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Exotic Behaviour of Infinite Hypermaps

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Abstract

This is a survey of infinite hypermaps, and of how they can be constructed by using examples and techniques from combinatorial group theory, with particular emphasis on phenomena which have no analogues for finite hypermaps.

Keywords: Hypermap, automorphism group, monodromy group.

Math. Subj. Class.: 05-02, 05C10, 20E06

1 Introduction

Hypermaps are natural generalisations of maps, representing surface embeddings of hypergraphs rather than graphs. In recent years, especially in the compact orientable case, they have assumed considerable importance in the theory of dessins d'enfants, where they correspond to algebraic curves defined over algebraic number fields. As in the case of maps, most of the theory has been developed in terms of finite hypermaps (those on compact surfaces), often using techniques from finite group theory such as the classification of finite simple groups, together with the classification of compact surfaces. By contrast, the group-theoretic and topological tools available in the infinite case are much weaker, and the only infinite hypermaps normally considered are the universal hypermaps of various types on the euclidean and hyperbolic planes (see §6). Although much of the basic theory of hypermaps extends without significant changes to the infinite case, there are a number of rather strange phenomena which can arise only for infinite hypermaps.

The basic idea is that a hypermap $\mathcal{H}$ (which we will generally assume to be oriented) can be regarded as a transitive permutation representation of a 2-generator group G, called the monodromy group of $\mathcal{H}$, and an orientably regular hypermap (one with maximum symmetry) can be regarded as the regular representation of G, which is then isomorphic to the automorphism group of $\mathcal{H}$. Any phenomenon which can happen inside a countable group can happen inside a 2-generator group, and hence there must be a hypermap exhibiting some

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analogous behaviour. More precisely, Higman, Neumann and Neumann [17] have shown that every countable group can be embedded in a 2-generator group, and hence in the automorphism group of an orientably regular hypermap. The aim here is to give some examples of 2-generator infinite groups with interesting (and occasionally surprising) properties, and where possible to describe the corresponding hypermaps.

For simplicity of exposition, this paper concentrates mainly on oriented hypermaps and orientation-preserving automorphisms, though these restrictions are removed in the last two sections. Similarly, many of the results are valid in the special case of maps; indeed, any hypermap can be regarded as a (bipartite) map on the same surface by means of the Walsh construction described in §2.

2 Oriented hypermaps

Cori introduced the theory of finite oriented hypermaps in [7]; he and Machì have given a detailed survey in [8], and connections between the algebraic, combinatorial and topological aspects of the theory are discussed by Corn and Singerman in [9]. Algebraically, an oriented hypermap $\mathcal{H}$ (finite or infinite) can be regarded as a 2-generator permutation group $G = \langle x, y\rangle$ acting transitively on a set Ω, or equivalently as a transitive permutation representation $\theta : F_2 \to \mathrm{Sym}\,\Omega$ of the free group F_2 of rank 2. The hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ are the cycles of x, y and $z := (xy)^{-1}$ on Ω, with incidence given by non-empty intersection. If $y^2 = 1$ then $\mathcal{H}$ is a map. The natural cyclic order within each cycle determines an orientation of the underlying surface associated with $\mathcal{H}$. The *monodromy group* $\mathrm{Mon}^+\mathcal{H}$ of $\mathcal{H}$ is the group $G = \theta(F_2)$ of permutations of Ω induced by F_2, and the (orientation-preserving) *automorphism group* $A = \mathrm{Aut}^+\mathcal{H}$ is the centraliser of G in $\mathrm{Sym}\,\Omega$. (We will introduce more general definitions of monodromy and automorphism groups, for unoriented hypermaps, in §8.)

We say that $\mathcal{H}$ is *orientably regular* (or *rotary*) if A acts transitively on Ω; this is equivalent to G acting regularly on Ω, in which case A and G can be regarded as the regular representations of G on itself by left and right multiplication, and hence $A \cong G$. In the case of an orientably regular hypermap, where we take $\Omega = G$, the hypervertices, hyperedges and hyperfaces correspond to the cosets of the subgroups $\langle x\rangle$, $\langle y\rangle$ and $\langle z\rangle$ in G, with orientation corresponding to multiplication by the corresponding generator, and incidence given by non-empty intersection.

There is a bijection between isomorphism classes of oriented hypermaps $\mathcal{H}$ and conjugacy classes of subgroups $H \le F_2$: these are the stabilisers of points in Ω, called the *hypermap subgroups*, and we have $A \cong N_{F_2}(H)/H$. In this correspondence, orientably regular hypermaps correspond to normal subgroups $H = \ker\theta$ of F_2, with $A \cong F_2/H \cong G$.

The *Walsh map* $\mathcal{W} = W(\mathcal{H})$ [32] is a useful way of visualising a hypermap $\mathcal{H}$. This is a bipartite map on the same surface as $\mathcal{H}$, with black and white vertices corresponding to the hypervertices and hyperedges of $\mathcal{H}$, edges (corresponding to elements of Ω) indicating incidences between them, and faces corresponding to the hyperfaces of $\mathcal{H}$. Then $\mathcal{H}$ is orientably regular if and only if the orientation- and colour-preserving automorphism group of $\mathcal{W}$ acts transitively on its edges. In this case, the map $\mathcal{W}$ is orientably regular if and only if there is an automorphism of G transposing x and y (and thus inducing a duality of $\mathcal{H}$), and $\mathcal{W}$ is regular if and only if there is also an automorphism inverting x and y (and thus reversing the orientations of $\mathcal{H}$ and $\mathcal{W}$).

A similar but more symmetric, and hence more elaborate, method of representing $\mathcal{H}$ is its *barycentric subdivision* $\mathcal{B} = B(\mathcal{H})$, a tripartite triangular map on the same surface. This has one vertex corresponding to each hypervertex, hyperedge and hyperface of $\mathcal{H}$, with incident pairs and triples of these joined by edges and faces of $\mathcal{B}$. Thus $\mathcal{B}$ is formed by stellating $\mathcal{W}$, putting a (red) vertex in each face of $\mathcal{W}$, and joining it by an edge to each black or white vertex incident with that face. Conversely, $\mathcal{W}$ can be reconstructed from $\mathcal{B}$ by deleting the red vertices and their incident edges.

3 Group-theoretic constructions

Here we briefly summarise two important group-theoretic constructions, namely free products and HNN extensions. For full details, see [25, Ch. IV].

