A note on automorphisms of halved Cayley graphs of Coxeter systems

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Abstract

We consider the halved Cayley graphs of Coxeter systems and show that every automorphism of such a graph can be uniquely extended to an automorphism of the corresponding Cayley graph.

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1   Introduction

Let W be a group generated by a finite set S whose elements are involutions. For distinct $s, s' \in S$ we denote by $m(s, s')$ the order of the element ss'. Then $m(s, s') = m(s', s)$ and the condition $m(s, s') = 2$ is equivalent to the commuting of s and s'. Suppose that (W, S) is a *Coxeter system*, i.e W is the quotient of the free group over S by the normal subgroup generated by all $(ss')^{m(s,s')}$ with $m(s, s') < \infty$.

The *Cayley graph* $\mathrm{C}(W, S)$ is the graph whose vertex set is W and $w, v \in W$ are adjacent vertices of the graph if $v = sw$ for a certain $s \in S$ (since S consists of involutions, the adjacency relation is symmetric). For the dihedral Coxeter system $\mathsf{I}_2(n)$ this graph is the $(2n)$-cycle and we get an infinite path if $n = \infty$. The Cayley graph of A_n is the *permutohedron* [6]. See [1, Figures 3.3] for the Cayley graph of H_3. Also, $\mathrm{C}(W, S)$ can be identified with the graph whose vertices are maximal simplices of the Coxeter complex $\Sigma(W, S)$ and two maximal simplices are adjacent vertices if their intersection consists of $|S| - 1$ elements [5].

In almost all cases, the automorphism group of $\mathrm{C}(W, S)$ is known. For every $w \in W$ the right multiplication $R_w : v \to vw$ is an automorphism of the graph. If the diagram of our Coxeter system does not contain adjacent edges labeled by ∞ then the automorphism

group of $C(W, S)$ is the semidirect product of W and the automorphism group of the diagram [1, Corollary 3.2.6].

The *length* $l(w)$ of an element $w \in W$ is the smallest number m such that w has an expression

$$w = s_1 \ldots s_m, \quad s_1, \ldots, s_m \in S. \tag{1.1}$$

It is clear that $l(w)$ is the distance between 1 and w in the Cayley graph. Since every right multiplication is an automorphism of the graph, the distance $d(w, v)$ between $w, v \in W$ is equal to $l(wv^{-1}) = l(vw^{-1})$. Recall the following remarkable property of Coxeter systems called the *exchange condition*: if (1.1) is a *reduced* expression, i.e. $l(w) = m$, then for every $s \in S$ satisfying $l(sw) \le m$ there exists $k \in \{1, \ldots, m\}$ such that

$$sw = s_1 \ldots \hat{s}_k \ldots s_m$$

(the symbol $\hat{}$ means that the corresponding term is omitted).

The group W can be presented as the disjoint union of the following subsets

$$W_1 := \{\, w \in W : l(w) \text{ is odd} \,\} \text{ and } W_2 := \{\, w \in W : l(w) \text{ is even} \,\}.$$

Using the exchange condition we establish the following:

- the distance between any two elements of W_i, $i \in \{1, 2\}$ is even,

- the distance between every element of W_1 and every element of W_2 is odd.

Note that W_2 is a subgroup of W. Consider the graph Γ_i, $i \in \{1, 2\}$ whose vertex set is W_i and two elements of W_i are adjacent vertices if the distance between them (in the Cayley graph) is equal to 2. The right multiplication R_w preserves both W_i in the case when $w \in W_2$. If $w \in W_1$ then R_w transfers W_1 to W_2 and conversely. The latter implies that Γ_1 and Γ_2 are isomorphic.

The main result of the note is the following.

Theorem 1.1. *If $|S| \ge 5$ then every isomorphism between Γ_i and Γ_j $i, j \in \{1, 2\}$ can be uniquely extended to an automorphism of the Cayley graph.*

The same fails if $|S| \in \{3, 4\}$ (Remark 3.2), but the statement holds for $|S| = 2$ (the Cayley graph is a cycle or an infinite path) and the case $|S| = 1$ is trivial.

Theorem 1.1 easily follows from the description of maximal 2-cliques of Cayley graph, i.e. maximal cliques of the halved Cayley graph, given in Lemma 2.5.

If S consists of n mutually commuting involutions then $C(W, S)$ is the n-dimensional hypercube graph H_n and every Γ_i is the half-cube graph $\frac{1}{2}H_n$. So, it is natural to ask which properties of the hypercube and half-cube graphs can be extended to $C(W, S)$ and Γ_i, respectively?

2   Maximal 2-cliques

Two vertices in a graph are said to be 2-*adjacent* if the distance between them is equal to 2. Recall that a *clique* is a subset of the vertex set, where any two distinct vertices are adjacent. We say that a subset in the vertex set is a 2-*clique* if any two distinct elements of this subset are 2-adjacent vertices.

Consider examples of 2-cliques in $C(W, S)$.

