ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 9 (2015) 287–300

Levels in bargraphs

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Received 20 January 2014, accepted 8 February 2015, published online 8 July 2015

Abstract

Bargraphs are lattice paths in $\mathbb{N}_0^2$, which start at the origin and terminate immediately upon return to the x-axis. The allowed steps are the up step $(0, 1)$, the down step $(0, -1)$ and the horizontal step $(1, 0)$. The first step is an up step and the horizontal steps must all lie above the x-axis. An up step cannot follow a down step and vice versa. In this paper we consider levels, which are maximal sequences of two or more adjacent horizontal steps. We find the generating functions that count the total number of levels, the leftmost x-coordinate and the height of the first level and obtain the generating function for the mean of these parameters. Finally, we obtain the asymptotics of these means as the length of the path tends to infinity.

Keywords: Bargraphs, levels, generating functions, asymptotics.

Math. Subj. Class.: 05A15, 05A16

1 Introduction

Bargraphs are lattice paths in $\mathbb{N}_0^2$, starting at the origin and ending upon first return to the x-axis. The allowed steps are the up step, $u = (0, 1)$, the down step, $d = (0, -1)$ and the horizontal step, $h = (1, 0)$. The first step has to be an up step and the horizontal steps must all lie above the x-axis. An up step cannot follow a down step and vice versa. It is clear that the number of down steps must equal the number of up steps. Related lattice paths such as Dyck paths and Motzkin paths have been studied extensively (see [4, 9]) whereas until now bargraphs which are fundamental combinatorial structures, have not attracted the same amount of interest.

*Corresponding author. This material is based upon work supported by the National Research Foundation under grant number 86329

†This material is based upon work supported by the National Research Foundation under grant number 81021

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Bousquet-Mélou and Rechnitzer in [2] and Geraschenko in [8] have studied bargraphs which were named skylines in the latter, and wall polyominoes as per the study of Feretić, in [6]. Bargraphs models arise frequently in statistical physics, see for example [3, 5, 10, 12, 15, 17]. In addition, bargraphs are commonly used in probability theory to represent frequency diagrams and are also related to compositions of integers [11].

In this paper, we consider levels, which are maximal sequences of two or more adjacent horizontal steps. We find different generating functions in each of the following sections where x counts the horizontal steps, y counts the up vertical steps and w counts one of the following parameters: the total number of levels and the horizontal position or the height of the first level. To facilitate these computations, we also find the generating function for paths with no levels.

The study of levels in bargraphs is related to the modelling of tethered polymers under pulling forces, see [13, 14]. These pulling forces have vertical and horizontal components and tend to be resisted by what is known as the stiffness of the polymers. The polymers undergo phase changes, called the stretched (adsorption) phase, where the polymer is stretched vertically. The free (desorbed) phase occurs only when the vertical force is zero. In the bargraph models of polymers positive or negative energy is added to points in levels on the bargraph (called stiffness sites), they tend to keep the polymer horizontal or cause it to bend.

As an example of a bargraph we have

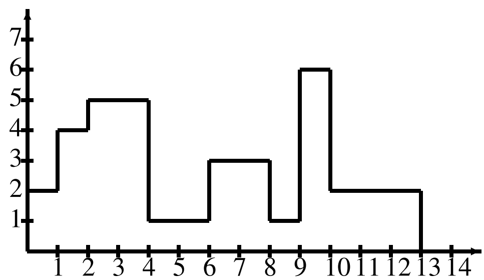

Figure 1: A bargraph with 12 up steps, 13 horizontal steps and 4 levels

Often in the lattice walk and polygon literature, "bargraphs" refer to polygon structures (which would be obtained from the objects considered here by joining the first and last vertices with horizontal steps). The objects discussed here are sometimes called "partially directed walks above a wall" depending on the context (in polymer modelling work for example).

The main tool for elucidating the statistics of interest in this study is a decomposition of bargraphs which is based on the first return to level one. This was described initially by Prellberg and Brak in [16] and more recently in [2], where it is called the *wasp-waist decomposition*. The present authors have also discussed it in [1].

