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# Interval Prediction of Order Statistics Based on Records by Employing Inter-Record Times: A Study Under Two Parameter Exponential Distribution

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**Abstract**

In this note, we propose a parametric inferential procedure for predicting future order statistics, based on record values, which takes inter-record times into account. We utilize the additional information contained in inter-record times for predicting future order statistics on the basis of observed record values from an independent sample. The two parameter exponential distribution is assumed to be the underlying distribution.

## 1 Introduction

Suppose $Y_1, \ldots, Y_m$ are independent and identically distributed (iid) observations from an absolutely continuous cumulative distribution function (cdf) $F$, possessing probability density function (pdf) $f$. The order statistics of the sample $Y_1, \ldots, Y_m$, represented by $Y_{1:m} < \cdots < Y_{m:m}$, are obtained by arranging the sample in an increasing order. Order statistics have been used in a wide range of applications, including robust statistical estimation, detection of outliers, characterization of probability distributions, goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials. A useful survey of available results until 2003 is given in the book of David and Nagaraja (2003).

Let $X_1, X_2, \ldots$ be a sequence of iid random variables, independent of and identically distributed to $Y_1$. An observation $X_j$ is called an upper (lower) record value if its value exceeds (resp. falls below) those of all the previous observations, that is the $n^{\text{th}}$ upper (resp. lower) record value, $U_n$ (resp. $L_n$), is defined as $X_{T_n}$, where $T_1 = 1$, with probability 1, and $T_n = \min\{j : \ j > T_{n-1}, \ X_j > X_{T_{n-1}}\}$ (resp. $T_n = \min\{j : \ j > T_{n-1}, \ X_j < X_{T_{n-1}}\}$), for $n > 1$. Throughout this paper we

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deal with upper record values for a predictive inference. Similar results can be obtained for the case of lower record values. The inter-record time statistic, defined as

$$\Delta_s = T_{s+1} - T_s, \; s \geq 1,$$

is the number of observations between $s^{\text{th}}$ and $(s+1)^{\text{th}}$ record values. For more details we refer the reader to Arnold et al. (1998). Record data arise in a wide variety of practical situations including industrial stress testing, finance, meteorological analysis, hydrology, seismology, sporting and athletic events, and mining surveys.

The problem of predicting future observations has been extensively studied in the literature and several parametric and non-parametric procedures are developed for prediction. In many practical data-analytic situations, one is interested in constructing a prediction interval on the basis of available observations. There are situations in which the available observations and the predictable future observation are of the same type. The prediction of future records on the basis of observed records from the same distribution and prediction of order statistics based on order statistics are studied, among others, by Dunsmore (1983), Nagaraja (1984), Chou (1988), Awad and Raqab (2000), Raqab and Balakrishnan (2008) and the references therein.

Recently, Ahmadi and Balakrishnan (2010), Ahmadi and MirMostafaee (2009), Ahmadi et al. (2010) and MirMostafaee and Ahmadi (2011), discussed the prediction of future records from a $Y$-sequence based on the order statistics observed from an independent $X$-sequence, and vice versa.

In predicting future order statistics on the basis of observed record statistics, sometimes the available observations also include inter-record times which can be utilized as additional information to improve the predictive inference. In other words, when both record values and the inter-record times are available, it would be nice to employ the information included in both records and record times. Feuerverger and Hall (1998) emphasized that "However, the record times and record values jointly contain considerably more information about $F$ than the record values alone." Actually, applying the additional information about record times is not a new subject and several authors focused on inference based on both record values and record times, see for example Samaniego and Whitaker (1986), Lin et al. (2003), Doostparast (2009), Doostparast and Balakrishnan (2013), Kızılaslan and Nadar (2014) and MirMostafaee et al. (2016).

In this paper, a two parameter exponential distribution, $Exp(\mu, \sigma)$, with pdf

$$f(x; \mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad x > \mu, \quad \mu \in \mathbb{R}, \; \sigma > 0, \tag{1.1}$$

is considered as the underlying distribution. We write $Z \sim Exp(\mu, \sigma)$ if the pdf of $Z$ can be expressed as (1.1). Note that $\mu$ and $\sigma$ are the location and scale parameters, respectively. Throughout this paper we assume that both parameters, $\mu$ and $\sigma$, are unknown.

