BLED WORKSHOPS
IN PHYSICS
VOL. 9, NO. 1
p. 50

Proceedings of the Mini-Workshop
Few-Quark States and the Continuum
Bled, Slovenia, September 15-22, 2008

Testing the usage of a generalization of the Witten-Veneziano relation in a bound-state approach to η and η′ mesons*

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Abstract. The results of the Dyson-Schwinger approach utilizing the Witten-Veneziano relation to obtain a description of the η and η′ mesons, are compared with the results obtained when Shore's generalization of the Witten-Veneziano relation is used instead. On the examples of three different model interactions, we find that irrespective of the concrete model dynamics, our Dyson-Schwinger approach is phenomenologically more successful in conjunction with the standard Witten-Veneziano relation than with the generalization valid, at least in principle, in all orders in the $1/N_c$ expansion.

1 Introduction

The complex of the η and η′ pseudoscalar mesons is an intriguing problem in the light-quark sector of the nonperturbative Quantum Chromodynamics (QCD). The mixing of the pertinent isospin-zero states should be such that the physical η meson is one of the (almost-)Goldstone bosons of the dynamical chiral symmetry breaking (DChSB) of QCD, whereas its partner η′ must be very massive (∼ 1 GeV) and remain such even in the chiral limit. For the correct η′ mass behavior, the non-abelian (gluon) axial anomaly of QCD is essential, and a way to extract its contribution is through the Witten-Veneziano (WV) relation [1,2].

We are particularly interested in the Dyson-Schwinger (DS) approach [3–8] to QCD and its modeling. In some variants of the DS approach (e.g., in Ref. [9]), the

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WV relation has been used to obtain the description of the η-η' complex. In the present paper, for three different DS models, we compare the usage of the WV relation with the usage of its recent generalization recently proposed by Shore [10,11]. The present paper in the Bled 2008 proceedings, is a shortened version of Ref. [12].

The DS approach [3–8] is the chirally well-behaved bound-state approach and thus the most suitable one to treat the light pseudoscalar mesons (those composed of the u, d and s quarks), for which DChSB is essential. One solves the DS equations (DSEs) for dressed quark propagators, which are then employed in Bethe-Salpeter equations (BSEs). Their solving yields quark-antiquark ($q\bar{q}'$) bound state amplitudes and corresponding masses $M_{q\bar{q}'}$.

To obtain the chiral behavior as in QCD, DS and BS equations must be solved in a consistent approximation. The rainbow-ladder approximation (RLA), where DChSB is well-understood, is still the most usual approximation in phenomenological applications. This also entails that in both DSE and BSE we employ the same effective interaction. Concretely, in the present paper we recall and utilize the results obtained *i)* in Refs. [13,14] by using the renormalization-group improved (RGI) interaction of Jain and Munczek [15], *ii)* in Ref. [9] by using the RGI gluon condensate-induced interaction [16], and *iii)* in Refs. [17,18] by using the separable interaction [19]. Such effective interactions must be modeled at least in the low-energy, nonperturbative regime in order to be phenomenologically successful – which above all means to be sufficiently strong in the low-momentum domain to yield DChSB. In the chiral limit (and *close* to it), light pseudoscalar (P) meson $q\bar{q}$ bound states ($P = \pi^{0,\pm}, K^{0,\pm}, \eta$) then simultaneously manifest themselves also as (*quasi-*)Goldstone bosons of DChSB. This enables one to work with the mesons as explicit $q\bar{q}$ bound states, while reproducing the results of the Abelian axial anomaly for the light pseudoscalars, i.e., the amplitudes for $P \to \gamma\gamma$ and $\gamma^\star \to P^0P^+P^-$. This is unique among the bound state approaches – e.g., see Refs. [5,20,22,21] and references therein. Nevertheless, one keeps the advantage of bound-state approaches that from the $q\bar{q}$ substructure one can calculate many important quantities (such as the pion, kaon and $s\bar{s}$ pseudoscalar decay constants: f_π, f_K and $f_{s\bar{s}}$) which are just parameters in most of other chiral approaches to the light-quark sector. The treatment [13,14,23,9] of the η-η' complex is remarkable in that it is very successful in spite of the limitations of RLA. (Very recently, during the work on the present paper, the first and still simplified DS treatments of η and η' beyond RLA appeared [24,25]. However, RLA treatments will probably long retain their usefulness in applications where simple modeling is desirable, as in the calculationally demanding finite-temperature calculations [18].) The RLA treatments [13,14,23,9,18] of the η-η' complex relied on the Witten-Veneziano (WV) relation [1,2]. Nevertheless, Shore achieved [10,11] what can be considered as a generalization of the WV relation, and the purpose of the present paper is exploring the usage of this generalization in the DS context.