It follows from the wasp-waist decomposition that the generating function $B(x, y)$ which counts all bargraphs is

$$B := B(x,y) = \frac{1-x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x}. \qquad (1.1)$$

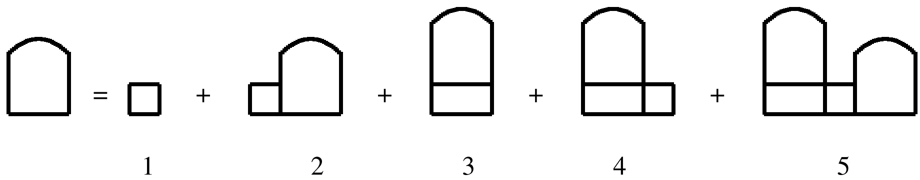

Figure 2: Wasp-waist decomposition of bargraphs

Here x counts the number of horizontal steps and y counts the number of up steps (see Theorem 1 in [1]) or [2, 7]).

The series expansion, $B(x, y)$ begins

$$x(y + y^2 + y^3 + y^4) + x^2(y + 3y^2 + 5y^3 + 7y^4) + x^3(y + 6y^2 + 16y^3 + 31y^4) \\ + x^4(y + \mathbf{10}y^2 + 40y^3 + 105y^4 + 219y^4).$$

The bold coefficient of x^4y^2 is illustrated below with the full set of 10 bargraphs with 4 horizontal steps and 2 vertical up steps.

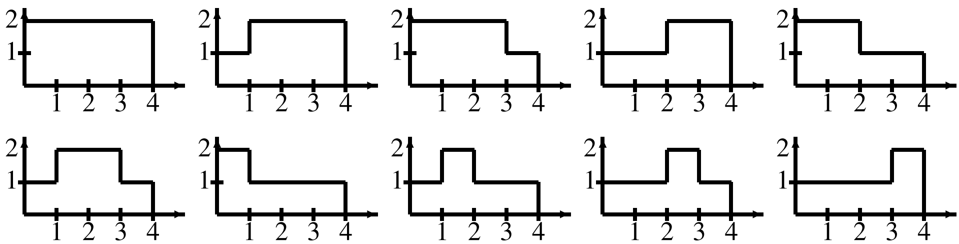

Figure 3: The 10 graphs with 4 horizontal steps and 2 vertical up steps

In [1, 2] the authors found an asymptotic expression for $B(z, z)$, where z marks the semi-perimeter of the bargraphs. This is known as the generating function for the isotropic case. The dominant singularity ρ is the positive root of

$$D := 1 - 4z + 2z^2 + z^4 = 0, \tag{1.2}$$

given by

$$\rho = \frac{1}{3}\left(-1 - \frac{4 \times 2^{2/3}}{(13 + 3\sqrt{33})^{1/3}} + \left(2(13 + 3\sqrt{33})\right)^{1/3}\right) = 0.295598\cdots. \tag{1.3}$$

We have $B(z, z) \sim -\frac{\sqrt{1-\rho-\rho^3}}{\sqrt{\rho}}\,(1 - \frac{z}{\rho})^{1/2}$ as $z \to \rho$. Hence

$$[z^n]B(z, z) \sim \frac{\sqrt{1 - \rho - \rho^3}}{2\sqrt{\pi\,\rho\,n^3}}\,\rho^{-n}. \tag{1.4}$$

The following definitions will be used:

A *level* in a bargraph is a maximal sequence of two or more adjacent horizontal steps denoted by h^r where $r \geq 2$. It is preceded and followed by either an up step or a down step. The *length of the level* is the number r of horizontal steps in the sequence. The *height* of a level is the y-coordinate of the horizontal steps in the sequence.

Thus, the graph in Figure 1 has four levels, three of length 2 and one of length 3.

In all the generating functions of the following sections, the horizontal steps are counted by x, the vertical up steps are counted by y and the parameter that is under investigation by w. In each section, we use $G(x, y, w)$ or $F(x, y, w)$ for the generating function where the definition of G or F applies only to the section under consideration.