Now, suppose that $Y_1, \cdots, Y_m$ constitute a future random sample from a two parameter exponential distribution, i.e. $Y_1, \cdots, Y_m \overset{iid}{\sim} Exp(\mu, \sigma)$ and $Y_{1:m} < \cdots < Y_{j:m}$ are the corresponding order statistics of this sample. In addition, $\bar{Y}_m = m^{-1} \sum_{i=1}^{m} Y_{i:m}$ denotes the mean of this future sample. If $Y_1, \cdots, Y_m$ denote the times to failure of $m$ independent units in a lifetime test, then $\bar{Y}_m$ can be interpreted as the mean time on test of these failed units. We assume that the available data include the observed upper record values, $U_1, \cdots, U_n$, given the inter-record times, $(\Delta_1, \ldots, \Delta_{n-1})$. We emphasize that these record values are assumed to be extracted from a sequence of iid random variables $\{X_j, j = 1, 2, \cdots\}$ where $X_j \sim Exp(\mu, \sigma)$ for $j = 1, 2, \cdots$. Moreover, the sequence $\{X_j, j = 1, 2, \cdots\}$ and the sample $\{Y_i, i = 1, \cdots, m\}$ are statistically independent. Note that $n$ is the number of the observed record values and depends on the experiment, however, $m$ is the sample size of the future observations and it can be considered arbitrary. In addition, $n$ and $m$ are unrelated. The problem of interest is to obtain conditional prediction intervals for $j^{\text{th}}$ future order statistic, $Y_{j:m}$, as well as for the mean, $\bar{Y}_m$, in a future sample on the basis of the available data. We compare our conditional prediction intervals with the unconditional ones proposed by Ahmadi and MirMostafaee (2009) and observe an improvement over the predictive inference without inter-record times. Therefore, we consider two cases: (a) The informative data contain only the upper record values, (b) The informative data contain the upper record values and the inter-record times, and then we observe that case (b) has some predictive inferential improvement in comparison with case (a).

The rest of the paper is organized as follows. Some general preliminaries are presented in Section 2. Conditional prediction intervals for the future $j^{\text{th}}$ order statistic, $Y_{j:m}$, and the mean of the future sample, $\bar{Y}_m$, based on record values of given inter-record times for the two parameter exponential distribution are studied in Sections 3 and 4. An illustrative example and some concluding remarks are involved in Sections 5 and 6. The R codes for computing some results of the paper are given in the appendix.

# 2 Preliminaries

In this section, we present some general preliminary results used in future sections. Given upper record values $u_1, \ldots, u_{n-1}$, which are observed and extracted from the sequence $\{X_j; j \geq 1\}$, inter-record times $\Delta_1, \ldots, \Delta_{n-1}$ are independent geometrically distributed random variables with success probabilities $\bar{F}(u_i),\ i = 1, \ldots, n-1$. Furthermore, the record values $U_1, \ldots, U_n$ form a Markov Chain with adjacent transition pdf equal to the left truncated pdf of the underlying distribution, see Arnold et al. (1998). Thus, the joint distribution of $\mathbf{U}_n = (U_1, \ldots, U_n)$ and $\boldsymbol{\Delta}_n = (\Delta_1, \ldots, \Delta_{n-1})$ is

$$f_{\mathbf{U}_n, \boldsymbol{\Delta}_n}(\mathbf{u}_n, \boldsymbol{\delta}_n) = \prod_{i=1}^{n-1} f(u_i)[F(u_i)]^{\delta_i - 1} f(u_n), \tag{2.1}$$

where $\mathbf{u}_n = (u_1, \ldots, u_n) \in \mathbb{X}^n$, in which $\mathbb{X}$ is the support of $X$ and $\boldsymbol{\delta}_n = (\delta_1, \ldots, \delta_{n-1}) \in \mathbb{N}^{n-1}$, see Samaniego and Whitaker (1986) and Arnold et al. (1998) page 169. We emphasize that $\boldsymbol{\Delta}_n$ contains $n-1$ positive integer-valued discrete random variables and $\boldsymbol{\delta}_n$ is the observed vector of $\boldsymbol{\Delta}_n$. By integrating (2.1) with respect to (w.r.t.) $u_1, \ldots, u_n$, we can easily prove the following result.

**Lemma 1** *The joint probability mass function of* $\Delta_1, \ldots, \Delta_{n-1}$ *is*

$$P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n) = \Pr(\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n) = \sum_{j=1}^{n-1} c_j(n, \boldsymbol{\delta}_n)[(a_1(n, j, \boldsymbol{\delta}_n) + 1)(a_1(n, j, \boldsymbol{\delta}_n) + a_n(n, j, \boldsymbol{\delta}_n) + 2)]^{-1},$$

*where*

$$c_j(n,\boldsymbol{\delta}_n) = (-1)^{n-j-1}\left[\prod_{j_1=0}^{j-2}\left(\sum_{t=n-j+1}^{n-j_1-1}\delta_t\right)\prod_{j_2=0}^{n-j-2}\left(\sum_{t=j_2+2}^{n-j}\delta_t\right)\right]^{-1},$$

$$a_1(n,j,\boldsymbol{\delta}_n) = \sum_{t=1}^{n-j}\delta_t - 1, \quad a_n(n,j,\boldsymbol{\delta}_n) = \sum_{t=n-j+1}^{n-1}\delta_t,$$

*in which we assume for $a > b$, $\sum_{t=a}^{b}\delta_t = 0$ and $\prod_{t=a}^{b}\delta_t = 1$.*

In this paper, we need the conditional distribution of $U_1$ and $U_n$ given by $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$ as follows.

