**IMFM**
Institute of Mathematics, Physics and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia

**Preprint series**
**Vol. 50 (2012), 1179**
ISSN 2232-2094

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Ljubljana, July 3, 2012

Preprint series, IMFM, ISSN 2232-2094, no. 1179, July 38, 2012

# Some Steiner Concepts on Lexicographic Products of Graphs*

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**Abstract**

The smallest tree that contains all vertices of a subset $W$ of $V(G)$ is called a Steiner tree. The number of edges of such a tree is the Steiner distance of $W$ and union of all Steiner trees of $W$ form a Steiner interval. Both of them are described for the lexicographic product in the present work. We also give a complete answer for the following invariants with respect to the Steiner convexity: the Steiner number, the rank, the hull number, and the Carathéodory number, and a partial answer for the Radon number. At the end we locate and repair a small mistake from [7].

*Work supported by the Ministry of Science of Slovenia and by the Ministry of Science and Technology of India under the bilateral India-Slovenia grants BI-IN/10-12-001 and INT/SLOVENIA-P-17/2009, respectively.

†Supported by the Ministry of Science of Slovenia under the grant P1-0297. The author is also with IMFM, Jadranska 19, 1000 Ljubljana, Slovenia.

‡Supported by the University Grants Commission, Govt. of India under the grant MRP(S)-864/10-11/KLCA042/UGC-SWRO.

**Key words**: Lexicographic product; Steiner convexity; Steiner set; Steiner distance

**2010 Math. Subj. Class.**: 05C05, 05C76

# 1 Introduction and preliminaries

Let $W$ be a subset of a set of vertices $V(G)$ of a graph $G$. A *Steiner tree* of $W$ is a minimum connected subgraph (if it exists) of $G$ that contains all vertices of $W$. Clearly Steiner tree is a tree. If $W$ has exactly two vertices $u$ and $v$, then its Steiner tree is a shortest $u,v$-path or a $u,v$-geodesic. Hence Steiner trees are natural generalization of geodesics. Similarly we have following generalizations. The number of edges on an $u,v$-geodesic is the usual *(geodetic) distance* $d_G(u,v)$ between $u$ and $v$ in $G$. The number of edges on a Steiner tree for $W$ is the *Steiner distance* $d_G(W)$ of $W$ in $G$. The *geodesic interval* $I_G(u,v)$ contains all vertices lying on an $u,v$-geodesic of $G$. The *Steiner interval* $SI_G(W)$ contains all vertices of a Steiner tree of $W$. A subset $C$ of $V(G)$ is *geodesically convex* if $I_G(u,v)$ is a subset of $C$ for every pair $u$ and $v$ from $C$. Similarly $C$ is *Steiner convex* if $SI_G(W)$ is a subset of $C$ for every $W \subseteq C$. It is easy to see that if $C$ induce a complete graph or if $C = V(G)$, then $C$ is a convex set for both geodesic and Steiner convexity. Such sets are called *trivial* (geodesic or Steiner) convex sets. For $W \subseteq V(G)$, let $I_G[W]$ be an union of intervals $I_G(u,v)$ for every pair $u,v \in W$, this is $I_G[W] = \cup_{u,v\in W} I_G(u,v)$. A set $W$ of vertices of $G$ is called a *geodetic set* if $I_G(W) = V(G)$. A geodetic set of minimum cardinality is a *minimum geodetic set* and its cardinality is the *geodetic number* $g(G)$ of $G$. If $SI_G(W) = V(G)$, then we call $W$ a *Steiner set* of $G$. A Steiner set of minimum cardinality is a *minimum Steiner set* and its cardinality is the *Steiner number* $s(G)$ of $G$. In an analogue fashion we can define the above concepts for the other convexities induced by different type of paths like induced path, detour path, any path, and the others; see [11].

The Steiner tree problem is a well-known problem with several applications. Its origin is in Euclidean (or other metric) space. In combinatorics Steiner trees play an important role in combinatorial optimization and application to combinatorial designs and transportation, to name just a few. For instance, see [2, 21] for some development in approximation algorithms. In general, the problem of finding a Steiner distance is NP-hard problem, see [17]. The beginning of graph theoretical approach to Steiner distance was probably made by Chartrand et. al. in [13]. Since then many papers appeared on the topic. For a small collection see [1, 4, 9, 13, 14, 20, 21, 22, 25] and the references therein. In particular, note that Steiner number was introduced by Chartrand and Zhang in [14]. In the second section we completely describe the Steiner distance, Steiner intervals, and the Steiner number with respect to lexicographic product of graphs.

In [4, 9] authors worked on a variation of Steiner problems on multi sets (instead of sets). Both definitions coincide for Steiner convexity, since we take all subsets of

$C$ in the definition of a Steiner convex set $C$ and not only $k$-subsets of $C$. (The latest is known as the *$k$-Steiner convexity.*) Not much is known about Steiner convexity. The pioneer work was done in [19] and continued in [5] on the so-called local Steiner convexity. For the general Steiner convexity, the non-trivial (geodesic and Steiner) convex sets of lexicographic product of graphs are characterized in [1]. We will use this result in the third section to describe some of the well-known convex invariants on lexicographic product graphs with respect to the Steiner convexity. The study of convexity invariants like the Carathéodory, the Helly, the Radon, the Hull numbers, and rank is one of the classical problems in combinatorial convexity. For the geodesic convexity an interesting observation due to Duchet in [16] states that these convexity invariants can be arbitrary in the sense that given any positive integers $c, h$, and $r$, there exists a finite graph whose geodesic convexity has Carathéodory, Helly, and Radon numbers $c, h$, and $r$, respectively. This result motivates to study these invariants for other graph convexities, see [3, 11, 12, 15], in particular in our case in third section the Steiner convexity.

In the last section we briefly discuss a small mistake from [7] and correct it, while in the remaining of this section we define the convexity invariants and introduce some notations.