2 Total number of levels

2.1 Generating function for the number of levels

A level is a sequence of two or more adjacent horizontal steps as defined in the previous section. Let $F(x, y, w)$ be the generating function where w marks the total number of levels. Using the wasp-waist decomposition in Figure 2, we have

$$F := F(x, y, w) = \underbrace{xy}_{1} + \underbrace{F_2}_{2} + \underbrace{yF}_{3} + \underbrace{xyF}_{4} + \underbrace{FF_2}_{5} \tag{2.1}$$

The numbers below the terms refer to the cases in the wasp-waist decomposition. This will be done throughout the paper. The generating function $F_2 := F_2(x, y, w)$ is the analogous function restricted to case 2. We use the following symbolic decomposition for F_2

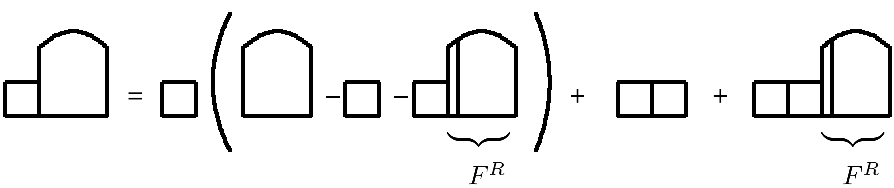

Figure 4: Decomposition for F_2

where F^R is the generating function for bargraphs in which the first column is of height 2 or more. The function F^R is easily obtained by considering all bargraphs except those starting with a column of height one. Thus

$$F^R = F - xy - F_2. \tag{2.2}$$

From Figure 4, we get

$$F_2 = x(F - xy - xF^R) + wx^2y + wx^2F^R. \tag{2.3}$$

So, combining equations (2.1), (2.2) and (2.3), we find

$$F = \frac{1}{2(x - x^2 + wx^2)} \Big(1 - x - y - xy + 2x^2y - 2wx^2y - \sqrt{4(-x + x^2 - wx^2)(xy - x^2y + wx^2y) + (1 - x - y - xy + 2x^2y - 2wx^2y)^2}\Big). \tag{2.4}$$

In order to find the generating function for the total number of levels in bargraphs, we differentiate F with respect to w and then put $w = 1$ to obtain

$$F_{Levels} := \left.\frac{\partial F}{\partial w}\right|_{w=1} = \frac{(1-x)(1-y)\left(1 - x - y - xy - \sqrt{(1-x-y-xy)^2 - 4x^2y}\right)}{2\sqrt{(1-x-y-xy)^2 - 4x^2y}},$$

where z marks the semiperimeter.

The series expansion begins

$$x^2(y + y^2 + y^3 + y^4) + x^3(y + 5y^2 + 9y^3 + 13y^4) + x^4(y + \mathbf{12}y^2 + 38y^3 + 79y^4).$$

There are in total 12 levels in our example in Figure 3. This is shown in bold in the series expansion.

2.2 Asymptotics in the isotropic case

We consider bargraphs with respect to the semiperimeter by substituting z for x and y in F to obtain

$$F_{Levels}(z, z) = \frac{(1-z)^2(1 - 2z - z^2 - \sqrt{1 - 4z + 2z^2 + z^4})}{2\sqrt{1 - 4z + 2z^2 + z^4}}.$$

In order to compute the asymptotics for the coefficients, we use singularity analysis as described in [7]. Let ρ be as in (1.2) and (1.3). We find that as $z \to \rho$

$$F_{Levels} \sim \frac{1 - 4\rho + 4\rho^2 - \rho^4}{4\sqrt{\rho(1 - \rho - \rho^3)}\sqrt{1 - \frac{z}{\rho}}}.$$

By singularity analysis we have

$$[z^n]F_{Levels} \sim \frac{1 - 4\rho + 4\rho^2 - \rho^4}{4\sqrt{\pi\, n}\sqrt{\rho\,(1 - \rho - \rho^3)}}\, \rho^{-n}.$$

Then after dividing by the asymptotic expression for the total number of bargraphs found in (1.4), we get the following result:

Theorem 2.1. *The average number of levels in bargraphs of semiperimeter n is asymptotic to*

$$\frac{1 - 4\rho + 4\rho^2 - \rho^4}{2(1 - \rho - \rho^3)}\, n = C\, n,$$

as $n \to \infty$ where $C = 0.117516\cdots$.