Example 2.1 (First type). Any two distinct elements of S are 2-adjacent and S is a 2-clique. Since the right multiplication R_w is an automorphism of the Cayley graph, Sw is a 2-clique for every $w \in W$.

Remark 2.2. Suppose that $S = Sw$. Then for any $s_1, s_2 \in S$ there exist $s_1', s_2' \in S$ such that $s_1 = s_1'w$ and $s_2 = s_2'w$. If $w \neq 1$ then $s_1 \neq s_1'$ and $s_2 \neq s_2'$. We have

$$s_1's_1 = w = s_2's_2 \quad \text{and} \quad s_2's_1's_1 = s_2$$

Since W cannot be generated by a proper subset of S, the latter means that $s_2 = s_1'$. Therefore, $S = \{s_1, s_2\}$ and $s_1s_2 = s_2s_1$. So, the equality $Sw = Sw'$ implies that $w = w'$ except the case when our Coxeter system is $I_2(2)$.

Example 2.3 (Second type). Let s, s', s'' be three distinct mutually commuting elements of S. Then $ss's''$ is 2-adjacent to s, s', s'' and $\{sw, s'w, s''w, ss's''w\}$ is a 2-clique for every $w \in W$.

Example 2.4 (Third type). Suppose that $s, s' \in S$ and $m(s, s') = 3$. Then $ss's = s'ss'$ and we denote this element by $w(s, s')$. It is 2-adjacent to s, s' and for every $w \in W$ the set $\{sw, s'w, w(s, s')w\}$ is a 2-clique.

Lemma 2.5. *Every maximal 2-clique of* $C(W, S)$ *is one of the 2-cliques described above.*

Remark 2.6. The n-dimensional hypercube graph contains only maximal 2-cliques of the first and second types if $n \geq 4$ [3]. In the case when $n = 3$, there are precisely two maximal 2-cliques of the second type and 2-cliques of the first type are not maximal.

To prove Lemma 2.5 we use the following properties of Coxeter systems:

(P1) for every $w \in W$ there is a subset $S_w \subset S$ such that every reduced expression of w is formed by all elements of S_w,

(P2) the group W cannot be generated by a proper subset of S.

Lemma 2.7. *If* $u \in W \setminus S$ *is 2-adjacent to three distinct* $s, s', s'' \in S$ *then* s, s', s'' *are mutually commuting and* $u = ss's''$.

Proof. Since u is 2-adjacent to s, s', s'' and $u \notin S$, there are three reduced expressions

$$u = s_1 s_2 s, \quad u = s_1' s_2' s', \quad u = s_1'' s_2'' s'',$$

where $s_1, s_2, s_1', s_2', s_1'', s_2'' \in S$. By (P1), we have $S_u = \{s, s', s''\}$ and

$$\{s_1, s_2\} = \{s', s''\}, \quad \{s_1', s_2'\} = \{s, s''\}, \quad \{s_1'', s_2''\} = \{s, s'\}.$$

Thus there are the following possibilities for the first and second expressions:

(1) $u = s''s's = s''ss'$,

(2) $u = s''s's = ss''s'$,

(3) $u = s's''s = s''ss'$,

(4) $u = s's''s = ss''s'$.

Case (1). The involutions s, s' are commuting and the third expression is

$$u = ss's'' = s'ss''. \tag{2.1}$$

Then $s''s's = u = s'ss''$ and $s's''s's = ss''$. We apply the exchange condition to $w = s''s's$ and get the following three possibilities:

- $s's = ss''$,
- $s''s = ss''$,
- $s''s' = ss''$.

The first and third contradict (P2). So, s and s'' are commuting. Similarly, the equality $s''s's = u = ss's''$ shows that $ss''s's = s's''$. Using the above arguments we establish that s' and s'' are commuting.

Case (2). The equality $s''s's = ss's'$ implies that $s's = s''ss''s'$. As in the previous case, we show that s and s' are commuting. Then the third expression is (2.1) which implies that $ss''s' = u = ss's''$ and s', s'' are commuting. The equality

$$s''ss' = s''s's = u = ss''s'$$

guarantees that s and s'' are commuting.

Case (3). We have $s's''s = s''ss'$ and $s''s's''s = ss'$. As above, this means that s, s' are commuting and the third expression is (2.1). Then $s's''s = u = s'ss''$ and s, s'' are commuting. The equality

$$s's''s = u = s''ss' = s''s's$$

shows that s' and s'' are commuting.

Case (4). Since $s's''s = ss''s'$, we have $s''s = s'ss''s'$ and $ss's''s = s''s'$. By the standard arguments, s'' is commuting with both s and s'. Then

$$s''ss' = ss''s' = u = s's''s = s''s's$$

which implies that s and s' are commuting.

<div align="right">□</div>

Remark 2.8. If s, s', s'' are distinct elements of S then each of the equalities

$$s''s's = ss''s', \quad s's''s = s''ss', \quad s's''s = ss''s'$$

implies that s and s' are commuting; moreover, the third equality guarantees that s, s', s'' are mutually commuting. See the cases (2)–(4) in the proof of Lemma 2.7.