**Lemma 2** *The conditional pdf of $U_1$ and $U_n$ given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$ is*

$$f_{U_1,U_n|\,\boldsymbol{\Delta}_n}(u_1,u_n|\,\boldsymbol{\delta}_n) = [P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)]^{-1}\sum_{j=1}^{n-1}c_j(n,\boldsymbol{\delta}_n)[F(u_1)]^{a_1(n,j,\boldsymbol{\delta}_n)}[F(u_n)]^{a_n(n,j,\boldsymbol{\delta}_n)}f(u_1)f(u_n),$$

*where $c_j(n,\boldsymbol{\delta}_n)$, $a_1(n,j,\boldsymbol{\delta}_n)$, $a_n(n,j,\boldsymbol{\delta}_n)$ and $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)$ are as in Lemma 1.*

The proof of Lemma 2 is straightforward by integrating (2.1) w.r.t. $u_2,\ldots,u_{n-1}$ and dividing the obtained equation by $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)$.

## 3 Conditional prediction intervals for order statistics

In this section, the goal is to find a conditional prediction interval for $Y_{j:m}$ when the observed $U_1,\ldots,U_n$ are available given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$ for the two parameter exponential distribution.

To this end, we consider the pivotal quantity

$$W_j = \frac{Y_{j:m} - U_1}{U_n - U_1}. \tag{3.1}$$

Note that the pivotal quantity $W_j$ is the same as the one considered by Ahmadi and MirMostafaee (2009). This quantity is location and scale invariant namely it is free of both unknown parameters i.e. the location parameter $\mu$ and the scale parameter $\sigma$. It is also a simple function of both observed and future statistics, so that the future statistic can be derived from it easily. Ahmadi and MirMostafaee (2009) found the unconditional distribution of $W_j$ while we present the conditional distribution of $W_j$ given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$, (i.e. the inter-record times are assumed to be known and fixed) in the following theorem.

**Theorem 1** *The conditional cdf of $W_j$ in* (3.1) *given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$ is for $w > 0$*

$$\begin{aligned} F_{W_j|\,\boldsymbol{\Delta}_n}(w|\boldsymbol{\delta}_n) &= \sum_{l=j}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)}\frac{\binom{m}{l}\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}}{(-1)^{j_2+j_3+j_4}P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)} \\ &\quad \times c_{j_1}(n,\boldsymbol{\delta}_n)[(j_2+m-l+j_3+j_4+2)((j_2+m-l)w+j_4+1)]^{-1}, \end{aligned}$$

*and for* $w < 0$

$$F_{W_j|\,\boldsymbol{\Delta}_n}(w|\boldsymbol{\delta}_n) = \sum_{l=j}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{\binom{m}{l}\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}}{(-1)^{j_2+j_3+j_4}P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)}$$
$$\times c_{j_1}(n,\boldsymbol{\delta}_n)[(j_2+m-l+j_3+j_4+2)(j_4+1-w(j_3+j_4+2))]^{-1},$$

*where* $a_1(n, j_1,\ \boldsymbol{\delta}_n)$ *and* $a_n(n, j_1,\ \boldsymbol{\delta}_n)$ *are defined in Lemma 1 and* $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)$ *is the joint mass function of* $\Delta_1, \dots, \Delta_{n-1}$ *which is also given in Lemma 1.*

**Proof:** Letting $J^*_{n,1} = (U_n - U_1)/\sigma$, $U^*_1 = (U_1 - \mu)/\sigma$ and $Y^*_{j:m} = (Y_{j:m} - \mu)/\sigma$, we may write

$$F_{W_j|\,\boldsymbol{\Delta}_n}(w|\boldsymbol{\delta}_n) = \int_0^\infty\int_0^\infty F_{Y^*_{j:m}}(vw+u) f_{U^*_1,J^*_{n,1}|\,\boldsymbol{\Delta}_n}(u,v|\boldsymbol{\delta}_n)\,\mathrm{d}u\,\mathrm{d}v. \tag{3.2}$$

For $t > 0$, we have

$$F_{Y^*_{j:m}}(t) = \sum_{l=j}^{m}\binom{m}{l}(1-e^{-t})^l e^{-(m-l)t}. \tag{3.3}$$

Also, from Lemma 2, we obtain

$$f_{U^*_1,J^*_{n,1}|\,\boldsymbol{\Delta}_n}(u,v|\ \boldsymbol{\delta}_n) = [P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)]^{-1}\sum_{j=1}^{n-1} c_j(n,\boldsymbol{\delta}_n)[1-e^{-u}]^{a_1(n,j,\boldsymbol{\delta}_n)}[1-e^{-(u+v)}]^{a_n(n,j,\boldsymbol{\delta}_n)}e^{-(2u+v)}. \tag{3.4}$$