3 Bargraphs with no levels

3.1 Generating function for the number of graphs with no level

Because we require it later, we begin by enumerating a special class of bargraphs, namely one in which an adjacent sequence of horizontal steps does not occur (i.e. the only sequences of horizontals are single). This is denoted by $F_0 := F(x, y, 0)$ where F is the generating function (2.4) from the previous section.

We use the wasp-waist decomposition in Figure 2 to obtain

$$F_0 = \underbrace{xy}_{1} + \underbrace{F_{0,2}}_{2} + \underbrace{yF_0}_{3} + \underbrace{yF_0x}_{4} + \underbrace{F_0F_{0,2}}_{5}. \tag{3.1}$$

Case 2 is explained below in Figure 5.

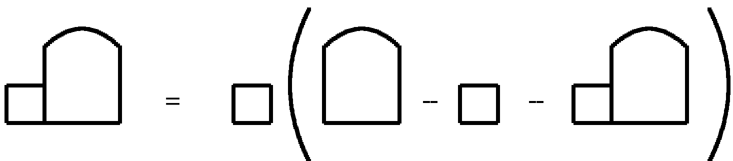

Figure 5: Explanation for case 2, decomposition of $F_{0,2}$

Thus

$$F_{0,2} = x\big(F_0 - xy - F_{0,2}\big),$$

which leads to

$$F_{0,2} = \frac{x\big(F_0 - xy\big)}{1 + x}. \tag{3.2}$$

The exclusions in case 2 are because we are not allowing adjacent horizontal steps.

Hence, from (3.1) and (3.2), we have:

$$F_0 = \underbrace{xy}_{1} + \underbrace{\frac{x(F_0 - xy)}{1 + x}}_{2} + \underbrace{yF_0}_{3} + \underbrace{yF_0x}_{4} + \underbrace{\frac{F_0x(F_0 - xy)}{1 + x}}_{5}.$$

Solving this for F_0, we obtain

$$F_0 = \frac{1 - y - 2xy - \sqrt{1 - y}\,\sqrt{1 - y - 4xy - 4x^2y}}{2x}. \tag{3.3}$$

The series expansion for F_0 begins

$$x(y + y^2 + y^3 + y^4) + x^2(2y^2 + 4y^3 + 6y^4) \\ + x^3(y^2 + 7y^3 + 18y^4) + x^4(6y^3 + 32y^4 + 92y^5).$$

Our example in Figure 3, shows that indeed there are no bargraphs having 4 horizontal and 2 up steps and no levels, which is confirmed by the lack of x^4y^2 term.

3.2 Asymptotics in the isotropic case

As before we substitute z for x and y in F_0 and obtain

$$F_0(z,z) = \frac{1 - z - 2z^2 - \sqrt{1-z}\sqrt{1 - z - 4z^2 - 4z^3}}{2z}.$$

Let τ be the dominant root of $1 - z - 4z^2 - 4z^3 = 0$, its value is

$$\tau = \frac{1}{12}\left(-4 + (224 - 24\sqrt{87})^{1/3} + 2(28 + 3\sqrt{87})^{1/3}\right) = 0.34781\cdots.$$

Using singularity analysis we have as $z \to \tau$

$$F_0(z,z) \sim -\frac{\sqrt{1-\tau}\sqrt{\tau(1 + 8\tau + 12\tau^2)}\sqrt{1 - \frac{z}{\tau}}}{2\tau}.$$

Extracting coefficients will yield the asymptotic number of bargraphs with no levels.

$$[z^n]F_0(z,z) \sim \frac{\sqrt{1-\tau}\sqrt{\tau(1 + 8\tau + 12\tau^2)}}{4\sqrt{\pi\, n^3}}\,\tau^{-n},$$

as $n \to \infty$.