Let $A$ be a subset of $V(G)$ for some graph $G$. The *geodesic convex hull* $\mathrm{ch}(A)$ of $A$ is the smallest geodesic convex set that contains $A$, while the *Steiner convex hull* $\mathrm{sch}(A)$ of $A$ is the smallest Steiner convex set that contains $A$. We will use $\mathrm{sch}(A)$ in the following definitions, since we are mainly interested in the Steiner convexity. The *Steiner Carathéodory number* of a graph $G$ is the smallest integer $c(G)$ (if it exists) such that for any finite subset $A$ of $V(G)$, $\mathrm{sch}(A)$ equals to $\cup\{\mathrm{sch}(S) : S \subseteq A, |S| \leq c(G)\}$. The *Radon number* of $G$ is the smallest integer $r(G)$ (if it exists) such that every $r(G)$-element set $S \subseteq V(G)$ admits a Radon partition $S_1$ and $S_2$, that is $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, and $\mathrm{sch}(S_1) \cap \mathrm{sch}(S_2) \neq \emptyset$. A subset $A$ of $V(G)$ is *Steiner convexly independent* if $a \notin \mathrm{sch}(A - \{a\})$ for every $a \in A$. Cardinality of a maximum convexly independent set is known as *Steiner rank,* $\mathrm{srank}(G)$ for short. A subset $A$ of $V(G)$ is called a *Steiner hull set* of $G$ if $\mathrm{sch}(A) = V(G)$, and a Steiner hull set of $G$ of the minimum cardinality is a *minimum Steiner hull set* in $G$. The cardinality of a minimum Steiner hull set in $G$ is called the *Steiner hull number* $\mathrm{sh}(G)$ of $G$. All these concepts are direct generalizations from the geodesic convexity. However they can be defined on general convexities. For more about this topic see the book [24].

We will use for a graph $G$ the standard notations $N_G(g)$ for the *open neighborhood* $\{g' : gg' \in E(G)\}$ and $\omega(G)$ for the order of a maximum complete subgraph. A *simplicial vertex* is a vertex $g$ whose $N_G(g)$ induce a complete graph. By $\lambda(G)$ we denote the number of all simplicial vertices of $G$. If a vertex is not simplicial it is called a $\Lambda$-vertex. For a subset $A$ of $V(G)$, we denote by $\langle A \rangle$ the subgraph of $G$ induced by $A$.

The *lexicographic product* of graphs $G$ and $H$ is the graph $G \circ H$ (also denoted by

$G[H]$) with the vertex set $V(G)\times V(H)$. Vertices $(g_1,h_1)$ and $(g_2,h_2)$ are adjacent if either $g_1g_2\in E(G)$ or $g_1=g_2$ and $h_1h_2\in E(H)$. The lexicographic product create a constant interest in the research community over the years. The intersection from past decade with other topics in this work can be found in [1, 6, 8, 23]. A product is called *non-trivial* if both factors are graphs on at least two vertices. It is easy to see that $G\circ H$ is connected if and only if $G$ is connected. For $h\in V(H)$ call $G^h=\{(g,h)\in G\circ H : g\in V(G)\}$ a *G-layer* in $G\circ H$ and for $g\in V(G)$ call ${}^gH=\{(g,h)\in G\circ H : h\in V(H)\}$ an *H-layer* in $G\circ H$. Note that $\langle G^h\rangle$ and $\langle {}^gH\rangle$ are isomorphic to $G$ and $H$, respectively. The map $p_G: V(G\circ H)\to V(G)$ defined by $p_G((g,h))=g$ is called a *projection map onto G*. Similarly we can define the *projection map onto H*. We can also project graphs which is clear from the domain. For more on lexicographic product or products in general we recommend [18].

## 2 The Steiner distance, interval, and number of $G\circ H$

First we state a useful lemma about relationship between Steiner trees in $G$ and in $G\circ H$ from [1].

**Lemma 1** *[1, Lemma 3.1] Let $g_1,\dots,g_k$ be different vertices of a connected graph $G$. For any (not necessarily different) vertices $h_1,\dots,h_k$ of a graph $H$, a Steiner tree of $g_1,\dots,g_k$ (in $G$) and a Steiner tree of $(g_1,h_1),\dots,(g_k,h_k)$ (in $G\circ H$) have the same size.*

The proof of this lemma imply even more. A tree $T$ is a Steiner tree of $(g_1,h_1),\dots,(g_k,h_k)$ in $G\circ H$ whenever $p_G(T)$ is a Steiner tree of $g_1,\dots,g_k$ in $G$.

**Theorem 2** *Let $G$ and $H$ be two non-trivial graphs and let $G$ be connected. For a subset $W$ of $V(G\circ H)$, $d_{G\circ H}(W)$ is equal to*

$$\begin{cases} d_G(p_G(W))+|W|-|p_G(W)| & : W\nsubseteq {}^gH,\ \text{for every } g\in V(G);\\ |W| & : W\subseteq {}^gH \text{ and } d_H(p_H(W))\geq |W|;\\ |W|-1 & : W\subseteq {}^gH \text{ and } d_H(p_H(W))=|W|-1.\end{cases}$$

**Proof.** If $d_H(p_H(W))=|W|-1$, then vertices of $p_H(W)$ induce a tree in $H$. If in addition $W\subseteq {}^gH$, then vertices of $W$ induce a tree in $\langle {}^gH\rangle$ and the proof is by nothing in this case. Assume next that $W\subseteq {}^gH$ and $d_H(p_H(W))\geq |W|$. Hence $W$ does not induce a connected subgraph of $G\circ H$ and $d_{G\circ H}(W)\geq |W|$. On the other hand, $d_{G\circ H}(W)\leq |W|$, since $W\cup\{(g',h)\}$ induce a connected subgraph of $G\circ H$ for every neighbor $g'$ of $g$ and any $h\in V(H)$.