The paper is organized as follows: in the next section, we recapitulate the procedures and results of our previous treatments [14,9,18] relying on the WV relation (11), and present in Table I also their extension to the scheme of the four decay constants (and two mixing angles) of η and η'. In Section 3, we expose the usage of the pertinent Shore's equations [10,11] in the context of DS approach. The last section concludes after giving the results of solving the pertinent equations.

2 η-η' mass matrix from Witten-Veneziano relation

All $q\bar{q}'$ model masses $M_{q\bar{q}'}$ $(q, q' = u, d, s)$ used in the present paper, and corresponding $q\bar{q}'$ bound-state amplitudes, were obtained in Refs. [13,14,9,26,17,18] in RLA, i.e., with an interaction kernel which (irrespective of how one models the dynamics) cannot possibly capture the effects of the non-Abelian, gluon axial anomaly. Thus, when we form the η-η' mass matrix

$$\hat{M}^2_{\rm NA} = \begin{bmatrix} M^2_{88} & M^2_{80} \\ M^2_{08} & M^2_{00} \end{bmatrix} , \tag{1}$$

in this case in the octet-singlet basis η_8-η_0 of the (broken) flavor-SU(3) states of isospin zero,

$$\eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}), \qquad\qquad \eta_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}), \tag{2}$$

this matrix (1), consisting of our calculated $q\bar{q}$ masses,

$$M^2_{88} \equiv \langle \eta_8 | \hat{M}^2_{\rm NA} | \eta_8 \rangle = \frac{2}{3}\,(M^2_{s\bar{s}} + \frac{1}{2} M^2_{u\bar{u}})\,, \tag{3}$$

$$M^2_{00} \equiv \langle \eta_0 | \hat{M}^2_{\rm NA} | \eta_0 \rangle = \frac{2}{3}\,(\frac{1}{2} M^2_{s\bar{s}} + M^2_{u\bar{u}})\,, \tag{4}$$

$$M^2_{80} \equiv \langle \eta_8 | \hat{M}^2_{\rm NA} | \eta_0 \rangle = M^2_{08} = \frac{\sqrt{2}}{3}(M^2_{u\bar{u}} - M^2_{s\bar{s}}) < 0, \tag{5}$$

is purely non-anomalous (NA), vanishing in the chiral limit. In the isospin limit, to which we adhere throughout, the pion is strictly decoupled from the gluon anomaly and $M_{u\bar{u}} = M_{d\bar{d}}$ is exactly our model pion mass M_π. Also the unphysical $s\bar{s}$ quasi-Goldstone's mass $M_{s\bar{s}}$ results from RLA BSE and does not include the contribution from the gluon anomaly. This is consistent with the fact that due to the Dashen-Gell-Mann-Oakes-Renner (DGMOR) relation, it is in a good approximation [13,14,9,18] given by $M^2_{s\bar{s}} = 2M^2_K - M^2_\pi$, i.e., by the kaon and pion masses protected from the anomaly by strangeness and/or isospin.

In our previous DS studies [13,14,9,26,17,18], to which we refer for all model details, the phenomenology of the non-anomalous sector was successfully reproduced, e.g., f_π, f_K, as well as the empirical masses M_π and M_K (see the upper part of Table 1), yielding a strongly non-diagonal $\hat{M}^2_{\rm NA}$ (1). Its diagonalization leads to the eigenstates known as the nonstrange-strange (NS-S) basis,

$$\eta_{\rm NS} = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\,, \qquad\qquad \eta_{\rm S} = s\bar{s}\,, \tag{6}$$

and to $\hat{M}^2_{\rm NA} = {\rm diag}[M^2_\pi, M^2_{s\bar{s}}]$. In contrast to these mass-squared eigenvalues, the experimental masses are such that $(M^2_\pi)_{exp}\lambda^2(M^2_\eta)_{exp}$, and η' is too heavy,

$(M_{\eta'})_{exp} = 958$ MeV, to be considered even as the $s\bar{s}$ quasi-Goldstone boson. This is the well-known $U_A(1)$ problem, resolved by the fact that the *complete* η-η' mass matrix $\hat{M}^2$ must contain the anomalous (A) part $\hat{M}^2_A$. That is, $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$.

However, $\hat{M}^2_A$ is inaccessible to RLA which yields our Goldstone pseudoscalars. In Refs. [13,14,9,17,18], $\hat{M}^2_A$ was extracted from lattice data through the WV relation [the second equality in Eq. (11)]. The main purpose of the present paper, instead, is to approach η and η' through Shore's [10,11] recent generalization of that relation.