Lemma 2.7 shows that for any three distinct mutually commuting $s, s', s'' \in S$ the 2-clique formed by s, s', s'' and $ss's''$ is maximal. Therefore, every 2-clique of the second type is maximal.

Lemma 2.9. *If $u \in W \setminus S$ is 2-adjacent to distinct $s, s' \in S$ then one of the following possibilities is realized:*

- $m(s, s') = 3$ *and* $u = w(s, s')$,

- s, s' *are commuting and* $u = s''s's$ *for a certain* $s'' \in S$.

Proof. Since u is 2-adjacent to s, s' and $u \notin S$, there are two reduced expressions

$$u = s_1 s_2 s \;\; \text{and} \;\; u = s_1' s_2' s',$$

where $s_1, s_2, s_1', s_2' \in S$. By (P1), we have $\{s, s_1, s_2\} = S_u = \{s', s_1', s_2'\}$. If $|S_u| = 2$ then $S_u = \{s, s'\}$ and $u = ss's = s'ss'$ which implies that $m(s, s') = 3$, i.e. the first possibility is realized.

If $|S_u| = 3$ then $S_u = \{s, s', s''\}$ and, as in the proof of Lemma 2.7, we have the following possibilities for the above expressions:

(1) $u = s''s's = s''ss'$,

(2) $u = s''s's = ss''s'$,

(3) $u = s's''s = s''ss'$,

(4) $u = s's''s = ss''s'$.

It is clear that s and s' are commuting in the case (1). By Remark 2.8, the same holds for the cases (2) – (4) and s, s', s'' are mutually commuting in the case (4). So, we get the second possibility. $\qquad\square$

By Lemma 2.9, for any $s, s' \in S$ satisfying $m(s, s') = 3$ the 2-clique formed by s, s' and $w(s, s')$ is maximal. Thus every 2-clique of the third type is maximal.

Proof of Lemma 2.5. Let C be a maximal 2-clique. For any distinct $u, u' \in C$ there exist $w \in W$ and $s, s' \in S$ such that $u = sw$ and $u' = s'w$. The maximal 2-clique Cw^{-1} contains s and s'. Thus we can suppose that C contains at least two distinct elements of S. Let s and s' be elements of S belonging to C. Suppose that $C \neq S$, i.e. there is $u \in C \setminus S$.

If there is a third element $s'' \in S$ contained in C then, by Lemma 2.7, s, s', s'' are mutually commuting and C is the 2-clique of the second type formed by s, s', s'' and $u = ss's''$. In the case when C contains precisely two elements of S, Lemma 2.9 shows that $m(s, s') = 3$ and $C = \{s, s', w(s, s')\}$ or s, s' are commuting and $u = s''s's$ for a certain $s'' \in S$. The latter means that the maximal 2-clique $Cs's$ contains s, s', s'', i.e. it coincides with S or $\{s, s', s'', ss's''\}$. Then C is a 2-clique of the first type or the second type. $\qquad\square$

3 Proof of Theorem 1.1

We consider the case when $i = j = 1$. Let $f : W_1 \to W_1$ be an automorphism of Γ_1. Then f preserves the family of maximal cliques of Γ_1. Every maximal clique of Γ_1 is a maximal 2-clique of $C(W, S)$ contained in W_1. By Lemma 2.5, there are precisely three types of such subsets. They contain $|S|$ vertices, 4 vertices and 3 vertices, respectively. The condition $|S| \geq 5$ guarantees that f preserves the types of maximal cliques.

If $w \in W_2$ then Sw is a maximal clique of Γ_1 and $f(Sw) = Sw'$ for a certain $w' \in W_2$. We set $f(w) := w'$ and get a bijective transformation of W.

If $w, v \in W$ are adjacent vertices of the Cayley graph then one of these vertices belongs to W_1 and the other is an element of W_2. Suppose that $v \in W_1$ and $w \in W_2$. Then $v \in Sw$

and $f(v) \in f(Sw) = Sf(w)$ which implies that $f(v)$ and $f(w)$ are adjacent vertices of the Cayley graph. The apply the same arguments to f^{-1} and establish that f is an automorphism of $\mathrm{C}(W, S)$.

The uniqueness of such extension follows from the fact that w is the unique vertex of the Cayley graph adjacent to all vertices from Sw (Remark 2.2).

Remark 3.1. A similar idea was exploited in [4, Section 4.8] for an alternative proof of Cooperstein–Kasikova–Shult's characterization of apartments in half-spin Grassmannians [2].

Remark 3.2. If $\mathrm{C}(W, S)$ is H_4 then there are automorphisms of $\Gamma_i = \frac{1}{2} H_4$ which change the types of 2-cliques contained in W_i. Such automorphisms are not extendable to automorphisms of H_4. Similarly, if (W, S) is the direct product of $\mathsf{I}_2(3)$ and the group spanned by an involution then every maximal 2-clique of $\mathrm{C}(W, S)$ is of the first or of the third type and there are automorphisms of Γ_i changing the types of 2-cliques contained in W_i.

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