Let now $W\nsubseteq {}^gH$ for every $g\in V(G)$. Without loss of generality we may assume that there are only $p$ number of $H$-layers ${}^{g_i}H$, $i\in\{1,\dots,p\}$, for which $W\cap V({}^{g_i}H)\neq\emptyset$. Let $(g_i,h_i)\in W\cap {}^{g_i}H$ for $i\in\{1,\dots,p\}$. By Lemma 1, a Steiner tree of $g_1,\dots,g_p$ (in $G$) and a Steiner tree of $(g_1,h_1),\dots,(g_p,h_p)$ (in $G\circ H$) have the same size. Let $T$ be a Steiner tree of $(g_1, h_1), \ldots, (g_p, h_p)$ (in $G \circ H$). Since $W \nsubseteq {}^gH$, every vertex of $(g_i, h_i)$, for $i \in \{1, \ldots, p\}$, has a neighbor $(g_i', h_i')$ on $T$ that is not in ${}^{g_i}H$. The remaining vertices of $W$ in each layer ${}^{g_i}H$, $i \in \{1, \ldots, p\}$, are also adjacent to $(g_i', h_i')$ (by the definition of the lexicographic product). Let $\overline{T}$ be a tree obtained from $T$ by adding all remaining vertices of $W$ and for every such vertex exactly one additional edge to a $(g_i', h_i') \in V(T)$. Clearly $\overline{T}$ has $d_G(p_G(W)) + |W| - |p_G(W)|$ edges. If $\overline{T}$ is not a Steiner tree of $W$, then there exists $T'$ which is a Steiner tree of $W$ with less edges than $\overline{T}$. The projection $p_G(T')$ is a tree containing $p_G(W)$ with less edges than $p_G(\overline{T})$. Since $p_G(\overline{T})$ and $p_G(T)$ have the same number of edges, we have a contradiction with the fact that $p_G(T)$ is a Steiner tree of $p_G(W)$. □

In the above theorem plays no role whether $H$ is connected or not. However we do not need the condition for $G$ to be connected. If $G$ is not connected the formula stays the same whenever all vertices of $p_G(W)$ are in the same component of $G$, while otherwise the Steiner distance does not exists. The same holds also for the next result.

**Theorem 3** *Let $G$ and $H$ be two non-trivial graphs and let $G$ be connected. For a subset $W$ of $V(G \circ H)$, $SI_{G\circ H}(W)$ is equal to*

$$\begin{cases} W \cup ((SI(P_G(W) - P_G(W)) \times V(H)) & : W \nsubseteq {}^gH \text{ for every } g \in G; \\ W \cup (N_G(g) \times V(H)) & : W \subseteq {}^gH \text{ and } d_H(p_H(W)) > |W|; \\ (\{g\} \times SI_H(p_H(W))) \cup (N_G(g) \times V(H)) & : W \subseteq {}^gH \text{ and } d_H(p_H(W)) = |W|; \\ W & : W \subseteq {}^gH \text{ and } d_H(p_H(W)) = |W| - 1. \end{cases}$$

**Proof.** If $W \subseteq {}^gH$ and $d_H(W) = |W| - 1$, then $W$ induce a tree in $G \circ H$ and hence $W$ itself is the Steiner interval. If $W \subseteq {}^gH$ and $d_H(p_H(W)) \geq |W|$, then $W \cup \{(g', h)\}$ induce a tree for every $g' \in N_g(g)$ and any $h \in V(H)$. Clearly this tree is a Steiner tree and hence $N_G(g) \times V(H) \subseteq SI_{G\circ H}(W)$. If in addition $d_H(p_H(W)) = |W|$, then there are also some Steiner trees completely contained in $\langle {}^gH \rangle$. Since $\langle {}^gH \rangle$ is isomorphic to $H$, we also have $\{g\} \times SI_H(p_H(W)) \subseteq SI_{G\circ H}(W)$. Suppose that there exists an additional vertex $(g', h)$ in $SI_{G\circ H}(W)$, where $(g', h) \notin (\{g\} \times SI_H(p_H(W))) \cup (N_G(g) \times V(H))$. Either $d_G(g, g') \geq 2$ or $g' = g$. If $d_G(g, g') \geq 2$, this tree contains more than $|W|$ edges. If $g' = g$, then $h \notin SI_H(p_H(W))$. The tree that contains $W$ and $(g, h)$ has more than $|W|$ edges and is not a Steiner tree for $W$.

Let now $d_H(W) > |W|$. If there exists a vertex $(g', h)$ in $SI_{G\circ H}(W)$, where $(g', h) \notin W \cup (N_G(g) \times V(H))$, then again either $d_G(g, g') \geq 2$ or $g' = g$. If $d_G(g, g') \geq 2$, we have the same contradiction. If $g' = g$, then $h \notin p_H(W)$. Since $d_H(W) > |W|$, there exists no vertex $(g, h)$ in ${}^gH$ such that $W \cup \{(g, h)\}$ induce a tree in $G \circ H$. Hence $(g, h)$ is not in $SI_{G\circ H}(W)$.

Let $W \nsubseteq {}^gH$ for every $g \in V(G)$. From the description of Steiner trees in the proof of Theorem 2, we can easily see the following. If $g \in p_G(W)$, then ${}^gH \cap$

$SI_{G\circ H}(W)$ contains only vertices of $W$. Otherwise, if $g \notin p_G(W)$ and $g$ is on some Steiner tree for $p_G(W)$ in $G$, then there exists a Steiner tree for $W$ in $G \circ H$ that contains $(g, h)$ for every $h \in V(H)$. Hence $[W \cup (SI(P_G(W) - P_G(W)) \times V(H))] \subseteq SI_{G\circ H}(W)$. If there would be an additional vertex in $SI_{G\circ H}(W)$, we would get the same contradiction as at the end of the proof of Theorem 2. □

The Steiner number of $G\circ H$ mainly depends on the number of vertices of $H$ which can be seen in the next theorem. But there is an exception. Let $H$ be a connected graph. If a subset $W$ of $V(H)$ induce a connected graph, then $SI_H(W) = W$. Hence every Steiner set $S$ of $H$, that is a proper subset of $V(H)$, does not induce a connected subgraph of $H$. Next we investigate those Steiner sets $S$ for which $d_H(S) = |S|$, called *perfect Steiner sets*. In other words, a Steiner set $S$ is a perfect Steiner set if $S \cup \{v\}$ induce a tree in $H$ for every $v \in V(H) - S$. There is no perfect Steiner set in a complete graph, since any proper subset of $V(K_n)$ induce a connected subgraph. So let $H$ be a connected non-complete graph and $v$ a vertex of minimum degree that is not a cut vertex. We claim that $S = V(H) - N_H(v)$ is a perfect Steiner set. Indeed, $S$ does not induce a connected subgraph of $H$ and $d(S) \geq |S|$. On the other hand $\langle S \cup \{u\}\rangle$ is connected for every $u \in N_H(v)$, since $v$ is not a cut vertex. Hence $d_H(S) = |S|$ and $SI_H(S) = V(H)$ and every non-complete connected graph admits a perfect Steiner set. However this is not necessarily a minimum perfect Steiner set. As an example observe a cube $Q_3$ and let vertices of $S$ induce two edges at distance 2. It is easy to see that this is a perfect Steiner set of cardinality 4, while the above description gives a perfect Steiner set of cardinality 5. Now we can describe the Steiner number of any nontrivial lexicographic product.

