperature $\theta_{*j}^{(k)}$ and the true field of movements $\boldsymbol{u}_{*j}^{(k)}$ are determined;
6) in the following period of time $j + 1$ the reference temperature and the position of the phase boundary will match those at the j- interval, i.e.

$$\theta_{\mathrm{j+1}}^{(\mathrm{k+1})} = \theta_{*\mathrm{j}}^{(\mathrm{k})},\ a_{\mathrm{j+1}}^{(\mathrm{k+1})} = a_{*\mathrm{j}}^{(\mathrm{k})}$$

Thus, the deduced algorithm for the solution of the equation of thermoviscoelasticity takes into accounts the previous history of all processes in each period of time and the motion of the boundary of phases, defined with the variation principle of Gibbs.

The numerical solution of the equations (1) and (2) for the shaping of the status of the strained state of the rising visco-elastic cylinder in presence of the phase transition due to the heat interaction of an exterior lateral area with the environment are obtained with the solution of the equation (5) and are shown in **Figures 1 to 3** for the following values b = 0.06 м, b_0 = 0.05 м, b_1 = 0.02 м, v_1 = 0.5, v_1 = 0.29.

In these figures the influence of heat interchange between the cylinder external surface and the environment on the law of motion for the phase boundary, the time change of the temperature field θ_2 (t,b) and the temporal θ_2 (t,r) relation for different points of the barrel

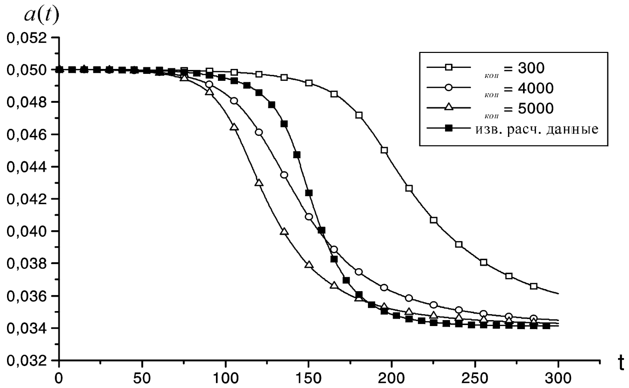

Figure 1: The law of motion of a phase boundary
Slika 1: Zakon o premikanju fazne meje

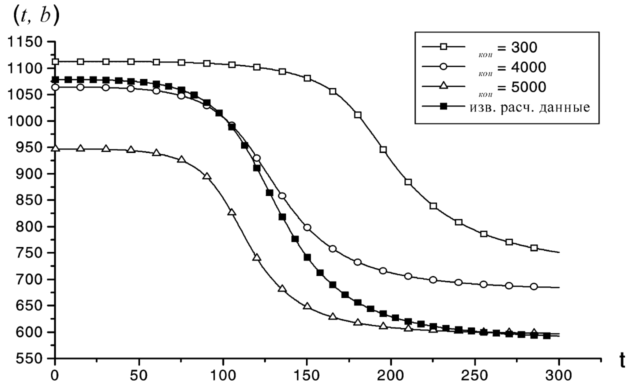

Figure 2: Change of temperature of a lateral area
Slika 2: Sprememba temperature lateralne površine

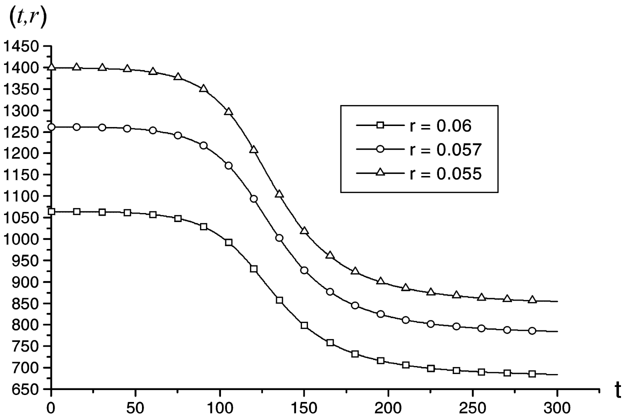

Figure 3: Change of temperature for different points of the cross section of the cylinder
Slika 3: Sprememba temperature za različne točke na prerezu valja