Before that, nevertheless, we review the usage of the WV relation in Refs. [13,14,9,17,18]. The expansion in the large number of colors, N_c, indicates that the leading approximation in that expansion describes the bulk of main features of QCD. The gluon anomaly is suppressed as $1/N_c$ and can be viewed as a perturbation in the large N_c expansion. In the SU(3) limit, it is coupled *only* to the singlet combination η_0 (2); only the η_0 mass receives, from the gluon anomaly, a contribution which, unlike quasi-Goldstone masses $M_{q\bar{q}'}$'s comprising $\hat{M}^2_{NA}$, does *not* vanish in the chiral limit. As discussed in Refs. [13,9], in the present bound-state context it is thus meaningful to include the effect of the gluon anomaly just on the level of a mass shift for the η_0 as the lowest-order effect, and retain the $q\bar{q}$ bound-state amplitudes and the corresponding mass eigenvalues $M_{q\bar{q}}$ as calculated by solving DSEs and BSEs with kernels in RLA.

References [13,14,9,17,18] thus break the $U_A(1)$ symmetry, and avoid the $U_A(1)$ problem, by shifting the η_0 (squared) mass by an amount denoted by 3β (in the notation of Refs. [14,9]). The complete mass matrix $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$ then contains the anomalous part $\hat{M}^2_A = \text{diag}[0, 3\beta]$, where the anomalous η_0 mass shift 3β is related to the topological susceptibility of the vacuum, but in the present approach must be treated as a parameter to be determined outside of our RLA model, i.e., fixed by phenomenology or taken from the lattice calculations [27]. (The possibility of employing an additional microscopic model for the gluon anomaly contribution, such as the one of Ref. [28], is presently not considered.)

The SU(3) flavor symmetry breaking and its interplay with the gluon anomaly [9] modifies $\hat{M}^2_A = \text{diag}[0, 3\beta]$ to

$$\hat{M}^2_A = \beta \begin{bmatrix} \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(1-X)(2+X) \\ \frac{\sqrt{2}}{3}(1-X)(2+X) & \frac{1}{3}(2+X)^2 \end{bmatrix} , \tag{7}$$

where X is the flavor symmetry breaking parameter. It is most often estimated as $X = f_\pi/f_{s\bar{s}} \sim 0.7 - 0.8$ (see, e.g., Refs. [30,29,14,9], although there are some other [14], of course related, estimates of X). Presently we also adopt $X = f_\pi/f_{s\bar{s}}$, which means that X is a calculated quantity in our approach. The employed models achieved good agreement with phenomenology [13,14,9,18], e.g., fitted the experimental value of $M^2_\eta + M^2_{\eta'}$ for β around 0.26 – 0.28 GeV^2. The anomaly contribution $\hat{M}^2_A$ then brings the complete M^2 rather close to a diagonal form for all considered models [13,14,9,18]; that is, to diagonalize M^2, only a relatively small rotation ($|\theta| \sim 13^\circ \pm 2^\circ$) of the η_8-η_0 basis states,

$$\eta = \cos\theta\, \eta_8 - \sin\theta\, \eta_0 \,, \qquad \eta' = \sin\theta\, \eta_8 + \cos\theta\, \eta_0 \,, \tag{8}$$

is needed to align them with the mass eigenstates, i.e., with the physical η and η'. In contrast to this, the η-η' mass matrix in the NS-S basis (6),

$$\hat{M}^2 = \begin{bmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S\eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{bmatrix} = \begin{bmatrix} M^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{bmatrix} \xrightarrow[\phi]{} \begin{bmatrix} M^2_\eta & 0 \\ 0 & M^2_{\eta'} \end{bmatrix} \tag{9}$$

is then strongly off-diagonal. The indicated diagonalization, given by

$$\eta = \cos\phi\,\eta_{NS} - \sin\phi\,\eta_S\,, \qquad \eta' = \sin\phi\,\eta_{NS} + \cos\phi\,\eta_S\,, \tag{10}$$

is thus achieved for a large NS-S state-mixing angle $\phi \sim 42^\circ \pm 2^\circ$. Of course, this is again in agreement with phenomenological requirements [14,9], since ϕ is fixed to the angle θ by the relation $\phi = \theta + \arctan\sqrt{2} = \theta + 54.74^\circ$.

The invariant trace of the mass matrix (9), together with $M^2_{s\bar{s}} = 2M^2_K - M^2_\pi$ (from the DGMOR relation), gives the first equality in

$$\beta\,(2 + X^2) = M^2_\eta + M^2_{\eta'} - 2M^2_K = \frac{6}{f^2_\pi}\,\chi_{YM}\,. \tag{11}$$

The second equality is the Witten-Veneziano (WV) relation [1,2] between the η, η' and kaon masses and χ_{YM}, the topological susceptibility of the pure gauge, Yang-Mills theory. Thus, β does not need to be a free parameter, but can be determined from lattice results on χ_{YM}, so that no fitting parameters are introduced. For the three models [15,16,19] utilized in our treatments [13,14,9,18] of η and η', the bare quark mass parameters and the interaction parameters were fixed already in the non-anomalous sector, by requiring the good pion and kaon phenomenology. (See the π and K masses and decay constants in Table 1.) Then, following Refs. [9,18] in adopting the central value of the weighted average of the recent lattice results on Yang-Mills topological susceptibility [31–33],

$$\chi_{YM} = (175.7 \pm 1.5\,\text{MeV})^4\,, \tag{12}$$

we have obtained the good descriptions of the η-η' phenomenology [13,14,9,18], exemplified by the first three columns (one for each DS models used) of the middle part of Table 1, giving the predictions for the η and η' masses and for the NS-S mixing angle ϕ.