**Theorem 4** *Let $G$ and $H$ be two non-trivial graphs and let $G$ be connected. If $A = min\{|W| : W$ is a perfect Steiner set in $H\}$, then $s(G \circ H)$ is equal to*

$$\begin{cases} A & : \text{if } G \text{ has a universal vertex and } H \text{ is connected and not complete;} \\ |V(H)| & : \text{if } G \text{ has a universal vertex and } H \text{ is not connected;} \\ \mathrm{s}(G)|V(H)| & : \text{otherwise.} \end{cases}$$

**Proof.** Suppose first that $G$ has a universal vertex $g$. If $H$ is connected and not complete, then there exists a perfect Steiner set of $H$ and a perfect Steiner set of minimum cardinality. Let $W$ be such a set, that is $A = |W|$. Clearly $S = \{g\} \times W$ is a subset of ${}^gH$ and $d_{G\circ H}(S) = |S| = A$. By Theorem 3, we have

$$\begin{aligned} SI_{G\circ H}(S) &= (\{g\} \times SI_H(p_H(W))) \cup (N_G(g) \times V(H)) = \\ &= (\{g\} \times V(H)) \cup ((V(G) - \{g\}) \times V(H)) = \\ &= V(G) \times V(H) = V(G \circ H) \end{aligned}$$

and $S$ is a Steiner set. Moreover, since $S$ does not induce a connected subgraph, we have $s(G \circ H) = |S| = A$.

Suppose now that $H$ is not connected. For $S = {}^gH$ we have $d(p_H(S)) = \infty > |S|$. By Theorem 3, we have

$$\begin{aligned} SI_{G\circ H}(S) &= S \cup (N_G(g) \times V(H)) = \\ &= (\{g\} \times V(H)) \cup ((V(G) - \{g\}) \times V(H)) = \\ &= V(G) \times V(H) = V(G \circ H) \end{aligned}$$

and $S$ is a Steiner set in this case. Let $S'$ be a smaller subset of $V(G \circ H)$ than $S$. For every $g' \in V(G)$ there exists an $h \in V(H)$, such that $(g', h) \notin S'$. Choose $g' \in p_G(S')$. We claim that ${}^{g'}H \nsubseteq SI(S')$. Indeed, if $S' \subset {}^{g'}H$, then by Theorem 3, (second and third possibility) and since $H$ is not connected, ${}^{g'}H$ is not entirely in $SI(S')$. So let $S' \nsubseteq {}^{g'}H$ for every $g' \in G$. By Theorem 3, (first option) again ${}^{g'}H$ is not in $SI(S')$. Hence $S'$ is not a Steiner set and $s(G \circ H) = |V(H)|$.

Next let $H$ be a complete graph. If $S'$ is a minimum Steiner set of $G$, then $S' \times V(H)$ clearly form a Steiner set in $G \circ H$ and $s(G \circ H) \leq s(G)|V(H)|$. Let $S$ be a minimum Steiner set of $G \circ H$. First note that $S \nsubseteq {}^gH$ for every $g \in G$, since $H$ is complete. Next if $g \in p_G(S)$, then all vertices of ${}^gH$ must be in $S$ by Theorem 3. If $p_G(S)$ is not a minimum Steiner set in $G$, we get a contradiction with Lemma 1 or on the other hand with minimality of $S$. Hence $s(G \circ H) = s(G)|V(H)|$.

Finally let $G$ be a graph without an universal vertex. For every Steiner set $S$ of $G \circ H$, $S \nsubseteq {}^gH$ for every $g \in V(G)$ by Theorem 3. By the same theorem for every $g \in p_G(S)$, the whole ${}^gH$ must be in $S$. Thus $s(G \circ H) \geq s(G)|V(H)|$ by Lemma 1. Since $S' \times V(H)$ is a Steiner set for a minimum Steiner set $S'$ of $G$, we have $s(G \circ H) = s(G)|V(H)|$ and the proof is complete. □

In [6] the geodetic number $g(G \circ H)$ was considered and $g(G \circ H)$ was bounded there between 2 and $3g(G)$. The Theorem 4 shows that s$(G \circ H)$ does not behave like $g(G \circ H)$, since it grows with the number of vertices of $H$ in most cases.

## 3 Invariants for $G \circ H$ with respect to Steiner convexity

In this section we discuss several invariants connected with convex sets on the lexicographic product with respect to Steiner convexity. Recall that a vertex $u$ of a graph $G$ is called a $\Lambda$-*vertex* if $u$ is not a simplicial vertex. An induced subgraph $Y$ of the lexicographic product $G \circ H$ is called $\Lambda$-*complete* if ${}^gH \cap Y = {}^gH$ holds for any $\Lambda$-vertex $g$ of $p_G(Y)$. The subgraph $K$ of $G$ is *(Steiner) convex* if $V(K)$ is Steiner convex in $V(G)$. The following theorem, recently proved in [1], is needed.

**Theorem 5** *[1, Theorem 3.2] Let $G \circ H$ be a non-trivial, connected lexicographic product. Then a proper non-complete induced subgraph $Y$ of $G \circ H$ is Steiner convex if and only if the following conditions hold:*
*(i) $p_G(Y)$ is Steiner convex in $G$,*

*(ii) $Y$ is $\Lambda$-complete, and*
*(iii) $H$ is complete.*

Also in [1] is the analogue version of this theorem (Theorem 2.1) for the geodesic convexity. The only difference is that one need to replace both words Steiner by geodesic. Hence a natural question appears whether also convex invariants have analogue solutions for both convexities in $G \circ H$. This question has a positive answer in case of all studied invariants here as we will see in next subsections.

### 3.1 Rank

Recall that $\mathrm{srank}(G)$ is the cardinality of a maximum convexly independent set with respect to Steiner convexity.