Hence, by substituting (3.4) and (3.3) in (3.2) and using the binomial expansions, we have for $w > 0$,

$$F_{W_j|\,\boldsymbol{\Delta}_n}(w|\boldsymbol{\delta}_n) = \sum_{l=j}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{\binom{m}{l}\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}c_{j_1}(n,\boldsymbol{\delta}_n)}{(-1)^{j_2+j_3+j_4}P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)}$$
$$\times\int_0^\infty\int_0^\infty e^{-(j_2+m-l+j_3+j_4+2)u}e^{-((j_2+m-l)w+j_4+1)v}\,\mathrm{d}u\,\mathrm{d}v,$$

and therefore we naturally arrive at the desired expression. Similarly, we may attain the expression for $F_{W_j|\,\boldsymbol{\Delta}_n}(w|\boldsymbol{\delta}_n)$ when $w < 0$ after substituting (3.4) and (3.3) in (3.2) by noting that the integral w.r.t. $u$ must be taken from $-vw$ to $\infty$. □

Let $w_\gamma(n,m,j;\boldsymbol{\delta}_n)$ be the $\gamma^{\text{th}}$ conditional quantile of $W_j$ given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$, i.e.

$$\Pr(W_j < w_\gamma(n,m,j;\boldsymbol{\delta}_n)|\ \boldsymbol{\Delta}_n = \boldsymbol{\delta}_n) = \gamma.$$

To find $100(1-\alpha)\%$ two-sided conditional prediction intervals for $Y_{j:m}$ based on record values given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$, we have to find the conditional quantiles $w_{\alpha_1}(n,m,j;\boldsymbol{\delta}_n)$ and $w_{1-\alpha_2}(n,m,j;\boldsymbol{\delta}_n)$, for $\alpha_1+\alpha_2 = \alpha$, $0 < \alpha_i < 1$, $i = 1, 2$, numerically.

Now, a $100(1-\alpha)\%$ conditional prediction interval for $Y_{j:m}$ based on record values given $\boldsymbol{\Delta}_n = \boldsymbol{\delta}_n$, is given by

$$(U_1 + w_{\alpha_1}(n,m,j;\boldsymbol{\delta}_n)(U_n-U_1), U_1 + w_{1-\alpha_2}(n,m,j;\boldsymbol{\delta}_n)(U_n-U_1)). \tag{3.5}$$

**Table 1:** The values of $w_{0.025}(3,m,j)$, $w_{0.975}(3,m,j)$, $w_{0.975}(3,m,j) - w_{0.025}(3,m,j)$, $w_{0.025}(3,m,j;\boldsymbol{\delta}_n)$, $w_{0.975}(3,m,j;\boldsymbol{\delta}_n)$, $w_{0.975}(3,m,j;\boldsymbol{\delta}_n) - w_{0.025}(3,m,j;\boldsymbol{\delta}_n)$, for $m = 10, 20$, $j = 5, 7, 10$ (for $m = 10$), $j = 12, 17, 20$ (for $m = 20$) and different values of $\boldsymbol{\delta}_n$.

