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From Italian domination in lexicographic product graphs to w-domination in graphs*

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Abstract

In this paper, we show that the Italian domination number of every lexicographic product graph $G \circ H$ can be expressed in terms of five different domination parameters of G. These parameters can be defined under the following unified approach, which encompasses the definition of several well-known domination parameters and introduces new ones.

Let $N(v)$ denote the open neighbourhood of $v \in V(G)$, and let $w = (w_0, w_1, \dots, w_l)$ be a vector of nonnegative integers such that $w_0 \geq 1$. We say that a function $f \colon V(G) \to \{0, 1, \dots, l\}$ is a w-dominating function if $f(N(v)) = \sum_{u \in N(v)} f(u) \geq w_i$ for every vertex v with $f(v) = i$. The weight of f is defined to be $\omega(f) = \sum_{v \in V(G)} f(v)$. The w-domination number of G, denoted by $\gamma_w(G)$, is the minimum weight among all w-dominating functions on G.

Specifically, we show that $\gamma_I(G \circ H) = \gamma_w(G)$, where $w \in \{2\} \times \{0, 1, 2\}^l$ and $l \in \{2, 3\}$. The decision on whether the equality holds for specific values of $w_0, \dots, w_l$ will depend on the value of the domination number of H. This paper also provides preliminary results on $\gamma_w(G)$ and raises the challenge of conducting a detailed study of the topic.

Keywords: Italian domination, w-domination, k-domination, k-tuple domination, lexicographic product graph.

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1 Introduction

Let G be a graph, l a positive integer, and $f\colon V(G) \to \{0,\dots,l\}$ a function. For every $i \in \{0,\dots,l\}$, we define $V_i = \{v \in V(G) : f(v) = i\}$. We will identify f with the subsets $V_0,\dots,V_l$ associated with it, and so we will use the unified notation $f(V_0,\dots,V_l)$ for the function and these associated subsets. The weight of f is defined to be

$$\omega(f) = f(V(G)) = \sum_{i=1}^{l} i|V_i|.$$

An *Italian dominating function* (IDF) on a graph G is a function $f(V_0, V_1, V_2)$ satisfying that $f(N(v)) = \sum_{u\in N(v)} f(u) \geq 2$ for every $v \in V_0$, where $N(v)$ denotes the open neighbourhood of v. Hence, $f(V_0, V_1, V_2)$ is an IDF if $N(v) \cap V_2 \neq \varnothing$ or $|N(v) \cap V_1| \geq 2$ for every $v \in V_0$. The *Italian domination number*, denoted by $\gamma_I(G)$, is the minimum weight among all IDFs on G. This concept was introduced by Chellali et al. in [6] under the name of Roman $\{2\}$-domination. The term “Italian domination” comes from a subsequent paper by Henning and Klostermeyer [13].

In this paper we show that the Italian domination number of every lexicographic product graph $G \circ H$ can be expressed in terms of five different domination parameters of G. These parameters can be defined under the following unified approach.

Let $w = (w_0,\dots,w_l)$ be a vector of nonnegative integers such that $w_0 \geq 1$. We say that $f(V_0,\dots,V_l)$ is a *w-dominating function* if $f(N(v)) \geq w_i$ for every $v \in V_i$. The *w-domination number* of G, denoted by $\gamma_w(G)$, is the minimum weight among all w-dominating functions on G. For simplicity, a w-dominating function f of weight $\omega(f) = \gamma_w(G)$ will be called a $\gamma_w(G)$-function.

This unified approach allows us to encompass the definition of several well-known domination parameters and introduce new ones. For instance, we would highlight the following particular cases of known domination parameters that we define here in terms of w-domination.

- The *domination number* of G is defined to be $\gamma(G) = \gamma_{(1,0)}(G) = \gamma_{(1,0,0)}(G)$. Obviously, every $\gamma_{(1,0,0)}(G)$-function $f(V_0, V_1, V_2)$ satisfies that $V_2 = \varnothing$ and V_1 is a dominating set of cardinality $|V_1| = \gamma(G)$, i.e., V_1 is a $\gamma(G)$-set.

- The *total domination number* of a graph G with no isolated vertex is defined to be $\gamma_t(G) = \gamma_{(1,1)}(G) = \gamma_{(1,1,w_2,\dots,w_l)}(G)$, for every $w_2,\dots,w_l \in \{0,1\}$. Notice that there exists a $\gamma_{(1,1,w_2,\dots,w_l)}(G)$-function $f(V_0, V_1,\dots,V_l)$ such that $V_i = \varnothing$ for every $i \in \{2,\dots,l\}$ and V_1 is a total dominating set of cardinality $|V_1| = \gamma_t(G)$, i.e., V_1 is a $\gamma_t(G)$-set.

- Given a positive integer k, the *k-domination number* of a graph G is defined to be $\gamma_k(G) = \gamma_{(k,0)}(G)$. In this case, V_1 is a k-dominating set of cardinality $|V_1| = \gamma_k(G)$, i.e., V_1 is a $\gamma_k(G)$-set. The study of k-domination in graphs was initiated by Fink and Jacobson [8] in 1984.

- Given a positive integer k, the *k-tuple domination number* of a graph G of minimum degree $\delta \geq k-1$ is defined to be $\gamma_{\times k}(G) = \gamma_{(k,k-1)}(G)$. In this case, V_1 is a k-tuple dominating set of cardinality $|V_1| = \gamma_{\times k}(G)$, i.e., V_1 is a $\gamma_{\times k}(G)$-set. In particular, $\gamma_{\times 1}(G) = \gamma(G)$ and $\gamma_{\times 2}(G)$ is known as the *double domination number* of G. This parameter was introduced by Harary and Haynes in [9].

- Given a positive integer k, the *k-tuple total domination number* of a graph G of minimum degree $\delta \geq k$ is defined to be $\gamma_{\times k,t}(G) = \gamma_{(k,k)}(G)$. In particular, $\gamma_{\times 1,t}(G) = \gamma_t(G)$ and $\gamma_{\times 2,t}(G)$ is known as the *double total domination number*, and V_1 is a double total dominating set of cardinality $|V_1| = \gamma_{\times 2,t}(G)$, i.e., V_1 is a $\gamma_{\times 2,t}(G)$-set. The k-tuple total domination number was introduced by Henning and Kazemi in [12].

- The *Italian domination number* of G is defined to be $\gamma_{_I}(G) = \gamma_{(2,0,0)}(G)$. As mentioned earlier, this parameter was introduced by Chellali et al. in [6] under the name of Roman $\{2\}$-domination number. The concept was studied further in [13, 16].

- The *total Italian domination number* of a graph G with no isolated vertex is defined to be $\gamma_{tI}(G) = \gamma_{(2,1,1)}(G)$. This parameter was introduced by Cabrera et al. in [4], and independently by Abdollahzadeh Ahangar et al. in [1], under the name of total Roman $\{2\}$-domination number. The total Italian domination number of lexicographic product graphs was studied in [5].

- The *$\{k\}$-domination number* of G is defined to be $\gamma_{\{k\}}(G) = \gamma_{(k,k-1,\ldots,1,0)}(G)$. This parameter was introduced by Domke et al. in [7] and studied further in [3, 15, 17].

Notice that the concept of Y-dominating function introduced by Bange et al. [2] is quite different from the concept of w-dominating function introduced in this paper. Given a set Y of real numbers, a function $f\colon V(G) \to Y$ is a Y-dominating function if $f(N[v]) = f(v) + \sum_{u \in N(v)} f(u) \geq 1$ for every $v \in V(G)$. The Y-domination number, denoted by $\gamma_{_Y}(G)$, is the minimum weight among all Y-dominating functions on G. Hence, if $Y = \{0, 1, \ldots, l\}$, then $\gamma_{_Y}(G) = \gamma_{(1,0,\ldots,0)}(G) = \gamma(G)$.

For the graphs shown in Figure 1 we have the following values.

- $\gamma_I(G_1) = \gamma_{(2,1,0)}(G_1) = \gamma_{(2,2,0)}(G_1) = 4 < 6 = \gamma_{(2,2,1)}(G_1) = \gamma_{(2,2,2)}(G_1)$.

- $\gamma_I(G_2) = \gamma_{(2,1,0)}(G_2) = \gamma_{(2,2,0)}(G_2) = \gamma_{(2,2,1)}(G_2) = \gamma_{(2,2,2)}(G_2) = 3$.

- $\gamma_I(G_3) = \gamma_{(2,1,0)}(G_3) = 6 < 8 = \gamma_{(2,2,0)}(G_3) = \gamma_{(2,2,1)}(G_3) = \gamma_{(2,2,2)}(G_3)$.

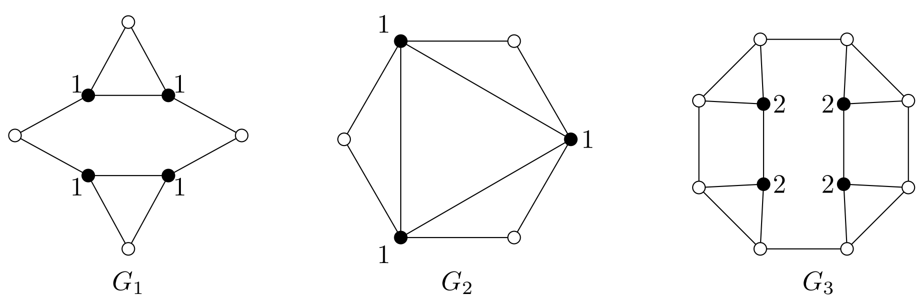

Figure 1: The labels of black-coloured vertices describe a $\gamma_{(2,1,0)}(G_1)$-function, a $\gamma_{(2,2,0)}(G_2)$-function and a $\gamma_{(2,2,2)}(G_3)$-function, respectively.

The remainder of the paper is organized as follows. In Section 2 we show that for any graph G with no isolated vertex and any nontrivial graph H with $\gamma(H) \neq 3$ or $\gamma_I(H) \neq 3$, the Italian domination number of $G \circ H$ equals one of the following parameters: $\gamma_{(2,1,0)}(G)$, $\gamma_{(2,2,0)}(G)$, $\gamma_{(2,2,1)}(G)$ or $\gamma_{(2,2,2)}(G)$. The specific value $\gamma_I(G \circ H)$ takes depends on the value of $\gamma(H)$. For the cases where $\gamma_I(H) = \gamma(H) = 3$, we show that $\gamma_I(G \circ H) = \gamma_{(2,2,2,0)}(G)$. Section 3 is devoted to providing some preliminary results on w-domination. We first describe some general properties of $\gamma_w(G)$ and then dedicate a subsection to each of the specific cases declared of interest in Section 2.

We assume that the reader is familiar with the basic concepts, notation and terminology of domination in graph. If this is not the case, we suggest the textbooks [10, 11, 14]. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 Italian domination in lexicographic product graphs

The *lexicographic product* of two graphs G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$.

Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_u. Moreover, the neighbourhood of $(x, y) \in V(G) \times V(H)$ will be denoted by $N(x, y)$ instead of $N((x, y))$. Analogously, for any function f on $G \circ H$, the image of (x, y) will be denoted by $f(x, y)$ instead of $f((x, y))$.

Lemma 2.1. *For any graph G with no isolated vertex and any nontrivial graph H with $\gamma_I(H) \neq 3$ or $\gamma(H) \neq 3$, there exists a $\gamma_I(G \circ H)$-function f satisfying that $f(V(H_u)) \leq 2$ for every $u \in V(G)$.*

Proof. Given an IDF f on $G \circ H$, we define the set $R_f = \{x \in V(G) : f(V(H_x)) \geq 3\}$. Let f be a $\gamma_I(G \circ H)$-function such that $|R_f|$ is minimum among all $\gamma_I(G \circ H)$-functions. Suppose that $|R_f| \geq 1$. Let $u \in R_f$ such that $f(V(H_u))$ is maximum among all vertices belonging to R_f. Suppose that $f(V(H_u)) > \gamma_I(H)$. In this case we take a $\gamma_I(H)$-function h and construct an IDF g defined on $G \circ H$ as $g(u, y) = h(y)$ for every $y \in V(H)$ and $g(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u\}$ and $y \in V(H)$. Obviously, $\omega(g) < \omega(f)$, which is a contradiction. Thus, $3 \leq f(V(H_u)) \leq \gamma_I(H_u) = \gamma_I(H)$. Now, we analyse the following two cases.