**Theorem 6** *Let $G$ and $H$ be two non-trivial graphs and let $G$ be connected. With respect to Steiner convexity is*

$$\mathrm{srank}(G \circ H) = \begin{cases} \mathrm{srank}(G)|V(H)| & : \text{if } H \text{ is complete;} \\ \omega(G)\omega(H) & : \text{otherwise.} \end{cases}$$

**Proof.** Suppose first that $H$ is not a complete graph. By Theorem 5, $V(G \circ H)$ has only trivial Steiner convex sets. Thus the only Steiner convex sets are subsets of $V(G)$ that induce a complete graph and $V(G \circ H)$. Let $A$ be a subset of $V(G \circ H)$ on at least three vertices which does not induce a complete graph. Let $(g, h)$ and $(g', h')$ be two non-adjacent vertices of $A$ and $(g'', h'')$ a third vertex. By Theorem 5, $\mathrm{sch}(A - (g'', h'')) = V(G \circ H)$. Thus all (maximum) Steiner convex sets induce complete graphs in $G \circ H$. Since vertices of a complete graph always form a Steiner convexly independent set, we need to find a maximum clique in $G \circ H$. Let $C \subseteq V(G \circ H)$ be a set that induce a maximum clique in $G \circ H$. Clearly $p_G(C) = C_G$ induce a maximum complete subgraph in $G$ and its size is $\omega(G)$. Moreover ${}^gH \cap C$ induce a complete subgraph in $\langle {}^gH \rangle$ for every $g \in p_G(C)$. If $|{}^gH \cap C| < \omega(H)$, we have a contradiction with the maximality of $C$, since $K_{\omega(G)} \circ K_{\omega(H)}$ is a complete graph. Hence $\mathrm{srank}(G \circ H) = \omega(G)\omega(H)$.

Let now $H$ be a complete graph. Let $\mathrm{srank}(G) = r$ and let $A_1$ be a Steiner convexly independent set with $|A_1| = r$. By the definition $a \notin \mathrm{sch}(A_1 - a)$ for every $a \in A_1$. We show next that $A_1 \times V(H)$ is Steiner convexly independent. If not, we can find an $(a_1, h_1) \in A_1 \times V(H)$ such that $(a_1, h_1) \in \mathrm{sch}(A_1 \times V(H) - \{(a_1, h_1)\})$. We can find $(b_1, \ell_1), \ldots, (b_m, \ell_m) \in A_1 \times V(H) - \{(a_1, h_1)\}$ whose Steiner tree contains $(a_1, h_1)$. Without loss of generality we can assume that $b_1, \ldots, b_r$, $r \leq m$, are in the projection of $(b_1, \ell_1), \ldots, (b_m, \ell_m)$ in $G$. By Lemma 1, the Steiner tree of $b_1, \ldots, b_r$ contains $a_1$. That is $a_1 \in \mathrm{sch}(A_1 - a_1)$, a contradiction, and $A_1 \times V(H)$ is maximal.

Since every Steiner convexly independent subset of $V(G \circ H)$ projects to a Steiner convexly independent subset of $V(G)$, clearly $A_1 \times V(H)$ is a maximum Steiner convexly independent set, otherwise we have a contradiction with $A_1$ being a maximum

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Steiner convexly independent set of $V(G)$. Hence $\text{srank}(G \circ H) = \text{srank}(G)|V(H)|$ and the proof is complete. □

The same theorem holds if we replace all terminology connected to Steiner convexity by the same terminology with respect to geodesic convexity. The only other change is in second paragraph of the proof, where we replace a Steiner tree by a shortest path in the case of geodesic convexity and conclusion follows the same lines.

### 3.2 Hull number

The hull number of lexicographic product of graphs with respect to geodetic convexity is described in [8]. Here we prove an analogue result for Steiner convexity. First we prove a lemma in order to prove the theorem.

**Lemma 7** *If $H$ is a complete graph, then every simplicial vertex of a graph $G \circ H$ belongs to any Steiner hull set of $G \circ H$.*

**Proof.** If a simplicial vertex $(g, h)$ is not in a Steiner hull set $A$, then it is not in any Steiner interval of any subset of $V(G \circ H)$ by Theorem 3. A contradiction with $A$ being a Steiner hull set. □

Recall that we denote the number of simplicial vertices of graph $G$ by $\lambda(G)$. It is a straightforward observation that $(g, h) \in V(G \circ H)$ is a simplicial vertex if and only if $g$ is a simplicial vertex of $G$ and $H$ is a complete graph.

**Theorem 8** *Let $G$ and $H$ be two non-trivial graphs and let $G$ be connected. With respect to Steiner convexity is*

$$sh(G \circ H) = \begin{cases} 2 & : H \text{ is not complete;} \\ \lambda(G)|V(H)| + h(G) - \lambda(G) & : H \text{ is complete.} \end{cases}$$

**Proof.** Let first $H$ be a non-complete graph. Since $G \circ H$ has no proper Steiner convex sets other than cliques by Theorem 5, the convex hull of any two non-adjacent vertices will be $G \circ H$. Hence in this case $\text{sh}(G \circ H) = 2$.

Let now $H$ be a complete graph. By Lemma 7, every simplicial vertex must be in any hull set and there are exactly $\lambda(G)|V(H)|$ simplicial vertices in $G \circ H$. Let $A = \{g_1, \ldots, g_{\lambda(G)}, \ell_1, \ldots, \ell_k\}$ be a minimal hull set of $G$, where $g_1, \ldots, g_{\lambda(G)}$ are simplicial vertices and $\ell_1, \ldots, \ell_k$ are $\Lambda$-vertices of $G$. We will prove that $\overline{A} = \{ {}^{g_1}H, \ldots, {}^{g_\lambda}H, (\ell_1, h_1), \ldots, (\ell_k, h_1)\}$ is a minimal hull set of $G \circ H$. First observe a subset $B = A \times \{h_1\}$. By Theorem 3, $SI(B)$ contains all vertices in all ${}^{g}H$ layers for which $g \notin A$. Since every $(\ell_i, h_1)$, for $i \in \{1, \ldots, k\}$, is a $\Lambda$-vertex, $(\ell_i, h_1)$ is adjacent to some non-adjacent vertices, say $(u_j, h_1)$ and $(u_m, h_1)$. If $(u_j, h_1)$ or $(u_m, h_1)$ are simplicial or not, they are in $\text{sch}(B)$. Now $SI[(u_j, h_1), (u_m, h_1)]$ will contain the layer ${}^{\ell_i}H$ and thus $\text{sch}(B)$ contains all vertices in ${}^{g}H$ for every $\Lambda$-vertex $g \in V(G)$.

