Quasi-configurations:
Building blocks for point – line configurations

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Abstract

We study point – line incidence structures and their properties in the projective plane. Our motivation is the problem of the existence of (n_4) configurations, still open for few remaining values of n. Our approach is based on quasi-configurations: point – line incidence structures where each point is incident to at least 3 lines and each line is incident to at least 3 points. We investigate the existence problem for these quasi-configurations, with a particular attention to 3|4-configurations where each element is 3- or 4-valent. We use these quasi-configurations to construct the first (37_4) and (43_4) configurations. The existence problem of finding (22_4), (23_4), and (26_4) configurations remains open.

Keywords: Projective arrangements, point – line incidence structure, (n_k) configurations.

Math. Subj. Class.: 52C30

1   Introduction

A *geometric (n_k) configuration* is a collection of n points and n lines in the projective plane such that each point lies on k lines and each line contains k points. We recommend our reader to consult Grünbaum's book [7] for a comprehensive presentation and an historical perspective on these configurations. The central problem studied in this book is to determine for a given k those numbers n for which there exist geometric (n_k) configurations. The answer is completely known for $k = 3$ (geometric (n_3) configurations exist if
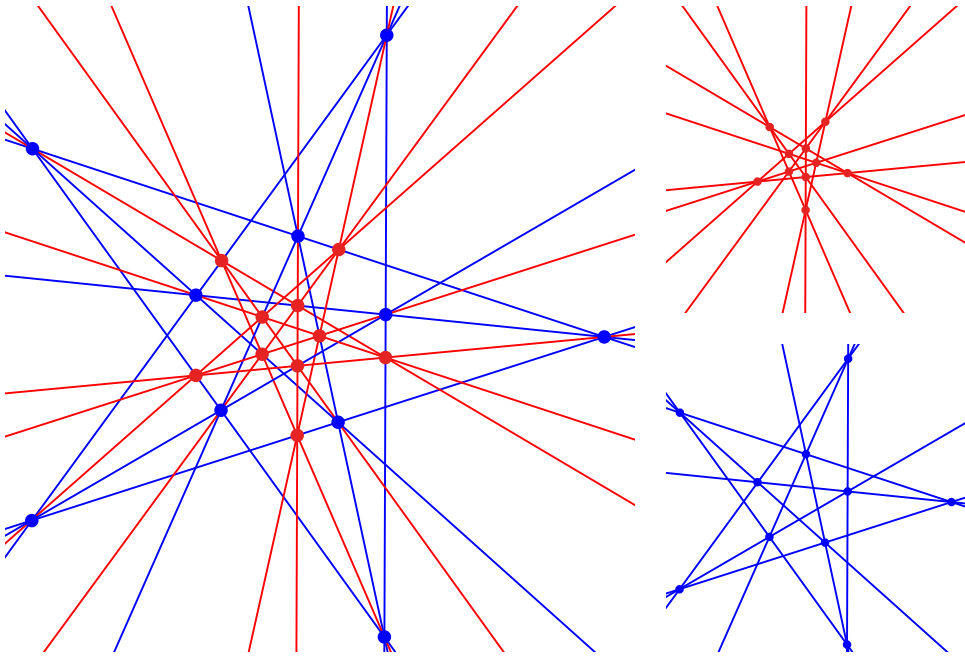
Figure 1: Splitting Grünbaum's geometric (20_4) configuration [6] into two (10_3) configurations.

and only if $n \geq 9$), partially solved for $n = 4$ (geometric (n_4) configurations exist if and only if $n = 18$ or $n \geq 20$ with a finite list of possible exceptions), and wide open for $k > 4$. Our contribution concerns $k = 4$, where we provide solutions for two former open cases: there exist geometric (37_4) and (43_4) configurations. Moreover, we study building blocks for constructing geometric (n_k) configurations that might be of some help for clarifying the final open cases (22_4), (23_4), and (26_4). Many aspects of our presentation appeared during our investigation of the case (19_4) in which there is no geometric (19_4) configuration, see [1, 2].

The approach of this paper is to construct geometric (n_4) configurations from smaller building blocks. For example, Grünbaum's geometric (20_4) configuration [6] can be constructed by superposition of two geometric (10_3) configurations as illustrated in Figure 1. To extend this kind of construction, we study an extended notion of point–line configurations, where incidences are not regular but still prescribed.

1.1   Point–line incidence structures

We define a *point–line incidence structure* as a set P of *points* and a set L of *lines* together with a point–line *incidence relation*, where two points of P can be incident with at most one line of L and two lines of L can be incident with at most one point of P. Throughout the paper, we only consider *connected* incidence structures, where any two elements of $P \sqcup L$ are connected via a path of incident elements.

For a point–line incident structure (P, L), we denote by p_i the number of points of P contained in i lines of L and similarly by ℓ_j the number of lines of L containing j points

of P. We find it convenient to encode these incidence numbers into a pair of polynomi-als $(\mathbf{P}(x), \mathbf{L}(y))$, called the *signature* of (P, L), and defined by

$$\mathbf{P}(x) := \sum_i p_i x^i \qquad \text{and} \qquad \mathbf{L}(y) := \sum_j \ell_j y^j.$$

For example, the point – line incidence structure represented in Figure 2 has signature $(8x^3 + 2x^4, 8y^3 + 2y^4)$. With these notations, the number of points and lines are given by $|P| = \mathbf{P}(1)$ and $|L| = \mathbf{L}(1)$, while the number of point – line incidences is given by $|\{(p, \ell) \in P \times L \mid p \in \ell\}| = \mathbf{P}'(1) = \mathbf{L}'(1)$.