Case 1. $f(V(H_u)) \geq 4$. Let $u' \in N(u)$ and $v \in V(H)$. We define a function f' on $G \circ H$ as $f'(u, v) = f'(u', v) = 2$, $f'(u, y) = f(u', y) = 0$ for every $y \in V(H) \setminus \{v\}$, and $f'(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$ and $y \in V(H)$. Notice that f' is an IDF on $G \circ H$ with $\omega(f') \leq \omega(f)$ and $|R_{f'}| < |R_f|$, which is a contradiction.

Case 2. $f(V(H_u)) = 3$. Suppose that $\gamma_I(H) \neq 3$. Since $\gamma_I(H) \geq 4$, there exist $u' \in N(u)$ and $v \in V(H)$ such that $f(u', v) \geq 1$. Hence, the function f' defined in Case 1 is an IDF on $G \circ H$ with $\omega(f') \leq \omega(f)$ and $|R_{f'}| < |R_f|$, which is again a contradiction.

Thus, $\gamma_I(H) = 3$, and so $\gamma(H) \neq 3$, which implies that $\gamma(H) = 2$. Let $\{v_1, v_2\}$ be a $\gamma(H)$-set. Let $u' \in N(u)$ and $v' \in V(H)$ such that $f(u', v') = \max\{f(u', y) : y \in V(H)\}$. Consider the function f' defined as $f'(u, v_1) = f'(u, v_2) = 1$, $f'(u, y) = 0$ for every $y \in V(H) \setminus \{v_1, v_2\}$, $f'(u', v') = \min\{2, f(u', v') + 1\}$, $f'(u', y) = 0$ for every $y \in V(H) \setminus \{v'\}$, and $f'(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$ and $y \in V(H)$.

Notice that f' is an IDF on $G \circ H$ with $\omega(f') \leq \omega(f)$ and $|R_{f'}| < |R_f|$, which is a contradiction.

Therefore, $R_f = \varnothing$, and the result follows. □

Theorem 2.2. *The following statements hold for any graph G with no isolated vertex and any nontrivial graph H with* $\gamma_I(H) \neq 3$ *or* $\gamma(H) \neq 3$.

(i) *If* $\gamma(H) = 1$, *then* $\gamma_I(G \circ H) = \gamma_{(2,1,0)}(G)$.

(ii) *If* $\gamma_2(H) = \gamma(H) = 2$, *then* $\gamma_I(G \circ H) = \gamma_{(2,2,0)}(G)$.

(iii) *If* $\gamma_2(H) > \gamma(H) = 2$, *then* $\gamma_I(G \circ H) = \gamma_{(2,2,1)}(G)$.

(iv) *If* $\gamma(H) \geq 3$, *then* $\gamma_I(G \circ H) = \gamma_{(2,2,2)}(G)$.

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I(G \circ H)$-function which satisfies Lemma 2.1. Let $f'(X_0, X_1, X_2)$ be the function defined on G by $X_1 = \{x \in V(G) : f(V(H_x)) = 1\}$ and $X_2 = \{x \in V(G) : f(V(H_x)) = 2\}$. Notice that $\gamma_I(G \circ H) = \omega(f) = \omega(f')$. We claim that f' is a $\gamma_{(w_0,w_1,w_2)}(G)$-function. In order to prove this and find the values of w_0, w_1 and w_2, we differentiate the following three cases.

Case 1. $\gamma(H) = 1$. Assume that $x \in X_0$. Since $f(V(H_x)) = 0$, for any $y \in V(H)$ we have that $f(N(x,y) \setminus V(H_x)) \geq 2$. Thus, $f'(N(x)) \geq 2$. Now, assume that $x \in X_1$, and let $(x,y) \in V_1$ be the only vertex in $V(H_x)$ such that $f(x,y) > 0$. Since $\gamma(H) = 1$, for any $z \in V(H) \setminus \{y\}$, we have that $f(N(x,z) \setminus V(H_x)) \geq 1$, which implies that $f'(N(x)) \geq 1$. Therefore, f' is a $(2,1,0)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,1,0)}(G)$.

Now, for any $\gamma_{(2,1,0)}(G)$-function $g(W_0, W_1, W_2)$ and any universal vertex v of H, the function $g'(W_0', W_1', W_2')$, defined by $W_2' = W_2 \times \{v\}$ and $W_1' = W_1 \times \{v\}$, is an IDF on $G \circ H$. Therefore, $\gamma_I(G \circ H) \leq \omega(g') = \omega(g) = \gamma_{(2,1,0)}(G)$.

Case 2. $\gamma(H) = 2$. As in Case 1 we conclude that $f'(N(x)) \geq 2$ for every $x \in X_0$. Now, assume that $x \in X_1$, and let $(x,y) \in V_1$ be the only vertex in $V(H_x)$ such that $f(x,y) > 0$. Since $\gamma(H) = 2$, there exists a vertex $z \in V(H)$ such that $(x,z) \in V_0 \setminus N(x,y)$. Hence, $f(N(x,z) \setminus V(H_x)) \geq 2$, which implies that $f'(N(x)) \geq 2$. Therefore, f' is a $(2,2,0)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,2,0)}(G)$.

Now, if $\gamma_2(H) > \gamma(H) = 2$, then for every $x \in X_2$, there exists $y \in V(H)$ such that $(x,y) \in V_0$ and $f(N(x,y) \cap V(H_x)) \leq 1$, which implies that $f(N(x,y) \setminus V(H_x)) \geq 1$, and so $f'(N(x)) \geq 1$. Hence, f' is a $(2,2,1)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,2,1)}(G)$.

On the other side, if $\gamma_2(H) = 2$, then for any $\gamma_{(2,2,0)}(G)$-function $g(W_0, W_1, W_2)$ and any $\gamma_2(H)$-set $S = \{v_1, v_2\}$, the function $g'(W_0', W_1', W_2')$, defined by $W_1' = (W_1 \times \{v_1\}) \cup (W_2 \times S)$ and $W_2' = \varnothing$, is an IDF on $G \circ H$. Therefore, $\gamma_I(G \circ H) \leq \omega(g') = \omega(g) = \gamma_{(2,2,0)}(G)$.

Finally, if $\gamma_2(H) > \gamma(H) = 2$ then we take a $\gamma_{(2,2,1)}(G)$-function $h(Y_0, Y_1, Y_2)$ and a $\gamma(H)$-set $S' = \{v_1', v_2'\}$, and construct a function $h'(Y_0', Y_1', Y_2')$ on $G \circ H$ by making $Y_1' = (Y_1 \times \{v_1'\}) \cup (Y_2 \times S')$ and $Y_2' = \varnothing$. Obviously, h' is an IDF on $G \circ H$, and so we can conclude that $\gamma_I(G \circ H) \leq \omega(h') = \omega(h) = \gamma_{(2,2,1)}(G)$.

Case 3. $\gamma(H) \geq 3$. In this case, for every $x \in V(G)$, there exists $y \in V(H)$ such that $f(N[(x,y)] \cap V(H_x)) = 0$. Hence, $f(N(x,y) \setminus V(H_x)) \geq 2$, which implies that $f'(N(x)) \geq 2$ for every $x \in V(G)$. Therefore, f' is a $(2,2,2)$-dominating function on G and, as a consequence, $\gamma_{_I}(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,2,2)}(G)$.

On the other side, for any $\gamma_{(2,2,2)}(G)$-function $g(W_0, W_1, W_2)$ and any $v \in V(H)$, the function $g'(W_0', W_1', W_2')$, defined by $W_2' = W_2 \times \{v\}$ and $W_1' = W_1 \times \{v\}$, is an IDF on $G \circ H$. Hence, $\gamma_{_I}(G \circ H) \leq \omega(g') = \omega(g) = \gamma_{(2,2,2)}(G)$.

According to the three cases above, the result follows. □

The following result considers the case $\gamma_{_I}(H) = \gamma(H) = 3$.

Theorem 2.3. *If H is a graph with $\gamma_{_I}(H) = \gamma(H) = 3$, then for any graph G,*

$$\gamma_{_I}(G \circ H) = \gamma_{(2,2,2,0)}(G).$$

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I(G \circ H)$-function, and $f'(X_0, X_1, X_2, X_3)$ the function defined on G by $X_1 = \{x \in V(G) : f(V(H_x)) = 1\}$, $X_2 = \{x \in V(G) : f(V(H_x)) = 2\}$ and $X_3 = \{x \in V(G) : f(V(H_x)) \geq 3\}$. We claim that f' is a $(2,2,2,0)$-dominating function on G.

Let $x \in X_0 \cup X_1 \cup X_2$. Since $f(V(H_x)) \leq 2$ and $\gamma(H) = 3$, there exists $y \in V(H)$ such that $f(N[(x,y)] \cap V(H_x)) = 0$. Thus, $f'(N(x)) \geq 2$ for every $x \in X_0 \cup X_1 \cup X_2$, which implies that f' is a $(2,2,2,0)$-dominating function on G. Therefore, $\gamma_{_I}(G \circ H) = \omega(f) \geq \omega(f') \geq \gamma_{(2,2,2,0)}(G)$.

On the other side, let $h(Y_0, Y_1, Y_2, Y_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function, h_1 a $\gamma_{_I}(H)$-function and $v \in V(H)$. We define a function g on $G \circ H$ by $g(x,v) = h(x)$ for every $x \in V(G) \setminus Y_3$, $g(x,y) = 0$ for every $x \in V(G) \setminus Y_3$ and $y \in V(H) \setminus \{v\}$, and $g(x,y) = h_1(y)$ for every $(x,y) \in Y_3 \times V(H)$. A simple case analysis shows that g is an IDF on $G \circ H$. Therefore, $\gamma_{_I}(G \circ H) \leq \omega(g) = \omega(h) = \gamma_{(2,2,2,0)}(G)$. □

The graph shown in Figure 2 satisfies $6 = \gamma_{(2,2,0)}(G) = \gamma_{(2,2,1)}(G) < 7 = \gamma_{(2,2,2,0)}(G) < \gamma_{(2,2,2)}(G) = 8$.

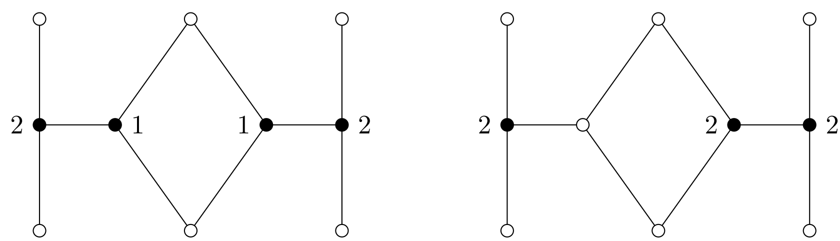

Figure 2: This figure shows two $\gamma_{(2,2,0)}(G)$-functions on the same graph. The function on the left is also a $\gamma_{(2,2,1)}(G)$-function.

3 Preliminary results on w-domination

In this section, we fix the notation $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$ for the sets of positive and nonnegative integers, respectively.

Throughout this section, we will repeatedly apply, without explicit mention, the following necessary and sufficient condition for the existence of a w-dominating function.

Remark 3.1. Let G be a graph of minimum degree δ and let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$. If $w_0 \geq \cdots \geq w_l$, then there exists a w-dominating function on G if and only if $w_l \leq l\delta$.

Proof. Let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$. If $w_l \leq l\delta$, then the function f, defined by $f(v) = l$ for every $v \in V(G)$, is a w-dominating function on G, as $V_l = V(G)$ and for any $x \in V_l$, $f(N(x)) \geq l\delta \geq w_l$.

