Inter Programming Models for the Target Visitation Problem

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The target visitation problem (TVP) is concerned with finding a route to visit a set of targets starting from and returning to some base. In addition to the distance traveled a tour is evaluated by taking also preferences into account which address the sequence in which the targets are visited. The problem thus is a combination of two well-known combinatorial optimization problems: the traveling salesman and the linear ordering problem. In this paper we point out some polyhedral properties and develop a branch-and-cut algorithm for solving the TVP to optimality. Some computational results are presented.

Povzetek: Prispevek obravnava iskanje poti v grafu, kjer je potrebno obiskati več ciljev v najboljšem vrstnem redu.

1 Introduction

Let $D_{n+1} = (V_{n+1}, A_{n+1})$ be the complete digraph on $n+1$ nodes where we set $V_{n+1} = \{0, 1, \ldots, n\}$. Furthermore let two types of arc weights be defined: weights d_{ij} (*distances*) for every arc (i,j), $0 \leq i, j \leq n$, and weights p_{ij} (*preferences*) associated with every arc (i,j), $1 \leq i, j \leq n$. The *target visitation problem (TVP)* consists of finding a Hamiltonian tour starting at node 0 visiting all remaining nodes (called *targets*) exactly once in some order and returning to node 0. Every tour can be represented by a permutation π of $\{1, 2, \ldots, n\}$ where $\pi(i) = j$ if target j is visited as i-th target. For convenience we also define $\pi(0) = 0$ and $\pi(n+1) = 0$.

So we are essentially looking for a traveling salesman tour, but for the TVP the profit of a tour depends on the two weights. Namely, the value of a tour is the sum of pairwise preferences between the targets corresponding to their visiting sequence minus the sum of distances traveled, i.e., it is calculated as

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{\pi(i)\pi(j)} - \sum_{i=0}^{n} d_{\pi(i)\pi(i+1)},$$

and the task is to find a tour of maximum value. So we have a multicriteria objective function.

The TVP was introduced in [4] and combines two classical combinatorial optimization problems: the *asymmetric traveling salesman problem (ATSP)* asking for a shortest Hamiltonian tour and the *linear ordering problem (LOP)* which is to find an acyclic tournament of maximum weight. (There is an obvious 1–1 correspondence between acyclic tournaments and linear orders of the node). Computational results of a genetic algorithm for problem instances with up to 16 targets have been reported in [1]. The original application of the TVP is the planning of routes for UAVs (unarmed aerial vehicles). But there is a wide field of applications, e.g. the delivery of relief supplies or any other routing problem where additional preferences should be considered (town cleaning, snow-plowing service, etc.).

Obviously, the TVP is NP-hard because it contains the traveling salesman problem ($p \equiv 0$) and the linear ordering problem ($d \equiv 0$) as special cases.

In this paper we present first polyhedral results for the TVP and develop an algorithm for solving it to optimality. In section 2 we introduce an integer programming model. Section 3 discusses some structural properties of the associated polytope. A branch-and-cut algorithm based on these results is described in section 4. The algorithm is then applied to a set of benchmark problems and the computational results are presented in section 5. A few remarks conclude the paper.

2 An integer programming model for the TVP

For convenience we first transform the problem to a Hamiltonian path problem and also get rid of the special base node. This transformation is well-known for the ATSP [7] and can be adapted for the TVP as follows.

The key idea is to exploit the fact that each tour has to start at the base and return to it and that no preferences are to be taken into account for the base. In the *TVP-path model* we leave out this node and just search for a Hamiltonian path which visits all targets exactly once.

Following [7] we make the following modifications.

(i) Transform the distance matrix by setting $d'_{ij} = d_{ij} - d_{i0} - d_{0j}$, for all pairs i and j of nodes, $1 \leq i, j \leq n$ $i \neq j$.

(ii) Change the computation of the distance part of the objective function to

$$\sum_{i=1}^{n-1} d'_{\pi(i)\pi(i+1)} - \sum_{i=1}^{n} d_{i0} - \sum_{i=1}^{n} d_{0i}.$$

The preferences are not affected by this change. From now on we consider the TVP as finding a Hamiltonian path in the complete digraph $D_n = (V_n, A_n)$ with additional preference costs to be taken into account. The path is described by a permutation π of $\{1, \ldots, n\}$ where $\pi(k)$ is the node at position k.