We distinguish three different levels of point – line incidence structures, in increasing generality:

Geometric Points and lines are ordinary points and lines in the real projective plane $\mathbb{P}$.

Topological Points are ordinary points in $\mathbb{P}$, but lines are *pseudolines*, *i.e.*, non-separating simple closed curves of $\mathbb{P}$ which cross pairwise precisely once.

Combinatorial Just an abstract incidence structure (P, L) as described above, with no ad-ditional geometric structure.

In this paper, we are mainly interested in the geometric level. We therefore omit the word geometric in what follows unless we have to distinguish different levels.

1.2 (n_k) configurations

One of the main problems in the theory of point – line incidence structures is to clarify the existence of regular point – line incidence structures. A *k-configuration* is a point – line incidence structure (P, L) where each point of P is contained in k lines of L and each line of L contains k points of P. In such a configuration, the number of points equals the number of lines, and thus it has signature (nx^k, ny^k). If we want to specify the number of points and lines, we call it an (n_k) *configuration*. We refer to the recent monographs of Grünbaum [7] and Pisanski and Servatius [8] for comprehensive presentations of these objects. Classical examples of regular configurations are Pappus' and Desargues' configu-rations, which are respectively (9_3) and (10_3) configurations. In the study of the existence of (n_4) configurations there are still a few open cases. Namely, it is known that (geometric) (n_4) configurations exists if and only if $n = 18$ or $n \geq 20$, with the possible exceptions of $n = 22, 23, 26, 37$ and 43 [5, 4, 2]. Different methods have been used to obtain the current results on the existence of 4-configurations:

(i) For $n \leq 16$, Bokowski and Schewe [3] used a counting argument based on Euler's formula to prove that there exist no (n_4) configuration, even topological.

(ii) For small values of n, one can search for all possible (n_4) configurations. For $n = 17$ or 18, one can first enumerate all combinatorial (n_4) configurations and search for ge-ometric realizations among them. This approach was used by Bokowski and Schewe in [4] to show that there is no (17_4) configuration and to produce a first (18_4) config-uration. Another approach, proposed in [1], is to enumerate directly all topological (n_4) configurations, and to search for geometric realizations among this restricted family. In this way, we showed that there are precisely two (18_4) configurations, that of [4] and another one [1], see Figure 3. For $n = 19$, we obtained in [1] all 4 028 topo-logical (19_4) configurations and the study of their realizability has led to the result that there is no geometric (19_4) configuration [2].

(iii) For larger values of n, one cannot expect a complete classification of (n_4) configurations. However, one can construct families of examples of 4-configurations. One of the key ingredients for such constructions is the use of symmetries. See Figure 1 for the smallest example obtained in this way, and refer to the detailed presentation in Grünbaum's recent monograph [7].

(iv) Finally, Bokowski and Schewe introduced in [4] a method to produce (n_4) configurations from deficient configurations. It consists in finding two point–line incidence structures (P, L) and (P', L') of respective signatures $(ax^3 + bx^4, cy^3 + dy^4)$ and $(cx^3 + ex^4, ay^3 + fy^4)$, where $a + b + c + e = a + c + d + f = n$, and a projective transformation which sends the 3-valent points of P to points contained in a 3-valent line of L', and at the same time the 3-valent lines of L to lines containing a 3-valent point of P'. This method was used to obtain the first examples of (29_4) and (31_4) configurations.

In this paper, we are interested in this very last method described above. We are going to study deficient configurations (see the notion of quasi-configuration and 3|4-configuration in the next subsection) for the use of them as building blocks for configurations. Our study has led in particular to first examples of (37_4) and (43_4) configurations. Thus the remaining undecided cases for the existence of (n_4) configurations are now only the cases $n = 22, 23$, and 26.

1.3   Quasi-configurations

A *quasi-configuration* (P, L) is a point–line incidence structure in which each point is contained in at least 3 lines and each line contains at least 3 points of P. In other words, the signature $(\mathbf{P}, \mathbf{L})$ of (P, L) satisfies $x^3 \mid \mathbf{P}(x)$ and $y^3 \mid \mathbf{L}(y)$. The term "quasi-configuration" for this concept was suggested by Grünbaum to the authors. As observed above, these quasi-configurations can sometimes be used as building blocks for larger point–line incidence structures.

In this paper, we investigate in particular 3|4-*configurations*, where each point of P is contained in 3 or 4 lines of L and each line of L contains 3 or 4 points of P. In other words, 3|4-configurations are point–line incidence structures whose signature is of the form $(ax^3 + bx^4, cx^3 + dx^4)$ for some $a, b, c, d \in \mathbb{N}$ satisfying $3a + 4b = 3c + 4d$. Note that their numbers of points and lines do not necessarily coincide. If it is the case, *i.e.*, if $a + b = c + d = n$, we speak of an $(n_{3|4})$ *configuration*. In this case, $a = c$ and $b = d$, the number of points and lines is $n = a + b = c + d$ and the number of incidences is $3a + 4b = 3c + 4d$.