| | $m$ | 10 | | | 20 | | |
|---|---|---|---|---|---|---|---|
| | $j$ | 5 | 7 | 10 | 12 | 17 | 20 |
| Unconditional | $w_{0.025}$ | -3.671 | -2.814 | -0.907 | -3.140 | -1.760 | -0.380 |
| | $w_{0.975}$ | 1.278 | 2.635 | 9.761 | 1.827 | 4.767 | 12.186 |
| | $w_{0.975} - w_{0.025}$ | **4.949** | **5.449** | **10.668** | **4.967** | **6.527** | **12.566** |
| $\boldsymbol{\delta}_n = (1,2)$ | $w_{0.025}$ | -1.097 | -0.500 | 0.249 | -0.651 | 0.055 | 0.464 |
| $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n) = 0.0833$ | $w_{0.975}$ | 1.766 | 3.502 | 11.868 | 2.459 | 6.108 | 14.586 |
| | $w_{0.975} - w_{0.025}$ | **2.863** | **4.002** | **11.619** | **3.110** | **6.053** | **14.122** |
| $\boldsymbol{\delta}_n = (1,3)$ | $w_{0.025}$ | -1.288 | -0.652 | 0.201 | -0.827 | -0.025 | 0.420 |
| $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n) = 0.05$ | $w_{0.975}$ | 1.290 | 2.627 | 9.481 | 1.786 | 4.675 | 11.690 |
| | $w_{0.975} - w_{0.025}$ | **2.578** | **3.279** | **9.280** | **2.613** | **4.700** | **11.270** |
| $\boldsymbol{\delta}_n = (1,4)$ | $w_{0.025}$ | -1.427 | -0.774 | 0.160 | -0.965 | -0.098 | 0.386 |
| $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n) = 0.0333$ | $w_{0.975}$ | 1.022 | 2.106 | 7.984 | 1.398 | 3.793 | 9.872 |
| | $w_{0.975} - w_{0.025}$ | **2.449** | **2.880** | **7.824** | **2.363** | **3.891** | **9.486** |
| $\boldsymbol{\delta}_n = (2,3)$ | $w_{0.025}$ | -2.181 | -1.267 | 0.045 | -1.538 | -0.320 | 0.324 |
| $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n) = 0.0167$ | $w_{0.975}$ | 1.027 | 2.413 | 10.212 | 1.502 | 4.669 | 12.787 |
| | $w_{0.975} - w_{0.025}$ | **3.208** | **3.680** | **10.167** | **3.040** | **4.989** | **12.463** |
| $\boldsymbol{\delta}_n = (2,4)$ | $w_{0.025}$ | -2.330 | -1.409 | -0.008 | -1.697 | -0.415 | 0.289 |
| $P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n) = 0.0119$ | $w_{0.975}$ | 0.823 | 1.976 | 8.880 | 1.193 | 3.896 | 11.163 |
| | $w_{0.975} - w_{0.025}$ | **3.153** | **3.385** | **8.888** | **2.890** | **4.311** | **10.874** |

Conditionally on $\boldsymbol{\delta}_n$, we get more information about the unknown parameters $\mu$ and $\sigma$, or generally more information about $F$, which leads to better prediction intervals for $Y_{j:m}$. It is noted that conditioning on inter-record times does not decrease the length of the prediction interval necessarily and increase or decrease in the location and scale of the interval depend on the values of $\boldsymbol{\delta}_n$. For the purpose of illustration, consider the conditional quantiles of $W_j$, which are computed and tabulated in Table 1, for $\alpha = 0.05$, $n = 3$, $m = 10, 20$, $j = 5, 7, 10$ ($m = 10$), $j = 12, 17, 20$ ($m = 20$) and some values of $\boldsymbol{\delta}_n$. The values of unconditional quantiles of $W_j$ in Table 1 are taken from Ahmadi and MirMostafaee (2009), Tables 3 and 4. By comparing the entries of Table 1, one can observe that for a few cases, the conditional prediction intervals have bigger lengths, especially when we predict the biggest future order statistic, i.e. $Y_{m:m}$. But note that in the most cases the conditional intervals are shorter than the unconditional ones for different values of $\boldsymbol{\delta}_n$, so we may conclude that generally the conditional prediction approach leads to shorter (and hence better) prediction intervals in average for different values of $\boldsymbol{\delta}_n$ and this can be considered as an improvement.

# 4 Conditional Prediction Intervals for the mean of future sample

The problem of constructing a conditional prediction interval for $\bar{Y}_m$ on the basis of observed $U_1, \ldots, U_n$, given $\mathbf{\Delta}_n = \boldsymbol{\delta}_n$, using the pivotal quantity

$$V_m = \frac{\bar{Y}_m - U_1}{U_n - U_1}, \tag{4.1}$$

is considered for the two parameter exponential distribution in this section. Note that the pivotal quantity $V_m$ has been also considered by Ahmadi and MirMostafaee (2009) and its unconditional distribution has been obtained by them. Moreover, $V_m$ is also location and scale invariant and therefore is free of the unknown location and scale parameters. The following theorem presents the conditional distribution function of $V_m$ given $\mathbf{\Delta}_n = \boldsymbol{\delta}_n$.

**Theorem 2** *The conditional distribution function of $V_m$ in (4.1) given $\mathbf{\Delta}_n = \boldsymbol{\delta}_n$ is*

$$\begin{aligned} F_{V_m|\,\mathbf{\Delta}_n}(x|\boldsymbol{\delta}_n) &= 1 - \sum_{l=0}^{m-1}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}}{(-1)^{j_3+j_4}P_{\mathbf{\Delta}_n}(\boldsymbol{\delta}_n)l!} \\ &\quad \times \frac{c_{j_1}(n,\boldsymbol{\delta}_n)x^{j_2}m^l\Gamma(l-j_2+1)\Gamma(j_2+1)}{(m+j_3+j_4+2)^{l-j_2+1}(mx+j_4+1)^{j_2+1}}, \end{aligned}$$