Now, suppose that $w_l > l\delta$. If g is a w-dominating function on G, then for any vertex v of degree δ we have $g(N(v)) \leq \delta l < w_l \leq w_{l-1} \leq \cdots \leq w_0$, which is a contradiction. Therefore, the result follows. □

We will show that in general the w-domination numbers satisfy a certain monotonicity. Given two integer vectors $w = (w_0, \dots, w_l)$ and $w' = (w'_0, \dots, w'_l)$, we say that $w \prec w'$ if $w_i \leq w'_i$ for every $i \in \{0, \dots, l\}$. With this notation in mind, we can state the next remark which is direct consequence of the definition of w-domination number.

Remark 3.2. Let G be a graph of minimum degree δ and let $w = (w_0, \dots, w_l), w' = (w'_0, \dots, w'_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_i \geq w_{i+1}$ and $w'_i \geq w'_{i+1}$ for every $i \in \{0, \dots, l-1\}$. If $w \prec w'$ and $w'_l \leq l\delta$, then every w'-dominating function is a w-dominating function and, as a consequence,

$$\gamma_w(G) \leq \gamma_{w'}(G).$$

We would emphasize the following remark on the specific cases of domination parameters considered in Section 2. Obviously, when we write $\gamma_{(2,2,2)}(G)$ or $\gamma_{(2,2,1)}(G)$, we are assuming that G has minimum degree $\delta \geq 1$.

Remark 3.3. The following statements hold.

(i) $\gamma_{_I}(G) = \gamma_{(2,0,0)}(G) \leq \gamma_{(2,1,0)}(G) \leq \gamma_{(2,2,0)}(G) \leq \gamma_{(2,2,1)}(G) \leq \gamma_{(2,2,2)}(G)$.

(ii) If $w_2 \in \{1, 2\}$, then $\gamma_{(1,0,w_2)}(G) = \gamma_{(1,0,0)}(G) = \gamma(G)$ and $\gamma_{(1,1,w_2)}(G) = \gamma_{(1,1,0)}(G) = \gamma_t(G)$.

(iii) For any integer $k \geq 3$, there exists an infinite family $\mathcal{H}_k$ of graphs such that for every graph $G \in \mathcal{H}_k$, $\gamma_{_I}(G) = \gamma_{(2,0,0)}(G) = \gamma_{(2,1,0)}(G) = \gamma_{(2,2,0)}(G) = \gamma_{(2,2,1)}(G) = \gamma_{(2,2,2)}(G) = k$.

(iv) There exists an infinite family of graphs such that $\gamma_{_I}(G) < \gamma_{(2,1,0)}(G) < \gamma_{(2,2,0)}(G) < \gamma_{(2,2,1)}(G) < \gamma_{(2,2,2)}(G)$.

In order to see that the remark above holds, we just have to construct families of graphs satisfying (iii) and (iv), as (i) is a particular case of Remark 3.2 and (ii) is derived from the definition of (w_0, w_1, w_2)-domination number. In the case of (iii), we construct a family $\mathcal{H}_k = \{G_{k,r} : r \in \mathbb{Z}^+\}$ as follows. Let $k \geq 3$ be an integer, and let N_r be the empty graph of order r. For any positive integer r we construct a graph $G_{k,r} \in \mathcal{H}_k$ from a complete graph K_k and $\binom{k}{2}$ copies of N_r, in such way that for each pair of different vertices $\{x, y\}$ of K_k we choose one copy of N_r and connect every vertex of N_r with x and y, making x and y vertices of degree $(k-1)(r+1)$ in $G_{k,r}$. For instance, the graph $G_{3,1}$ is isomorphic to the graph G_2 shown in Figure 1. It is readily seen that

$\gamma_{_I}(G_{k,r}) = \gamma_{(2,2,2)}(G_{k,r}) = k$. On the other hand, in the case of (iv), we consider the family of cycles of order $n \geq 10$ with $n \equiv 1 \pmod 3$. For these graphs we have that $\gamma_I(C_n) < \gamma_{(2,1,0)}(C_n) < \gamma_{(2,2,0)}(C_n) < \gamma_{(2,2,1)}(C_n) < \gamma_{(2,2,2)}(C_n)$. The specific values of $\gamma_{(w_0,w_1,w_2)}(C_n)$ will be given in Subsections 3.1 – 3.4.

Next we show a class of graphs where $\gamma_{(w_0,\ldots,w_l)}(G) = w_0\gamma(G)$ whenever $l \geq w_0 \geq \cdots \geq w_l$. To this end, we need to introduce some additional notation and terminology. Given two graphs G_1 and G_2, the *corona product graph* $G_1 \odot G_2$ is the graph obtained from G_1 and G_2, by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining by an edge every vertex from the i^{th}-copy of G_2 with the i^{th}-vertex of G_1. For every $x \in V(G_1)$, the copy of G_2 in $G_1 \odot G_2$ associated to x will be denoted by $G_{2,x}$. It is well known that $\gamma(G_1 \odot G_2) = |V(G_1)|$ and, if G_1 does not have isolated vertices, then $\gamma_t(G_1 \odot G_2) = \gamma(G_1 \odot G_2) = |V(G_1)|$.

Theorem 3.4. *Let $G \cong G_1 \odot G_2$ be a corona graph where G_1 does not have isolated vertices, and let $w = (w_0,\ldots,w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$. If $l \geq w_0 \geq \cdots \geq w_l$ and $|V(G_2)| \geq w_0$, then*

$$\gamma_w(G) = w_0\gamma(G).$$

Proof. Since G_1 does not have isolated vertices, the upper bound $\gamma_w(G) \leq w_0|V(G_1)| = w_0\gamma(G)$ is straightforward, as the function f, defined by $f(x) = w_0$ for every vertex $x \in V(G_1)$ and $f(x) = 0$ for every $x \in V(G) \setminus V(G_1)$, is a w-dominating function on G.

On the other hand, let f be a $\gamma_w(G)$-function and suppose that there exists $x \in V(G_1)$ such that $f(V(G_{2,x})) + f(x) \leq w_0 - 1$. In such a case, $f(N[y]) \leq w_0 - 1$ for every $y \in V(G_{2,x})$, which is a contradiction, as $|V(G_2)| \geq w_0$. Therefore, $\gamma_w(G) = \omega(f) \geq w_0|V(G_1)| = w_0\gamma(G)$. □

Proposition 3.5. *Let G be a graph of order n. Let $w = (w_0,\ldots,w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$. If G' is a spanning subgraph of G with minimum degree $\delta' \geq \frac{w_l}{l}$, then*

$$\gamma_w(G) \leq \gamma_w(G').$$

Proof. Let $E^- = \{e_1,\ldots,e_k\}$ be the set of all edges of G not belonging to the edge set of G'. Let $G'_0 = G$ and, for every $i \in \{1,\ldots,k\}$, let $X_i = \{e_1,\ldots,e_i\}$ and $G'_i = G - X_i$, the edge-deletion subgraph of G induced by $E(G) \setminus X_i$. Since any w-dominating function on G'_i is a w-dominating function on G'_{i-1}, we can conclude that $\gamma_w(G'_{i-1}) \leq \gamma_w(G'_i)$. Hence, $\gamma_w(G) = \gamma_w(G'_0) \leq \gamma_w(G'_1) \leq \cdots \leq \gamma_w(G'_k) = \gamma_w(G')$. □

From Proposition 3.5 we obtain the following result.

Corollary 3.6. *Let G be a graph of order n and $w = (w_0,\ldots,w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$.*

- *If G is a Hamiltonian graph and $w_l \leq 2l$, then $\gamma_w(G) \leq \gamma_w(C_n)$.*
- *If G has a Hamiltonian path and $w_l \leq l$, then $\gamma_w(G) \leq \gamma_w(P_n)$.*

In order to derive lower bounds on the w-domination number, we need to state the following useful lemma.

Lemma 3.7. *Let G be a graph with no isolated vertex, maximum degree Δ and order n. For any w-dominating function $f(V_0,\dots,V_l)$ on G such that $w_0 \geq \cdots \geq w_l$,*

$$\Delta\omega(f) \geq w_0 n + \sum_{i=1}^{l}(w_i - w_0)|V_i|.$$

Proof. The result follows from the simple fact that the contribution of any vertex $x \in V(G)$ to the sum $\sum_{x\in V(G)} f(N(x))$ equals $\deg(x)f(x)$, where $\deg(x)$ denotes the degree of x. Hence,

$$\begin{aligned}\Delta\omega(f) &= \Delta \sum_{x\in V(G)} f(x)\\ &\geq \sum_{x\in V(G)} \deg(x)f(x)\\ &= \sum_{x\in V(G)} f(N(x))\\ &\geq w_0|V_0| + \sum_{i=1}^{l} w_i|V_i|\\ &= w_0 n + \sum_{i=1}^{l}(w_i - w_0)|V_i|.\end{aligned}$$

Therefore, the result follows. □

Corollary 3.8. *The following statements hold for $k, l \in \mathbb{Z}^+$ and a graph G with minimum degree $\delta \geq 1$, maximum degree Δ and order n.*

(i) *If $k \leq l\delta + 1$ and $w = (\underbrace{k+l-1, k+l-2, \dots, k-1}_{l+1})$, then $\gamma_w(G) \geq \left\lceil \frac{(k+l-1)n}{\Delta+1} \right\rceil$.*

(ii) *If $k \leq l\delta$ and $w = (\underbrace{k, \dots, k}_{l+1})$, then $\gamma_w(G) \geq \lceil \frac{kn}{\Delta} \rceil$.*

(iii) *If $k \leq l\delta + 1$ and $w = (\underbrace{k, k-1, \dots, k-1}_{l+1})$, then $\gamma_w(G) \geq \left\lceil \frac{kn}{\Delta+1} \right\rceil$.*

(iv) *Let $w = (w_0, \dots, w_l)$ with $w_0 \geq \cdots \geq w_l$. If $l\delta \geq w_l$, then $\gamma_w(G) \geq \left\lceil \frac{w_0 n}{\Delta + w_0} \right\rceil$.*

In the next subsections we shall show that lower bounds above are tight. Corollary 3.8 implies the following known bounds.

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta+1} \right\rceil, \quad \gamma_t(G) \geq \left\lceil \frac{n}{\Delta} \right\rceil, \quad \gamma_{I}(G) \geq \left\lceil \frac{2n}{\Delta+2} \right\rceil, \quad \gamma_{tI}(G) \geq \left\lceil \frac{2n}{\Delta+1} \right\rceil,$$
$$\gamma_k(G) \geq \left\lceil \frac{kn}{\Delta+k} \right\rceil, \quad \gamma_{\times k}(G) \geq \left\lceil \frac{kn}{\Delta+1} \right\rceil, \quad \gamma_{\{k\}} \geq \left\lceil \frac{kn}{\Delta+1} \right\rceil, \quad \gamma_{\times k,t}(G) \geq \left\lceil \frac{kn}{\Delta} \right\rceil.$$

It is readily seen that $\gamma_{(w_0,\dots,w_l)}(G) = 1$ if and only if $w_0 = 1$, $w_1 = 0$ and $\gamma(G) = 1$. Next we characterize the graphs with $\gamma_{(w_0,\dots,w_l)}(G) = 2$.

Theorem 3.9. *Let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$. For a graph G of order at least three, $\gamma_{(w_0,\dots,w_l)}(G) = 2$ if and only if one of the following conditions holds.*

(i) $w_2 = 0$, $\gamma(G) = 1$ *and either* $w_0 = 2$ *or* $w_0 = w_1 = 1$.

(ii) $w_0 = 1$, $w_1 = 0$ *and* $\gamma(G) = 2$.

(iii) $w_0 = 1$, $w_1 = 1$ *and* $\gamma_t(G) = 2$.

(iv) $w_0 = 2$, $w_1 = 0$ *and* $\gamma_2(G) = 2$.

(v) $w_0 = 2$, $w_1 = 1$ *and* $\gamma_{\times 2}(G) = 2$.