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But then $\mathrm{sch}(\overline{A})$ contains the whole vertex set $V(G \circ H)$ and is a hull set. Suppose that a hull set $S$ is smaller than $\overline{A}$. Clearly $S$ must contain all simplicial vertices of $G \circ H$. Thus there are less than $k$ $\Lambda$-vertices in $S$ and $p_G(S)$ is a hull set of $G$ with cardinality less than $|A|$, contradicting the minimality of $A$. Hence $\overline{A}$ is a minimum hull set and $\mathrm{sh}(G \circ H) = \lambda(G)|V(H)| + h(G) - \lambda(G)$. □

### 3.3 Carathéodory number

First we need a lemma which is a direct consequence of Lemma 1 and Theorem 5.

**Lemma 9** *Let $H$ be a complete graph and $G$ a graph. If $F$ is a subset of $V(G \circ H)$, then $p_G(\mathrm{sch}(F)) = \mathrm{sch}(p_G(F))$.*

**Proof.** Clearly $\mathrm{sch}(p_G(F)) \subseteq p_G(\mathrm{sch}(F))$, since $\mathrm{sch}(p_G(F))$ is the smallest Steiner convex set that contains $p_G(F)$ and $p_G(\mathrm{sch}(F))$ is a convex set by Theorem 5 that contains all vertices of $p_G(F)$. On the other hand let $g \in p_G(\mathrm{sch}(F))$. Either $(g, h) \in F$ for some $h \in V(H)$ or $(g, h)$ is on some Steiner tree of $F_1 \subseteq F$. Clearly $(g, h) \in F$ imply that $g \in \mathrm{sch}(p_G(F))$. If $(g, h)$ is on some Steiner tree of $F_1 \subseteq F$, then $g$ is on a Steiner tree of $p_G(F_1)$ by Lemma 1. Hence also $p_G(\mathrm{sch}(F)) \subseteq \mathrm{sch}(p_G(F))$ holds and we have an equality. □

The Carathéodory number with respect to the Steiner convexity of $G \circ H$ is completely described by the following result.

**Theorem 10** *Let $G$ and $H$ be two non-trivial graphs and let $G$ be connected. The Carathéodory number with respect to the Steiner convexity is*

$$c(G \circ H) = \begin{cases} c(G) & : H \text{ is complete and } G \text{ is not;} \\ 2 & : H \text{ is not complete;} \\ 1 & : G \text{ and } H \text{ are complete.} \end{cases}$$

**Proof.** If $G$ and $H$ are complete graphs, then $G \circ H$ is complete and clearly $c(G \circ H) = 1$. If $H$ is not complete, then $G \circ H$ contains no proper non-complete Steiner convex set by Theorem 5. Furthermore if $F$ contains any two non-adjacent vertices $(g, h_1)$ and $(g, h_2)$ of $G \circ H$, then $\mathrm{sch}(F) = V(G \circ H)$ as well as $\mathrm{sch}(\{(g, h_1), (g, h_2)\}) = V(G \circ H)$. Thus $c(G \circ H) = 2$.

Let $H$ be a complete graph and $G$ a non-complete graph. Suppose that $c(G \circ H) < c(G)$. Let $F'$ be any subset of $V(G)$. Set $F = F' \times \{h\}$ for some $h \in V(H)$. We can find a collection of subsets $S_1, \ldots, S_t$ of $F$ of cardinality at most $c(G \circ H)$, such that the equality $\mathrm{sch}(F) = \cup_{i=1}^{t} \mathrm{sch}(S_i)$ holds. Subsets $p_G(S_1), \ldots, p_G(S_t)$ of $p_G(F)$ have cardinality at most $c(G \circ H) < c(G)$. By Lemma 9

$$\begin{aligned} \mathrm{sch}(p_G(F)) &= p_G(\mathrm{sch}(F)) = p_G\left(\cup_{i=1}^{t} \mathrm{sch}(S_i)\right) = \\ &= \cup_{i=1}^{t}(p_G(\mathrm{sch}(S_i))) = \cup_{i=1}^{t}(\mathrm{sch}(p_G(S_i))), \end{aligned}$$

which is a contradiction with minimality of Carathéodory number of $G$. Thus $c(G) \leq c(G \circ H)$.

On the other hand, let $F$ be any subset of $V(G \circ H)$. For $F' = p_G(F)$ there exists a family $S'_1, \ldots, S'_k$ of subsets of cardinality at most $c(G)$ for which $\mathrm{sch}(F') = \cup_{i=1}^{k} \mathrm{sch}(S'_i)$ holds. Let $\mathcal{S}_i$, $i \in \{1, \ldots, k\}$, be a family of following subsets of cardinality $|S'_i| \leq c(G)$: $S \in \mathcal{S}_i$ if for every $g \in S'_i$ there is exactly one vertex $(g, h) \in F$ in $S$. We claim that $\mathrm{sch}(F) = A$ for $A = \cup_{i=1}^{k} \cup_{S \in \mathcal{S}_i} \mathrm{sch}(S)$. If $(g, h) \in \mathrm{sch}(F)$, then $g \in p_G(\mathrm{sch}(F))$ and $g \in \mathrm{sch}(p_G(F)) = \mathrm{sch}(F')$ by Lemma 9. Hence $g \in \mathrm{sch}(S'_i)$ for some $i \in \{1, \ldots, k\}$. The way families $\mathcal{S}_i$ are defined, there exists $S \in \mathcal{S}_i$, such that $(g, h) \in \mathrm{sch}(S)$. Thus $(g, h) \in A$ and $\mathrm{sch}(F) \subseteq A$. Let now $(g, h) \in A$, more accurate let $(g, h) \in \mathrm{sch}(S)$ and $S \in \mathcal{S}_i$ for some $i \in \{1, \ldots, k\}$. By Lemma 9 again we have

$$p_G(\mathrm{sch}(S)) = \mathrm{sch}(p_G(S)) = \mathrm{sch}(S'_i) \subseteq \mathrm{sch}(F') = \mathrm{sch}(p_G(F)) = p_G(\mathrm{sch}(F)).$$

Hence $g \in p_G(\mathrm{sch}(F))$. If $g$ is a $\Lambda$-vertex, then $(g, h)$ is in $\mathrm{sch}(F)$ by Theorem 5, since $\mathrm{sch}(F)$ is Steiner convex and thus also $\Lambda$-complete. If $g$ is a simplicial vertex, then $(g, h)$ must be in $S$ by Lemma 7 and hence also in $F$ and clearly also in $\mathrm{sch}(F)$. Thus $A \subseteq \mathrm{sch}(F)$ and the proof is complete. □

In [10, Theorem 4], a family of graphs $G_k$, $k \geq 2$, was constructed with the property that $G_k$ has a $k$-Steiner convex subset that is not $(k+1)$-Steiner convex. This family is an example for graphs with large Steiner Carathéodory number, since $c(G_k) > k$.