We think of an $(n_{3|4})$ configuration as a deficient (n_4) configuration. A good measure on $(n_{3|4})$ configurations is the number of missing incidences a. We say that an $(n_{3|4})$ configuration is *optimal* if it contains the maximal number of point–line incidences among all $(n_{3|4})$ configurations. One objective is to study and classify optimal $(n_{3|4})$ configurations for small values of n.

Example 1.1. Figure 2 shows an incidence structure with signature $(8x^3 + 2x^4, 8y^3 + 2y^4)$. It is a $10_{3|4}$-configuration: the 3-valent elements are colored red while the 4-valent elements are colored blue. We will see in Figure 7 that this 3|4-configuration is not optimal.
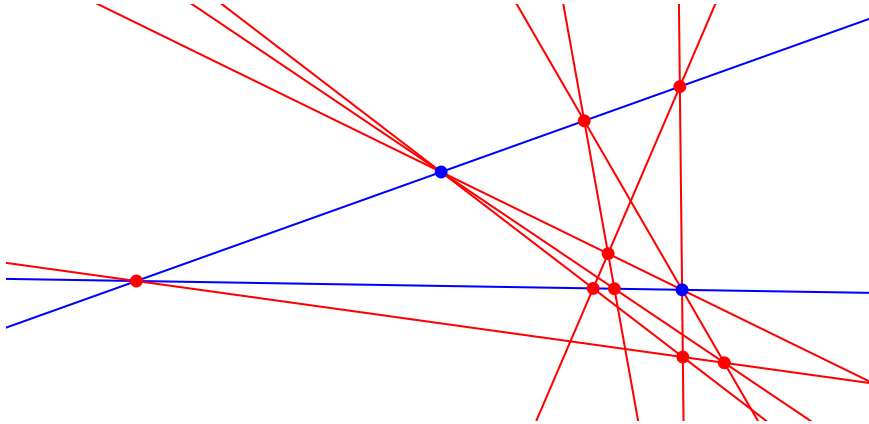
Figure 2: A quasi-configuration with signature $(8x^3 + 2x^4, 8y^3 + 2y^4)$.

1.4 Overview

The paper is divided into two parts. In Section 2, we illustrate how quasi-configurations (in particular 3|4-configurations) can be used as building blocks to construct (n_4) configurations, and we obtain in particular examples of (37_4) and (43_4) configurations. In Section 3, we present a counting obstruction for the existence of topological quasi-configurations, and we study optimal $(n_{3|4})$ configurations with few points and lines.

2 Constructions

We discuss here different ways to obtain new point–line incidence structures from old ones. We are in particular interested in the construction of new quasi-configurations from old ones. We use these techniques to provide the first (37_4) and (43_4) configurations.

2.1 Operations on point–line incidence structures

To construct new point–line incidence structures from old ones, we will use the following operations, illustrated in Section 2.2:

Deletion Deleting elements from a point–line incidence structure yields a smaller incidence structure. Note that deletions do not necessarily preserve connectedness or quasi-configurations. We can however use deletions in 4-configurations to construct 3|4-configurations if no remaining element is incident to two deleted elements.

Addition As illustrated by the example of Grünbaum's (20_4) configuration [6] in Figure 1, certain point–line incidence structures can be obtained as the disjoint union of two smaller incidence structures (P, L) and (P', L'). In particular, we obtain an (n_4) configuration if (P, L) and (P', L') are 3|4-configurations, if each 3-valent element of (P, L) is incident to precisely one 3-valent element of (P', L') and conversely, and if no other incidences appear.

Splitting The reverse operation of addition is splitting: given a point–line incidence structure, we can split it into two smaller incidence structures. We can require additionally
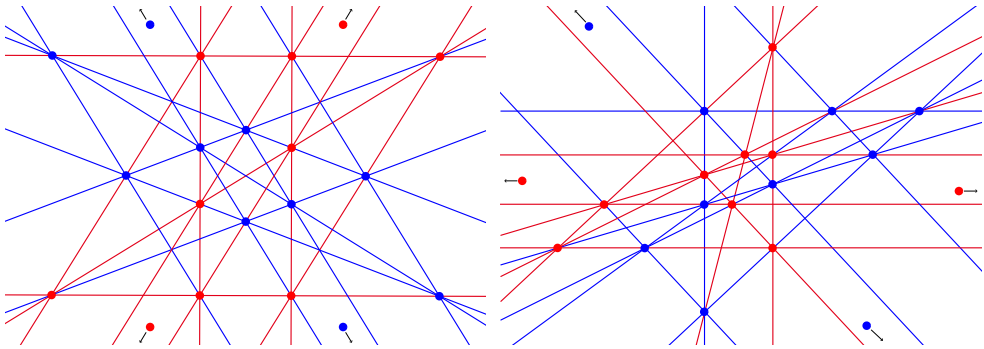
Figure 3: Splittings of the two geometric (18_4) configurations [4, 1] into two $(9_{3|4})$ configurations. The rightmost (18_4) configuration even splits into two (9_3) configurations. The points which seem isolated are in fact at infinity in the direction pointed by the corresponding arrow, and are incident to the 4 lines parallel to that direction.

       the two resulting incidence structures to be quasi-configurations or even regular configurations. For example, the two geometric (18_4) configurations [4, 1] as well as Grünbaum's (20_4) configuration [6] are splittable into 3|4-configurations, see Figures 1 and 3.