Proof. Assume first that $\gamma_{(w_0,\dots,w_l)}(G) = 2$ and let $f(V_0, \dots, V_l)$ be a $\gamma_{(w_0,\dots,w_l)}(G)$-function. Notice that $w_0 \in \{1, 2\}$ and $|V_2| \in \{0, 1\}$. If $|V_2| = 1$, then $w_2 = 0$ and $V_i = \varnothing$ for every $i \neq 0, 2$. Hence, $\gamma(G) = 1$ and either $w_0 = 2$ or $w_0 = w_1 = 1$. Therefore, (i) follows.

Now we consider the case $V_2 = \varnothing$. Notice that V_1 is a dominating set of cardinality two, $w_1 \in \{0, 1\}$ and $V_i = \varnothing$ for every $i \neq 0, 1$.

Assume first that $w_0 = 1$ and $w_1 = 0$. If $\gamma(G) = 1$, then $\gamma_{(w_0,\dots,w_l)}(G) = 1$, which is a contradiction. Hence, $\gamma(G) = 2$ and so (ii) follows. For $w_0 + w_1 \geq 2$ we have the following possibilities.

If $w_0 = w_1 = 1$, then V_1 is a total dominating set of cardinality two, and so $\gamma_t(G) = 2$. Therefore, (iii) follows.

If $w_0 = 2$ and $w_1 = 0$, then V_1 is a 2-dominating set of cardinality two, which implies that $\gamma_2(G) = 2$. Therefore, (iv) follows.

If $w_0 = 2$ and $w_1 = 1$, then V_1 is a double dominating set of cardinality two, and this implies that $\gamma_{\times 2}(G) = 2$. Therefore, (v) follows.

Conversely, if one of the five conditions holds, then it is easy to check that $\gamma_{(w_0,\dots,w_l)}(G) = 2$, which completes the proof. □

In order to establish the following result, we need to define the following parameter.

$$\nu_{(w_0,\dots,w_l)}(G) = \max\{|V_0| : f(V_0, \dots, V_l) \text{ is a } \gamma_{(w_0,\dots,w_l)}(G)\text{-function}\}.$$

In particular, for $l = 1$ and a graph G of order n, we have that $\nu_{(w_0,w_1)}(G) = n - \gamma_{(w_0,w_1)}(G)$.

Theorem 3.10. *Let G be a graph of minimum degree δ and order n. The following statements hold for any $(w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ with $w_0 \geq \cdots \geq w_l$.*

(i) *If there exists $i \in \{1, \dots, l-1\}$ such that $i\delta \geq w_i$, then*

$$\gamma_{(w_0,\dots,w_l)}(G) \leq \gamma_{(w_0,\dots,w_i)}(G).$$

(ii) *If $l \geq i + 1 \geq w_0$, then*

$$\gamma_{(w_0,\dots,w_i,0,\dots,0)}(G) \leq (i + 1)\gamma(G).$$

(iii) *Let $k, i \in \mathbb{Z}^+$ such that $l \geq ki$, and let $(w'_0, w'_1, \dots, w'_i) \in \mathbb{Z}^+ \times \mathbb{N}^l$. If $i\delta \geq w'_i$ and $w_{kj} = kw'_j$ for every $j \in \{0, 1, \dots, i\}$, then*

$$\gamma_{(w_0,\dots,w_l)}(G) \leq k\gamma_{(w'_0,\dots,w'_i)}(G).$$

(iv) *Let $k \in \mathbb{Z}^+$ and $\beta_1, \dots, \beta_k \in \mathbb{Z}^+$. If $l\delta \geq k + w_l > k$ and $w_0 + k \geq \beta_1 \geq \cdots \geq \beta_k \geq w_1 + k$, then*

$$\gamma_{(w_0+k,\beta_1,\dots,\beta_k,w_1+k,\dots,w_l+k)}(G) \leq \gamma_{(w_0,\dots,w_l)}(G) + k(n - \nu_{(w_0,\dots,w_l)}(G)).$$

(v) *If $l\delta \geq w_l \geq l \geq 2$, then*

$$\gamma_{(w_0,\dots,w_l)}(G) \leq l\gamma_{(w_0-l+1,w_l-l+1)}(G).$$

(vi) *If $\delta \geq 1$, $w_0 \leq l - 1$ and $w_{l-1} \geq 1$, then*

$$\gamma_{(w_0,\dots,w_{l-2},1)}(G) \leq \gamma_{(w_0,\dots,w_{l-1},0)}(G).$$

Proof. If there exists $i \in \{1, \dots, l-1\}$ such that $i\delta \geq w_i$, then for any $\gamma_{(w_0,\dots,w_i)}(G)$-function $f(V_0, \dots, V_i)$ we define a $(w_0, \dots, w_l)$-dominating function $g(W_0, \dots, W_l)$ by $W_j = V_j$ for every $j \in \{0, \dots, i\}$ and $W_j = \varnothing$ for every $j \in \{i+1, \dots, l\}$. Hence, $\gamma_{(w_0,\dots,w_l)}(G) \leq \omega(g) = \omega(f) = \gamma_{(w_0,\dots,w_i)}(G)$. Therefore, (i) follows.

Now, assume $l \geq i + 1 \geq w_0$. Let S be a $\gamma(G)$-set. Let f be the function defined by $f(v) = i + 1$ for every $v \in S$ and $f(v) = 0$ for the remaining vertices. Since f is a $(w_0, \dots, w_i, 0 \dots, 0)$-dominating function, we conclude that $\gamma_{(w_0,\dots,w_i,0\dots,0)}(G) \leq \omega(f) = (i+1)|S| = (i+1)\gamma(G)$, which implies that (ii) follows.

In order to prove (iii), assume that $l \geq ki$, $i\delta \geq w'_i$ and $w_{kj} = kw'_j$ for every $j \in \{0, \dots, i\}$. Let $f'(V'_0, \dots, V'_i)$ be a $\gamma_{(w'_0,\dots,w'_i)}(G)$-function. We construct a function $f(V_0, \dots, V_l)$ as $f(v) = kf'(v)$ for every $v \in V(G)$. Hence, $V_{kj} = V'_j$ for every $j \in \{0, \dots, i\}$, while $V_j = \varnothing$ for the remaining cases. Thus, for every $v \in V_{kj}$ with $j \in \{0, \dots, i\}$ we have that $f(N(v)) = kf'(N(v)) \geq kw'_j = w_{kj}$, which implies that f is a $(w_0, \dots, w_l)$-dominating function, and so $\gamma_{(w_0,\dots,w_l)}(G) \leq \omega(f) = k\omega(f') = k\gamma_{(w'_0,\dots,w'_i)}(G)$. Therefore, (iii) follows.

Now, assume that $l\delta \geq k + w_l > k$ and $w_0 + k \geq \beta_1 \geq \cdots \geq \beta_k \geq w_1 + k$. Let $g(W_0, \dots, W_l)$ be a $\gamma_{(w_0,\dots,w_l)}(G)$-function. We construct a function $f(V_0, \dots, V_{l+k})$ as $f(v) = g(v) + k$ for every $v \in V(G) \setminus W_0$ and $f(v) = 0$ for every $v \in W_0$. Hence, $V_{j+k} = W_j$ for every $j \in \{1, \dots, l\}$, $V_0 = W_0$ and $V_j = \varnothing$ for the remaining cases. Thus, if $v \in V_{j+k}$ and $j \in \{1, \dots, l\}$, then $f(N(v)) \geq g(N(v)) + k \geq w_j + k$, and if $v \in V_0$, then $f(N(v)) \geq g(N(v)) + k \geq w_0 + k$. This implies that f is a $(w_0 + k, \beta_1, \dots, \beta_k, w_1 + k, \dots, w_l + k)$-dominating function, and so

$$\begin{aligned}\gamma_{(w_0+k,\beta_1,\dots,\beta_k,w_1+k,\dots,w_l+k)}(G) &\leq \omega(f) = \omega(g) + k\sum_{j=1}^{l} |W_j| \\ &= \gamma_{(w_0,\dots,w_l)}(G) + k(n - |W_0|) \\ &\leq \gamma_{(w_0,\dots,w_l)}(G) + k(n - \nu_{(w_0,\dots,w_l)}(G)).\end{aligned}$$

Therefore, (iv) follows.

Furthermore, if $l\delta \geq w_l \geq l \geq 2$, then by applying (iv) for $k = l-1$, we deduce that

$$\begin{aligned}\gamma_{(w_0,\ldots,w_l)}(G) &\leq \gamma_{(w_0-l+1,w_l-l+1)}(G) + (l-1)(n - \nu_{(w_0-l+1,w_l-l+1)}(G)) \\ &= l\gamma_{(w_0-l+1,w_l-l+1)}(G).\end{aligned}$$

Therefore, (v) follows.

From now on, let $\delta \geq 1$, $w_0 \leq l-1$ and $w_{l-1} \geq 1$. Let $f(V_0,\ldots,V_l)$ be a $\gamma_{(w_0,\ldots,w_{l-1},0)}(G)$-function. Assume first $V_l = \varnothing$. Since $w_{l-1} \geq 1$, we have that f is a $(w_0,\ldots,w_{l-2},1)$-dominating function on G, which implies that (vi) follows. Assume now that there exists $v \in V_l$. If $f(N(v)) \geq l-1$, then the function f', defined by $f'(v) = l-1$ and $f'(x) = f(x)$ for every $x \in V(G) \setminus \{v\}$, is a $(w_0,\ldots,w_{l-1},0)$-dominating function with $\omega(f') < \omega(f)$, which is a contradiction. Hence, $f(N(v)) \leq l-2$ for every $v \in V_l$. Since $\delta \geq 1$, for each vertex $x \in V_l$, we fix one vertex $x' \in N(x)$ and we form a set S from them such that $|S| \leq |V_l|$. Let g be the function defined by $g(x) = f(x)+1$ for any $x \in S$, $g(y) = l-1$ for any $y \in V_l$, and $g(z) = f(z)$ for the remaining vertices of G. Since $g(N(x)) \geq l-1 \geq w_i$ for every $x \in S$ and $i \in \{0,\ldots,l-2\}$, $g(N(y)) \geq 1$ for every $y \in V_{l-1} \cup V_l$, and $g(N(z)) \geq w_i$ for every $z \in V_i \setminus (S \cup V_{l-1} \cup V_l)$ and $i \in \{0,\ldots,l-2\}$, we conclude that g is a $(w_0,\ldots,w_{l-2},1)$-dominating function on G. Therefore, $\gamma_{(w_0,\ldots,w_{l-2},1)}(G) \leq \omega(g) \leq \omega(f) = \gamma_{(w_0,\ldots,w_{l-1},0)}(G)$, which completes the proof of (vi). □

In the next subsections we consider several applications of Theorem 3.10 where we show that the bounds are tight. For instance, the following particular cases will be of interest.

Corollary 3.11. *Let G be a graph of minimum degree δ, and let $k, l, w_2, \ldots, w_l \in \mathbb{Z}^+$ with $k \geq w_2 \geq \cdots \geq w_l$.*

(i) *If $\delta \geq k$ and $w = (k+1, k, w_2, \ldots, w_l)$, then $\gamma_w(G) \leq \gamma_{\times k}(G)$.*

(ii) *If $\delta \geq k$ and $w = (k, k, w_2, \ldots, w_l)$, then $\gamma_w(G) \leq \gamma_{\times k,t}(G)$.*

(iii) *If $l\delta \geq k \geq l \geq 2$ and $w = (\underbrace{k+1, k, \ldots, k}_{l+1})$, then $\gamma_w(G) \leq l\gamma_{\times(k-l+2)}(G)$.*

(iv) *If $l\delta \geq k \geq l \geq 2$ and $w = (\underbrace{k, k, \ldots, k}_{l+1})$, then $\gamma_w(G) \leq l\gamma_{\times(k-l+1),t}(G)$.*

(v) *If $l \geq k$, $\delta \geq 1$ and $w = (\underbrace{k, \ldots, k}_{l+1})$, then $\gamma_w(G) \leq k\gamma_t(G)$.*

Proof. If $\delta \geq k$, then by Theorem 3.10(i) we conclude that (i) and (ii) follows.