The lowest part of the table, below the second horizontal dividing line, contains the results on the quantities (θ_0, θ_8, etc.) defined in the scheme with four η and η' decay constants and two mixing angles, introduced and explained in the following Section 3. Table 1 also compares these results of ours (in the first three columns) with the corresponding results of Shore's approach [10,11], in which the *experimental* values of the meson masses M_π, M_K, M_η, and $M_{\eta'}$, as well as the decay constants f_π and f_K (in contrast to our $q\bar{q}$ bound-state model predictions for these quantities) are used as inputs enabling the calculation of various decay constants in the η-η' complex and the two mixing angles θ_0 and θ_8 (corresponding to $\phi = 38.24^\circ$ in our approach).

3 Usage of Shore's equations in DS approach

The WV relation was derived in the lowest-order approximation in the large N_c expansion. However, considerations by Shore [10,11] contain what amounts to

from Ref.	[14] & WV	[9] & WV	[18] & WV	Shore [10,11]	Experiment
M_π	137.3	135.0	140.0		$(138.0)^{\text{isospin}}_{\text{average}}$
M_K	495.7	494.9	495.0		$(495.7)^{\text{isospin}}_{\text{average}}$
$M_{s\bar{s}}$	700.7	722.1	684.8		
f_π	93.1	92.9	92.0		92.4 ± 0.3
f_K	113.4	111.5	110.1		113.0 ± 1.0
$f_{s\bar{s}}$	135.0	132.9	119.1		
M_η	568.2	577.1	542.3		547.75 ± 0.12
$M_{\eta'}$	920.4	932.0	932.6		957.78 ± 0.14
ϕ	41.42°	39.56°	40.75°	(38.24°)	
θ_0	−2.86°	−5.12°	−6.80°	−12.3°	
θ_8	−22.59°	−24.14°	−20.58°	−20.1°	
f_0	108.8	107.9	101.8	106.6	
f_8	122.6	121.1	110.7	104.8	
f_η^0	5.4	9.6	12.1	22.8	
$f_{\eta'}^0$	108.7	107.5	101.1	104.2	
f_η^8	113.2	110.5	103.7	98.4	
$f_{\eta'}^8$	-47.1	-49.5	-38.9	-37.6	

Table 1. The results of employing the WV relation (11) in our DS approach for the three dynamical models used in Refs. [14,9,18], compared with the results of Shore's analysis [10,11] and with the experimental results. The first column was obtained by the WV-recalculation of the results of Ref. [14], which in turn used the Jain-Munczek *Ansatz* for the gluon propagator [15]. Column 2: the results based on Ref. [9], which used the OPE-inspired, gluon-condensate-enhanced gluon propagator [16]. Column 3: the results based on Ref. [18], which utilized the separable *Ansatz* for the dressed gluon propagator [19]. Column 4: The results of Shore [10,11], who used the lattice result $\chi_{YM} = (191\,\text{MeV})^4$ of Ref. [32], and not the weighted average (12), in contrast to us. Column 5: the experimental values. All masses and decay constants are in MeV, and angles are in degrees. For more details, see text.

the generalization of the WV relation, which is valid to all orders in $1/N_c$. Among the relations he derived through the inclusion of the gluon anomaly in DGMOR relations, the following are pertinent for the present paper:

$$(f_{\eta'}^0)^2 M_{\eta'}^2 + (f_\eta^0)^2 M_\eta^2 = \frac{1}{3}\left(f_\pi^2 M_\pi^2 + 2 f_K^2 M_K^2\right) + 6A\,, \tag{13}$$

$$f_{\eta'}^0 f_{\eta'}^8 M_{\eta'}^2 + f_\eta^0 f_\eta^8 M_\eta^2 = \frac{2\sqrt{2}}{3}\left(f_\pi^2 M_\pi^2 - f_K^2 M_K^2\right)\,, \tag{14}$$

$$(f_{\eta'}^8)^2 M_{\eta'}^2 + (f_\eta^8)^2 M_\eta^2 = -\frac{1}{3}\left(f_\pi^2 M_\pi^2 - 4 f_K^2 M_K^2\right)\,, \tag{15}$$

where A is the full QCD topological charge parameter, which is presently unknown, but in the large N_c limit, it reduces to YM topological susceptibility: $A = \chi_{YM} + \mathcal{O}(1/N_c)$. Besides f_π, they contain f_K and the *four* decay constants [34–36], $f_{\eta'}^0$, f_η^8, f_η^0, and $f_{\eta'}^8$, associated with the two pseudoscalars η and η'.