Superposition Slightly more general than addition is the superposition, where we allow the two point–line incidence structures (P, L) and (P', L') to share points or lines. For example, we can superpose two 2-valent vertices to make one 4-valent vertex. This idea is used in our construction of (37_4) and (43_4) configurations below.

2.2   Examples of constructions

We now illustrate the previous operations and produce 4-configurations from smaller point–line incidence structures. We start with a simple example.

Example 2.1 (A (38_4) configuration)**.** It was shown in [2] that no topological (19_4) configuration can be geometrically realized with points and lines in the projective plane. However, Figure 4 (left) shows a geometric realization of a topological (19_4) configuration where one line has been replaced by a circle. Forgetting this circle, we obtain a 3|4-configuration with signature $(15x^4 + 4x^3, 18x^4)$. We take two opposite copies of this 3|4-configuration (colored purple and red in Figure 4 (right)) and add two lines (colored green in Figure 4 (right)) each incident to two points in each copy. We obtain a (38_4) configuration.

    Using similar ideas, we observe that it is always possible to produce a 4-configuration from any 3|4-configuration.

Example 2.2 (Any 3|4-configuration generates a 4-configuration)**.** From a 3|4-configuration with signature $(ax^3 + bx^4, cy^3 + dy^4)$, we can construct an (n_4) configuration where $n = 16a + 16b + 4c = 4a + 16c + 16d$ as follows:

(i) We take four translated copies of the 3|4-configuration and add suitable parallel lines through all 3-valent points.
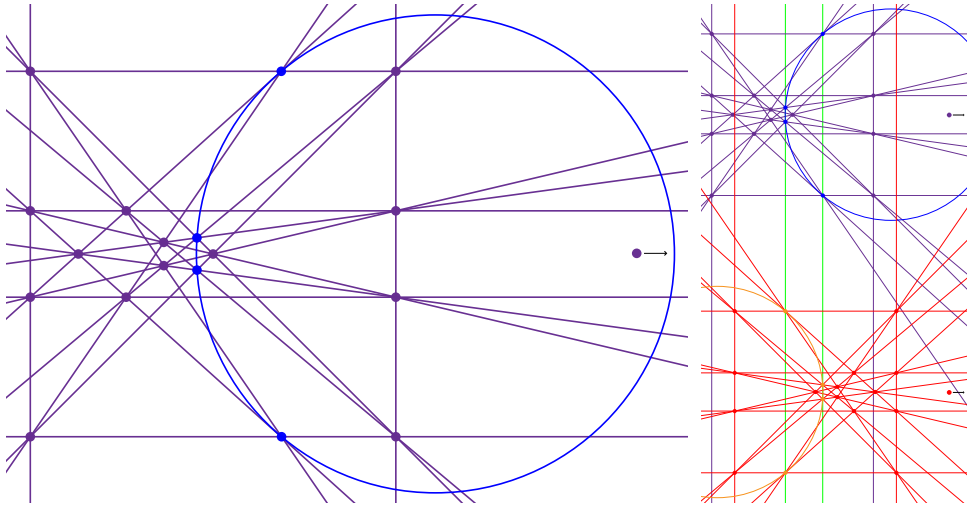
Figure 4: (Left) A geometric realization of a topological (19_4) configuration where one line has been replaced by a circle. (Right) A (38_4) configuration built from two copies of this incidence structure. The construction is explained in full detail in Example 2.1.

(ii) We take the geometric dual of the resulting $3|4$-configuration (remember that geometric duality transforms a point p of the projective plane into the line formed by all points orthogonal to p and conversely).

(iii) We take again four translated copies of this dual $3|4$-configuration and add suitable parallel lines through all 3-valent vertices.

Of course, we can try to obtain other 4-configurations from $3|4$-configurations. This approach was used by Bokowski and Schewe [4] to construct (29_4) and (31_4) configurations from the $(14_{3|4})$, $(15_{3|4})$ and $(16_{3|4})$ configurations of Figure 9. We refer to their paper [4] for an explanation. Here, we elaborate on the same idea to construct two new relevant (n_4) configurations.

Example 2.3 (First (43_4) configuration). To construct a $((n+m)_4)$ configuration from an (n_4) configuration and an (m_4) configuration, we proceed as follows (see Figure 5):

(i) We delete two points not connected by a line in the (n_4) configuration and consider the eight resulting 3-valent lines (colored blue in Figure 5 (top left) and orange in Figure 5 (top right)).

(ii) We add four points (colored green in Figure 5), each incident with precisely two 3-valent lines. All points and lines are now 4-valent again, except the four new 2-valent points.

(iii) We do the same operations in the (m_4) configuration.

(iv) Finally, we use a projective transformation that maps the set of four 2-valent points in the first quasi-configuration onto the set of four 2-valent points in the second quasi-configuration. This transformation superposes the 2-valent points to make them 4-valent.

If this transformation does not superpose other elements than the 2-valent ones and does not create additional unwanted incidences, it yields the desired $((n + m)_4)$ configuration. This construction is illustrated on Figure 5, where we obtain a (43_4) configuration from a (25_4) configuration [7] and a (18_4) configuration [1].

Unfortunately, the method from the previous example cannot provide a (37_4) configuration since there is no (n_4) configuration for $n \leq 17$ [4] and for $n = 19$ [2]. We therefore need another method, which we describe in the following example.