Let H and K be groups, assumed to be disjoint, with presentations $H = \langle\, X \mid R\,\rangle$ and $K = \langle\, Y \mid S\,\rangle$, where X and Y are sets of generators, and R and S are sets of defining relations. The *free product* [25, §IV.1] of H and K is the group $G = H * K$ with presentation $\langle\, X \cup Y \mid R \cup S\,\rangle$; it is, up to isomorphism, independent of the choice of presentations of H and K.

Each element $g \in G$ can be written as a word

$$g_1 g_2 \ldots g_n \quad (n \geq 0) \tag{3.1}$$

where $g_i \in H \cup K$ for each i. A word (3.1) is *reduced* if $g_i \neq 1$ for each i, and successive terms g_i and g_{i+1} do not belong to the same factor H or K: such a pair could be replaced in (3.1) with their product in H or K, giving a representation of g by a shorter word, so it follows that each $g \in G$ can be represented by a reduced word, with the empty word (of length $n = 0$) representing the identity element $g = 1$. The normal form theorem for free products [25, Theorem IV.1.2] states that this reduced word is unique, and if it has length $n > 0$ then $g \neq 1$. It follows from this that H and K are embedded isomorphically as subgroups of G. The torsion theorem for free products [25, Theorem IV.1.6] states that the torsion elements (those of finite order) in G are the conjugates of those in H or K.

An *HNN extension* [17], [25, §IV.2] is constructed from a group H and an isomorphism $\phi : A \to B$ between two subgroups A and B of H; we define

$$G = \langle H, t \mid a^t = \phi(a) \text{ for all } a \in A\rangle,$$

meaning that $G = F/N$ where $F = H * \langle t\rangle$ is the free product of H and an infinite cyclic group $\langle t\rangle$, and N is the normal closure in F of the elements $a^t\phi(a)^{-1} = t^{-1}at\phi(a)^{-1}$ where $a \in A$. Equivalently, if H has a presentation $\langle\, X \mid R\,\rangle$, then G has a generating set $X \cup \{t\}$, and its defining relations consist of those in R together with the relations $a^t = \phi(a)$ for all $a \in A$.

Each element $g \in G$ can be written as a word

$$h_1 t^{e_1} h_2 t^{e_2} \ldots t^{e_{n-1}} h_n \quad (n \geq 0) \tag{3.2}$$

where $h_i \in H$ and $e_i = \pm 1$ for all i; the case $n = 0$ corresponds to the empty word, representing $g = 1$. A word (3.2) is *reduced* if it contains no subword $t^{-1}h_i t$ with $h_i \in A$ or $th_i t^{-1}$ with $h_i \in B$: such a subword could be replaced with $\phi(h_i)$ or $\phi^{-1}(h_i)$ respectively, leading to a represention of g by a shorter word, so every element is represented by a reduced word. The normal form theorem for HNN extensions [25, Theorem IV.2.1] states that this reduced word is unique, and if $n \geq 1$ then $g \neq 1$. Thus H and $\langle t\rangle$ are isomorphically embedded as subgroups of G, with conjugation by t inducing the isomorphism ϕ. The torsion theorem for HNN extensions [25, Theorem IV.2.4] states that the torsion elements of G are the conjugates in G of the torsion elements of H.

4 Counting hypermaps

For each $n \in \mathbf{N}$ there are only finitely many non-isomorphic oriented hypermaps $\mathcal{H}$ with $|\Omega| = n$; there is at least one such hypermap for each n (since the symmetric group S_n is a 2-generator group), so the total number of finite oriented hypermaps (up to isomorphisms) is $\aleph_0$. In fact, if instead we use the regular representation of the cyclic group C_n we see that the total number of finite orientably regular hypermaps is also $\aleph_0$. When we consider infinite hypermaps, however, we will see that the corresponding number is $2^{\aleph_0}$ in each case. To prove this, we first need some classic group-theoretic results. The proofs are outlined here for completeness.

The following result is due to G. Higman, B. H. Neumann and H. Neumann [17]:

Theorem 1. *Every countable group C can be embedded in a 2-generator group G, such that G has an element of finite order n if and only if C has such an element.*

Proof. In the free group $F_2 = \langle a, b\rangle$ of rank 2, the elements $a_i = b^{-i}ab^i$ $(i \geq 0)$ freely generate a free subgroup A. Similarly, if C is generated by $c_1, c_2, \ldots$ then in the free product $H = F_2 * C$ the elements $b_0 = b$ and $b_i = c_i a^{-i} b a^i$ $(i \geq 1)$ freely generate a free subgroup B (since their images in F_2 do, when we map C to 1). Using the isomorphism $\phi : A \to B$, $a_i \mapsto b_i$ we can therefore form an HNN extension

$$G = \langle H, t \mid a_i^t = b_i \ (i \geq 0)\rangle.$$

The group C is embedded in H and hence in G. The subgroup $\langle a, t\rangle$ of G generated by a $(= a_0)$ and t also contains

$$b = b_0 = a_0^t \quad \text{and} \quad c_i = b_i a^{-i} b^{-1} a^i = a_i^t a^{-i} b^{-1} a^i = (b^{-i}ab^i)^t a^{-i} b^{-1} a^i$$

for all $i \geq 1$; the elements a, b, c_i and t generate G, which is thus a 2-generator group. The assertion about elements of finite order follows from the torsion theorems for free products and HNN extensions. □

The first part of Theorem 1 immediately implies that every countable group can be embedded in the automorphism group of some orientably regular hypermap, thus justifying the claim made in §1. However, there is no *single* 2-generator group containing a copy of every countable group, and hence there is no *single* orientably regular hypermap with this property: a 2-generator group, beng countable, has only countably many 2-generator subgroups, whereas B. H. Neumann [28] has proved the following result:

Theorem 2. *There are $2^{\aleph_0}$ mutually non-isomorphic 2-generator groups.*

Proof. Let $C_P = \oplus_{p \in P} C_p$ where P is any set of prime numbers. This is a countable group (since the elements of this direct sum have finite support), so by Theorem 1 it is embedded in a 2-generator group G_P which has an element of prime order p if and only if C_P has one, that is, if and only if $p \in P$. The groups G_P, for distinct sets P, are therefore mutually non-isomorphic, and there are $2^{\aleph_0}$ of them since this is the number of sets P of prime numbers. □