Adding Eqs. (13) and (15), one gets the relation

$$(f^0_{\eta'})^2 M^2_{\eta'} + (f^0_\eta)^2 M^2_\eta + (f^8_\eta)^2 M^2_\eta + (f^8_{\eta'})^2 M^2_{\eta'} - 2 f^2_K M^2_K = 6A \qquad (16)$$

which is the analogue of the standard WV formula (11), to which it reduces in the large N_c limit where $A \to \chi_{YM}$, the $f^0_{\eta'}, f^8_\eta, f_K \to f_\pi$ limit, and the limit of vanishing subdominant decay constants (since η and η' are dominantly η_8 and η_0, respectively), i.e., $f^0_\eta, f^8_{\eta'} \to 0$. Nevertheless, we will need to use not just this single equation, but the three equations (13)-(15) from Shore's generalization.

These four η and η' decay constants are often parameterized in terms of two decay constants, f_8 and f_0, and two mixing angles, θ_8 and θ_0:

$$f^8_\eta = \cos\theta_8\, f_8 \,, \quad f^0_\eta = -\sin\theta_0\, f_0 \,, \quad f^8_{\eta'} = \sin\theta_8\, f_8 \,, \quad f^0_{\eta'} = \cos\theta_0\, f_0 \,. \qquad (17)$$

This is the so-called two-angle mixing scheme, which shows explicitly that it is inconsistent to assume that the mixing of the decay constants follows the pattern (8) of the mixing of the states η_8 and η_0 [34–36,30,37,29].

The advantage of our model is that, as we shall see, we are able to calculate the f_8 and f_0 parts of the physical decay constants (17) from the $q\bar{q}$ substructure. However, we cannot keep the full generality of Shore's approach, which allows for the mixing with the gluonic pseudoscalar operators, and therefore employs the definition [10,11] of the decay constants which, in general, due to the gluonic contribution, differs from the following standard definition through the matrix elements of the axial currents $A^{a\,\mu}(x)$:

$$\langle 0|A^{a\,\mu}(x)|P(p)\rangle = i f^a_P\, p^\mu e^{-ip\cdot x}, \qquad a = 8, 0; \qquad P = \eta, \eta' \,. \qquad (18)$$

Nevertheless, Shore's definition [10,11] coincides with the above standard one in the non-singlet channel, where there cannot be any admixture of the pseudoscalar gluonic component. Similarly, since our BS solutions (from Refs. [13,14,9,18]) are the pure $q\bar{q}$ states, without any gluonic components, using Shore's definition would not help us calculate the gluon anomaly influence on the decay constants. We thus employ the standard definitions (18), also used by, e.g., Gasser, Leutwyler, and Kaiser [34–36], as well as by Feldmann, Kroll, and Stech (FKS) [30,37,29].

Equivalently to $f^0_{\eta'}, f^8_\eta, f^0_\eta$, and $f^8_{\eta'}$, defined by Eq. (18), one has four related but different constants $f^{NS}_{\eta'}, f^{NS}_\eta, f^S_\eta$, and $f^S_{\eta'}$, if instead of octet and singlet axial currents ($a = 8, 0$) in Eq. (18) one uses the nonstrange-strange axial currents ($a = NS, S$)

$$A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^{8\,\mu}(x) + \sqrt{\frac{2}{3}} A^{0\,\mu}(x) = \frac{1}{2}\left[\bar{u}(x)\gamma^\mu\gamma_5 u(x) + \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right], \qquad (19)$$

$$A^\mu_S(x) = -\sqrt{\frac{2}{3}} A^{8\,\mu}(x) + \frac{1}{\sqrt{3}} A^{0\,\mu}(x) = \frac{1}{\sqrt{2}} \bar{s}(x)\gamma^\mu\gamma_5 s(x) \,. \qquad (20)$$

The relation between the two equivalent sets is thus

$$\begin{bmatrix} f^{NS}_\eta & f^S_\eta \\ f^{NS}_{\eta'} & f^S_{\eta'} \end{bmatrix} = \begin{bmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \,. \qquad (21)$$

Of course, this other quartet of η and η' decay constants can also be parameterized in terms of other two constants and two other mixing angles:

$$f_\eta^{NS} = \cos\phi_{NS}\, f_{NS}\ , \qquad f_\eta^{S} = -\sin\phi_S\, f_S\ , \qquad f_{\eta'}^{NS} = \sin\phi_{NS}\, f_{NS}\ , \qquad f_{\eta'}^{S} = \cos\phi_S\, f_S\ , \tag{22}$$

where f_{NS} and f_S are given by the matrix elements

$$\langle 0|A^\mu_{NS}(x)|\eta_{NS}(p)\rangle = i f_{NS}\, p^\mu e^{-ip\cdot x}\ , \qquad \langle 0|A^\mu_{S}(x)|\eta_{S}(p)\rangle = i f_{S}\, p^\mu e^{-ip\cdot x}\ , \tag{23}$$

while $\langle 0|A^\mu_{NS}(x)|\eta_S(p)\rangle = 0$ and $\langle 0|A^\mu_{S}(x)|\eta_{NS}(p)\rangle = 0$.