Example 2.4 (First (37_4) configuration). To construct a $((n+m-1)_4)$ configuration from an (n_4) configuration and an (m_4) configuration, we proceed as follows (see Figure 6):

  (i) We delete two points on the same line (colored green in Figure 6) of the (n_4) configuration and consider the six resulting 3-valent lines (colored blue in Figure 6 (top left) and orange in Figure 6 (top right)).

 (ii) We add three points (colored green in Figure 6), each incident with precisely two 3-valent lines. All points and lines are now 4-valent again, except the initial 2-valent line and the three new 2-valent points.

(iii) We do the same operations in the (m_4) configuration.

 (iv) Finally, we use a projective transformation that maps the set of four 2-valent elements in the first quasi-configuration onto the set of four 2-valent elements in the second quasi-configuration. This transformation superposes the 2-valent elements to make them 4-valent.

If this transformation does not superpose other elements than the 2-valent ones and does not create additional unwanted incidences, it yields the desired $((n + m - 1)_4)$ configuration. This construction is illustrated on Figure 6, where we obtain a (37_4) configuration from a (20_4) configuration [7] and a (18_4) configuration [1].

We invite the reader to try his own constructions, similar to the constructions of Examples 2.3 and 2.4, using the operations on point – line incidence structures described above. In this way, one can obtain many (n_4) configurations for various values of n. Additional features can even be imposed, such as non-trivial motions or symmetries. We have however not been able to find answers to the following question.

Question 1. Can we create a (22_4) configuration by glueing two quasi-configurations with 11 points and lines each? More generally, can we construct (22_4), (23_4), or (26_4) configurations by superposition of smaller quasi-configurations?

3   Obstructions and optimal 3|4-configurations

In this section, we further investigate point – line incidence structures and 3|4-configurations. We start with a necessary condition for the existence of topological incidence structures with a given signature. For this, we extend to all topological incidence structures an argument of Bokowski and Schewe [3] that was used to prove the non-existence of (15_4) configurations. We obtain the following inequality.

Proposition 3.1. *Let (P, L) be topological incidence structure with signature $(\mathbf{P}, \mathbf{L})$. Then*