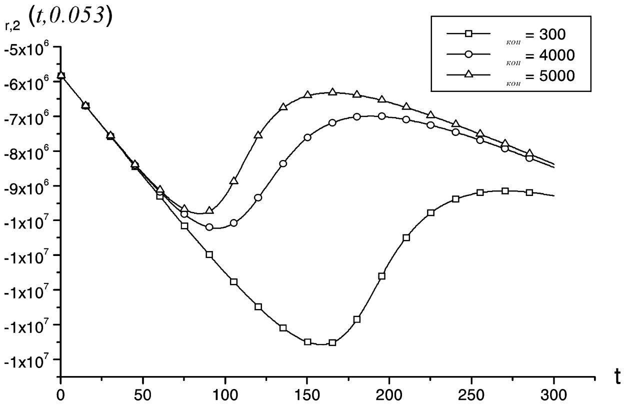

Figure 4: Allocation of radial stresses
Slika 4: Radialne napetosti na različnih prerezih

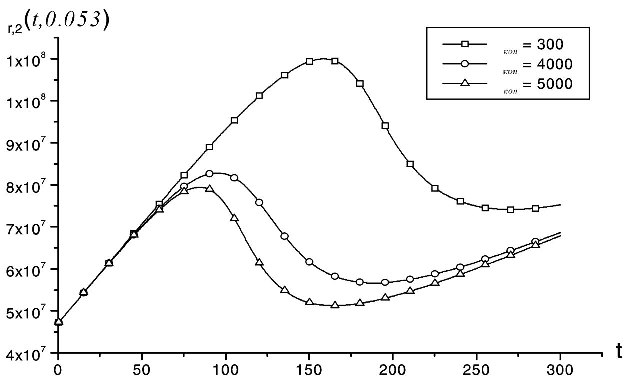

Figure 5: Allocation of horizontal stresses
Slika 5: Vodoravne napetosti na različnih prerezih

cross section at fixed value $\alpha_{\text{кон}} = 4 \cdot 10^3$ Bm/(м$^2 \cdot$ K) are presented.

The functionability of the algorithm is checked with comparison of the obtained results with data in ref. [7,8,9] for the motion of the phase boundary at $\alpha_{\text{кон}} = 3 \cdot 10^3$ Bm/(м$^2 \cdot$ K) (material - pig iron) and for the temperature change of an external surface at $100 < \alpha_{\text{кон}} < 5 \cdot 10^3$ Bm/(м$^2 \cdot$ K).

In **Figures 4 and 5** the time evolution of the radial and horizontal pressure $\alpha_{\text{кон}}$ for the cross section r = 0.053 м of the solid phase of the cylinder at different values is shown. With the increase of the intensity of the heat interchange of an exterior surface with the environment the stresses increase also. Also the increase of an absolute value of growth $\alpha_{\text{кон}}$ with safe characteristic of stress in time is watched.

The analysis of the obtained results shows that the modification of heat interchange with the environment and the geometrical shape influence the process of crystallization and, consequently, and also the temperature and the stress field.

3 THERMOMECHANICAL STATUS OF A TWO-PHASE ROD IN DEPENDENCE OF HEATING AND THE TEMPERATURE CHANGE

We shall consider a straight-line rod of length l. Let's enter in a undeformed configuration the Cartesian coordinate system (x_1, x_2, x_3) with the axis x_3 in the axis of the rod. One extremity $x = 0$ of the rod is rigidly restrained, and the other $x = l$ is free. Let's assume, that the matter of the rod consists of two phases - hyperthermal and cold. The heat conduction of the cold rod is described with equations of state for the growth of a visco-elastic field in a hyperthermal - elastic body. Let's assume that θ^0 is the phase transition temperature, therefore, at $\theta_2 < \theta^0$ an element of the material is in solid and at $\theta_1 > \theta^0$ - in hyperthermal phase. Let us further assume, that up to a strain level the matter of the rod is a solid phase in the range $\omega_2(t) = \{l_0 \le x \le l\}$, and it is a hyperthermal phase in the range $\omega_1(t) = \{0 \le x \le l_0\}$. The heat interaction of side and end surfaces of a solid phase of the rod with the environment is characterized with the convection coefficients for heat rejection, $\alpha_{1\text{кон}}$, $\alpha_{2\text{кон}}$ and irradiation. The hyperthermal phase is supposed to be heat-insulated.

If at the instant $t = 0$ the longitudinal pressure load of intensity $P(t)$ is applied the removal of heat to the environment begins on the free end of the rod at $x = l$. Parallely, part of the rod solidifies and its length varies accordingly to law $x = a = a(t)$.

Let us define the law of motion of the phase boundary with $a = a(t)$, of the temperature with $\theta_i = \theta_i(t,x)$ (i = 1,2) and the strained state in the rod due to an axial stress.