*for $x > 0$, and*

$$\begin{aligned} F_{V_m|\,\mathbf{\Delta}_n}(x|\boldsymbol{\delta}_n) &= \sum_{j_1=1}^{n-1}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{(-1)^{j_3+j_4}\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}c_{j_1}(n,\boldsymbol{\delta}_n)}{P_{\mathbf{\Delta}_n}(\boldsymbol{\delta}_n)(2+j_3+j_4)[j_4+1-(2+j_3+j_4)x]} \\ &\quad - \sum_{l=0}^{m-1}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_5=0}^{l-j_2} \frac{\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}}{(-1)^{j_3+j_4+j_5}P_{\mathbf{\Delta}_n}(\boldsymbol{\delta}_n)l!} \\ &\quad \times \frac{c_{j_1}(n,\boldsymbol{\delta}_n)x^{j_2+j_5}m^l\Gamma(l-j_2+1)\Gamma(j_2+j_5+1)}{j_5!(m+j_3+j_4+2)^{l-j_2-j_5+1}[j_4+1-(j_3+j_4+2)x]^{j_2+j_5+1}}, \end{aligned}$$

*for $x < 0$, where $a_1(n, j_1, \boldsymbol{\delta}_n)$ and $a_n(n, j_1, \boldsymbol{\delta}_n)$ are given in Lemma 1*

**Proof:** Let $J^*_{n,1} = (U_n - U_1)/\sigma$, $U^*_1 = (U_1 - \mu)/\sigma$ and $\bar{Y}^*_m = (\bar{Y}_m - \mu)/\sigma$. Note that

$$F_{V_m|\,\mathbf{\Delta}_n}(x|\boldsymbol{\delta}_n) = \int_0^\infty \int_0^\infty F_{\bar{Y}^*_m}(vx+u) f_{U^*_1,J^*_{n,1}|\,\mathbf{\Delta}_n}(u,v|\boldsymbol{\delta}_n)\,\mathrm{d}u\,\mathrm{d}v, \tag{4.2}$$

where $f_{U^*_1,J^*_{n,1}|\,\mathbf{\Delta}_n}(u,v|\,\boldsymbol{\delta}_n)$ is given in (3.4). Since $m\bar{Y}^*_m \sim \Gamma(m,1)$, that is for $t > 0$

$$F_{\bar{Y}^*_m}(t) = 1 - \sum_{l=0}^{m-1} \frac{(mt)^l e^{-mt}}{l!}, \tag{4.3}$$

so by substituting (3.4) and (4.3) in (4.2) and using the binomial expansions, we get for $x<0$

$$
\begin{aligned}
F_{V_m|\,\boldsymbol{\Delta}_n}(x|\boldsymbol{\delta}_n) &= \sum_{j_1=1}^{n-1}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}c_{j_1}(n,\boldsymbol{\delta}_n)}{(-1)^{j_3+j_4}P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)} \\
&\times \int_0^\infty \int_{-vx}^\infty e^{-(j_3+j_4+2)u}e^{-(j_4+1)v}\,\mathrm{d}u\,\mathrm{d}v \\
&- \sum_{l=0}^{m-1}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}}{(-1)^{j_3+j_4}P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)l!} \\
&\times c_{j_1}(n,\boldsymbol{\delta}_n)x^{j_2}m^l \int_0^\infty \int_{-vx}^\infty e^{-(m+j_3+j_4+2)u}e^{-(mx+j_4+1)v}u^{l-j_2}v^{j_2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \sum_{j_1=1}^{n-1}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)} \frac{(-1)^{j_3+j_4}\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}c_{j_1}(n,\boldsymbol{\delta}_n)}{P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)(2+j_3+j_4)[j_4+1-(2+j_3+j_4)x]} \\
&- \sum_{l=0}^{m-1}\sum_{j_1=1}^{n-1}\sum_{j_2=0}^{l}\sum_{j_3=0}^{a_1(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_4=0}^{a_n(n,j_1,\,\boldsymbol{\delta}_n)}\sum_{j_5=0}^{l-j_2} \frac{\binom{a_1(n,j_1,\,\boldsymbol{\delta}_n)}{j_3}\binom{a_n(n,j_1,\,\boldsymbol{\delta}_n)}{j_4}\binom{l}{j_2}}{(-1)^{j_3+j_4+j_5}P_{\boldsymbol{\Delta}_n}(\boldsymbol{\delta}_n)l!} \\
&\times \frac{c_{j_1}(n,\boldsymbol{\delta}_n)x^{j_2+j_5}m^l\Gamma(l-j_2+1)}{j_5!(m+j_3+j_4+2)^{l-j_2-j_5+1}} \int_0^\infty e^{-(j_4+1-(j_3+j_4+2)x)v}v^{j_2+j_5}\,\mathrm{d}v
\end{aligned}
$$

and therefore we naturally attain the desired result. Similarly, we may deduce the desired expression for $F_{V_m|\,\boldsymbol{\Delta}_n}(x|\boldsymbol{\delta}_n)$ when $x>0$. □