$$\mathbf{P}''(1) + 2\mathbf{P}'(1) - \mathbf{L}(1)^2 + \mathbf{L}(1) - 6\mathbf{P}(1) + 6 \leq 0.$$
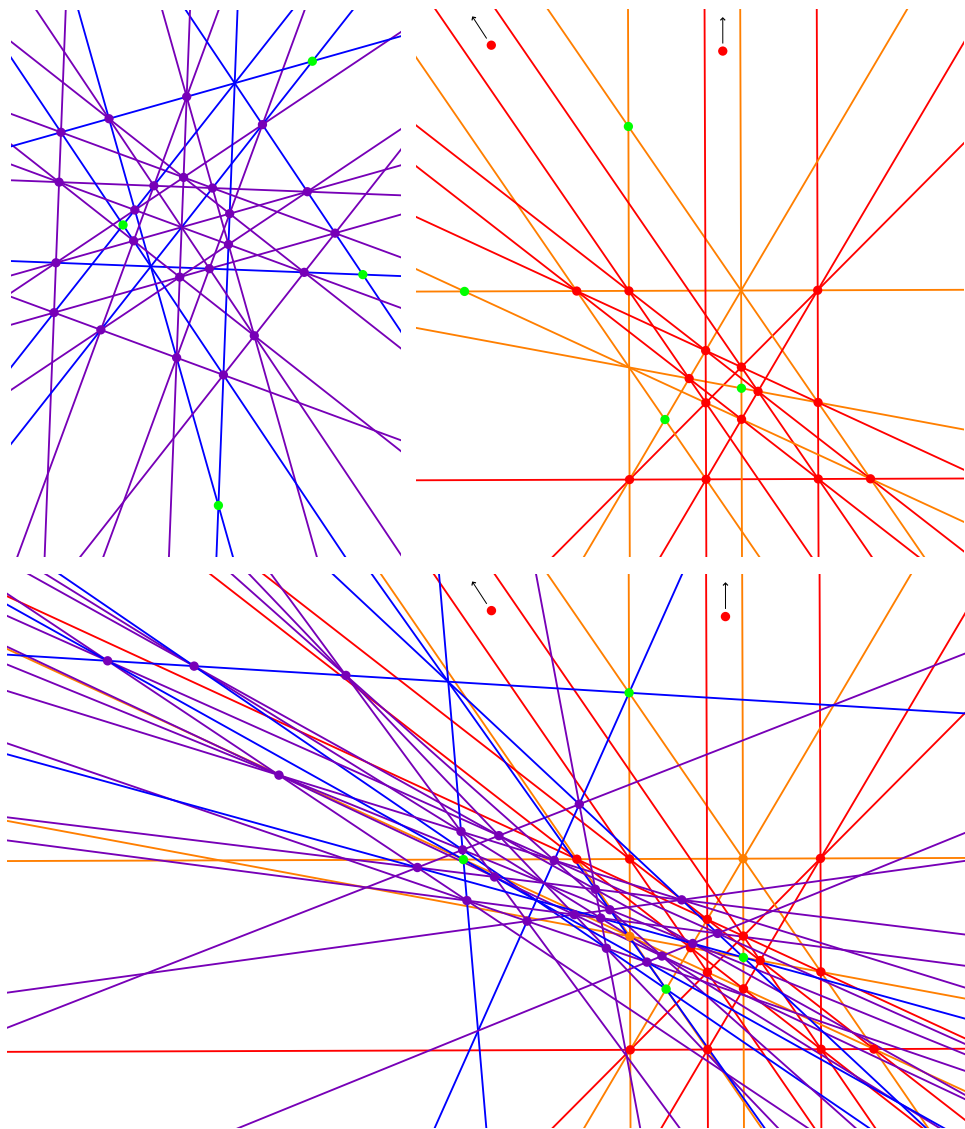
Figure 5: A (43_4) configuration built from deficient (25_4) and (18_4) configurations. The construction is explained in full detail in Example 2.3.
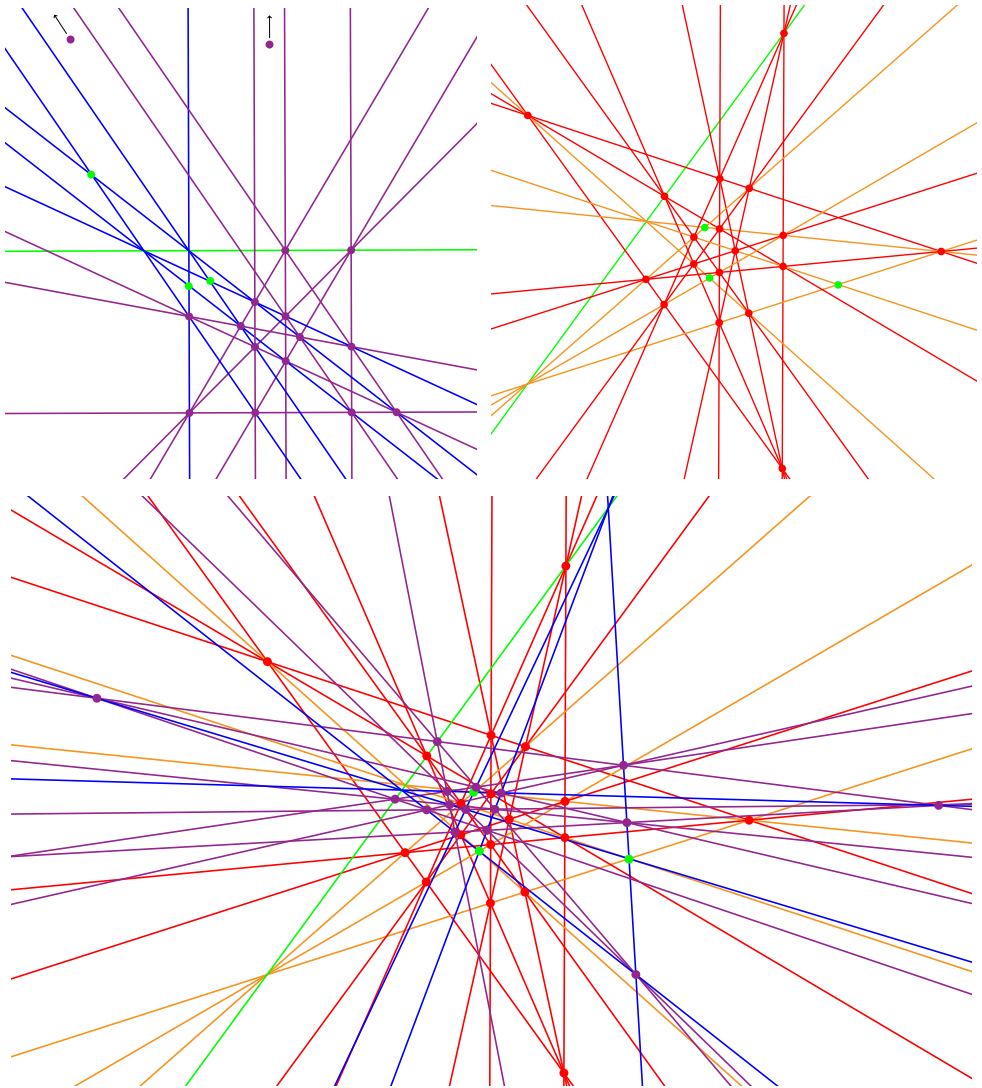
Figure 6: A (37_4) configuration built from deficient (20_4) and (18_4) configurations. The construction is explained in full detail in Example 2.4.

Proof. Let p_i be the number of i-valent points and ℓ_j the number of j-valent lines in the incidence structure (P, L). The signature $(\mathbf{P}, \mathbf{L})$ is $\mathbf{P}(x) := \sum_i p_i x^i$ and $\mathbf{L}(y) := \sum_j \ell_j y^j$.