For $n = 100$, there are 3.20775×10^{42} bargraphs whereas the asymptotics give 3.24376×10^{42}.

4 Horizontal position of the first level

4.1 Generating function for the mean

Now we derive a generating function G_x for bargraphs in which the leftmost x-coordinate of the first level is counted by w. In the case where the bargraph has no level, we define the horizontal position to be 0. In Figure 6, the start of the first level is the point with coordinates $(2, 5)$ and therefore the x-coordinate of the start of the first level here is 2.

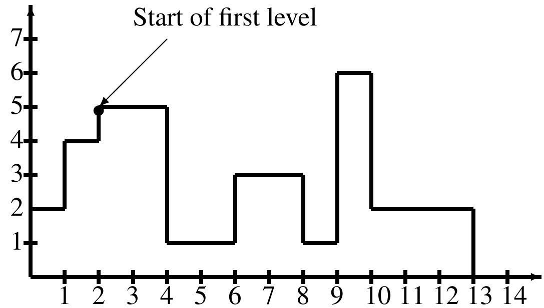

Figure 6: Horizontal position of the start of the first level

By the wasp-waist decomposition we have

$$G_x = \underbrace{xy}_{1} + \underbrace{F_2}_{2} + \underbrace{yG_x}_{3} + \underbrace{yG_x x}_{4} + \underbrace{F_5}_{5} \tag{4.1}$$

To calculate the generating function for case 2, we use Figure 7 below. The part labelled L in Figure 7 indicates a bargraph with at least one level.

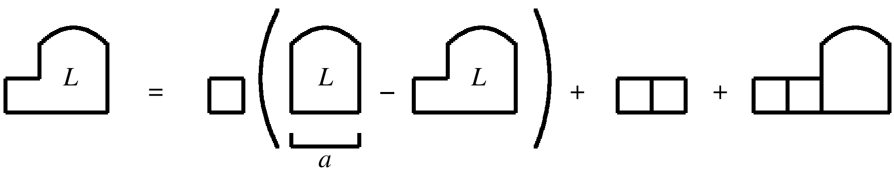

Figure 7: Decomposition for $F_{L,2}$

Note that $F_{L,2}$ is the generating function for case 2, (paths which have at least 1 level).

The generating function for the graph labelled "a" in Figure 7 is therefore $G_x - F_0$, since F_0 is the generating function for graphs with no levels from Section 3.

Thus, using Figure 7, we have:

$$F_{L,2} := F_{L,2}(x, y, w) = wx\big(G_x - F_0 - F_{L,2}\big) + x^2y + x^2B$$

where B is the generating function for all bargraphs from equation (1.1). Hence,

$$F_{L,2} = \frac{wxG_x - wxF_0 + x^2y + x^2B}{1 + wx}, \tag{4.2}$$

and from (3.2)

$$F_{0.2} = \frac{x}{1+x}(F_0 - xy) = \frac{x\left(-xy + \frac{1-y-2xy+\sqrt{1-y}\sqrt{1-y-4xy-4x^2y}}{2x}\right)}{1+x}.$$

So, for case 2

$$F_2 = F_{L,2} + F_{0,2}.$$

Thus finally, the decomposition for case 5 requires Figure 8 below:

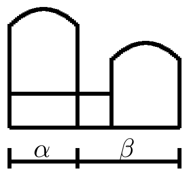

Figure 8: Case 5

For case 5, we have the concatenation of two bargraphs labelled α and β. There are three cases depending on whether the graphs α and β have levels or not.

i. Graph α has levels with generating function $y(G_x - F_0)$, in which case the generating function for β is $\frac{xB}{y}$.

ii. Neither graph has levels, thus the generating function is $F_0F_{0,2}$ where $F_{0,2}$ is as in (3.2) or

iii. Graph α has no levels but graph β has, so the generating function is $F_0(xw, y)F_2$ where $F_0(w) := F_0(xw, y)$ indicates that x has been replaced by xw in $F_0(x, y)$.