We introduce two types of variables. The sequence in which the targets are visited is represented by binary *ATSP variables* x_{ij} for $1 \leq i, j \leq n,\ i \neq j$, with the interpretation

$$x_{ij} := \begin{cases} 1 & \text{if } i = \pi(k) \text{ and } j = \pi(k+1) \\ & \text{for some } 1 \leq k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that some target i is visited before some target j is modeled with binary *LOP variables* w_{ij} for $1 \leq i, j \leq n,\ i \neq j$, with the definition

$$w_{ij} := \begin{cases} 1 & \text{if } i = \pi(k) \text{ and } j = \pi(l) \\ & \text{for some } 1 \leq k < l \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

An obvious idea for obtaining an IP model of the TVP is to combine well-known IP formulations for the ATSP and the LOP. This combination gives the following integer programming model.

$$\max \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} w_{ij} - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij} x_{ij} \tag{1}$$

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = n-1, \tag{2}$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1,$$
$$S \subseteq \{1, \ldots, n\}, 2 \leq |S| \leq n-1, \tag{3}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} \leq 1, \qquad 1 \leq j \leq n, \tag{4}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} \leq 1, \qquad 1 \leq i \leq n, \tag{5}$$

$$w_{ij} + w_{jk} + w_{ki} \leq 2, \qquad 1 \leq i, j, k \leq n,\ i \neq j \neq k, \tag{6}$$

$$w_{ji} + w_{ij} = 1, \qquad 1 \leq i, j \leq n,\ i \neq j \tag{7}$$

$$x_{ij} - w_{ij} \leq 0, \qquad 1 \leq i, j \leq n,\ i \neq j \tag{8}$$

$$x_{ij} \in \{0, 1\}, \quad 1 \leq i, j \leq n,\ i \neq j \tag{9}$$

$$w_{ij} \in \{0, 1\}, \quad 1 \leq i, j \leq n,\ i \neq j \tag{10}$$

Constraints (2)–(5) model the directed Hamiltonian paths where inequalities (3) are the *subtour elimination constraints*. Acyclic tournaments are modelled by the *3-dicycle inequalities* (6) and the *tournament equations* (7). Inequalities (8) connect the solutions of both problems. Together with the integrality conditions (9) and (10) this obviously constitutes a 0/1 model of the TVP.

At first want to prove the correctness of the model.

Lemma 1. *The model presented in (1) - (10) is a correct IP model for the TVP).*

Proof. At first we have to prove that every feasible solution fulfills the model. Since (2) - (5), (9) is a well known model for the ATSP and (6) - (7), (10) is a well known model for the LOP it is sufficient to show that the values of x_{ij} do match with the values w_{ij} or in equal that both types of variables describe the same TVP-path. To assure this it is sufficient to prove the following two facts:

a) $x_{ij} = 1 \Rightarrow w_{ij} = 1$

b) $w_{ij} = 1 \Rightarrow i$ must be visited before j in the path describe by the x-variables

Because (8) must be fulfilled a) is obvious. To prove b) we assume j is visited before i in the path. That means the are existing indizees $k_1, \ldots, k_l$ so that $j, k_1, \ldots, k_n, i$ is a part of the path. So it follows that $x_{j,k_1} = x_{k_1,k_2} = \cdots = x_{k_l,i} = 1$. With a) we get that $w_{j,k_1} = w_{k_1,k_2} = \cdots = w_{k_l,i} = 1$. Because of (6) and (7) we can than iteratively conclude that $w_{j,k_2} = 1,\ w_{j,k_3} = 1, \ldots, w_{j,i} = 1$. But this is a contradiction to our assumption.

It remains to show that every feasible solution of (1) - (10) is a correct TVP-path. It is clear that every feasible integer solution must induce a feasible linear ordering and a feasible TSP tour. Because of the facts we mentioned above it is clear that the two feasible solutions must match which each other.

□

As an interesting fact we note that the subtour elimination constraints are actually not needed. If (w, x) satisfies (2) and (4)–(10), but not all inequalities (3) then there is some subtour on $k \geq 2$ nodes. W.l.o.g. we can assume that the node set is $\{1, 2, \ldots, k\}$ and the subtour is given as $\{(1, 2), (2, 3), \ldots, (k-1, k), (k, 1)\}$. Hence $x_{12} = x_{23} = ... = x_{k-1,k} = x_{k1} = 1$, implying because of (8) that $w_{12} = w_{23} = ... = w_{k-1,k} = w_{k1} = 1$. This is a contradiction to the requirement that the w-variables represent an acyclic tournament.

So we can eliminate the exponentially many constraints (3) and obtain a TVP formulation with a polynomial number (cubic in n) of constraints.

For our algorithm it will be useful to calculate the position of a node i in the path. This can easily be done using the LOP variables. The value $n - \sum_{\substack{j=1 \\ j i\!\!\!/}}^{n} w_{ij}$ gives the position of node i.