If $l\delta \geq k \geq l \geq 2$, then by Theorem 3.10(v) we deduce that

$$\gamma_{(\underbrace{k+1, k, \ldots, k}_{l+1})}(G) \leq l\gamma_{(k-l+2,k-l+1)}(G) = l\gamma_{\times(k-l+2)}(G).$$

Hence, (iii) follows. By analogy we derive (iv), as $\gamma_{(k-l+1,k-l+1)}(G) = l\gamma_{\times(k-l+1),t}(G)$.

Finally, if $l \geq k$ and $\delta \geq 1$, then by Theorem 3.10(iii) we deduce that

$$\gamma_{(\underbrace{k, \ldots, k}_{l+1})}(G) \leq k\gamma_{(1,1)}(G) = k\gamma_t(G).$$

Therefore, (v) follows. □

3.1 Preliminary results on $(2,2,2)$-domination

Theorem 3.12. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n}{\Delta} \right\rceil \leq \gamma_{(2,2,2)}(G) \leq 2\gamma_t(G).$$

Furthermore, if G has minimum degree $\delta \geq 2$, then

$$\gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G).$$

Proof. From Corollary 3.8 we deduce the lower bound. The upper bound $\gamma_{(2,2,2)}(G) \leq 2\gamma_t(G)$ follows by Corollary 3.11(v), while, if $\delta \geq 2$, then we apply Corollary 3.11(ii) to deduce that $\gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G)$. Therefore, the result follows. □

The bounds above are tight. For instance, for the graphs G_2 and G_3 shown in Figure 1 we have that $\lceil \frac{2n}{\Delta} \rceil = \gamma_{(2,2,2)}(G_2) = \gamma_{\times 2,t}(G_2) = 3$ and $\gamma_{(2,2,2)}(G_3) = 2\gamma_t(G_3) = 8$. Notice that every graph $G_{k,r}$ belonging to the infinite family $\mathcal{H}_k$ constructed after Remark 3.3 satisfies the equality $\gamma_{(2,2,2)}(G_{k,r}) = \gamma_{\times 2,t}(G_{k,r}) = k$. Furthermore, from Theorem 3.4 we have that for any corona graph $G \cong G_1 \odot G_2$, where G_1 does not have isolated vertices, $\gamma_{(2,2,2)}(G) = 2\gamma(G) = 2\gamma_t(G)$.

Notice that by Theorem 3.12 we have that $\gamma_{(2,2,2)}(G) \geq \lceil \frac{2n}{\Delta} \rceil \geq 3$ for every graph G with no isolated vertex. Next we characterize all graphs with $\gamma_{(2,2,2)}(G) = 3$. To this end, we need to establish the following lemma.

Lemma 3.13. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2)}(G) = \gamma_{\times 2,t}(G)$.

(ii) *There exists a $\gamma_{(2,2,2)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_2 = \varnothing$.*

Proof. If $\gamma_{(2,2,2)}(G) = \gamma_{\times 2,t}(G)$, then for any $\gamma_{\times 2,t}(G)$-set D, the function $g(W_0, W_1, W_2)$, defined by $W_1 = D$ and $W_0 = V(G) \setminus D$, is a $\gamma_{(2,2,2)}(G)$-function. Therefore, (ii) follows.

Conversely, if there exists a $\gamma_{(2,2,2)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_2 = \varnothing$, then V_1 is a double total dominating set of G, and so $\gamma_{\times 2,t}(G) \leq |V_1| = \omega(f) = \gamma_{(2,2,2)}(G)$. Therefore, Theorem 3.12 leads to $\gamma_{(2,2,2)}(G) = \gamma_{\times 2,t}(G)$. □

Theorem 3.14. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2)}(G) = 3$.

(ii) $\gamma_{\times 2,t}(G) = 3$.

Proof. Assume first that $\gamma_{(2,2,2)}(G) = 3$, and let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(G)$-function. Suppose that there exists $u \in V_2$. Since $f(N(u)) \geq 2$, we deduce that $\gamma_{(2,2,2)}(G) \geq 4$, which is a contradiction. Hence, $V_2 = \varnothing$ and by Lemma 3.13 we conclude that $\gamma_{\times 2,t}(G) = 3$.

Conversely, if $\gamma_{\times 2,t}(G) = 3$, then G has minimum degree $\delta \geq 2$ and so Theorem 3.12 leads to $3 \leq \lceil \frac{2n}{\Delta} \rceil \leq \gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G) = 3$. Therefore, $\gamma_{(2,2,2)}(G) = 3$. □

Next we consider the case of graphs with $\gamma_{(2,2,2)}(G) = 4$.

Theorem 3.15. *For a graph G, $\gamma_{(2,2,2)}(G) = 4$ if and only if at least one of the following conditions holds.*

(i) $\gamma_{\times 2,t}(G) = 4$.

(ii) $\gamma_t(G) = 2$ *and G has minimum degree* $\delta = 1$.

(iii) $\gamma_t(G) = 2$ *and* $\gamma_{\times 2,t}(G) \geq 4$.

Proof. Assume $\gamma_{(2,2,2)}(G) = 4$. Notice that G does not have isolated vertices. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(G)$-function. If $V_2 = \varnothing$, then by Lemma 3.13 we obtain that $\gamma_{\times 2,t}(G) = \gamma_{(2,2,2)}(G) = 4$, and so (i) follows.

From now on, assume that $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$ and, as a result, V_2 is a total dominating set of G, which implies that $\gamma_t(G) = 2$. On the other side, if $|V_2| = 1$, then $|V_1| = 2$ and both vertices belonging to V_1 are adjacent to the vertex of weight two, and every $v \in V_0$ satisfies $N(v) \cap V_2 \neq \varnothing$ or $V_1 \subseteq N(v)$. This implies that the union of V_2 with a singleton subset of V_1 forms a total dominating set of G, and again $\gamma_t(G) = 2$. Now, if $\delta \geq 2$, then Theorem 3.12 leads to $4 = \gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G)$. Hence, by Theorem 3.14 we conclude that either $\delta = 1$ or $\gamma_{\times 2,t}(G) \geq 4$. Therefore, either (ii) or (iii) holds.

Conversely, if $\gamma_{\times 2,t}(G) = 4$, then G has minimum degree $\delta \geq 2$ and by Theorem 3.12 we have that $3 \leq \gamma_{(2,2,2)}(G) \leq 4$. Hence, by Theorem 3.14 we deduce that $\gamma_{(2,2,2)}(G) = 4$. Finally, if $\gamma_t(G) = 2$, then Theorem 3.12 leads to $3 \leq \gamma_{(2,2,2)}(G) \leq 4$. Therefore, if $\delta = 1$ or $\gamma_{\times 2,t}(G) \geq 4$, then Theorem 3.14 leads to $\gamma_{(2,2,2)}(G) = 4$. □

Theorem 3.12 implies the next result.

Corollary 3.16. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,2)}(C_n) = n.$$

In order to give the value of $\gamma_{(2,2,2)}(P_n)$, we recall the following well-known result.

Proposition 3.17 ([14]). *For any integer $n \geq 3$,*

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4, \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod 4, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Lemma 3.18. *If $P_n = u_1 u_2 \dots u_n$ is a path of order $n \geq 6$, then there exists a $\gamma_{(2,2,2)}(P_n)$-function f such that $f(u_n) = f(u_{n-3}) = 0$ and $f(u_{n-1}) = f(u_{n-2}) = 2$.*

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(P_n)$-function such that $|V_2|$ is maximum. Since u_n is a leaf, $f(u_{n-1}) = 2$. Notice that $f(u_n) + f(u_{n-2}) \geq 2$. Hence, we can assume that $f(u_{n-2}) = 2$ and $f(u_n) = 0$. Now, if $f(u_{n-3}) > 0$, then we can define a $(2,2,2)$-dominating function f' by $f'(u_{n-3}) = 0$, $f'(u_{n-5}) = \min\{2, f(u_{n-5}) + f(u_{n-3})\}$ and $f'(u_i) = f(u_i)$ for the remaining cases. Since $\omega(f') \leq \omega(f) = \gamma_{(2,2,2)}(P_n)$, either f' is a $\gamma_{(2,2,2)}(P_n)$-function with $f'(u_{n-3}) = 0$ or $f(u_{n-3}) = 0$. In both cases the result follows. □

Proposition 3.19. *For any integer* $n \geq 3$*,*

$$\gamma_{(2,2,2)}(P_n) = 2\gamma_t(P_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1,3 \pmod 4, \\ n+2 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Since Theorem 3.12 leads to $\gamma_{(2,2,2)}(P_n) \leq 2\gamma_t(P_n)$, we only need to prove that $\gamma_{(2,2,2)}(P_n) \geq 2\gamma_t(P_n)$. We proceed by induction on n. It is easy to check that $\gamma_{(2,2,2)}(P_n) = 2\gamma_t(P_n)$ for $n = 3, 4, 5, 6$. This establishes the base case. Now, we assume that $n \geq 7$ and $\gamma_{(2,2,2)}(P_k) \geq 2\gamma_t(P_k)$ for $k < n$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(P_n)$-function which satisfies Lemma 3.18, and let f' be the restriction of f to $V(P_{n-4})$, where $P_n = u_1u_2 \dots u_n$ and $P_{n-4} = u_1u_2 \dots u_{n-4}$. Hence, by applying the induction hypothesis,

$$\gamma_{(2,2,2)}(P_n) = \omega(f) = \omega(f') + 4 \geq \gamma_{(2,2,2)}(P_{n-4}) + 4 \geq 2\gamma_t(P_{n-4}) + 4 \geq 2\gamma_t(P_n).$$

To conclude the proof we apply Proposition 3.17. □

3.2 Preliminary results on $(2, 2, 1)$-domination

Theorem 3.20. *For any graph* G *with no isolated vertex, order* n *and maximum degree* Δ*,*

$$\left\lceil \frac{2n + \gamma_t(G)}{\Delta + 1} \right\rceil \leq \gamma_{(2,2,1)}(G) \leq \min\{3\gamma(G), 2\gamma_t(G)\}.$$

Furthermore, if G *has minimum degree* $\delta \geq 2$*, then*

$$\gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G).$$

Proof. In order to prove the upper bound $\gamma_{(2,2,1)}(G) \leq 2\gamma_t(G)$, we apply Remark 3.2 and Theorem 3.12, i.e., $\gamma_{(2,2,1)}(G) \leq \gamma_{(2,2,2)} \leq 2\gamma_t(G)$.

Now, let S be a $\gamma(G)$-set. Since G does not have isolated vertex, for each vertex $x \in S$ such that $N(x) \cap S = \varnothing$, we fix one vertex $x' \in N(x)$ and we form a set S' from them. Hence, $S \cup S'$ is a total dominating set and $|S \cup S'| = |S| + |S'| \leq 2\gamma(G)$. Notice that the function $g(X_0, X_1, X_2)$ defined by $X_2 = S$ and $X_1 = S'$, is a $(2, 2, 1)$-dominating function on G. Thus, $\gamma_{(2,2,1)}(G) \leq \omega(g) = 2|S| + |S'| \leq 3\gamma(G)$, and so $\gamma_{(2,2,1)}(G) \leq \min\{2\gamma_t(G), 3\gamma(G)\}$.

On the other side, if G has minimum degree $\delta \geq 2$, then by Corollary 3.11(ii) we have that $\gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G)$.

In order to prove the lower bound, let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(G)$-function. Since $V_1 \cup V_2$ is a total dominating set, $\gamma_t(G) \leq |V_1| + |V_2|$. Furthermore, from Lemma 3.7 we have, $2n - |V_2| \leq \Delta\gamma_{(2,2,1)}(G)$, which implies that $2n + \gamma_t(G) \leq 2n + |V_1| + |V_2| \leq \Delta\gamma_{(2,2,1)}(G) + |V_1| + 2|V_2| = (\Delta + 1)\gamma_{(2,2,1)}(G)$. Therefore, the lower bound follows. □

The bounds above are tight. For instance, the graph in Figure 3 satisfies $\gamma_{(2,2,1)}(G) = 3\gamma(G) = 9$. Next we show that the remaining two bounds are also achieved.