Since the incidence structure is topological, we can draw it on the projective plane such that no three pseudolines pass through a point which is not in P. We call *additional 2-crossings* the intersection points of two lines of L which are not points of P. We consider the lifting of this drawing on the 2-sphere. We obtain a graph embedded on the sphere, whose vertices are all points of P together with all additional 2-crossings, whose edges are the segments of lines of L located between two vertices, and whose faces are the connected components of the complement of L. Let f_0, f_1 and f_2 denote respectively the number of vertices, edges and faces of this map. Denoting by $\deg(p)$ the number of lines of L containing a point $p \in P$ and similarly by $\deg(\ell)$ the numbers of points of P contained in a line $\ell \in L$, we have

$$f_0 = 2\binom{\mathbf{L}(1)}{2} - 2\sum_{p \in P}\left(\binom{\deg(p)}{2} - 1\right) = \mathbf{L}(1)(\mathbf{L}(1) - 1) + 2\mathbf{P}(1) - \sum_i i(i-1)p_i,$$

$$f_1 = 2\sum_{\ell \in L}\deg(\ell) + 2f_0 - 2\mathbf{P}(1) = 2\sum_j j\ell_j + 2f_0 - 2\mathbf{P}(1) = 2\sum_i ip_i + 2f_0 - 2\mathbf{P}(1),$$

$$f_2 = f_1 - f_0 + 2.$$

Moreover, since no face is a digon, we have $3f_2 \leq 2f_1$. Replacing f_2 and f_1 by the above expressions, we obtain

$$0 \geq 3f_2 - 2f_1 = f_1 - 3f_0 + 6 = 2\sum_i ip_i - 4\mathbf{P}(1) - f_0 + 6$$

$$= \sum_i i(i+1)p_i - \mathbf{L}(1)(\mathbf{L}(1) - 1) - 6(\mathbf{P}(1) - 1),$$

and thus the desired inequality. □

Corollary 3.2. *If $(ax^3 + bx^4, ay^3 + by^4)$ is the signature of a topological incidence structure, then*

$$-(a + b)^2 + 7a + 15b + 6 \leq 0.$$

The following table provides the minimum value of b for which there could exist a topological incidence structure with signature $(ax^3 + bx^4, ay^3 + by^4)$:

a	0	1	2	3	4	5	6	7
$b_{\min}$	16	14	13	11	9	8	6	3

Proof. Direct from Proposition 3.1 with $\mathbf{P}(x) = ax^3 + bx^4$ and $\mathbf{L}(y) = ay^3 + by^4$. □

For example, there is no topological (15_4) configuration [3] and no incidence structure with signature $(7x^3 + 2x^4, 7y^3 + 2y^4)$. Compare to Example 1.1 which shows that a configuration with signature $(8x^3 + 2x^4, 8y^3 + 2y^4)$ exists.

Corollary 3.3. *A $(n_{3|4})$ configuration has at most $I_{\max} := \min\left(4n, \left\lfloor\dfrac{n^2 + 17n - 6}{8}\right\rfloor\right)$ incidences. The values of $I_{\max}$ appear in the following table:*

n	7	8	9	10	11	12	13	14	15	16
$I_{\max}$	20	24	28	33	37	42	48	53	59	64

Proof. Consider an $(n_{3|4})$ configuration with signature $(ax^3 + bx^4, ay^3 + by^4)$ where $a + b = n$. The number of incidences is $I := 3a + 4b$. It can clearly not exceed $4n$. For the second term in the minimum, we apply Corollary 3.2 to get

$$0 \geq -(a + b)^2 + 7a + 15b + 6$$
$$= -(a + b)^2 + 8(3a + 4b) - 17(a + b) + 6$$
$$= -n^2 + 8I - 17n + 6. \qquad \square$$

Corollary 3.4. *There is no topological $(n_{3|4})$ configuration if $n \leq 8$.*

Proof. If $n \leq 7$, there is no topological $(n_{3|4})$ configuration since it should have at least $3n$ incidences, which is larger than the upper bound of Corollary 3.3. If $n = 8$, a $(8_{3|4})$ configuration should be a (8_3) configuration by Corollary 3.3. But the only combinatorial (8_3) configuration is not topological. $\qquad \square$

To close this section, we exhibit optimal $(n_{3|4})$ configurations for small values of n, *i.e.*, $(n_{3|4})$ configurations which maximize the number of point–line incidences.

Proposition 3.5. *For $9 \leq n \leq 13$, the bound of Corollary 3.3 is tight: there exists $(n_{3|4})$ configurations with $\left\lfloor \frac{n^2 + 17n - 6}{8} \right\rfloor$ incidences.*

Proof. For $n = 13$, we consider the $(13_{3|4})$-configuration of Figure 7. The homogeneous coordinates of its points and lines are given by

$$P := L := \left\{ \begin{bmatrix} i \\ j \\ 1 \end{bmatrix} \,\middle|\, i, j \in \{-1, 0, 1\} \right\} \cup \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

For $n = 10, 11$ or 12, we obtain $(n_{3|4})$ configurations by removing suitable points and lines in our $(13_{3|4})$ configuration. The resulting configurations are illustrated in Figure 7. (Note that for $n = 10$, we even have two dual ways to suitably remove three points and three lines from our $(13_{3|4})$ configuration: either we remove three 3-valent points and the three 4-valent lines containing two of these points, or we remove three 3-valent lines and the three 4-valent points contained in two of these lines). Finally, for $n = 9$ we use the bottom rightmost $(9_{3|4})$ configuration of Figure 7. $\qquad \square$

As a curiosity, we give another example of an optimal $(12_{3|4})$ configuration which contains Pappus' and Desargues' configurations simultaneously. See Figure 8.

Observe that optimal $(n_{3|4})$ configurations are given by (n_4) configurations for large n, and that the only remaining cases for optimal $(n_{3|4})$ configurations are for $n = 14, 15, 16, 17, 19, 22, 23,$ and 26. We have represented in Figure 9 some $(15_{3|4})$ and $(16_{3|4})$ configurations which we expect to be optimal, although they do not reach the theoretical upper bound of Corollary 3.3. Observe also that deleting the circle in Figure 4 (left) and adding one line through two of the resulting 3-valent points provides a $(19_{3|4})$ configuration with 74 incidences, which is almost optimal since there is no (19_4) configuration [1, 2]. To conclude, we thus leave the following question open.