Note that because of the tournament equations we can substitute an LOP variable w_{ij}, $j > i$, by $1 - w_{ji}$. Now the 3-dicycle inequalities are turned into $1 \geq w_{ij} + w_{jk} - w_{ik} \geq 0$ for all $1 \leq i < j < k \leq n$ and the part of the objective function for the LOP variables reads $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [(p_{ij} - p_{ji}) w_{ij} + p_{ji}]$.

2.1 Extended formulation of the basic model

The use of extended formulations is a common technique with is used to strengthen the LP Formulation of a combinatorial optimization problem. The key idea of this approach is to add new variables and constraints to a given IP formulation so that the gap between the solution of the LP relaxation and the optimal integral solution becomes much smaller.

In the case of TVP we can obtain an extended formulation for the TVP by adding three-indexed variables, which are a generalization of the linear ordering variables to the standard model. In detail this new variables w_{ijk} are than defined as follows:

$$w_{ijk} := \begin{cases} 1 & \text{if } i = \pi(a),\, j = \pi(b) \text{ and } k = \pi(c) \\ & \text{for some } 1 \leq a < b < c \leq n-1\,, \\ 0 & \text{otherwise.} \end{cases}$$

So as one see this new type of variables is a straight forward extension of the w_{ij}-variables. In the objective function we assign zero coefficients to the new variables. In order to extend our standard model we also need to introduce two new classes of constraints to make sure that the solution of the new variables match with the old $x_i j$ and $w_i j$ variables. In detail the extended formulation looks a follows:

$$\max \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} w_{ij} - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij} x_{ij} \tag{11}$$

$$s.t.$$

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = n-1, \tag{12}$$

$$\sum_{i=1}^{n} x_{ij} \leq 1,\ 1 \leq j \leq n, \tag{13}$$

$$\sum_{j=1}^{n} x_{ij} \leq 1,\ 1 \leq i \leq n, \tag{14}$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1,$$

$$S \subseteq \{1, \ldots, n\}, 2 \leq |S| \leq n-1, \tag{15}$$

$$w_{ij} + w_{jk} + w_{ki} \leq 2,$$

$$1 \leq i, j, k \leq n,\ i < j, i < kj \neq k, \tag{16}$$

$$w_{ij} + w_{jik} + w_{jki} + w_{kji} = 1$$

$$1 \leq i, j, k \leq n, i < j \tag{17}$$

$$x_{ij} - w_{ijk} - w_{kij} \leq 0\ 1 \leq i, j, k \leq n, i < j \tag{18}$$

$$x_{ij} - w_{ij} \leq 0,\ 1 \leq i, j \leq n, \tag{19}$$

$$x_{ij} \in \{0, 1\},\ 1 \leq i, j \leq n, \tag{20}$$

$$w_{ij} \in \{0, 1\},\ 1 \leq i, j \leq n. \tag{21}$$

$$w_{ijk} \in \{0, 1\},\ 1 \leq i, j, k \leq n. \tag{22}$$

3 The edge-node-formulation

The key idea of the next model is to combine the w and x variables of the -Model to new three index variables which states the relation between a node n and an fixed edge (i, j). More precisely we define:

$$w_{ij}^{k} := \begin{cases} 1 & \text{if } k = \pi(a),\, i = \pi(b) \text{ and } j = \pi(b+1) \\ & \text{for some } 1 \leq a < b \leq n-1\,, \\ 0 & \text{otherwise.} \end{cases}$$

and analogously :

$$w_{k}^{ij} := \begin{cases} 1 & \text{if } i = \pi(a),\, j = \pi(a+1) \text{ and } k = \pi(b) \\ & \text{for some } 1 \leq a < b \leq\,, \\ 0 & \text{otherwise.} \end{cases}$$

A first IP-Model can then be develop out of the basic model by transforming the inequalities/equations to inequalities/equations with the new variables:

$$max \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} p_{ij} (\sum_{m=1, m \neq j}^{n} w_{mj}^{i}) + \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} (w_{ij}^{\Omega} + w_{\Omega}^{ij}) \tag{23}$$

$$s.t.$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (w_{ij}^{\Omega} + w_{\Omega}^{ij}) = n-1 \tag{24}$$

$$\sum_{i=1}^{n} (w_{ij}^{\Omega} + w_{\Omega}^{ij}) \leq 1\ \ j \in V \tag{25}$$

$$\sum_{j=1}^{n} (w_{ij}^{\Omega} + w_{\Omega}^{ij}) \leq 1\ \ i \in V \tag{26}$$

$$\sum_{l=1}^{n} w_{lj}^{i} + \sum_{l=1}^{n} w_{lk}^{j} + \sum_{l=1}^{n} w_{li}^{k} + (w_{ik}^{\Omega} + w_{\Omega}^{ik}) \leq 2\ \ i, j, k \in V \tag{27}$$

$$\tag{28}$$

Please note that that $\Omega \in V$ and it could be chosen arbitrarily for each summand in (23) -(26) but $\neq j$ and $\neq i$. It is the same in (27) but here Ω must not be equal i or k