Corollary 3.21. *Let* G *be a graph with no isolated vertex, order* n *and maximum degree* Δ*. If* $\gamma_t(G) < \frac{n+\Delta+1}{\Delta+1/2}$*, then*

$$\gamma_{(2,2,1)}(G) = 2\gamma_t(G) \qquad \text{or} \qquad \gamma_{(2,2,1)}(G) = \left\lceil \frac{2n + \gamma_t(G)}{\Delta + 1} \right\rceil.$$

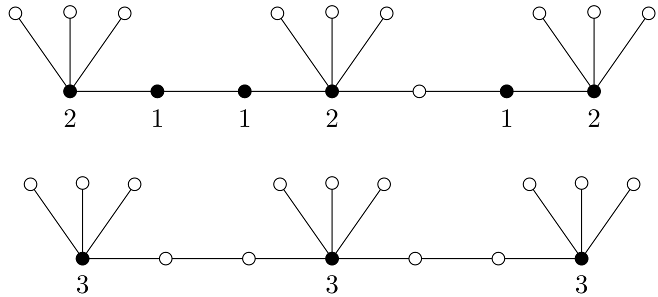

Figure 3: This figure shows a $\gamma_{(2,2,1)}(G)$-function and a $\gamma_{(2,2,2,0)}(G)$-function on the same graph.

Proof. If $\gamma_{(2,2,1)}(G) \neq \left\lceil \frac{2n+\gamma_t(G)}{\Delta+1} \right\rceil$ and $\gamma_{(2,2,1)}(G) \neq 2\gamma_t(G)$, then by Theorem 3.20 we deduce that $\left\lceil \frac{2n+\gamma_t(G)}{\Delta+1} \right\rceil + 1 \leq \gamma_{(2,2,1)}(G) \leq 2\gamma_t(G) - 1$, which implies that $\gamma_t(G) \geq \frac{n+\Delta+1}{\Delta+1/2}$. Therefore, the result follows. □

For the graphs G_2 and G_3 illustrated in Figure 1 we have that $\gamma_t(G_2) = 2 < \frac{22}{9} = \frac{n+\Delta+1}{\Delta+1/2}$ and $\gamma_t(G_3) = 4 < \frac{32}{7} = \frac{n+\Delta+1}{\Delta+1/2}$. Notice that, $\gamma_{(2,2,1)}(G_2) = 3 = \left\lceil \frac{2n+\gamma_t(G_2)}{\Delta+1} \right\rceil$ and $\gamma_{(2,2,1)}(G_3) = 8 = 2\gamma_t(G_3)$.

Below we characterize the graphs with $\gamma_{(2,2,1)}(G) = 3$.

Theorem 3.22. *For a graph G with no isolated vertex, the following statements are equivalent.*

(i) $\gamma_{(2,2,1)}(G) = 3$.

(ii) $\gamma(G) = 1$ *or* $\gamma_{\times 2,t}(G) = 3$.

Proof. Assume first that $\gamma_{(2,2,1)}(G) = 3$, and let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(G)$-function. If $V_2 \neq \varnothing$, then V_2 is a dominating set of cardinality one. Hence, $\gamma(G) = 1$. Now, if $V_2 = \varnothing$, then V_1 is a double total dominating set of cardinality three. Thus, $\gamma_{\times 2,t}(G) = 3$.

On the other side, by Theorem 3.20 we have that $3 \leq \left\lceil \frac{2n+\gamma_t(G)}{\Delta+1} \right\rceil \leq \gamma_{(2,2,1)}(G) \leq 3\gamma(G)$. Hence, if $\gamma(G) = 1$, then $\gamma_{(2,2,1)}(G) = 3$. Now, if $\gamma_{\times 2,t}(G) = 3$, then G has minimum degree $\delta \geq 2$ and by Theorem 3.20 we have that $\gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G) = 3$. Therefore, $\gamma_{(2,2,1)}(G) = 3$. □

Next we consider the case of graphs with $\gamma_{(2,2,1)}(G) = 4$.

Theorem 3.23. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,1)}(G) = 4$.

(ii) $\gamma_t(G) = \gamma(G) = 2$ *or* $\gamma_{\times 2,t}(G) = 4$.

Proof. Assume $\gamma_{(2,2,1)}(G) = 4$. Notice that G does not have isolated vertices and, by Theorem 3.20, we have that $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(G)$-function. If $V_2 =$

$\varnothing$, then V_1 is a double total dominating set of cardinality four. Hence, $3 \leq \gamma_{\times 2,t}(G) \leq |V_1| = 4$, and Theorem 3.22 implies that $\gamma_{\times 2,t}(G) = 4$.

From now on, assume that $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$ and, as a result, V_2 is a total dominating set of G, which implies that $\gamma_t(G) = \gamma(G) = 2$. Now, if $|V_2| = 1$, then $|V_1| = 2$ and both vertices belonging to V_1 are adjacent to the vertex of weight two, and every $v \in V_0$ satisfies $N(v) \cap V_2 \neq \varnothing$ or $V_1 \subseteq N(v)$. This implies that the union of V_2 with a singleton subset of V_1 forms a total dominating set of G, and again $\gamma_t(G) = \gamma(G) = 2$.

Conversely, if $\gamma_{\times 2,t}(G) = 4$, then G has minimum degree $\delta \geq 2$ and by Theorem 3.20 we have that $3 \leq \gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G) = 4$. Hence, by Theorem 3.22 we deduce that $\gamma_{(2,2,1)}(G) = 4$. Finally, if $\gamma_t(G) = 2$, then Theorem 3.20 leads to $3 \leq \gamma_{(2,2,1)}(G) \leq 4$. Therefore, if $\gamma(G) = 2$ then by Theorem 3.22 we conclude that $\gamma_{(2,2,1)}(G) = 4$. □

Lemma 3.24. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,1)}(P_n) \leq \begin{cases} n - \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } n \equiv 1, 2 \pmod 7, \\ n - \left\lfloor \frac{n}{7} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof. First we show how to construct a $(2, 2, 1)$-dominating function f on P_n for $n \in \{2, \ldots, 8\}$.

- $n = 2$: $f(u_1) = 2$ and $f(u_2) = 1$.
- $n = 3$: $f(u_1) = 0$, $f(u_2) = 2$ and $f(u_3) = 1$.
- $n = 4$: $f(u_1) = f(u_4) = 0$ and $f(u_2) = f(u_3) = 2$.
- $n = 5$: $f(u_1) = f(u_5) = 0$, $f(u_2) = f(u_4) = 2$ and $f(u_3) = 1$.
- $n = 6$: $f(u_1) = f(u_6) = 0$, $f(u_2) = f(u_5) = 2$ and $f(u_3) = f(u_4) = 1$.
- $n = 7$: $f(u_1) = f(u_4) = f(u_7) = 0$, $f(u_2) = f(u_6) = 2$ and $f(u_3) = f(u_5) = 1$.
- $n = 8$: $f(u_1) = f(u_4) = f(u_8) = 0$, $f(u_2) = f(u_6) = f(u_7) = 2$ and $f(u_3) = f(u_5) = 1$.

We now proceed to describe the construction of f for any $n = 7q + r$, where $q \geq 1$ and $0 \leq r \leq 6$. We partition $V(P_n) = \{u_1, \ldots, u_n\}$ into q sets of cardinality 7 and for $r \geq 1$ one additional set of cardinality r, in such a way that the subgraph induced by all these sets are paths.

For any $r \neq 1$, the restriction of f to each of these q paths of length 7 corresponds to the weights associated above with P_7, while for the path of length r (if any) we take the weights associated above with P_r. The case $r = 1$ and $q \geq 2$ is slightly different, as for the first $q - 1$ paths of length 7 we take the weights associated above with P_7 and for the last 8 vertices of P_n we take the weights associated above with P_8.

Notice that, for $n \equiv 1, 2 \pmod 7$, we have that $\gamma_{(2,2,1)}(P_n) \leq \omega(f) = 6q + r + 1 = n - \left\lfloor \frac{n}{7} \right\rfloor + 1$, while for $n \not\equiv 1, 2 \pmod 7$ we have $\gamma_{(2,2,1)}(P_n) \leq \omega(f) = 6q + r = n - \left\lfloor \frac{n}{7} \right\rfloor$. Therefore, the result follows. □

Lemma 3.25. *Let $P_7 = x_1 \ldots x_7$ be a subgraph of C_n and $X = \{x_1, \ldots, x_7\}$. If f is a $(2, 2, 1)$-dominating function on C_n, then*

$$f(X) \geq 6.$$

Proof. Notice that $f(\{x_1, x_2, x_3\}) \geq 2$ and $f(\{x_4, x_5, x_6, x_7\}) \geq 3$ as f is a $(2,2,1)$-dominating function. If $f(\{x_1, x_2, x_3\}) \geq 3$, then we are done. Hence, we assume that $f(\{x_1, x_2, x_3\}) = 2$. In this case, it is not difficult to deduce that $f(\{x_4, x_5, x_6, x_7\}) \geq 4$, which implies that $f(X) \geq 6$, as desired. Therefore, the proof is complete. □

Lemma 3.26. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,1)}(C_n) \geq \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \textit{if } n \equiv 1, 2 \pmod 7, \\ n - \lfloor \frac{n}{7} \rfloor & \textit{otherwise.} \end{cases}$$

Proof. It is easy to check that $\gamma_{(2,2,1)}(C_n) = n$ for every $n \in \{3, 4, 5, 6\}$. Now, let $n = 7q + r$, with $0 \leq r \leq 6$ and $q \geq 1$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(C_n)$-function.

If $r = 0$, then by Lemma 3.25 we have that $\omega(f) \geq 6q = n - \lfloor \frac{n}{7} \rfloor$. From now on we assume that $r \geq 1$. By Proposition 3.5 and Lemma 3.24 we deduce that $\gamma_{(2,2,1)}(C_n) \leq \gamma_{(2,2,1)}(P_n) < n$, which implies that $V_2 \neq \varnothing$, otherwise there exists $u \in V(C_n) = V_0 \cup V_1$ such that $N(u) \cap V_0 \neq \varnothing$ and so $|N(u) \cap V_1| \leq 1$, which is a contradiction. Let $x \in V_2$ and, without loss of generality, we can label the vertices of C_n in such a way that $x = u_1$, and $u_2 \in V_1 \cup V_2$ whenever $r \geq 2$. We partition $V(C_n)$ into $X = \{u_1, \dots, u_r\}$ and $Y = \{u_{r+1}, \dots, u_n\}$. Notice that Lemma 3.25 leads to $f(Y) \geq 6q$.

Now, if $r \in \{1, 2\}$, then $f(X) \geq r + 1$, which implies that $\omega(f) \geq r + 1 + 6q = n - \lfloor \frac{n}{7} \rfloor + 1$. Analogously, if $r = 3$, then $f(X) \geq r$ and so $\omega(f) \geq r + 6q = n - \lfloor \frac{n}{7} \rfloor$.

Finally, if $r \in \{4, 5, 6\}$, then as f is a $(2,2,1)$-dominating function we deduce that $f(X) \geq r$, which implies that $\omega(f) \geq r + 6q = n - \lfloor \frac{n}{7} \rfloor$. □

The following result is a direct consequence of Proposition 3.5 and Lemmas 3.24 and 3.26.