In the NS-S basis, it is possible to recover a scheme with a single mixing angle ϕ through the application of the Okubo-Zweig-Iizuka (OZI) rule [30,37,29]. For example, $f_{NS}f_S\sin(\phi_{NS}-\phi_S)$ differs from zero just by an OZI-suppressed term [29]. Neglecting this term thus implies $\phi_{NS}=\phi_S$. (Refs. [30,37,29] denote $f_{NS}, f_S, \phi_{NS}, \phi_S$ by, respectively, f_q, f_s, ϕ_q, ϕ_s.) In general, neglecting the OZI-suppressed terms, i.e., application of the OZI rule, leads to the so-called FKS scheme [30,37,29], which exploits a big practical difference between the (in principle equivalent) parameterizations (17) and (22): while θ_8 and θ_0 differ a lot from each other and from the octet-singlet *state* mixing angle $\theta \approx (\theta_8+\theta_0)/2$, the NS-S decay-constant mixing angles are very close to each other and both can be approximated by the state mixing angle: $\phi_{NS}\approx\phi_S\approx\phi$. Therefore one can deal with only this one angle, ϕ, and express the physical η-η' decay constants as

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} f_{NS}\cos\phi & -f_S\sin\phi \\ f_{NS}\sin\phi & f_S\cos\phi \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} . \tag{24}$$

This relation is valid also in our approach, where η and η' are the simple η_{NS}-η_S mixtures (10). In our present DS approach, mesons are pure $q\bar{q}$ BS solutions, without any gluonium admixtures, which are prominent possible sources of OZI violations. Therefore, our decay constants are calculated quantities, $f_{NS}=f_{u\bar{u}}=f_{d\bar{d}}=f_\pi$ and $f_S=f_{s\bar{s}}$, in agreement with the OZI rule. Our DS approach is thus naturally compatible with the FKS scheme, and we can use the η and η' decay constants (24) with our calculated $f_{NS}=f_\pi$ and $f_S=f_{s\bar{s}}$ in Shore's equations (13)-(15).

4 Results and conclusions

All quantities appearing on the right-hand side of Eqs. (13)-(15), namely M_π, M_K, f_π, and f_K, are calculated in our DS approach [14,9,18] (for the three dynamical models [15,16,19]), *except* the full QCD topological charge parameter A. Since it is at present unfortunately not yet known, we follow Shore and approximate it by the Yang-Mills topological susceptibility χ_{YM}.

On the left-hand side of Eqs. (13)-(15), the model results for $f_{NS}=f_\pi$ and $f_S=f_{s\bar{s}}$ and Eq. (24) reduce the unknown part of the four η and η' decay constants f_η^0, $f_{\eta'}^0$, f_η^8, and $f_{\eta'}^8$, down to the mixing angle ϕ. The three Shore's equations (13)-(15) can then be solved for ϕ, M_η and $M_{\eta'}$, providing us with the upper three lines

Inputs:	from Ref. [14]		from Ref. [9]		from Ref. [18]	
$\chi_{YM}^{1/4}$	175.7	191	175.7	191	175.7	191
M_η	485.7	499.8	482.8	496.7	507.0	526.2
$M_{\eta'}$	815.8	931.4	818.4	934.9	868.7	983.2
ϕ	46.11°	52.01°	46.07°	51.85°	40.86°	47.23°
θ_0	1.84°	7.74°	1.39°	7.17°	−6.69°	−0.33°
θ_8	−17.90°	−12.00°	−17.6°	−11.85°	−20.47°	−14.11°
f_0	108.8	108.8	107.9	107.9	101.8	101.8
f_8	122.6	122.6	121.1	121.1	110.7	110.7
f_η^0	-3.5	-14.7	-2.6	-13.5	11.9	0.6
$f_{\eta'}^0$	108.8	107.9	107.9	107.1	101.1	101.8
f_η^8	116.7	119.9	115.4	118.5	103.7	107.4
$f_{\eta'}^8$	-37.7	-25.5	-37.6	-24.9	-38.7	-27.0