The mathematical solution of the equation for the change of the conditions of phase transition requires the calculations of unknown characteristics using the following equations:

$$\varepsilon_i(t,x) = \frac{\partial u_i(t,x)}{\partial x}, \; \frac{\partial \sigma_i(t,x)}{\partial x} - r_i P(t) = 0 \quad (\text{i} = 1,2),$$

$$\sigma_i(t,x) = E_1 \varepsilon_1(t,x)$$

$$\sigma_2(t,x) = E_2(t - \tau^*(x))(\varepsilon_2(t,x) -$$

$$-\int_{\tau^*(x)}^{t} \varepsilon_2(\tau,x) R(t-\tau^*(x),\ \tau-\tau^*(x))\mathrm{d}\tau \qquad (6)$$

$$\rho_1 c_1 \frac{\partial \theta_1(t,x)}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_1 \frac{\partial \theta_1(t,x)}{\partial x}\right)$$

$$\rho_2 c_2 \frac{\partial \theta_2(t,x)}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_2 \frac{\partial \theta_2(t,x)}{\partial x}\right) + \Phi(\theta_2,\alpha_2) \qquad (7)$$

In the equations (6) and (7) the index "1" is related to arguments of the matter in the hyperthermal, and the index "2" - in the solid phase. In the equations are: E_1 - the Young's modulus; $E_2(t-\tau^*(r))$ - the elastic and momentary strain module; $R(t-\tau^*(x),\tau-\tau^*(x))$ - the core of a relaxation of a visco-elastic material; $\tau^*(x)$ - the moment of transition of an element of the material to the solid modular status; $\rho_i, c_i, \lambda_i = \lambda_i(\theta_i)$, $\alpha_i = \alpha_i(\theta_i)$ - the density, the thermal capacity and the thermal conductivity of the material related to the coefficient of heat rejection [5], which describes the intensity of convection and radiative heat interchange with the environment; $\theta_i(t,x)$ - the allocation of temperature in the processed phase of the rod and $\Phi(\theta_2 \alpha_2) = -\frac{\alpha_{2,\text{кои}}}{h\lambda_2}\left[\theta_2(t_k,x)\right] - -\frac{\varepsilon_{12}\sigma_0}{h\lambda_2}\left[\theta_2^4(t_k,x)-\theta_{\text{cp}}^4\right] = -\frac{\alpha_2}{h\lambda_2}\left[\theta_2(t_k,x)-\theta_{\text{cp}}\right]$ - the power of interior effluents of heat in a solid phase which takes into account also the convection and radiative heat interchange of the lateral area of the rod with the environment.

The boundary and the initial conditions of the equations are:

$$\sigma_2(t,l)=0, \quad u_1(t,0)=0,$$

$$\sigma_1(t,a(t)) = \sigma_2(t,a(t)), \quad u_1(t,a(t)) = u_2(t,a(t)) \qquad (8)$$

$$\theta_1(t,0)=f, \ \ \theta_2(t,a(t)) = \theta_1(t,a(t)) - \Delta) = \theta^0$$

$$-\lambda_2 \left.\frac{\partial \theta_2(t,x)}{\partial x}\right|_{x=l} = \alpha_{1,\text{кои}}\left[\theta_2(t,x)-\theta_{\text{cp}}\right]\Big|_{x=l} + \varepsilon_{12}\sigma_0\theta_2^4(t,l) = \alpha_1\left[\theta_2(t,l)-\theta_{\text{cp}}\right],$$

$$-\lambda_1 \left.\frac{\partial \theta_1(t,x)}{\partial x}\right|_{x=a(t)} = -\lambda_2 \left.\frac{\partial \theta_2(t,x)}{\partial x}\right|_{x=a(t)} + \rho_1\mu\frac{\mathrm{d}a(t)}{\mathrm{d}t} \qquad (9)$$

$$\theta_1(t,x)\big|_{t=0} = \theta_1(0,x),\ \theta_2(t,x)\big|_{t=0} = \theta_2(0,x) \qquad (10)$$

The equations (6) to (10) are the formulation of the thermoviscoelasticity for the phase boundary deplacement in the two-phase rod in self contained shape. With difference to the solutions in ref. [2,10] the volume is considered also. The equations for heat conduction in the relations (7), (9), (10) allow for the non linearity of first kind with regard of the relation of the heating properties of a material in dependence upon the temperature. For the solution of the equations (6) to (10) the approximated analytical method offered in the present paper is used.