Again the same result and proof can be stated if we replace Steiner terminology by the geodesic one. Note only that in the proof Lemma 9, we cannot refer to Lemma 1. However the geodesic version of Lemma 1 is an easy task for the students. Hence we leave the details to the reader.

### 3.4 Radon number

Clearly $r(G) > \omega(G)$, since in a complete subgraph $K_{\omega(G)}$ for every partition $S_1$ and $S_2$ we have $\mathrm{sch}(S_1) \cap \mathrm{sch}(S_2) = \emptyset$. Also for complete graphs the Radon number does not exist.

**Theorem 11** *Let $G$ and $H$ be two non-trivial graphs, at most one complete and let $G$ be connected. The Radon number with respect to Steiner convexity is*
*(i) $r(G \circ H) = \omega(G)\omega(H) + 1$ if $H$ is not complete;*
*(ii) $\max\{r(G), \omega(G)\omega(H) + 1\} \leq r(G \circ H) \leq (r(G) - 1)\omega(H) + 1$ if $H$ is complete.*

**Proof.** Since $\omega(G \circ H) = \omega(G)\omega(H)$, we always have $r(G \circ H) \geq \omega(G)\omega(H) + 1$. For $(i)$, let $H$ be a non-complete graph. If $S$ is a subset of $V(G \circ H)$ of cardinality $\omega(G)\omega(H) + 1$, then there exists two non-adjacent vertices $(g, h)$ and $(g', h')$. Sets $S_1 = \{(g, h), (g', h')\}$ and $S_2 = S - S_1$ form a Radon partition. Indeed, $S_1 \cup S_2 = S$,

$S_1 \cap S_2 = \emptyset$, and $\mathrm{sch}(S_1) \cap \mathrm{sch}(S_2) \neq \emptyset$, since $\mathrm{sch}(S_1) = V(G \circ H)$ by Theorem 5. Hence $r(G \circ H) = \omega(G)\omega(H) + 1$ if $H$ is a non-complete graph.

For $(ii)$, let $H$ be a complete graph, which implies that $G$ is not. Choose any subset $S$ of $V(G \circ H)$ of cardinality $(r(G) - 1)|V(H)| + 1$. The cardinality of $p_G(S)$ is at least $r(G)$. Let $S'$ be any subset of $p_G(S)$ of cardinality $r(G)$. Let $S'_1$ and $S'_2$ form a Radon partition for $S'$. By $S_1$ we denote all vertices of $S$ that projects to $S'_1$ and $S_2 = S - S_1$. Clearly $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. Let $v \in \mathrm{sch}(S'_1) \cap \mathrm{sch}(S'_2)$. Since $S'_1 \cap S'_2 = \emptyset$, then $v \notin S_1$ or $v \notin S_2$. Without loss of generality we may assume that $v \notin S_1$. But then the whole layer ${}^vH$ is in $\mathrm{sch}(S_1)$. Thus $\mathrm{sch}(S_1) \cap \mathrm{sch}(S_2) \neq \emptyset$ and $S_1$ and $S_2$ form a Radon partition of $S$. Hence $r(G \circ H) \leq (r(G) - 1)|V(H)| + 1$. For the lower bound we may assume that $r(G) > \omega(G)\omega(H) + 1$. Let $S$ be a subset of $V(G)$ of cardinality $r(G) - 1$ that does not admit a Radon partition. Observe $S' = S \times \{h\}$. If $S'$ admits a Radon partition $S'_1$ and $S'_2$, then also $S = p_G(S')$ admits a Radon partition $S_1 = p_G(S'_1)$ and $S_2 = p_G(S'_2)$ which is not possible. Hence $r(G \circ H) \geq r(G)$ and the proof is complete. □

While in most cases one can expect that $r(G) < \omega(G)\omega(H) + 1$, this is not always the case. Let $K_n^+$ be a graph obtained from $K_n$ by subdividing each edge by a vertex. It is easy to see that a set $S$ of all original vertices of $K_n$ does not admit a Radon partition and is thus $r(K_n^+) \geq n + 1$. For $G \cong K_{2n+1}^+$ and $H \cong K_\ell$, $\ell \leq n$, we have $r(G) > \omega(G)\omega(H) + 1$.

The following corollary is a direct application of $(ii)$ in Theorem 11.

**Corollary 12** *Let $H$ be a complete graph and $G$ a connected graph. If $r(G) = \omega(G) + 1$, then $r(G \circ H) = \omega(G)\omega(H) + 1$ (with respect to the Steiner convexity).*

As an example that the lower bound of $(ii)$ in Theorem 11 is not always attained let $G \cong C_{2k+1}$, $k \geq 2$, and $H \cong K_n$. It is easy to see that $r(C_{2k+1}) = 4$ and we have $2n + 1 \leq r(G \circ H) \leq 3n + 1$ by $(ii)$ of Theorem 11. Let $uv \in E(C_{2k+1})$ and $w$ the antipodal vertex of $u$ and $v$ on $C_{2k+1}$. For $S = \{(w, h)\} \cup (\{v, u\} \times V(H))$, where $h \in V(H)$, it is easy to see that $S$ has no Radon partition. Since $|S| = 2n + 1$ we have $2n + 1 < r(G \circ H)$. Moreover, every subset of $V(G \circ H)$ of cardinality $2n + 2$ has a Radon partition and we have $r(C_{2k+1} \circ K_n) = 2n + 2$ (we leave the details to the reader).