Corollary 3. *The free group F_2 has $2^{\aleph_0}$ conjugacy classes of subgroups, including $2^{\aleph_0}$ normal subgroups.*

Proof. Each of the $2^{\aleph_0}$ 2-generator groups G_P described above determines a normal subgroup N_P of the free group F_2, with $G_P \cong F_2/N_P$, so we have at least $2^{\aleph_0}$ conjugacy classes of subgroups of the form $\{N_P\}$. On the other hand, being countable, F_2 has $2^{\aleph_0}$ subsets, so it has at most $2^{\aleph_0}$ subgroups and hence at most $2^{\aleph_0}$ conjugacy classes of subgroups. □

Corollary 4. *There are $2^{\aleph_0}$ mutually non-isomorphic oriented hypermaps, including $2^{\aleph_0}$ orientably regular hypermaps.*

Proof. There is one oriented hypermap for each conjugacy class of subgroups of F_2, with orientably regular hypermaps corresponding to normal subgroups. □

We can in fact prove a slightly stronger result than Corollary 4. As shown by James [19], the group of all operations on oriented hypermaps (such as dualities and reflection) can be identified with the outer automorphism group $\mathrm{Out}\,F_2 = \mathrm{Aut}\,F_2/\mathrm{Inn}\,F_2$ of F_2 acting on conjugacy classes of subgroups of F_2. Since this action preserves normality of subgroups, it preserves orientable regularity of hypermaps. Now $\mathrm{Out}\,F_2$ is a countable group (isomorphic to $GL_2(\mathbf{Z})$, in fact, see [25, Ch. I, Prop. 4.5]), so we have:

Corollary 5. *The group $\mathrm{Out}\,F_2$ of operations has $2^{\aleph_0}$ orbits on oriented hypermaps, including $2^{\aleph_0}$ orbits consisting of orientably regular hypermaps.*

In its action on normal subgroups N of F_2, $\mathrm{Out}\,F_2$ preserves their quotient groups F_2/N, so it preserves the automorphism groups of the corresponding orientably regular hypermaps. For example, for different sets P the normal subgroups N_P in the proof of Corollary 3 have mutually non-isomorphic quotient groups $G_P = F_2/N_P$, so the corresponding orientably regular hypermaps $\mathcal{H}$ lie in $2^{\aleph_0}$ distinct orbits $\mathcal{O}_P$ of $\mathrm{Out}\,F_2$, with $\mathrm{Aut}^+\mathcal{H} \cong G_P$ having an element of prime order p if and only if $p \in P$.

There is a connection here with another topic from combinatorial group theory, namely T-systems, introduced by B. H. Neumann and H. Neumann in [29]. If G is any group, then the orientably regular hypermaps $\mathcal{H}$ with $\mathrm{Aut}^+\mathcal{H} \cong G$ correspond to the normal subgroups N of F_2 with $F_2/N \cong G$. These are the kernels of the epimorphisms $F_2 \to G$, and two such epimorphisms have the same kernel (and thus correspond to isomorphic hypermaps) if and only if they differ by an automorphism of G. Thus these hypermaps $\mathcal{H}$ correspond to the orbits of $\mathrm{Aut}\,G$ on generating pairs for G. Now there is also a natural action of $\mathrm{Aut}\,F_2$ on epimorphisms $F_2 \to G$, and hence on generating pairs for G, permuting the orbits of $\mathrm{Aut}\,G$. A T-system of such pairs is an orbit of the permutation group generated by $\mathrm{Aut}\,F_2$ and $\mathrm{Aut}\,G$, so two hypermaps $\mathcal{H}$ are equivalent under hypermap operations if and only if they correspond to generating pairs in the same T-system for G.

The automorphism group G of any orientably regular hypermap, being a 2-generator group, is countable, so there are at most $\aleph_0$ epimorphisms $F_2 \to G$ and hence at most $\aleph_0$ kernels N of such epimorphisms, or equivalently orientably regular hypermaps $\mathcal{H}$ with automorphism group G. In some cases N and hence $\mathcal{H}$ are unique: for instance, if $G = \mathbf{Z}^2$ then N is the commutator subgroup F_2' of F_2, and $\mathcal{H}$ is the hypermap described later in Example 6. In some other cases there are $\aleph_0$ kernels N and hence hypermaps $\mathcal{H}$: for instance, if $G = \mathbf{Z}$ then these subgroups N are the inverse images of the cyclic subgroups $\langle(i,j)\rangle = \langle(-i,-j)\rangle$ of $\mathbf{Z}^2$ under the abelianising epimorphism $F_2 \to F_2/F_2' \cong \mathbf{Z}^2$, with i and j coprime integers; now $GL_2(\mathbf{Z})$ acts transitively on such vectors (i,j), so the corresponding hypermaps $\mathcal{H}$ are all equivalent under hypermap operations. By contrast, it follows from work of Dunwoody and Pietrowski [11] that if G is the group $\langle x, y \mid x^2 = y^3 \rangle$ of the trefoil knot, isomorphic to the three-string braid group $B_3 = \langle a, b \mid aba = bab \rangle$, the corresponding hypermaps form $\aleph_0$ orbits under operations (see also Example 9).

In this section we have proved that there are uncountably many orientably regular hypermaps. In §6 we will show that this remains true if we restrict attention to hypermaps of certain fixed types, and if we impose various algebraic conditions on their automorphism groups. These results are not constructive, in the sense that they do not provide explicit descriptions of the hypermaps or their groups, so first, in the next section, we will study a few specific examples of infinite orientably regular hypermaps.

5 Baumslag-Solitar groups and hypermaps

The *Baumslag-Solitar groups* [1] are given by the presentation

$$G = BS(p, q) = \langle x, y \mid (x^p)^y = x^q \rangle,$$

where p and q are non-zero integers. By inverting each side of the defining relation, if necessary, we may assume that $p \geq 1$.

These groups are examples of HNN extensions. In the notation of §3 we take H to be an infinite cyclic group $\langle x \rangle$, with A and B the subgroups generated by x^p and x^q, we take ϕ to be the isomorphism $A \to B$ defined by $x^p \mapsto x^q$, and we let y play the role of t. The resulting HNN extension G is $BS(p, q)$, in which x and y are elements of infinite order.

Let $\mathcal{H} = \mathcal{H}(p, q)$ be the orientably regular hypermap corresponding to (G, x, y), and let $\mathcal{W} = \mathcal{W}(p, q)$ be the corresponding Walsh map.