According to [1] the tight-strained state in a rod on each period of time can be determined with the solution of the contact task of two bodies occupying the areas $\omega_1^j(t_k)$ and $\omega_2^j(t_k)$ determined with the equations:

$$\sigma_1^j(t_k,x) = -P(t_k)\left[\rho_2(l-a_j(t_k)) + \rho_1(a_j(t_k)-x)\right],$$

$$\sigma_2^j(t_k,x) = -\rho_2 P(t_k)(l-x),$$

$$u_1^j(t,x) = -\frac{P(t_k)}{E_1}\left[\rho_2(l-a_j(t_k))x - \frac{\rho_1}{2}\left[(a_j(t_k)-x)^2 - (a_j(t_k))^2\right]\right]$$

$$u_2^j(t_k,x) = \frac{\rho_2 P(t_k)}{2E_2}\left[(l-x)^2 - (l-a_j(t_k))^2\right] - -\frac{P(t_k)a_j(t_k)}{E_1}\left[\rho_2(l-a_j(t_k)) + \frac{\rho_1 a_j(t_k)}{2}\right]$$

where the index j means the fitting to j-period of time and; $a_j(t_k)$ - the law of motion of a phase boundary.

According to [4] the temperature field for the *j*-period of time is:

$$\theta_1^j(t_k,x) = \frac{1}{\lambda_1}B(\theta^0-\theta_{\text{cp}})\left[\frac{\alpha_1 - B}{\alpha_1 - B - e^{2B_1(l-a_j(t_k))}(B+\alpha_1)} - -\frac{\alpha_1 + B}{\alpha_1 + B - e^{-2B_1(l-a_j(t_k))}(\alpha_1 - B)}\right]x + f,$$

$$\theta_2^j(t,x) = \frac{(\alpha_1 - B)(\theta^0-\theta_{\text{cp}})e^{B_1 x}}{e^{B_1 a_j(t_k)}\left[\alpha_1 - B\right] - e^{B_1(2l-a_j(t_k))}\left[B+\alpha_1\right]} - -\frac{(\alpha_1 + B)(\theta^0-\theta_{\text{cp}})e^{-B_1 x}}{e^{-B_1(2l-a_j(t_k))}\left[\alpha_1 - B\right] - e^{-B_1 a_j(t_k)}\left[\alpha_1 + B\right]} + \theta_{\text{cp}},$$

where $B = \sqrt{\frac{\alpha_2\lambda_2}{h}}$, $B_1 = \sqrt{\frac{\alpha_2}{\lambda_2 h}}$.

For the definition of the function $a(t)$ and the connection of temperature and mechanical fields we shall, as earlier, take advantage of the variation principle of Gibbs, according to which the point $a_j = a_j(t_k)$ describing j-position of a phase boundary in the *j*-interval conveys the maximum rating of the entropy function.

As result and similarly to the previous task, we will have a differential equation for the law of motion of a phase boundary with the solution in **Figure 6, 8 and 9**.

In **Figure 6a** the relation of temperature of a solid phase in a two-phase rod as dependence upon the size of the solid phase and of the time is shown. The relation of temperature versus time in the section $x = 0.38$ is shown in **Figure 6b**, and the law of motion of the phase boundary in **Figure 6c**. In **Figure 7** the stress distribution in the solid rod in dependence of the cooling time is shown.

In case of combined effect of heat interaction of side and end surfaces with the environment and of phase transition, the results are shown in a **Figure 8, 9**.

It is clear from **Figure 8** that in presence of heat interchange of a lateral area with the environment the solidification time of a material is essentially decreased.