Proposition 3.27. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,1)}(C_n) = \gamma_{(2,2,1)}(P_n) = \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \textit{if } n \equiv 1, 2 \pmod 7, \\ n - \lfloor \frac{n}{7} \rfloor & \textit{otherwise.} \end{cases}$$

3.3 Preliminary results on $(2,2,0)$-domination

Theorem 3.28. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n}{\Delta + 1} \right\rceil \leq \gamma_{(2,2,0)}(G) \leq 2\gamma(G).$$

Furthermore, if G has minimum degree $\delta \geq 2$, then

$$\gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G).$$

Proof. The upper bound $\gamma_{(2,2,0)}(G) \leq \omega(g) = 2\gamma(G)$ is derived by we applying Theorem 3.10(ii) for $i = 1$ and $l = 2$. Furthermore, if G has minimum degree $\delta \geq 2$, then by Corollary 3.11(ii) we have that $\gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G)$.

Now, let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(G)$-function. From Lemma 3.7 we deduce that $2(n - |V_2|)) \leq \Delta\gamma_{(2,2,0)}(G)$, which implies that $2n \leq 2n + |V_1| \leq (\Delta + 1)\gamma_{(2,2,0)}(G)$. Therefore, the result follows. □

Theorem 3.28 implies that, if $\gamma(G) = \frac{n}{\Delta+1}$, then $\gamma_{(2,2,0)}(G) = \frac{2n}{\Delta+1}$. It is easy to see that a graph satisfies $\gamma(G) = \frac{n}{\Delta+1}$ if and only if there exists a $\gamma(G)$-set S which is a 2-packing[1] and every vertex in S has degree Δ. The upper bound $\gamma_{(2,2,0)}(G) \leq 2\gamma(G)$ is achieved for the graph G shown in Figure 2, which satisfies $\gamma_{(2,2,0)}(G) = 2\gamma(G) = 6$. Furthermore, by Theorem 3.4 we have that for any corona graph $G \cong G_1 \odot G_2$, where G_1 does not have isolated vertices, $\gamma_{(2,2,0)}(G) = 2\gamma(G)$.

As shown in Theorem 3.9, for a graph G, $\gamma_{(2,2,0)}(G) = 2$ if and only if $\gamma(G) = 1$. Now we consider the case $\gamma_{(2,2,0)}(G) = 3$.

Theorem 3.29. *For a graph G, $\gamma_{(2,2,0)}(G) = 3$ if and only if $\gamma_{\times 2,t}(G) = \gamma(G) + 1 = 3$.*

Proof. Assume $\gamma_{(2,2,0)}(G) = 3$. By Theorem 3.9 we have that $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(G)$-function. If $|V_2| = 1$ then $|V_1| = 1$, and as f is a $(2, 2, 0)$-dominating function we deduce that $N[V_2] = V(G)$, i.e., $\gamma(G) = 1$, which is a contradiction. Thus, $V_2 = \varnothing$ and $|V_1| = 3$. Notice that V_1 is a double total dominating set and since $\gamma(G) \geq 2$, it follows that $3 \leq \gamma(G) + 1 \leq \gamma_{\times 2,t}(G) \leq |V_1| = 3$. Hence, $\gamma_{\times 2,t}(G) = \gamma(G) + 1 = 3$, as required.

Conversely, assume $\gamma_{\times 2,t}(G) = \gamma(G) + 1 = 3$. Since G has minimum degree at least two, Theorem 3.28 leads to $2 \leq \gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G) = 3$, and so Theorem 3.9 implies that $\gamma_{(2,2,0)}(G) = 3$, which completes the proof. □

Theorem 3.30. *For a graph G, $\gamma_{(2,2,0)}(G) = 4$ if and only if one of the following conditions holds.*

(i) *$G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.*

(ii) *$\gamma_{\times 2,t}(G) = 4$.*

(iii) *$\gamma(G) = 2$ and G has minimum degree one.*

(iv) *$\gamma(G) = 2$ and $\gamma_{\times 2,t}(G) \geq 4$.*

Proof. If K_1 is a component of G, then by Theorem 3.9 we conclude that $\gamma_{(2,2,0)}(G) = 4$ if and only if $G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.

From now on, we consider the case where G is a graph with no isolated vertex. Assume $\gamma_{(2,2,0)}(G) = 4$ and let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(G)$-function. If $V_2 = \varnothing$, then V_1 is a double total dominating set of G. In this case, G has minimum degree $\delta \geq 2$ and by Theorem 3.28 we have that $\gamma_{\times 2,t}(G) \leq |V_1| = 4 = \gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G)$. Hence (ii) follows.

Now, assume that $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$, and so $\gamma(G) \leq 2$. Now, if $|V_2| = 1$, then $|V_1| = 2$ and both vertices belonging V_1 are adjacent to the vertex of weight two, and every $v \in V_0$ satisfies $N(v) \cap V_2 \neq \varnothing$ or $V_1 \subseteq N(v)$. This implies that the union of V_2 with a singleton subset of V_1 forms a dominating set of G, and again $\gamma(G) \leq 2$. Thus, from Theorem 3.9 we deduce that $\gamma(G) = 2$. Furthermore, if $\delta \geq 2$, then by Theorem 3.28 we have that $\gamma_{\times 2,t}(G) \geq \gamma_{(2,2,0)} = 4$. Therefore, either (iii) or (iv) holds.

Conversely, if $\gamma_{\times 2,t}(G) = 4$, then Theorem 3.28 leads to $2 \leq \gamma_{(2,2,0)} \leq \gamma_{\times 2,t}(G) = 4$. Hence, by Theorems 3.9 and 3.29 we deduce that $\gamma_{(2,2,0)}(G) = 4$. Analogously, if $\gamma(G) = 2$ and $\delta \geq 1$, then Theorem 3.28 leads to $2 \leq \gamma_{(2,2,0)} \leq 2\gamma(G) = 4$. Thus, by Theorem 3.9 we have that $3 \leq \gamma_{(2,2,0)} \leq 4$. In particular, if $\delta = 1$ or $\gamma_{\times 2,t}(G) \geq 4$, then Theorem 3.29 leads to $\gamma_{(2,2,0)}(G) = 4$, which completes the proof. □

[1] A set $S \subseteq V(G)$ is a 2-packing if $N[u] \cap N[v] = \varnothing$ for every pair of different vertices $u, v \in S$.

Lemma 3.31. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,0)}(G) = 2\gamma(G)$.

(ii) *There exists a $\gamma_{(2,2,0)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_1 = \varnothing$.*

Proof. First, we assume that $\gamma_{(2,2,0)}(G) = 2\gamma(G)$ and let D be a $\gamma(G)$-set. Hence, the function $f(V_0, V_1, V_2)$, defined by $V_2 = D$ and $V_0 = V(G) \setminus D$, is a $\gamma_{(2,2,0)}(G)$-function which satisfies (ii), as desired.

Finally, we assume that there exists a $\gamma_{(2,2,0)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_1 = \varnothing$. This implies that V_2 is a dominating set of G. Hence, $\gamma_{(2,2,0)}(G) \leq 2\gamma(G) \leq 2|V_2| = \gamma_{(2,2,0)}(G)$, and the desired equality holds, which completes the proof. □

The following result provides the $(2, 2, 0)$-domination number of paths and cycles.

Proposition 3.32. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,0)}(P_n) = \gamma_{(2,2,0)}(C_n) = 2\left\lceil \frac{n}{3} \right\rceil.$$

Proof. We first prove that $\gamma_{(2,2,0)}(C_n) \geq 2\left\lceil \frac{n}{3} \right\rceil$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(C_n)$-function. If $V_1 = \varnothing$, then by Lemma 3.31 it follows that $\gamma_{(2,2,0)}(C_n) = 2\gamma(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$. If $V_1 \neq \varnothing$, then $1 + 2|V_2| \leq |V_1| + 2|V_2| = \gamma_{(2,2,0)}(C_n) \leq 2\gamma(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$, which leads to $|V_2| \leq \left\lceil \frac{n}{3} \right\rceil - 1$. By Lemma 3.7 we have that $\gamma_{(2,2,0)}(C_n) \geq n - |V_2| \geq n - \left\lceil \frac{n}{3} \right\rceil + 1 \geq 2\left\lceil \frac{n}{3} \right\rceil$, as desired.

Therefore, by the inequality above, Proposition 3.5 and Theorem 3.28 we deduce that $2\lceil \frac{n}{3} \rceil \leq \gamma_{(2,2,0)}(C_n) \leq \gamma_{(2,2,0)}(P_n) \leq 2\gamma(P_n) = 2\lceil \frac{n}{3} \rceil$. Thus, we have equalities in the inequality chain above, which implies that the result follows. □

3.4 Preliminary results on $(2, 1, 0)$-domination

Given a graph G, we use the notation $L(G)$ and $S(G)$ for the sets of leaves and support vertices, respectively.

Theorem 3.33. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n}{\Delta + 1} \right\rceil \leq \gamma_{(2,1,0)}(G) \leq \min\{\gamma_{\times 2}(G) - |L(G)| + |S(G)|, 2\gamma(G)\}.$$

Proof. If $f(V_0, V_1, V_2)$ is a $\gamma_{(2,1,0)}(G)$-function, then from Lemma 3.7 we conclude that $2n - |V_1| - 2|V_2| \leq \Delta\gamma_{(2,1,0)}(G)$. Hence, $2n \leq \Delta\gamma_{(2,1,0)}(G) + \omega(f) = (\Delta+1)\gamma_{(2,1,0)}(G)$. Therefore, the lower bound follows.

Let D be a $\gamma_{\times 2}(G)$-set. Notice that $S(G) \cup L(G) \subseteq D$. Since $|N[v] \cap D| \geq 2$ for every $v \in V(G)$, the function $g(V_0, V_1, V_2)$ defined by $V_1 = D \setminus (L(G) \cup S(G))$ and $V_2 = S(G)$, is a $(2, 1, 0)$-dominating function. Hence, $\gamma_{(2,1,0)}(G) \leq \omega(g) = \gamma_{\times 2}(G) - |L(G)| + |S(G)|$.

By Remark 3.2, $\gamma_{(2,1,0)}(G) \leq \gamma_{(2,2,0)}(G)$, hence the upper bound $\gamma_{(2,1,0)}(G) \leq 2\gamma(G)$ is derived from Theorem 3.28. Therefore, $\gamma_{(2,1,0)}(G) \leq \min\{\gamma_{\times 2}(G) - |L(G)| + |S(G)|, 2\gamma(G)\}$. □

The bounds above are tight. For instance, for the graph G_1 shown in Figure 1 we have that $\gamma_{(2,1,0)}(G_1) = \left\lceil \frac{2n}{\Delta+1} \right\rceil = \gamma_{\times 2}(G_1) = 2\gamma(G_1) = 4$. As an example of graph of minimum degree one where $\gamma_{(2,1,0)}(G) = \gamma_{\times 2}(G) - |L(G)| + |S(G)|$ we take the graph G obtained from a star graph $K_{1,r}$, $r \geq 3$, by subdividing one edge just once. In such a case, $\gamma_{(2,1,0)}(G) = 4 = \gamma_{\times 2}(G) - |L(G)| + |S(G)|$. Another example is the graph shown in Figure 2 which satisfies $\gamma_{(2,1,0)}(G) = \gamma_{\times 2}(G) - |L(G)| + |S(G)| = 6$.

Notice that $\gamma_{(2,1,0)}(G) \geq \left\lceil \frac{2n}{\Delta+1} \right\rceil \geq 2$. As shown in Theorem 3.9, $\gamma_{(2,1,0)}(G) = 2$ if and only if $\gamma(G) = 1$. Next we characterize the graph satisfying $\gamma_{(2,1,0)}(G) = 3$.

Theorem 3.34. *For a graph G, $\gamma_{(2,1,0)}(G) = 3$ if and only if $\gamma_{\times 2}(G) = \gamma(G) + 1 = 3$.*

Proof. Assume $\gamma_{(2,1,0)}(G) = 3$. By Theorem 3.9 we have that $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,1,0)}(G)$-function. If $|V_2| = 1$ then $N[V_2] = V(G)$, i.e., $\gamma(G) = 1$, which is a contradiction. Thus, $V_2 = \varnothing$ and $|V_1| = 3$, which implies that V_1 is a double dominating set. Hence, $3 \leq \gamma(G) + 1 \leq \gamma_{\times 2}(G) \leq |V_1| = 3$. Therefore, $\gamma_{\times 2}(G) = \gamma(G) + 1 = 3$.