Thus

$$G_x = \underbrace{xy}_{1} + \underbrace{F_2}_{2} + \underbrace{yG_x}_{3} + \underbrace{yG_xx}_{4} + \underbrace{\big((G_x - F_0)xB + F_0F_{0,2} + F_0(xw, y)F_2\big)}_{5}$$

where in all but one case, the parameters have been omitted.

We solve for G_x, leading to

$$G_x(x, y, w) = \frac{-\frac{Bx^2F_0}{wx+1} + BxF_0 - \frac{Bx^2}{wx+1} - F_0F_{0,2} - \frac{x^2yF_0(w)}{wx+1} + \frac{wx(F_0)^2}{wx+1} + \frac{F_0wx}{wx+1} - F_{0,2} - \frac{x^2y}{wx+1} - xy}{Bx + \frac{wxF_0(w)}{wx+1} + \frac{wx}{wx+1} + xy + y - 1} \tag{4.3}$$

where

$$F_0(w) = \frac{1 - y - 2wxy - \sqrt{1-y}\sqrt{1 - y - 4wxy - 4w^2x^2y}}{2wx}.$$

Remark: We note that from (3.3) $F_0(w)|_{w=1} = F_0$.

Now, in order to find the mean horizontal position, we calculate:

$$\begin{aligned}
&\left.\frac{\partial G_x}{\partial w}\right|_{w=1} \\
&= \frac{x}{\left(Bx(x+1)+F_0x+(x+1)^2y-1\right)^2} \\
&\times \{F_0((x+1)F_0'+F_0+1)(F_{0,2}+xy+y-1)+F_{0,2}((x+1)F_0'+F_0+1) \\
&+B^2x^2(x(-F_0'+F_0+1)-F_0')+Bx(2x^2y(-F_0'+F_0+1)-(y-1)F_0' \\
&+x(F_0^2-3yF_0'+F_0'+F_0y+F_0+y))+xy(x^2y(-F_0'+F_0+1)-yF_0'+2F_0' \\
&+x(F_0^2-2yF_0'+2F_0'+F_0y+F_0+y)+F_0+1)\Big\}
\end{aligned}$$

where

$$\begin{aligned}
F_0' &= \left.\frac{\partial F_0(w)}{\partial w}\right|_{w=1} \\
&= \frac{y\left(-2x\sqrt{y-1}+\sqrt{(2x+1)^2y-1}-\sqrt{y-1}\right)-\sqrt{(2x+1)^2y-1}+\sqrt{y-1}}{2x\sqrt{(2x+1)^2y-1}}.
\end{aligned} \tag{4.4}$$

The series expansion of $\left.\frac{\partial G_x}{\partial w}\right|_{w=1}$ begins

$$x^3\left(2y^2+4y^3+6y^4\right)+x^4\left(\mathbf{5}y^2+25y^3+60y^4\right).$$

In our example in Figure 3, the sum of the horizontal positions of the first levels is 5.

4.2 Asymptotics in the isotropic case

Using singularity analysis and computer algebra we find that

$$\left.\frac{\partial G_x}{\partial w}\right|_{w=1} \sim -2\,c_1(\rho)\sqrt{\rho\,(1-\rho-\rho^3)}\left(1-\frac{z}{\rho}\right)^{1/2}$$

where ρ is as in (1.3) and

$$\begin{aligned}
c_1(\rho) = &\frac{1-\rho}{\left((-1+\rho)\rho^2+\sqrt{-1+\rho}\sqrt{Y(\rho)}\right)^3\sqrt{Y(\rho)}} \\
&\times\Big(\sqrt{-1+\rho}(1-\rho-12\rho^2-4\rho^3+13\rho^4+27\rho^5+18\rho^6+18\rho^7+4\rho^8) \\
&\qquad +(-1+\rho+4\rho^2+8\rho^3-5\rho^4+\rho^5-6\rho^6-2\rho^7)\sqrt{Y(\rho)}\Big)
\end{aligned}$$

as $z \to \rho$ and $Y(\rho) = -1+\rho+4\rho^2+4\rho^3$.