Example 6. Let $G = BS(1, 1)$. This is the simplest of the Baumslag-Solitar groups, and while it does not exhibit any particularly unusual behaviour, it does give rise to a number of interesting maps and hypermaps.

An equivalent presentation for this group is

$$G = \langle x, y \mid xy = yx \rangle,$$

so G is a free abelian group of rank 2. Each element $g \in G$ can be written uniquely in the form $g = x^i y^j$ where $i, j \in \mathbf{Z}$, giving an isomorphism $G \to \mathbf{Z}^2,\ g \mapsto (i, j)$. The Cayley graph C of G with respect to the generators x, y can be identified with the 1-skeleton of the standard square tessellation of $\mathbf{R}^2$, with edges labelled x and y pointing in the first and second coordinate directions (to the right and upwards). The hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ correspond to the horizontal, vertical and diagonal lines $x_2 = c$, $x_1 = c$ and $x_1 - x_2 = c$ ($c \in \mathbf{Z}$). (Thus the underlying surface $\mathcal{S}$ of $\mathcal{H}$ is *not* the euclidean plane on which we imagine C drawn.) Each hypervertex meets each hyperedge exactly once, so the Walsh map $\mathcal{W}$ is an embedding in $\mathcal{S}$ of the complete bipartite graph $K_{\aleph_0,\aleph_0}$ of countably infinite degree. This map is edge-transitive since $\mathcal{H}$ is orientably regular; in fact $\mathcal{W}$ is regular since there are automorphisms of G transposing and inverting x and y.

Every 2-generator abelian group is a quotient of G, so every orientably regular hypermap with an abelian orientation-preserving automorphism group is a quotient of $\mathcal{H}$. For example, if we reduce G mod (n) by factoring out the (normal) subgroup N generated by x^n and y^n, the corresponding quotient $\mathcal{H}/N$ of $\mathcal{H}$ is the Fermat hypermap of genus $(n-1)(n-2)/2$, with automorphism group $\mathbf{Z}_n^2$ [23]; its Walsh map is the standard embedding of $K_{n,n}$, constructed by Biggs and White [3, §5.6.7] as a Cayley map for $\mathbf{Z}_{2n}$. More generally, taking the quotient by $\langle x^n, y^m \rangle$ gives an edge-transitive embedding of $K_{m,n}$.

The Cayley graph C' of G with respect to the generators x, y and z can also be embedded in the euclidean plane, namely as the 1-skeleton of the standard triangular tessellation, with edges labelled x, y and z making angles 0, $2\pi/3$ and $4\pi/3$ with the first coordinate axis. In this model, the hypervertices, hyperedges and hyperfaces can be identified with the corresponding three families of parallel lines in the tessellation. Each pair from different families meet exactly once, so the barycentric subdivision $\mathcal{B}$ of $\mathcal{H}$ is a triangular embedding in $\mathcal{S}$ of the complete tripartite graph $K_{\aleph_0,\aleph_0,\aleph_0}$ of countably infinite degree; again, this is a regular map. Adding to G the relations $x^n = y^n = 1$ (which imply that $z^n = 1$) we get a finite quotient of this map, namely a regular (and minimum genus) embedding of $K_{n,n,n}$, the stellation of the standard embedding of $K_{n,n}$ considered earlier.

Example 7. Let $p = 1$ and $q = -1$, so

$$G = BS(1, -1) = \langle x, y \mid x^y = x^{-1} \rangle.$$

This is a semidirect product $\mathbf{Z} : \mathbf{Z}$ of a normal subgroup $X = \langle x \rangle \cong \mathbf{Z}$ by a complement $Y = \langle y \rangle \cong \mathbf{Z}$, with y inverting X by conjugation. As in Example 6, each $g \in G$ has the unique form $x^i y^j$ where $i, j \in \mathbf{Z}$. The Cayley graph C for G with respect to x and y is as in Example 6, except that x-edges in alternate horizontal lines $x_2 = c$ point to the right or left as c is even or odd. The Walsh map $\mathcal{W}$ is again an edge-transitive embedding of the complete bipartite graph $K_{\aleph_0,\aleph_0}$; in this case $\mathcal{W}$ is not orientably regular, since no automorphism of G can transpose x and y.

Example 8. Let

$$G = BS(1, q) = \langle x, y \mid x^y = x^q \rangle.$$

where $q \geq 2$. Let x_r denote x^{y^r} for each $r \in \mathbf{Z}$, so by iterating the relation $x^y = x^q$ we have $x_r = x^{q^r}$ for all $r \geq 0$. Thus

$$x_r = x_s^{q^{r-s}}$$

if $r \geq s \geq 0$, so conjugating by negative powers of y we see that this equation holds whenever $r \geq s$. This implies that x_r and x_s commute for all r and s, so the normal closure X^G of $X = \langle x \rangle$ in G, which is generated by the elements x_r $(r \in \mathbf{Z})$, is abelian. This group is isomorphic to the additive group $\mathbf{A} = \mathbf{Z}[\frac{1}{q}] = \{aq^r \mid a, r \in \mathbf{Z}\}$ of q-adic rationals, with each x_r corresponding to q^r. Using the equations $yx_r = x_{r-1}y$ and $y^{-1}x_r = x_{r+1}y^{-1}$, each word in x and y can be rewritten in the unique form $x^i y^j$ where $i \in \mathbf{A}$ and $j \in \mathbf{Z}$; here, if $i = aq^r$ then x^i denotes the element x_r^a of X^G. Thus G is a semidirect product of $X^G \cong \mathbf{A}$ by $Y = \langle y \rangle \cong \mathbf{Z}$, with the action of y by conjugation on X^G corresponding to multiplication by q on $\mathbf{A}$.

Let C be the Cayley graph of G with respect to x and y. The vertices $x^i y^j$ can be identified with the points $(i, j) \in \mathbf{A} \times \mathbf{Z} \subset \mathbf{R}^2$. At each such vertex, there is an edge labelled y from (i, j) to $(i, j + 1)$, and (since $x^i y^j x = x^i x_{-j} y^j$) an edge labelled x from (i, j) to $(i + q^{-j}, j)$. This is therefore *not* an embedding of C in the plane: instead the relation $x^y = x^q$ determines a set of overlapping rectangles $[i, i + q^{-j}] \times [j, j + 1]$ for $i \in \mathbf{A}$ and $j \in \mathbf{Z}$, with an edge labelled y along each of the the vertical sides, q edges labelled x along the top, and one along the bottom.