To find conditional prediction interval for $\bar{Y}_m$ based on records given $\boldsymbol{\Delta}_n=\boldsymbol{\delta}_n$, we have to find the conditional quantiles of $V_m$ given $\boldsymbol{\Delta}_n=\boldsymbol{\delta}_n$, $v_{\alpha_1}(n,m;\boldsymbol{\delta}_n)$ and $v_{1-\alpha_2}(n,m;\boldsymbol{\delta}_n)$, for $\alpha_1+\alpha_2=\alpha$, $0<\alpha_i<1$, $i=1,2$, numerically, where

$$\Pr(V_m<v_\gamma(n,m;\boldsymbol{\delta}_n)|\ \boldsymbol{\Delta}_n=\boldsymbol{\delta}_n)=\gamma.$$

A $100(1-\alpha)\%$ conditional prediction interval for $\bar{Y}_m$ based on record values given $\boldsymbol{\Delta}_n=\boldsymbol{\delta}_n$ then is

$$(U_1+v_{\alpha_1}(n,m;\boldsymbol{\delta}_n)(U_n-U_1),U_1+v_{1-\alpha_2}(n,m;\boldsymbol{\delta}_n)(U_n-U_1)). \tag{4.4}$$

An illustrative example has been presented in Section 5.

# 5 An illustrative example

In this section, we illustrate the proposed procedures by considering a real data set. A rock crushing machine has to be reset if, at any operation, the size of rock being crushed

**Table 2:** 95% CPIs and UPIs for $Y_{12:20}$, $Y_{20:20}$ and $\bar{Y}_{20}$ for Example 1.

| | CPI | UPI |
|---|---|---|
| $Y_{12:20}$ | (0, 24.17836) | (0, 54.061745) |
| $Y_{20:20}$ | (13.290315, 183.67385) | (0, 307.85602) |
| $\bar{Y}_{20}$ | (0, 26.233175) | (0, 61.32183) |

is larger than any that has been crushed before. The following data given by Dunsmore (1983) are the sizes dealt with up to the third time that the machine has been reset:

9.3, 0.6, 24.4, 18.1, 6.6, 9.0, 14.3, 6.6, 13.0, 2.4, 5.6, 33.8.

The record values were the sizes at the operation when resetting was necessary. Dunsmore (1983) assumed that these data follow an $Exp(0, \sigma)$ distribution. Clearly, we have

$$U_1 = 9.3, \;\; U_2 = 24.4, \;\; U_3 = 33.8,$$

$$T_1 = 1, \;\; T_2 = 3, \;\; T_3 = 12,$$

$$\Delta_1 = 2, \quad \text{and} \quad \Delta_2 = 9.$$

Consider a future sample of size $m = 20$. We want to find equi-tailed 95% conditional prediction intervals (CPIs) for $Y_{12:20}$, $Y_{20:20}$ and $\bar{Y}_{20}$ using (3.5) and (4.4) and compare these intervals with unconditional ones (UPIs). The results are given in Table 2. Note that some lower bounds have got negative values, which were replaced by zero. We can see that the conditional prediction intervals are shorter than the corresponding unconditional ones.

## 6 Concluding remarks

In this paper, we found prediction intervals for the future order statistics based on record values, given record time statistics, when the underlying distribution is two parameter exponential. These intervals have the advantage of utilizing more information embedded in the observed sequence in comparison with their corresponding unconditional ones obtained by Ahmadi and MirMostafaee (2009). These ideas can be extended to the nonparametric and the Bayesian context. The conditional point predictors are also of interest. Work on these problems is currently under process and we hope to report these findings in future papers.

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# Appendix

Here, we present the R codes for computing the conditional cumulative distribution functions of $W_j$, (see Theorem 1) and $V_m$ (see Theorem 2). R functions for computing the unconditional cumulative distribution functions of $W_j$ and $V_m$ (see Ahmadi and MirMostafaee, 2009) are also given.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%  cjn function %%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

cjn=function(n,j,delta){
z=(-1)^(n-j-1)
z1=n-j+1
z2=j-2
z4=n-j-2
z5=n-j
s=1
if(z2>=0 & z1>=0){
for(j1 in 0:z2){
z3=n-j1-1
ss=ifelse(z3>=z1,sum(delta[z1:z3]),0)
s=s*ss
}}
t=1
if(z4>=0){
for(j2 in 0:z4){
z6=j2+2
tt=ifelse(z5>=z6,sum(delta[z6:z5]),0)
t=t*tt
}}
return(z/t/s)
```

```
}



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%% Mass probability of Delta %%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pdelta=function(n,delta){
n1=n-1
pdel=0
for(jj in 1:n1){
nj=n-jj
nj1=n-jj+1
A=cjn(n,jj,delta)
a1=ifelse(nj>=1,sum(delta[1:nj]),0)-1
an=ifelse(n1>=nj1,sum(delta[nj1:n1]),0)
C=(a1+1)*(a1+an+2)
pdel=pdel+A/C
}
return(pdel)
}