The coefficient is

$$[z^n]\left.\frac{\partial G_x}{\partial w}\right|_{w=1} \sim \frac{c_1(\rho)\sqrt{\rho\,(1-\rho-\rho^3)}}{\sqrt{\pi\,n^3}}\,\rho^{-n}.$$

After dividing by the asymptotic number of bargraphs we get

Theorem 4.1. *The average horizontal position of the first level in bargraphs is asymptotic to the constant*

$$2\,\rho\, c_1(\rho) = 2.38298, \quad \textit{as } n \to \infty.$$

For $n = 200$, the exact average is $2.35787\cdots$.

5 Height of the first level

5.1 Generating function for the mean

Let $G_y(x, y, w)$ be the generating function for the y-coordinate of the first level for bargraphs where w marks this coordinate. If there are no levels then there is no w, so we have a contribution to w^0. As in the previous section, the first level in Figure 9 begins at the point $(2, 5)$, with y-coordinate 5.

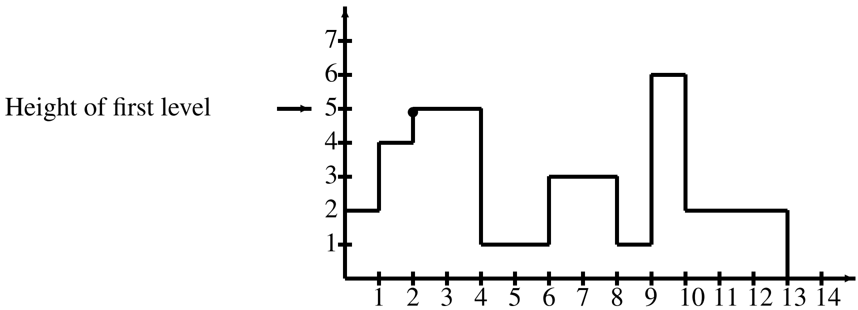

Figure 9: Height of the first level

Using the wasp-waist decomposition, this yields:

$$G_y = \underbrace{xy}_{1} + \underbrace{F_2}_{2} + \underbrace{F_3}_{3} + \underbrace{xF_3}_{4} + \underbrace{F_5}_{5}. \tag{5.1}$$

Considering case 2 separately, we have for F_2:

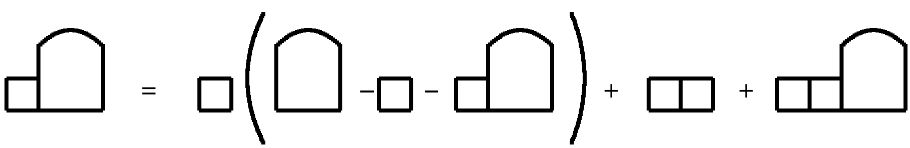

Figure 10: Decomposition for F_2

Thus

$$F_2 = x(G_y - xy - F_2) + x^2yw + x^2wB.$$

So

$$F_2 = \frac{x(G_y - xy + xyw + xwB)}{1+x} \tag{5.2}$$

and

$$F_3 = yw\big(G_y - F_0\big) + yF_0 \tag{5.3}$$

where the first and second terms distinguish between the cases where there are levels (which are therefore multiplied by w) and no levels.

Also separately, for the last case F_5 we can use Figure 8. If α has levels, then the generating functions for α and β are $w\big(G_y - F_0\big)$ and $xB(x, y)$ respectively. On the other hand, if α has no levels, the generating functions are yF_0 and F_2.

Thus

$$F_5 = w(G_y - F_0)xB(x, y) + yF_0F_2. \tag{5.4}$$

Substituting (5.2), (5.3), and (5.4) in (5.1) and solving for G_y, we obtain

$$G_y = \frac{T}{Bwx + \frac{F_0x}{x+1} + w(x+1)y + \frac{x}{x+1} - 1} \tag{5.5}$$

where

$$\begin{aligned} T = &-\frac{BF_0wx^2}{x+1} + BF_0wx - \frac{Bwx^2}{x+1} - \frac{F_0wx^2y}{x+1} + F_0w(x+1)y \\ &+ \frac{F_0x^2y}{x+1} - F_0(x+1)y - \frac{wx^2y}{x+1} + \frac{x^2y}{x+1} - xy. \end{aligned}$$

The generating function for the sum of the heights of the first levels is obtained from the derivative of G_y with respect to w and then setting $w = 1$.