As in the previous subsections of this section we can state analogue results for the geodesic convexity if we replace Steiner terminology by geodesic one.

## 4 Erratum

While going through the hull number, rank, and Steiner number of lexicographic product, we found a small error in the Proposition 14 of the paper [7], and we have rectified the proposition. We state the definition which is used in that paper and proposition as it is. But first we define the strong product.

The *strong product* of graphs $G$ and $H$ is the graph $G \boxtimes H$ with the vertex set $V(G) \times V(H)$. Vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if either $g_1g_2 \in E(G)$ and $h_1 = h_2$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$.

**Definition 13** *Let $S$ be a set of vertices in a graph $G$. Then, $S$ is said to satisfy the condition*
*($A$) if, for every vertex $x \in S$, there exists two vertices $y, z \in S \setminus x$ such that $x \in I[y, z]$.*
*($B$) if there are two vertices $x, y \in S$ such that $x \notin I[S \setminus x]$ and $y \notin I[S \setminus y]$*

**Proposition 14** ***[7, Proposition 5]*** *Let $G$ be a graph with a minimum geodetic set $S$ satisfying the condition ($A$). Then, for every positive integer $n$, $g(G \boxtimes K_n) = g(G)$.*

**Example 15 (Counter example for Proposition 14)** *Let $G = C_6$ and $H = K_n$. Let $uv$ and $xy$ be two opposite edges of $C_6$. Set $S = \{u, x, v, z\}$ is the minimum geodetic set that satisfies the condition ($A$) and $g(G) = 2$. But we can see that $g(G \boxtimes H) = 4 > g(G)$.*

We modify the Proposition 14 as follows.

**Proposition 16** *Let $G$ be a graph with a minimum geodetic set $S$ satisfying the condition ($A$). Then, for every positive integer $n$, $g(G \boxtimes K_n) = |S|$.*

**Proof.** We can see that $|S|$ need not be $g(G)$ as in Example 15. The proof follows from the proof of the Proposition 5 in [7] by replacing $g(G)$ by $|S|$. □

## References


[1] B. S. Anand, M. Changat, S. Klavžar, I. Peterin, Convex Sets in Lexicographic Products of Graphs, Graphs Combin. 28 (2012) 77–84.

[2] S. Arora, Approximation schemes for NP-hard geometric optimization problems: a survey, Math. Program. 97 (2003) 43–69.

[3] K. Balakrishnan, M. Changat, Hull numbers of path convexities on graphs, in: Convexity in Discrete Structures (M. Changat, S. Klavžar, H. M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 47–55.

[4] B. Brešar, M. Changat, J. Mathews, I. Peterin, P. G. Narasimha-Shenoi, A. Tepeh Horvat, Steiner intervals, geodesic intervals, and betweenness, Discrete Math. 309 (2009) 6114–6125.

[5] B. Brešar, T. Gologranc, On a local 3-Steiner convexity, European J. Combin. 32 (2011) 1222–1235.

[6] B. Brešar, T. Kraner Šumenjak, A. Tepeh, The geodetic number of the lexicographic product of graphs, Discrete Math. 311 (2011) 1693–1698.

[7] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, On the geodetic and the hull numbers in strong product graphs, Comput. Math. Appl. 60 (2010) 3020–3031.

[8] S. R. Canoy Jr., G. B. Cagaanan, On the Hull Number of the Composition of Graphs, Ars Combin. 75 (2005) 113–119.

[9] M. Changat, A. K. Lakshmikuttyamma, J. Mathews, I. Peterin, P. G. Narasimha-Shenoi, A. Tepeh, A note on 3-Steiner intervals and betweenness, Discrete Math. 311 (2011) 2601–2609.

[10] M. Changat, J. Mathews, P. G. Narasimha-Shenoi, I. Peterin, $n$-ary transit functions in graphs, Discuss. Math. Graph Theory 30 (2010) 671–685.

[11] M. Changat, H. M. Mulder, G. Sierksma, Convexities Related to Path Properties on Graphs, Discrete Math. 290 (2005) 117–131.

[12] M. Changat, J. Mathew, On Triangle path convexity in graphs, Discrete Math. 206 (1999) 91–95.

[13] G. Chartrand, O. R. Oellermann, S. Tian, H. B. Zou, Steiner distance in graphs, Časopis Pro. Pest. Mat. 114 (1989) 399–410.

[14] G. Chartrand, P. Zhang, The Steiner Number of a Graph, Discrete Math. 242 (2002) 41-54.

[15] P. Duchet, Convex sets in graphs II. Minimal path convexity, J. Combin. Theory Ser. B 44 (1988) 307–316.

[16] P. Duchet, Discrete Convexity: retractions and morphisms and partition problem, in: Proc. of the conf. on graph connections (R. Balakrishnan, H. M. Mulder and A. Vijayakumar eds.), New Delhi (1999) 87–90.

[17] M. R. Garey, D. S. Johnson, *Computers and Intractability: A Guide to NP-completnes*, Freeman, New York, 1979.

[18] R. Hammack, W. Imrich, S. Klavžar, *Handbook of Product Graphs, Second Edition*, CRC Press, Boca Raton, FL, 2011.

[19] M. Henning, M. H. Nielsen, O. R. Oellermann, Local Steiner convexity, European J. Combin. 30 (2009) 1186–1193.

[20] C. Hernando, T. Jiang, M. Mora, I.M. Pelayo, C. Seara, On the Steiner, geodetic and hull numbers of graphs, Discrete Math. 293 (2005) 139-154.

[21] A. Moss, Y. Rabani, Approximation algorithms for constrained node weighted Steiner tree problems, SIAM J. Comput. 37 (2007) 460-481.

[22] O. R. Oellermann, M. L. Puertas, Steiner intervals and Steiner geodetic numbers in distance-hereditary graphs, Discrete Math. 307 (2007) 88–96.

[23] I. Peterin, Pre-hull number and lexicographic product, Discrete Math. 312 (2012) 2153–2157.

[24] M. L. J. van de Vel, Theory of Convex Structures, North Holland, Amsterdam, 1993.

[25] H. G. Yeh, C.-Y. Chiang, S.-H. Peng, Steiner centers and Steiner medians of graphs, Discrete Math. 308 (2008) 5298–5307.