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%% conditional cdf of W      %%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Fw=function(n,j,m,w,delta){
n1=n-1
pw=0
for(l in j:m){
for(j1 in 1:n1){
for(j2 in 0:l){
nj1=n-j1+1
nj=n-j1
a1=ifelse(nj>=1,sum(delta[1:nj]),0)-1
an=ifelse(n1>=nj1,sum(delta[nj1:n1]),0)
for(j3 in 0:a1){
for(j4 in 0:an){
A=choose(m,l)*choose(a1,j3)*choose(an,j4)*choose(l,j2)
*((-1)^(j2+j3+j4))*cjn(n,j1,delta)/pdelta(n,delta)
B=j2+m-l+j3+j4+2
if(w<0) C=B*(j4+1-w*(j3+j4+2))
if(w>=0) C=B*(w*(j2+m-l)+j4+1)
pw=pw+A/C
}}}}}
```

```
return(pw)
}


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%% conditional cdf of V      %%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Fv=function(n,m,v,delta){
pv=0
n1=n-1
m1=m-1
if(v>=0){
for(l in 0:m1){
for(j1 in 1:n1){
for(j2 in 0:l){
nj1=n-j1+1
nj=n-j1
a1=ifelse(nj>=1,sum(delta[1:nj]),0)-1
an=ifelse(n1>=nj1,sum(delta[nj1:n1]),0)
for(j3 in 0:a1){
for(j4 in 0:an){
A=choose(a1,j3)*choose(an,j4)*choose(l,j2)/factorial(l)
/pdelta(n,delta)*((-1)^(j3+j4))
B=cjn(n,j1,delta)*(v^j2)*(m^l)*gamma(l-j2+1)*gamma(j2+1)
/((m+j3+j4+2)^(l-j2+1))/((m*v+j4+1)^(j2+1))
pv=pv+A*B
}}}}}}
if(v>=0) pv=1-pv
pv1=0
pv2=0
if(v<0){
for(j1 in 1:n1){
nj1=n-j1+1
nj=n-j1
a1=ifelse(nj>=1,sum(delta[1:nj]),0)-1
an=ifelse(n1>=nj1,sum(delta[nj1:n1]),0)
for(j3 in 0:a1){
for(j4 in 0:an){
A=((-1)^(j3+j4))*choose(a1,j3)*choose(an,j4)*cjn(n,j1,delta)
/pdelta(n,delta)/(2+j3+j4)/(j4+1-v*(2+j3+j4))
pv1=pv1+A
}}}
for(l in 0:m1){
for(j1 in 1:n1){
for(j2 in 0:l){
```

```
nj1=n-j1+1
nj=n-j1
a1=ifelse(nj>=1,sum(delta[1:nj]),0)-1
an=ifelse(n1>=nj1,sum(delta[nj1:n1]),0)
for(j3 in 0:a1){
for(j4 in 0:an){
lj2=l-j2
for(j5 in 0:lj2){
A=choose(a1,j3)*choose(an,j4)*choose(l,j2)/factorial(l)
/pdelta(n,delta)*((-1)^(j3+j4+j5))
B=cjn(n,j1,delta)*(v^(j2+j5))*(m^l)*gamma(l-j2+1)
*gamma(j2+j5+1)/factorial(j5)/((m+j3+j4+2)^(l-j2-j5+1))
/((j4+1-v*(j3+j4+2))^(j2+j5+1))
pv2=pv2+A*B
}}}}}}
pv=pv1-pv2
}
return(pv)
}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%% unconditional cdf of W    %%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

FwU=function(n,j,m,w){
pw=0
if(w<0) pw=(m-j+1)*((1-w)^(1-n))/(m+1)
if(w>=0){
ss=0
j1=j-1
for(i in 0:j1){
ss=ss+choose(j1,i)*((-1)^i)*((1+w*(m-j+i+1))^(1-n))
/(m-j+i+1)/(m-j+i+2)
}
pw=1-j*choose(m,j)*ss
}
return(pw)
}

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%% unconditional cdf of V    %%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

FvU=function(n,m,v){
pv=0
if(v<0) pv=((1-v)^(1-n))/((1+1/m)^m)
if(v>=0){
```

```
m1=m-1
s1=0
s2=0
for(i in 0:m1){
nn=n+i-2
s1=s1+choose(nn,i)*((1-1/(m*v+1))^i)*((1/(m*v+1))^(n-1))
*((m/(m+1))^(m-i))
s2=s2+choose(nn,i)*((1-1/(m*v+1))^i)*((1/(m*v+1))^(n-1))
}
pv=s1+1-s2
}
return(pv)
}
```