The hyperedges of $\mathcal{H}$ correspond to the cosets $gY = x^i y^j Y = x^i Y$ of Y in G; distinct elements x^i lie in distinct cosets, so we can identify the hyperedges with the elements $i \in$

$\mathbf{A}$. Hypervertices correspond to cosets $gX = x^iy^jX = y^jx^{iq^j}X$ of X, with $y^jx^{iq^j}X = y^{j'}x^{i'q^{j'}}X$ if and only if $j = j'$ and $iq^j - i'q^{j'} \in \mathbf{Z}$, so they can be identified with the elements of the disjoint union of the groups $\mathbf{A}/\langle q^{-j}\rangle$ for all $j \in \mathbf{Z}$. Just as $\mathbf{A}$ is a dense subgroup of the line $\mathbf{R}$, each $\mathbf{A}/\langle q^{-j}\rangle$ is a dense subgroup of the circle $\mathbf{R}/\langle q^{-j}\rangle$, the circumference q^{-j} decreasing geometrically towards 0 as j increases. Hyperedges and hypervertices are incident if and only if the corresponding group elements are related by the natural projection $\mathbf{A} \to \mathbf{A}/\langle q^{-j}\rangle$.

Let us define $H_r = \langle x_r\rangle = \langle x^{q^r}\rangle \leq G$ for each $r \in \mathbf{Z}$, and $\mathcal{H}_r = \mathcal{H}/H_r$, a non-regular quotient of $\mathcal{H}$ since the subgroup H_r is not normal in G. Now x_r acts on $\mathcal{H}$ by left-multiplication $g \mapsto x_r^{-1}g$, so the hyperedges of $\mathcal{H}_r$ correspond to double cosets $H_rgY = H_rx^iy^jY = H_rx^iY$, the hypervertices correspond to double cosets $H_rgX = H_rx^iy^jX$, and their incidences correspond to cosets $H_rg = H_rx^iy^j$.

One can form $\mathcal{H}_r$ from $\mathcal{H}$ by replacing the hyperedge set $\mathbf{A}$ of $\mathcal{H}$ with $\mathbf{A}/\langle q^r\rangle$, and in the hypervertex set replacing each $\mathbf{A}/\langle q^{-j}\rangle$ with $\mathbf{A}/\langle q^r\rangle$ if $-j > r$, so the large circles now have constant circumference q^r, while the small circles still have circumference $q^{-j} \to 0$. As before, incidence is given by the natural projection.

The index q normal inclusions

$$\cdots < H_1 < H_0 < H_{-1} < \cdots$$

give rise to q-sheeted regular coverings

$$\cdots \to \mathcal{H}_1 \to \mathcal{H}_0 \to \mathcal{H}_{-1} \to \cdots .$$

Since $H_r^y = H_{r+1}$ for all r, these hypermaps $\mathcal{H}_r$ are all isomorphic to each other. (Of course, simple counting arguments imply that a finite hypermap can never be isomorphic to a proper quotient or covering of itself.)

For each r we have $\mathrm{Aut}^+\mathcal{H}_r \cong N_G(H_r)/H_r = X^G/H_r \cong \mathbf{A}/\langle q^r\rangle$. These groups are all isomorphic to each other, and in particular they are isomorphic to the group $\mathbf{Z}_{q^\infty}$, the direct limit of the sequence of natural embeddings $\mathbf{Z}_q \to \mathbf{Z}_{q^2} \to \mathbf{Z}_{q^3} \to \cdots$.

Example 9. Let $G = BS(2,3)$. This is a well-known example of a non-Hopfian group. We say that a group G is *Hopfian* if the only normal subgroup N with $G/N \cong G$ is the identity subgroup, or equivalently, if every epimorphism $G \to G$ is an automorphism. To prove that G is non-Hopfian, one first checks that $x \mapsto x^2$, $y \mapsto y$ defines an epimorphism θ from G to G: it is a homomorphism since $((x\theta)^2)^{y\theta} = (x^4)^y = x^6 = (x\theta)^3$, and it is an epimorphism since the image contains x^2 and y, and thus contains $(x^2)^y = x^3$ and hence also $x = x^3x^{-2}$; now the commutator $g = [x^y, x]$ is in $\ker\theta$ since $(x\theta)^{y\theta} = (x^2)^y = x^3$ commutes with $x\theta = x^2$, and $g \neq 1$ by the normal form theorem for HNN extensions. (In fact, Baumslag and Solitar showed that $B(p,q)$ is Hopfian if and only if one of p and q divides the other or they have the same prime factors.)

This implies that, although the orientably regular hypermap $\mathcal{H}$ corresponding to (G,x,y) is not isomorphic to its regular quotient $\overline{\mathcal{H}} = \mathcal{H}/\ker\theta$, nevertheless $\mathrm{Aut}^+\mathcal{H} \cong G \cong G/\ker\theta \cong \mathrm{Aut}^+\overline{\mathcal{H}}$. By iterating this we get a sequence

$$\mathcal{H} \to \overline{\mathcal{H}} \to \overline{\overline{\mathcal{H}}} \to \cdots$$

of mutually non-isomorphic orientably regular hypermaps, with

$$\mathrm{Aut}^+\mathcal{H} \cong \mathrm{Aut}^+\overline{\mathcal{H}} \cong \mathrm{Aut}^+\overline{\overline{\mathcal{H}}} \cong \cdots .$$

In fact, Baumslag and Solitar showed that G and

$$G^* := \langle x, y \mid (x^2)^y = x^3, ([x, y]^2 y^{-1})^2 = 1 \rangle$$

are non-isomorphic groups, and each is an epimorphic image of the other. This implies that there is a sequence

$$\mathcal{H}_0 \to \mathcal{H}_0^* \to \mathcal{H}_1 \to \mathcal{H}_1^* \to \mathcal{H}_2 \to \mathcal{H}_2^* \to \cdots$$

of coverings of orientably regular hypermaps, where $\mathcal{H}_0$ and $\mathcal{H}_0^*$ are the hypermaps corresponding to G and G^*, with $\mathrm{Aut}^+\mathcal{H}_i \cong G$ and $\mathrm{Aut}^+\mathcal{H}_i^* \cong G^*$ for all i.