Table 2. The results of the three DS models obtained through Shore's equations (13)-(15) for the two values of χ_{YM} approximating A: $(175.7\text{MeV})^4$ and $(191\text{MeV})^4$. Columns 1 and 2: The results when the non-anomalous inputs for Eqs. (13)-(15), namely $M_\pi, M_K, f_\pi = f_{NS}, f_{s\bar{s}} = f_S$ and f_K, are taken from Ref. [14], which uses Jain–Munczek *Ansatz* interaction [15]. Columns 3 and 4: The results for the non-anomalous inputs from Ref. [9] using OPE-inspired interaction nonperturbatively dressed by gluon condensates [16]. Columns 5 and 6: The results for the inputs from Ref. [18] using the separable *Ansatz* interaction [19]. All masses and decay constants, as well as $\chi_{YM}^{1/4}$, are in MeV, and angles are in degrees.

of Table 2. For each of the three different dynamical models which we used in our previous DS studies [13,14,9,26,17,18], these results are displayed for $\chi_{YM} = (175.7\,\text{MeV})^4$ as in Refs. [9,18] and for $\chi_{YM} = (191\,\text{MeV})^4$ [32] (adopted by Shore [10,11]). The lower part of the table, displaying various additional results, is then readily obtained through Eq. (24) and/or the following useful relations [29,14]:

$$f_8 = \sqrt{\frac{1}{3}f_{NS}^2 + \frac{2}{3}f_S^2}\,, \quad \theta_8 = \phi - \arctan\left(\frac{\sqrt{2}f_S}{f_{NS}}\right)\,, \tag{25}$$

$$f_0 = \sqrt{\frac{2}{3}f_{NS}^2 + \frac{1}{3}f_S^2}\,, \quad \theta_0 = \phi - \arctan\left(\frac{\sqrt{2}f_{NS}}{f_S}\right)\,. \tag{26}$$

Note that f_0 and f_8 do not result from solving of Eqs. (13)-(15), but are the calculated predictions of a concrete dynamical DS model, independently of Shore's equations.

For all three quite different (RGI [15,16] and non-RGI [19]) dynamical models which we used in our previous DS studies [13,14,9,26,17,18], the situation with the results turns out to be rather similar. The most conspicuous feature is that η and η' masses are both much too low when the weighted average $\chi_{YM} = (175.7 \pm 1.5\,\text{MeV})^4$ of Refs. [31–33] is used, in contrast to the results from the standard WV relation, displayed in Table 1. If we single out just the highest of these values $(191\,\text{MeV})^4$ [32]), the masses improve somewhat. However, other results are spoiled – e.g., the mixing angle ϕ becomes too high to enable agreement with the experimental results on $\eta, \eta' \to \gamma\gamma$ decays, which require $\phi \sim 40°$ [9].

When we turn to the lower parts of Tables 1 and 2, where the results for the η and η' decay constants, and the corresponding two mixing angles θ_0 and θ_8, are given, we notice a feature common to all our results, as well as Shore's (also given in Table 1). The diagonal ones, $f^0_{\eta'}$ and f^8_{η}, are all of the order of f_π, being larger by some 10% to 30%. The off-diagonal ones, $f^8_{\eta'}$ and f^0_{η}, are, on the other hand, in general strongly suppressed. This is expected, as η' is mostly singlet, and η is mostly octet. The feature that may be surprising is that Shore's results (which, to be sure, were obtained [10,11] in quite a different way from ours) are more similar to our results obtained through the standard WV relation, than to our results obtained through Shore's Eqs. (13)-(15).

All in all, inspection and comparison of the results in Table 2 with the results (in Table 1) from the analogous calculations but using the standard WV relation to construct the complete η-η' mass matrix, leads to the conclusion that the DS approach with the standard WV relation (11) is phenomenologically more successful, yielding the masses closer to the experimental ones. This may seem surprising, as Shore's generalization is in principle valid to all orders in $1/N_c$, while the standard WV relation is a lowest order $1/N_c$ result. Nevertheless, one must be aware that we do not yet have at our disposal the full QCD topological charge parameter A, and that we (along with Shore) had to use its lowest $1/N_c$ approximation, χ_{YM}. Also, we should recall from Sections 1 and 2 that the very usage of the RLA assumed that the anomaly is implemented on the level of the anomalous mass only, as a lowest order $1/N_c$ correction [13,14,9,17,18]. Thus, with respect to the orders in $1/N_c$, the usage of the standard WV relation is consistent in the present formulation of our DS approach, whereas the usage of Shore's generalization is not, which is probably the cause of its lesser phenomenological success. However, the usage of Shore's generalization in the DS context as exposed here, will likely find its application at finite temperatures. Namely, there it may help alleviate the difficulties met due to the usage of the standard WV relation in the DS approach at $T > 0$, as discussed in Ref. [18].