Conversely, assume $\gamma_{\times 2}(G) = \gamma(G) + 1 = 3$. Notice that G has minimum degree $\delta \geq 1$ and so by Theorems 3.9 and 3.33 we have that $3 \leq \gamma_{(2,1,0)}(G) \leq \gamma_{\times 2}(G) = 3$, which implies that $\gamma_{(2,1,0)}(G) = 3$. □

Next we consider the case of graphs with $\gamma_{(2,1,0)}(G) = 4$.

Theorem 3.35. *For a graph G, $\gamma_{(2,1,0)}(G) = 4$ if and only if one of the following conditions is satisfied.*

(i) *$G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.*

(ii) *$\gamma_{\times 2}(G) = 4$.*

(iii) *$\gamma(G) = 2$ and $\gamma_{\times 2}(G) \geq 4$.*

Proof. If K_1 is a component of G, then by Theorem 3.9 we conclude that $\gamma_{(2,1,0)}(G) = 4$ if and only if $G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.

From now on, we consider the case where G is a graph with no isolated vertex. Assume $\gamma_{(2,1,0)}(G) = 4$. By Theorem 3.33 we deduce that $\gamma_{\times 2}(G) \geq 4$ and $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,1,0)}(G)$-function. If $V_2 = \varnothing$, then V_1 is a double dominating set of G, which implies that $\gamma_{\times 2}(G) \leq |V_1| = 4$. Hence, (ii) follows. From now on, assume $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$ and so, V_2 is a dominating set of G, which implies that $\gamma(G) = 2$. If $|V_2| = 1$, then for every $v \in V_1$ we have that $V_2 \cup \{v\}$ is a dominating set of G. Hence, $\gamma(G) = 2$. Therefore, (iii) follows.

Conversely, if (ii) or (iii) holds, then by Theorems 3.33 we have that $2 \leq \gamma_{(2,1,0)}(G) \leq 4$. Therefore, by Theorems 3.9 and 3.34 we deduce that $\gamma_{(2,1,0)}(G) = 4$, which completes the proof. □

The formulas on the $\{k\}$-dominating number of cycles and paths were obtained in [17]. We present here the particular case of $k = 2$, as $\gamma_{\{2\}}(G) = \gamma_{(2,1,0)}(G)$.

Proposition 3.36 ([17])**.** *For any integer $n \geq 3$,*

$$\gamma_{\{2\}}(C_n) = \left\lceil \frac{2n}{3} \right\rceil \qquad \text{and} \qquad \gamma_{\{2\}}(P_n) = 2\left\lceil \frac{n}{3} \right\rceil.$$

3.5 Preliminary results on $(2, 2, 2, 0)$-domination

The following result is a direct consequence of Theorem 3.10(i), (ii) and (vi).

Corollary 3.37. *For any graph G with no isolated vertex,*

$$\gamma_{(2,2,1)}(G) \leq \gamma_{(2,2,2,0)}(G) \leq \min\{3\gamma(G), \gamma_{(2,2,2)}(G)\}.$$

The bounds above are tight. For instance, every graph $G_{k,r}$ belonging to the infinite family $\mathcal{H}_k$ constructed after Remark 3.3 satisfies the equalities $\gamma_{(2,2,1)}(G_{k,r}) = \gamma_{(2,2,2)}(G_{k,r}) = \gamma_{(2,2,2,0)}(G_{k,r}) = k$. In contrast, the graph shown in Figure 2 satisfies $\gamma_{(2,2,1)}(G) = 6 < 7 = \gamma_{(2,2,2,0)}(G) < 8 = \gamma_{(2,2,2)}(G)$. Moreover, Figure 3 illustrates a graph G with $\gamma_{(2,2,1)}(G) = \gamma_{(2,2,2,0)}(G) = 3\gamma(G) = 9$.

In order to characterize the graphs with $\gamma_{(2,2,2,0)}(G) \in \{3, 4\}$, we need to establish the following lemma.

Lemma 3.38. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2,0)}(G) = \gamma_{(2,2,2)}(G)$.

(ii) *There exists a $\gamma_{(2,2,2,0)}(G)$-function $f(V_0, V_1, V_2, V_3)$ such that $V_3 = \varnothing$.*

Proof. If $\gamma_{(2,2,2,0)}(G) = \gamma_{(2,2,2)}(G)$, then for any $\gamma_{(2,2,2)}(G)$-function $f(V_0, V_1, V_2)$, there exists a $\gamma_{(2,2,2,0)}(G)$-function $g(W_0, W_1, W_2, W_3)$ defined by $W_0 = V_0$, $W_1 = V_1$, $W_2 = V_2$ and $W_3 = \varnothing$. Therefore, (i) implies (ii).

Conversely, if there exists a $\gamma_{(2,2,2,0)}(G)$-function $f(V_0, V_1, V_2, V_3)$ such that $V_3 = \varnothing$, then the function $g(W_0, W_1, W_2)$, defined by $W_0 = V_0$, $W_1 = V_1$ and $W_2 = V_2$, is a $(2, 2, 2)$-dominating function on G, and so $\gamma_{(2,2,2)}(G) \leq \omega(g) = \omega(f) = \gamma_{(2,2,2,0)}(G)$. Therefore, Corollary 3.37 leads to $\gamma_{(2,2,2,0)}(G) = \gamma_{(2,2,2)}(G)$, which completes the proof. □

Theorem 3.39. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2,0)}(G) = 3$.

(ii) $\gamma(G) = 1$ *or* $\gamma_{\times 2,t}(G) = 3$.

Proof. Assume first that $\gamma_{(2,2,2,0)}(G) = 3$, and let $f(V_0, V_1, V_2, V_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function. Notice that $|V_3| \in \{0, 1\}$. If $|V_3| = 1$, then $V_1 \cup V_2 = \varnothing$, which implies that V_3 is a dominating set of cardinality one. Hence, $\gamma(G) = 1$.

If $V_3 = \varnothing$, then by Lemma 3.38 we have that $\gamma_{(2,2,2)}(G) = \gamma_{(2,2,2,0)}(G) = 3$, and by Theorem 3.14 we deduce that $\gamma_{\times 2,t}(G) = 3$.

Conversely, if $\gamma(G) = 1$, then Corollary 3.37 leads to $3 \leq \gamma_{(2,2,2,0)}(G) \leq 3\gamma(G) = 3$. Moreover, if $\gamma_{\times 2,t}(G) = 3$, then G has minimum degree $\delta \geq 2$ and so Theorem 3.10(i) leads to $3 \leq \gamma_{(2,2,2,0)}(G) \leq \gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G) = 3$. Therefore, $\gamma_{(2,2,2,0)}(G) = 3$. □

Theorem 3.40. *For a graph G, $\gamma_{(2,2,2,0)}(G) = 4$ if and only if at least one of the following conditions holds.*

(i) $\gamma_{\times 2,t}(G) = 4$.

(ii) $\gamma(G) = \gamma_t(G) = 2$ *and G has minimum degree $\delta = 1$.*

(iii) $\gamma(G) = \gamma_t(G) = 2$ *and* $\gamma_{\times 2,t}(G) \geq 4$.

Proof. Assume $\gamma_{(2,2,2,0)}(G) = 4$. Let $f(V_0, V_1, V_2, V_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function. Hence, $|V_3| \in \{0, 1\}$. If $|V_3| = 1$, then V_3 is a dominating set of cardinality one. Hence, $\gamma(G) = 1$, which is a contradiction with Theorem 3.39. Hence, $V_3 = \varnothing$, and so, Lemma 3.38 leads to $\gamma_{(2,2,2)}(G) = \gamma_{(2,2,2,0)}(G) = 4$. Thus, by Theorems 3.15 and 3.39 we deduce (i) – (iii).

Conversely, if conditions (i) – (iii) hold, then by Theorem 3.14 we have that $\gamma_{(2,2,2)}(G) = 4$. Corollary 3.37 leads to $3 \leq \gamma_{(2,2,2,0)}(G) \leq \gamma_{(2,2,2)}(G) = 4$. Notice that if $\delta \geq 2$, then $\gamma(G) \geq 2$ and $\gamma_{\times 2,t}(G) \geq 4$. Hence, Theorem 3.39 leads to $\gamma_{(2,2,2,0)}(G) = 4$. □

Proposition 3.41. *For any integer* $n \geq 3$,

$$\gamma_{(2,2,2,0)}(C_n) = n.$$

Proof. By Corollaries 3.16 and 3.37 we have that $\gamma_{(2,2,2,0)}(C_n) \leq \gamma_{(2,2,2)}(C_n) = n$. We only need to prove that $\gamma_{(2,2,2,0)}(C_n) \geq n$. Let $f(V_0, V_1, V_2, V_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function such that $|V_3|$ is minimum. If $V_3 = \varnothing$, then by Lemma 3.38 and Corollary 3.16 we conclude that $\gamma_{(2,2,2,0)}(C_n) = n$. Assume $V_3 \neq \varnothing$. If $v \in V_3$, then $N(v) \subseteq V_0$ as otherwise, by choosing one vertex $u \in N(v) \setminus V_0$, the function f' defined by $f'(v) = 2$, $f'(u) = \min\{2, f(u) + 1\}$ and $f'(x) = f(x)$ for the remaining vertices, is a $(2, 2, 2, 0)$-dominating function with $\omega(f') \leq \omega(f)$ and $|V_3'| < |V_3|$, which is a contradiction. Hence, $\sum_{x \in V_3} f(N[x]) = 3|V_3|$. Now, we observe that

$$2 \sum_{x \in V(C_n) \setminus N[V_3]} f(x) \geq \sum_{x \in V(C_n) \setminus N[V_3]} \left(\sum_{u \in N(x)} f(u) \right) \geq 2(n - 3|V_3|).$$

Therefore,

$$\begin{aligned} \gamma_{(2,2,2,0)}(C_n) = \omega(f) &= \sum_{x \in V_3} f(N[x]) + \sum_{x \in V(C_n) \setminus N[V_3]} f(x) \\ &\geq 3|V_3| + (n - 3|V_3|) = n, \end{aligned}$$

and the result follows. □

Proposition 3.42. *For any integer* $n \geq 3$,

$$\gamma_{(2,2,2,0)}(P_n) = \begin{cases} 6 & \text{if } n = 5, \\ n & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that $\gamma_{(2,2,2,0)}(P_n) = n$ for $n = 3, 4, 6, 7, 8$, and also $\gamma_{(2,2,2,0)}(P_5) = 6$. From now on, assume $n \geq 9$. By Propositions 3.5 and 3.41 we have that $n = \gamma_{(2,2,2,0)}(C_n) \leq \gamma_{(2,2,2,0)}(P_n)$. Hence, we only need to prove that $\gamma_{(2,2,2,0)}(P_n) \leq n$. To this end, we proceed to construct a $(2, 2, 2, 0)$-dominating function $f(V_0, V_1, V_2, V_3)$ on $P_n = v_1 v_2 \ldots v_n$ such that $\omega(f) = n$.

- If $n \equiv 0 \pmod 3$, then we set $V_3 = \bigcup_{i=1}^{n/3} \{v_{3i-1}\}$ and $V_0 = V(G) \setminus V_3$.
- If $n \equiv 1 \pmod 3$, then we set $V_3 = \bigcup_{i=1}^{(n-4)/3} \{v_{3i-1}\}$, $V_2 = \{v_{n-2}, v_{n-1}\}$ and $V_0 = V(G) \setminus (V_2 \cup V_3)$.

- If $n \equiv 2 \pmod 3$, then we set $V_3 = \bigcup_{i=1}^{(n-8)/3}\{v_{3i-1}\}$, $V_2 = \{v_{n-6}, v_{n-5}, v_{n-2}, v_{n-1}\}$ and $V_1 = \varnothing$.

Notice that in the three cases above, f is a $(2, 2, 2, 0)$-dominating function of weight $\omega(f) = n$, as required. Therefore, the proof is complete. □

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