4 Three distance model

Another idea for constructing an IP-model for the TVP has been made by Prof. E. Fernandez from the UPC Barcelona. The key idea of this approach is the use of distance variables. In Detail we define Variables z_{ij}^t which describe the fact whether there is as path of length t between i an j or not. More formally we state:

$$z_{ij}^t = \begin{cases} 1 & \text{if } i \text{ the solution contains a path with } t \text{ arcs} \\ & \text{from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

The advantage of this model is that we only have to deal here with one type of variables. Since we are not longer distinguish between distance and ordering variables we have to adjust the coefficients in the following way:

$$w_{ij}^t = \begin{cases} c_{ij} - d_{ij} & \text{if } t = 1, \\ c_{ij} & \text{otherwise.} \end{cases}$$

With are now able to formulated a TVP model with distance variables:

The formulation is:

$$\max \sum_{i \in N} \sum_{j \in N \setminus \{i\}} \sum_{t \in N \setminus \{n\}} \overline{w}_{ij}^t z_{ij}^t - \sum_{i \in N} (d_{i0} + d_{0i}) \quad (29)$$

$$\sum_{i=1}^{n} z_{ij}^1 \leq 1 \qquad j \in N, \quad (30)$$

$$\sum_{j=1}^{n} z_{ij}^1 \leq 1 \qquad i \in N, \quad (31)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij}^k = n - k \qquad k \in V \quad (32)$$

$$z_{ij}^{t_1} + z_{jk}^{t_2} \leq z_{ik}^{t_1+t_2} + 1,$$
$$i, j, k \in N, t_1, t_2 \in N \setminus \{n\},$$
$$i \neq j, j \neq k, i \neq k, t_1 + t_2 < n, \quad (33)$$

$$\sum_{t=1}^{n-1} z_{ij}^t + z_{ji}^t = 1 \qquad i, j \in N, i \neq j, \quad (34)$$

$$z_{ij}^t \in \{0, 1\} \qquad i, j \in N, i \neq j, t \in N \setminus \{n\}. \quad (35)$$

Also we only have one Type of variable now the $z_{i,j}^1$ variables still play a special role, for example in the objective function. On the other hand we again have a cubic number of variables

5 Conclusions

The target visitation problem turned out to be a very difficult and therefore challenging problem. The present paper gives some first results. More research is needed. An improvement of the simple heuristic used here can be accomplished along well-known lines. It is more interesting to find ways for improving the upper bound. The IP model already seems to be at its limits for fairly small problem instances unless some additional insight into the polyhedral structure can be obtained. Alternate optimization approaches like branch-and-bound with combinatorial bounds, dynamic or semidefinite programming should be devised and their limits should be explored. Furthermore it should be investigated further how the balance between the distance and the preference part of the object function influences the difficulty of problem instances.

References

[1] A. Arulselvan, C.W. Commander and P.M. Pardalos: *A hybrid genetic algorithm for the target visitation problem*, Technical report, University of Florida.

[2] T. Christof and A. Loebel: *PORTA: POlyhedron Representation Transformation Algorithm, Version 1.4*, http://comopt.ifi.uni-heidelberg.de/software/PORTA/index.html.

[3] M. Grötschel, M. Jünger and G. Reinelt: *Facets of the linear ordering polytope*, Math. Programming 33 (1985), 43–60.

[4] D.A. Grundel and D.E. Jeffcoat: Formulation and solution of the target visitation problem, In: Proceedings of the AIAA 1st Intelligent Systems Technical Conference, 2004.

[5] A. Hildenbrand: *Benchmark instances for the TVP*, http://comopt.ifi.uni-heidelberg.de/people/hilden brandt/TVP/template.html.

[6] M. Jünger and S. Thienel: *The ABACUS System for branch-and-cut-and-price algorithms in integer programming and combinatorial optimization*, Software: Practice and Experience, 30 (2000), 1325–1352.

[7] M. Queyranne and Y. Wang: *Hamiltonian path and symmetric travelling salesman polytopes*, Math. Programming 58 (1993), 89–110.

[8] G. Reinelt: *TSPLIB – Benchmark instances for the TSP*, http://comopt.ifi.uni-heidelberg.de/software/TSPLIB95/.