Question 2. What are the optimal $(14_{3|4})$ configurations? Are the $(15_{3|4})$ and $(16_{3|4})$ configurations in Figure 9 optimal? Is there a $(19_{3|4})$ configuration with 75 incidences.
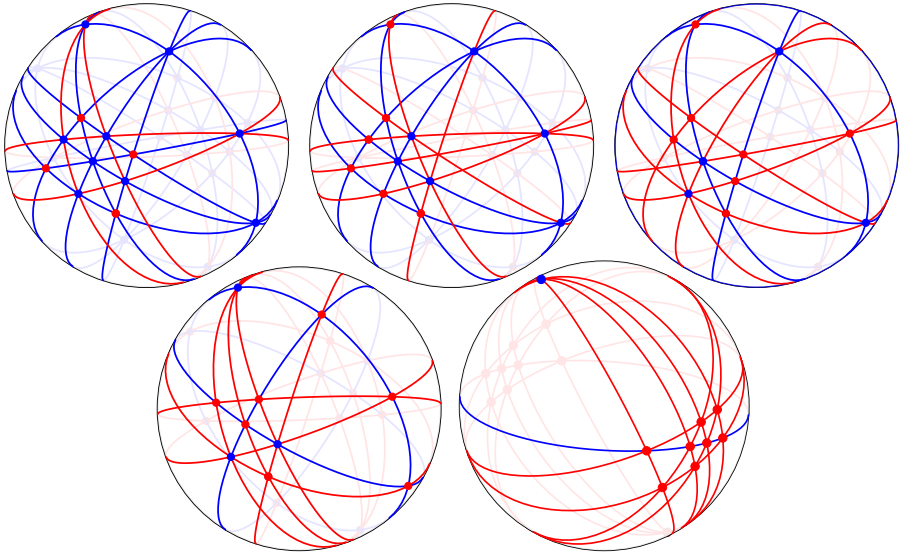
Figure 7: Optimal $(n_{3|4})$ configurations, for $n = 13, 12, 11, 10, 9$. They have respectively 48, 42, 37, 33, and 28 point – line incidences. The 3-valent elements are colored red while the 4-valent elements are colored blue.
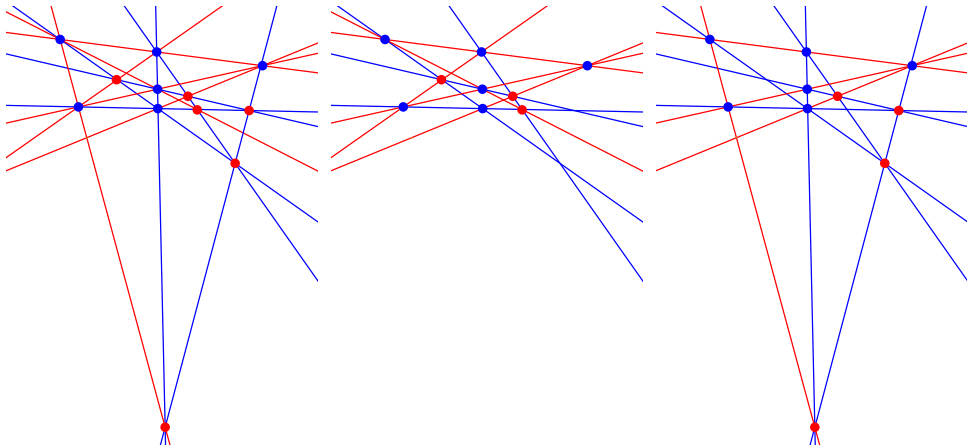

Figure 8: An optimal $(12_{3|4})$ configuration (left) which contains simultaneously Pappus' configuration (middle) and Desargues' configuration (right). In the $(12_{3|4})$ configuration, the 3-valent elements are colored red while the 4-valent elements are colored blue. In the Pappus' and Desargues' subconfigurations, all elements are 3-valent, but we keep the color to see the correspondence better.
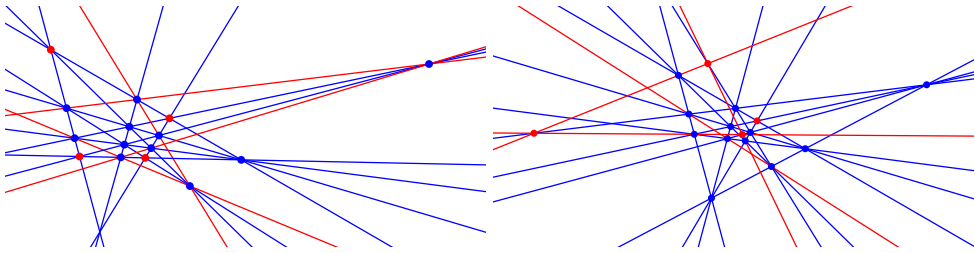
Figure 9: Apparently optimal $(15_{3|4})$ and $(16_{3|4})$ configurations. They have 56 and 60 point – line incidences respectively. The 3-valent elements are colored red while the 4-valent elements are colored blue.

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