Brunner [4] has shown that G is generated by x^{2^n} and y for each $n \geq 0$, and that the kernels of the corresponding epimorphisms $F_2 \to G$ are inequivalent under $\mathrm{Aut}\, F_2$, so the group $\mathrm{Out}\, F_2$ of hypermap operations has $\aleph_0$ orbits on orientably regular hypermaps with automorphism group G (see §4). For another example of this phenomenon see [5].

6 Infinite quotients of triangle groups

The *type* of a hypermap $\mathcal{H}$ is the triple $\tau = (l, m, n)$ where l, m and n are the least common multiples of the valencies of its hypervertices, hyperedges and hyperfaces; these can take any values from $\{1, 2, 3, \ldots, \infty\}$. We say that $\mathcal{H}$ has *type dividing* τ if these least common multiples (and hence all the corresponding valencies) divide l, m and n; by convention, every natural number divides ∞. For example, maps are simply hypermaps of type dividing $(\infty, 2, \infty)$.

An oriented hypermap $\mathcal{H}$ has type dividing τ if and only if the permutation representation $\theta : F_2 \to \mathrm{Sym}\,\Omega$ in §2 can be factored through the triangle group

$$\Delta(\tau) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle.$$

It follows that oriented hypermaps of type dividing τ correspond to conjugacy classes of subgroups of $\Delta(\tau)$, and orientably regular hypermaps correspond to normal subgroups of $\Delta(\tau)$, with the quotient isomorphic to $\mathrm{Aut}^+\mathcal{H}$.

One can realise $\Delta(\tau)$ as the orientation-preserving automorphism group of the *universal hypermap* $\mathcal{U}(\tau)$ of type τ. Let $\mathcal{S}$ be the sphere, the euclidean plane or the hyperbolic plane as $l^{-1} + m^{-1} + n^{-1} > 1, = 1$ or < 1, and let ABC be a triangle in $\mathcal{S}$ with internal angles $\pi/l, \pi/m$ and π/n. (If, for instance, $l = \infty$ we take A to be an ideal vertex on the boundary of $\mathcal{S}$, and AB and AC to be parallel lines meeting at A with internal angle 0.) The extended triangle group $\Delta[\tau]$ is the group generated by the reflections of $\mathcal{S}$ is the sides of ABC, and $\Delta(\tau)$ is the subgroup of index 2 consisting of orientation-preserving isometries of $\mathcal{S}$. Its generators X, Y and Z are rotations around A, B and C through $2\pi/l, 2\pi/m$ and $2\pi/n$. The images of ABC under $\Delta[\tau]$ form a triangular tessellation of $\mathcal{S}$, and this is the barycentric subdivision of $\mathcal{U}(\tau)$, with the images of A, B and C representing the hypervertices, hyperedges and hyperfaces. Any oriented hypermap $\mathcal{H}$ of type dividing τ is isomorphic to the quotient of $\mathcal{U}(\tau)$ by a subgroup of $\Delta(\tau)$. This imposes on $\mathcal{H}$ a Riemann surface structure, compact if and only if $\mathcal{H}$ is finite. Belyĭ's Theorem [2] implies that the Riemann surfaces obtained from finite hypermaps are those defined, as projective algebraic curves, over algebraic number fields, leading to Grothendieck's theory of *dessins d'enfants* [15], [23], [34]; at present there is no corresponding characterisation of the non-compact Riemann surfaces obtained from infinite hypermaps.

We call $\Delta(\tau)$ and τ *spherical, euclidean* or *hyperbolic* as $\mathcal{S}$ is the sphere, the euclidean plane or the hyperbolic plane. Spherical triangle groups are all finite, and are solvable apart from $\Delta(2,3,5) \cong A_5$, while euclidean triangle groups, although infinite, are abelian by cyclic, so neither class have particularly complicated quotients. By contrast, hyperbolic triangle groups are in some sense 'large enough' to have many interesting quotients, and these correspond to interesting orientably regular hypermaps of various hyperbolic types. Here we are able to demonstrate this for most hyperbolic triangle groups, but first we must describe some possible exceptions.

We define a hyperbolic triple $\tau = (l,m,n)$, and the associated triangle group $\Delta(\tau)$, to have *small periods* if τ can be obtained from one of the following thirty triples by permuting l, m and n:

$$\begin{array}{llll} (2,3,n) & \text{for} \quad 7 \le n \le 15, & (2,4,n) & \text{for} \quad 5 \le n \le 10, \\ (3,3,n) & \text{for} \quad 4 \le n \le 10, & (3,4,n) & \text{for} \quad 4 \le n \le 5, \\ (4,4,n) & \text{for} \quad 4 \le n \le 5, & (l,5,5) & \text{for} \quad 2 \le l \le 5. \end{array}$$

The following result is proved in [21], as part of a more general result about Fuchsian groups:

Theorem 10. *Let $\Delta(\tau)$ be a hyperbolic triangle group which does not have small periods. Then every countable group can be embedded in*

(1) *a simple quotient of $\Delta(\tau)$, and*

(2) *a non-Hopfian quotient of $\Delta(\tau)$.*

This shows that the simple and non-Hopfian quotients of $\Delta(\tau)$ are arbitrarily complicated. By imitating an argument due to P. M. Neumann [30], we can deduce that there are uncountably many of them:

Corollary 11. *Let $\Delta(\tau)$ be a hyperbolic triangle group which does not have small periods. Then $\Delta(\tau)$ has*

(1) $2^{\aleph_0}$ *non-isomorphic simple quotients, and*

(2) $2^{\aleph_0}$ *non-isomorphic non-Hopfian quotients.*

Proof. By Theorem 2 there are $2^{\aleph_0}$ non-isomorphic 2-generator groups; each such group is countable, so by Theorem 10 it can be embedded in a simple (respectively non-Hopfian) quotient G of $\Delta(\tau)$. Since $\Delta(\tau)$ is finitely generated, G is countable and hence has only countably many 2-generator subgroups, so $\Delta(\tau)$ must have $2^{\aleph_0}$ non-isomorphic simple (respectively non-Hopfian) quotients. □

Theorem 10 shows that for each hyperbolic triple τ without small periods, the orientation-preserving automorphism group $\mathrm{Aut}^+\,\mathcal{H}$ of an orientably regular hypermap $\mathcal{H}$ of type dividing τ can be arbitrarily complicated, even if we require it to be simple or non-Hopfian. Corollary 11 implies that in either case there are $2^{\aleph_0}$ non-isomorphic orientably regular hypermaps $\mathcal{H}$ of type dividing τ, each having no proper regular quotients or many regular quotients respectively. Having mutually non-isomorphic automorphism groups, these hypermaps are mutually inequivalent under hypermap operations.