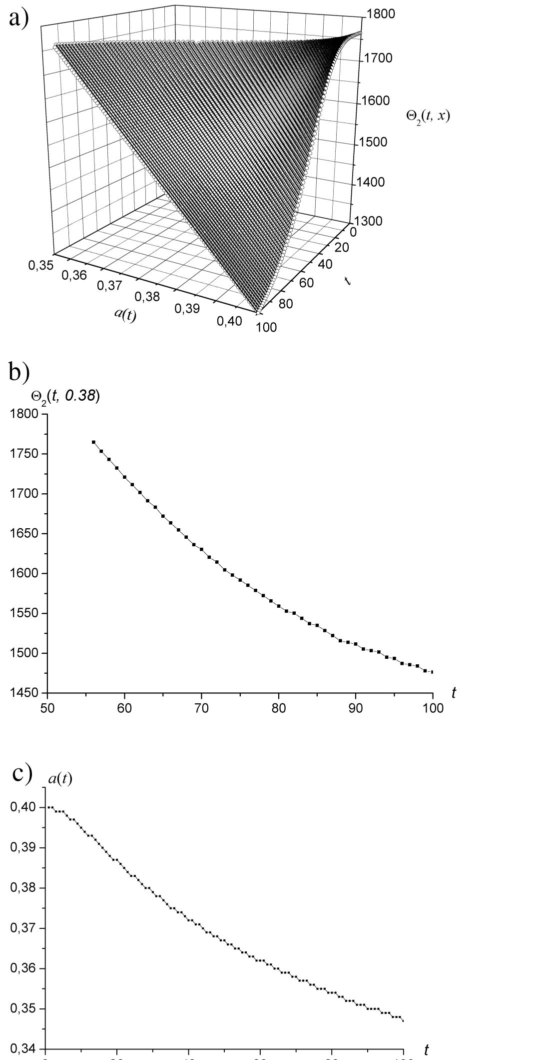

Figure 6: The temperature field in the solid phase of the rod: a - location of temperature in the solid phase, b - relation temperature versus time in the section x = 0.38, c - law of motion of the phase boundary

Slika 6: Temperaturno polje v trdni palici: a - lokacija temperature v palici, b - odnos temperature, čas na prerezu x = 0,38, c - zakon o premikanju meje med fazama

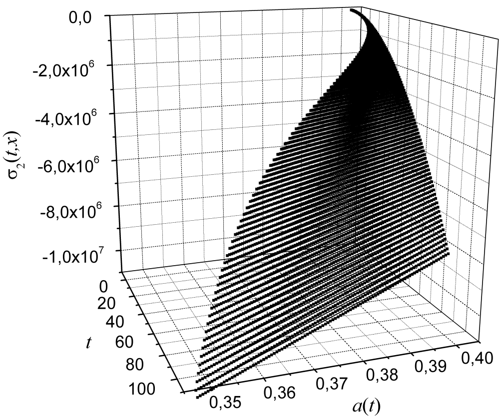

Figure 7: The stress distribution in solid of rod

Slika 7: Porazdelitev napetosti v trdni palici

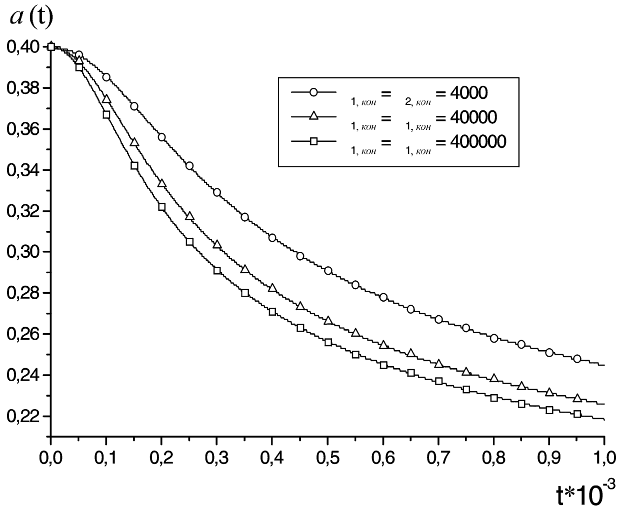

Figure 8: The law of motion of a phase boundary for different intensity of cooling

Slika 8: Zakon o premikanju fazne meje za različno intenzivnost ohlajanja

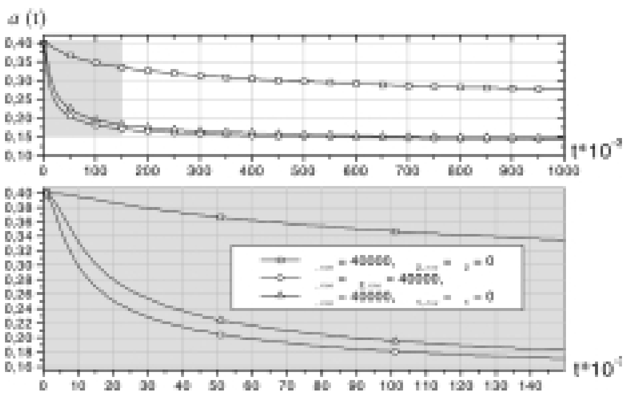

Figure 9: The law of motion of a phase boundary for different cooling conditions

Slika 9: Zakon o premikanju fazne meje za različne pogoje ohlajanja

The curves in **Figure 9** show that the heat interchange from the lateral area of a rod has a greater influence on the crystallization process, than the heat interchange at the end, as already determined with computation and verified with experimental data [8].

4 CONCLUSIONS

The developed algorithm and the obtained solutions can be used for the simulation of nonlinear thermo-mechanical processes related to phase transitions and allow to take into account the interaction of fields of different physical nature.

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