Using the following substitutions

$$\begin{cases} X(x, y) = -1 + (1+2x)^2y, \\ Y(x, y) = (-1+y)(-1+x^2(-1+y)+y+2x(1+y)), \end{cases} \tag{5.6}$$

we have

$$\begin{aligned} &\left.\frac{\partial G_y}{\partial w}\right|_{w=1} \\ &= \frac{\left(-1+x+y-xy+\sqrt{Y(x,y)}\right)}{\left(x^2(1-y)+x\sqrt{Y(x,y)}-\sqrt{X(x,y)}\sqrt{y-1}+\sqrt{Y(x,y)}\right)^2} \\ &\times \left(4x^2(y-1)y + x\left(-2\sqrt{X(x,y)}\sqrt{y-1}y + \sqrt{X(x,y)}\sqrt{y-1} + 4y^2 - 3y - 1\right)\right. \\ &\qquad \left. + y\left(-\sqrt{X(x,y)}\sqrt{y-1} + y - 1\right)\right). \end{aligned} \tag{5.7}$$

The series expansion of $\left.\frac{\partial G_y}{\partial w}\right|_{w=1}$ begins

$$x^2\left(y + 2y^2 + 3y^3 + 4y^4\right) + x^3\left(y + 8y^2 + 21y^3 + 40y^4\right) + x^4\left(y + \mathbf{15}y^2 + 71y^3 + 198y^4\right).$$

Figure 3 illustrates that the sum of the heights of the first levels is 15 as shown in bold above.

5.2 Asymptotics in the isotropic case

Substituting z for both x and y in the above equation (5.7) and using $X(z,z) := X(z) = -1 + z + 4z^2 + 4z^3$ and $Y(z,z) := Y(z) = 1 - 4z + 2z^2 + z^4$, we obtain

$$\begin{aligned}
&\left.\frac{\partial G_y}{\partial w}\right|_{w=1} \\
&= \frac{\left(-1 + 2z - z^2 + \sqrt{Y(z)}\right)}{\left((1-z)z^2 - \sqrt{z-1}\sqrt{X(z)} + \sqrt{Y(z)} + z\sqrt{Y(z)}\right)^2} \\
&\quad \times \Big[4z^3(z-1) + z\left(-1 + z - \sqrt{z-1}\sqrt{X(z)}\right) \\
&\qquad + z\left(-1 - 3z + 4z^2 + \sqrt{z-1}\sqrt{X(z)} - 2z\sqrt{z-1}\sqrt{X(z)}\right)\Big] \\
&\sim -2\,c_2(\rho)\sqrt{\rho(1-\rho-\rho^3)}\left(1 - \frac{z}{\rho}\right)^{1/2},
\end{aligned}$$

by using computer algebra as $z \to \rho$, where

$$c_2(\rho) = 2\rho\frac{\left(-2 + 2\rho + \rho^2 - \rho^3 + \sqrt{-1+\rho}\sqrt{X(\rho)}\right)\left(1 + \rho - 2\rho^3 + \rho\sqrt{-1+\rho}\sqrt{X(\rho)}\right)}{\left(\rho^2(-1+\rho) + \sqrt{-1+\rho}\sqrt{X(\rho)}\right)^3}.$$

Hence

$$[z^n]\left.\frac{\partial G_y}{\partial w}\right|_{w=1} \sim \frac{c_2(\rho)\sqrt{\rho(1-\rho-\rho^3)}}{\sqrt{\pi\, n^3}}\rho^{-n} \quad \text{as } n \to \infty$$

where

Thus after dividing by the asymptotic number of bargraphs we obtain

Theorem 5.1. *The average height of the first level in bargraphs is asymptotic to the constant*

$$2\,\rho\, c_2(\rho) \approx 6.15883\cdots, \quad \textit{as } n \to \infty.$$

For $n = 300$, the exact average is $6.00066\cdots$.

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