The proof of Theorem 10 uses results from small cancellation theory [25, Ch. V], where the required hypotheses are too restrictive to apply to the hyperbolic triangle groups with small periods. Apart from this, there is little evidence to suggest that the behaviour of such hyperbolic triangle groups should be exceptional, so we conjecture that the above results apply to *all* hyperbolic triangle groups. In support of this, we note that Wilson [33] has shown that $\Delta(2,3,n)$ satisfies Theorem 10(1) (and hence Corollary 11(1)) for all $n \geq 7$; his method, developed with Lucchini and Tamburini in [24] for $\Delta(2,3,7)$, uses matrix groups over rings as quotients.

In [21], small cancellation theory is also used to prove:

Theorem 12. *If $\Delta(\tau)$ is a hyperbolic triangle group which does not have small periods, then $\Delta(\tau)$ has $\aleph_0$ non-isomorphic finitely presented quotients with unsolvable word problem.*

Of course, there are (up to isomorphism) only countably many finitely presented groups, so in this case one cannot hope to obtain $2^{\aleph_0}$ non-isomorphic quotients, as we do in Corollary 11. This result implies that for each corresponding triple τ there are $\aleph_0$ non-isomorphic orientably regular hypermaps of type dividing τ which are finitely presented but have an unsolvable word problem. Here we say that an orientably regular hypermap $\mathcal{H}$ is *finitely presented* if the corresponding normal subgroup of F_2 in the normal closure of a finite subset $F \subset F_2$, so that any relation in $\mathcal{H}$ of the form $\alpha g = \alpha$ for all $\alpha \in \Omega$ (where $g \in F_2$) is a consequence of the finite set of relations $\alpha f = \alpha$ for all $\alpha \in \Omega$ where $f \in F$. The *word problem* for $\mathcal{H}$ asks, for any given $g \in F_2$, whether or not $\mathcal{H}$ has a relation $\alpha g = \alpha$ for all $\alpha \in \Omega$; it is *solvable* if there is a deterministic algorithm to decide this question. As before, it is conjectured that Theorem 12 applies to hyperbolic triangle groups in general, not just those without small periods.

7 Tarski monster groups and hypermaps

Given an infinite orientably regular hypermap $\mathcal{H}$, one might expect to obtain coverings (not necessarily regular) $\mathcal{H} \to \overline{\mathcal{H}} = \mathcal{H}/H$ of many different degrees $|H|$ by choosing appropriate subgroups $H \leq G = \mathrm{Aut}^+\mathcal{H}$. This is not always possible, as shown by Ol'shanskii's construction of so-called *Tarski monster* groups [31]:

Theorem 13. *For each sufficiently large prime p there is an infinite group G in which every proper subgroup has order p.*

Here it is sufficient to take $p > 10^{75}$. Any two elements $x, y \in G$ which are not powers of each other must generate G, since they cannot be contained in a proper subgroup; now x, y and xy must have order p, so G is a quotient of the triangle group $\Delta(p,p,p)$, and is therefore the automorphism group of an orientably regular infinite hypermap $\mathcal{H}$ of type (p,p,p). The only proper quotients of $\mathcal{H}$ are the p-sheeted coverings $\mathcal{H} \to \overline{\mathcal{H}} = \mathcal{H}/H$ where $1 < H < G$ and $|H| = p$; there is one (up to isomorphism) for each conjugacy class of such subgroups H. Since H is a maximal subgroup of G, $\overline{\mathcal{H}}$ has no proper quotients.

8 Unoriented hypermaps

Our final examples are based on Grigorchuk's group, first introduced in [12] as an explicit example of a finitely generated infinite periodic group. This group has many other remarkable properties, and as shown in [20], it arises as the automorphism group of an infinite orientable map. Since some of its elements reverse the orientation, we first need to extend our general remarks about hypermaps in §2 to remove the restriction that orientation should be preserved, or even that it should exist.

An orientably regular hypermap $\mathcal{H}$ is *regular* (or *reflexible*) if it is isomorphic to its mirror image, with the reverse orientation, or equivalently its orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{H}$ has an automorphism inverting its canonical generators x and y. The corresponding hypermap subgroup $N \leq F_2$ is then a normal subgroup of the group

$$\Delta = \Delta[\infty, \infty, \infty] = \langle r_0, r_1, r_2 \mid r_i^2 = 1 \rangle \cong C_2 * C_2 * C_2, \tag{8.1}$$

which contains F_2 as its even subgroup $\Delta(\infty, \infty, \infty)$ of index 2 generated by r_1r_2 and r_2r_0; the full automorphism group $\mathrm{Aut}\,\mathcal{H}$ of $\mathcal{H}$, including orientation-reversing elements, is isomorphic to Δ/N, and the automorphism of $\mathrm{Aut}^+\mathcal{H}$ inverting x and y is induced by conjugation by the image of r_2.

More generally, the theory of unoriented hypermaps (possibly non-orientable or with boundary, see [18]) is similar to that for oriented hypermaps outlined in §2, with the above group Δ replacing F_2, so that hypermaps now correspond to transitive permutation representations of Δ acting on flags, or equivalently to conjugacy classes of subgroups $H \leq \Delta$. The elements r_0, r_1 and r_2 permute the flags of a hypermap $\mathcal{H}$ by changing the hypervertex, hyperedge or hyperface of each flag while preserving the other two constituents; the monodromy group $\mathrm{Mon}\,\mathcal{H}$ is the group of permutations of the flags which they generate, and the automorphism group $\mathrm{Aut}\,\mathcal{H}$ is its centraliser. The hypervertices, hyperedges and hyperfaces of $\mathcal{H}$ are identified with the orbits of the dihedral groups $\langle r_1, r_2\rangle$, $\langle r_2, r_0\rangle$ and $\langle r_0, r_1\rangle$, and incidence corresponds to non-empty intersection. Regular hypermaps, with $\mathrm{Aut}\,\mathcal{H}$ transitive on flags, correspond to regular permutation groups $\mathrm{Mon}\,\mathcal{H}$, or equivalently to normal subgroups H of Δ. Subgroups H contained in F_2 correspond to orientable hypermaps without boundary.