Acknowledgments: D.H. and D.Kl. acknowledge the support of the project No. 119-0982930-1016 of MSES of Croatia. D.Kl. also acknowledges the hospitality and support through senior associateship of International Centre for Theoretical Physics at Trieste, Italy, where the presented research was started. D.Kl. also thanks the LIT of JINR for its hospitality in Dubna, Russia, in August 2007. D.Ke. acknowledges the support of the Croatian MSES project No. 098-0982887-2872. Yu.K. thanks for support from Deutsche Forschungsgemeinschaft (DFG) under grant No. BL 324/3-1, and the work of D.B. was supported by the Polish Ministry of Science and Higher Education under grant No. N N 202 0953 33.

References

1. E. Witten, Nucl. Phys. **B156**, 269 (1979).
2. G. Veneziano, Nucl. Phys. **B159**, 213 (1979).
3. For example, see Refs. [4–8] for recent reviews and references on various applications of the DS approach.

4. R. Alkofer and L. von Smekal, Phys. Rept. **353**, 281 (2001).
5. C. D. Roberts, arXiv:nucl-th/0007054.
6. C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. **45**, S1 (2000).
7. A. Holl, C. D. Roberts, and S. V. Wright, arXiv:nucl-th/0601071.
8. C. S. Fischer, J. Phys. G **32**, R253 (2006).
9. D. Kekez and D. Klabučar, Phys. Rev. D **73**, 036002 (2006).
10. G. M. Shore, Nucl. Phys. **B744**, 34 (2006).
11. G. M. Shore, hep-ph/0701171.
12. D. Horvatić, D. Blaschke, Yu. Kalinovsky, D. Kekez and D. Klabučar, arXiv:0710.5650 [hep-ph], accepted for publication in European Physical Journal A.
13. D. Klabučar and D. Kekez, Phys. Rev. D **58**, 096003 (1998).
14. D. Kekez, D. Klabučar, and M. D. Scadron, J. Phys. G **26**, 1335 (2000).
15. P. Jain and H. J. Munczek, Phys. Rev. D **48**, 5403 (1993).
16. D. Kekez and D. Klabučar, Phys. Rev. D **71**, 014004 (2005).
17. D. Horvatić, D. Blaschke, D. Klabučar, and A. E. Radzhabov, arXiv:hep-ph/0703115.
18. D. Horvatić, D. Klabučar and A. E. Radzhabov, Phys. Rev. D **76**, 096009 (2007) [arXiv:0708.1260 [hep-ph]].
19. D. Blaschke, G. Burau, Y. L. Kalinovsky, P. Maris, and P. C. Tandy, Int. J. Mod. Phys. A **16**, 2267 (2001).
20. D. Kekez, B. Bistrović, and D. Klabučar, Int. J. Mod. Phys. A **14**, 161 (1999).
21. B. Bistrović and D. Klabučar, Phys. Lett. B **478**, 127 (2000); Phys. Rev. D **61**, 033006 (2000).
22. R. Alkofer and C. D. Roberts, Phys. Lett. B **369**, 101 (1996).
23. D. Kekez and D. Klabučar, Phys. Rev. D **65**, 057901 (2002).
24. O. Lakhina and P. Bicudo, arXiv:0708.0984.
25. M. S. Bhagwat, L. Chang, Y. X. Liu, C. D. Roberts, and P. C. Tandy, arXiv:0708.1118.
26. D. Blaschke, D. Horvatić, D. Klabučar, and A. E. Radzhabov, hep-ph/0703188, in *Proceedings of the Mini-Workshop "Progress in Quark Models", Bled, Slovenia, 2006*, editors B. Golli, M. Rosina, and S. Širca (DMFA - založništvo, Ljubljana 2006), p. 20.
27. B. Alles, M. D'Elia, and A. Di Giacomo, Nucl. Phys. **B494**, 281 (1997).
28. D. Blaschke, H. P. Pavel, V. N. Pervushin, G. Ropke, and M. K. Volkov, Phys. Lett. B **397**, 129 (1997).
29. Th. Feldmann, Int. J. Mod. Phys. A **15**, 159 (2000).
30. Th. Feldmann, P. Kroll, and B. Stech, Phys. Rev. D **58**, 114006 (1998).
31. B. Lucini, M. Teper, and U. Wenger, Nucl. Phys. **B715**, 461 (2005).
32. L. Del Debbio, L. Giusti, and C. Pica, Phys. Rev. Lett. **94**, 032003 (2005).
33. B. Alles, M. D'Elia, and A. Di Giacomo, Phys. Rev. D **71**, 034503 (2005).
34. J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).
35. H. Leutwyler, Nucl. Phys. Proc. Suppl. **64**, 223 (1998).
36. R. Kaiser and H. Leutwyler, hep-ph/9806336, in *Proceedings of the Workshop "Nonperturbative methods in quantum field theory", Adelaide, Australia, 1998*, editors A.W. Schreiber, A.G. Williams, and A.W. Thomas (World Scientific, Singapore 1998).
37. Th. Feldmann, P. Kroll, and B. Stech, Phys. Lett. B **449**, 339 (1999).