9 The Grigorchuk map

If a group G has a finite generating set $X = X^{-1}$ then let $\gamma_X(n)$ denote the number of elements of G which can be represented as words of length at most n in the elements of X. Although the values of the function γ_X depend on the choice of X, its asymptotic rate of growth as $n \to \infty$ is independent of X, so we call this the rate of growth of G. For many years it was conjectured that every finitely generated group has polynomial or exponential growth [27]; Grigorchuk's group [12], [13], [14] was the first counterexample, having intermediate growth (faster than polynomial but slower than exponential). There are excellent surveys of the construction and properties of this group in [6] and [16, Ch.VIII].

It is convenient to define Grigorchuk's group G as a group of automorphisms of the binary rooted tree T whose vertices are the words w of length $l \geq 0$ in the alphabet $\{0, 1\}$, with each w adjacent to wi for $i = 0, 1$. Automorphisms a, b, c and d of T are defined to act as follows, fixing any vertices not specified:

- a transposes $0w$ and $1w$ for all w;
- b transposes 1^m00w and 1^m01w for all $m \equiv 0, 1 \bmod (3)$;
- c transposes 1^m00w and 1^m01w for all $m \equiv 0, 2 \bmod (3)$;
- d transposes 1^m00w and 1^m01w for all $m \equiv 1, 2 \bmod (3)$.

Then G is the subgroup of $\mathrm{Aut}\,T$ generated by a, b, c and d. By inspection,

$$a^2 = b^2 = c^2 = d^2 = bcd = 1,$$

so G has generators a, b and c which satisfy

$$a^2 = b^2 = c^2 = (bc)^2 = 1. \tag{8.2}$$

It follows from (8.1) and (8.2) that there are epimorphisms $\Delta \to \Delta[\infty, 2, \infty] \to G$ with $r_0 \mapsto b$, $r_1 \mapsto a$ and $r_2 \mapsto c$, so G is the monodromy group (and hence also the full automorphism group) of an unoriented regular map $\mathcal{G}$, the *Grigorchuk map*, with a, b and c acting on flags of $\mathcal{G}$ by changing their edge, vertex and face respectively. These flags can be identified with the elements of G, permuted regularly. In their actions as automorphisms, a, b and c are reflections of $\mathcal{G}$, changing the edge, vertex and face of one specific flag. The vertices correspond to the cosets of $\langle a, c\rangle$ in G, and their valency is the order of ac, namely 8. Similarly, the faces of $\mathcal{G}$ are all 16-gons, since this is the order of ab, so $\mathcal{G}$ is a hypermap of type $(8, 2, 16)$, i.e. a map of type $\{16, 8\}$ in the notation of Coxeter and Moser [10, Ch. 8]. The Petrie polygons (closed zig-zag paths) have length equal to the order of $abc = ad$, namely 4, so $\mathcal{G}$ is a quotient of the map $\{16, 8\}_4$, the universal map of type $\{16, 8\}$ with Petrie length 4 [10, §8.6]. The group G is not finitely presented, but Lysionok [26] has given a recursive presentation in which all the relations have even length in the generating set $\{a, b, c\}$, so $\mathcal{G}$ is an orientable map without boundary.

In any map one can define the distance between two vertices to be their distance in the embedded graph, that is, the least number of edges in any path in the graph joining them. One can then define the asymptotic rate of growth of a map to be that of the function which counts the vertices at distance at most n from a given vertex. If the map is vertex-transitive this is independent of the choice of base vertex. Most familiar examples of infinite maps, such as the universal maps of particular types on the euclidean or hyperbolic planes (see §6), have polynomial or exponential growth. However, since $\mathcal{G}$ has finite type its rate of growth is equivalent to that of G, so $\mathcal{G}$ has intermediate growth; it is perhaps the first known example of such a map. See [20] for more on rates of growth of maps.

It is well known that a finite map (or hypermap) is regular if and only if its automorphism group and monodromy group are isomorphic. In the infinite case, this condition is necessary but insufficient, and what is required is that these two groups should be isomorphic, not just as abstract groups, but as permutation groups on the flags. The following example, based on Grigorchuk's group and taken from [22], illustrates the problem.

Example 14. The generator a of G transposes the subtrees T_i $(i = 0, 1)$ of T spanned by the vertices iw, while b, c and d preserve them, so G has a subgroup $N = \langle b, c, d, b^a, c^a, d^a\rangle$ of index 2 preserving them. Under the isomorphism $T_0 \to T$, $iw \mapsto w$, these six generators of N act on T_0 as $a, a, 1, c, d$ and b act on T, so this isomorphism induces an epimorphism $N \to G$; its kernel K_0 is the subgroup of N fixing T_0. Similarly the action of N on T_1 induces an epimorphism $N \to G$ with kernel $K_1 = K_0^a$, the subgroup fixing T_1. Each K_i is normal in N but not in G (for instance $d \in K_0$ but $d^a \notin K_0$), so it has normaliser $N_G(K_i) = N$. The map $\mathcal{G}_i = \mathcal{G}/K_i$ therefore has automorphism group $\mathrm{Aut}\,\mathcal{G}_i \cong N_G(K_i)/K_i = N/K_i \cong G$. The core K_i^* of each K_i in G is $K_0 \cap K_1$, which fixes T_0 and T_1, so $K_i^* = 1$ and hence $\mathcal{G}_i$ has monodromy group $\mathrm{Mon}\,\mathcal{G}_i \cong G$. Thus $\mathrm{Aut}\,\mathcal{G}_i \cong \mathrm{Mon}\,\mathcal{G}_i$, but $\mathcal{G}_i$ is not regular since K_i is not normal in G: for instance, $\mathrm{Aut}\,\mathcal{G}_i$ has two orbits on edges of $\mathcal{G}_i$, and two on flags.

Further properties of $\mathcal{G}$ are considered in [20]. For instance, G is residually finite, meaning that its normal subgroups of finite index have trivial intersection, and it has no non-trivial normal subgroups of infinite index, so $\mathcal{G}$ has infinitely many regular quotients, all of them (except $\mathcal{G}$ itself) finite. By modifying the conditions on m in the definitions of b, c and d, one can construct uncountably many groups, and hence uncountably many maps, with properties similar to those described here.

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