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Popravni količniki za izračun Youngovega modula iz resonančnega upogibnega nihanja

Correction Coefficients for Calculating the Young's Modulus from the Resonant Flexural Vibration

Igor Štubňa - Anton Trník
(Constantine the Philosopher University, Nitra)

Enostavna enačba, dobljena iz poenostavljene diferencialne enačbe upogibnega nihanja za primer z enakomernim prerezom, nam ne da točne vrednosti Youngovega modula oz. hitrosti zvoka, če je razmerje dolžine in premera (oz. dolžine in debeline) vzorca manjše od 20. Napako lahko odpravimo z množenjem izmerjene resonančne frekvence ali izračunanega Youngovega modula s popravnim količnikom. V prispevku predstavljamo nekaj enačb za popravne količnike in novo enačbo ter jih primerjamo z enačbami Ameriškega združenja za preizkušanje in materiale (American Society for Testing and Materials - ASTM).

(Ključne besede: nihanja upogibna, frekvence resonančne, moduli Youngovi, izračuni)

A simple formula derived from the simplified differential equation of flexural vibration of a sample with a uniform cross-section does not give exact values for the Young's modulus or the velocity of sound if the ratio of the length to the diameter (or the length to the thickness) of the sample is less than 20. The error can be eliminated by multiplying the measured resonant frequency or the calculated Young's modulus by a correction coefficient. Some formulae for the correction coefficients as well as a new formula are presented and compared with the ASTM formulae.

(Keywords: flexural vibrations, resonant frequencies, Young's modulus, calculations)

0 INTRODUCTION

One of the best methods for determining the velocity of sound or the Young's modulus of solids is based on the resonant flexural vibrating of a sample of a cylindrical or a prismatic shape. It is easy to excite this vibration, and its magnitude is sufficiently high. Its resonant frequency is less than the resonant frequency of a longitudinal vibration of a sample with the same length. These properties of the flexural vibration make it preferable for measuring the Young's modulus or the velocity of sound.

For measuring purposes the most suitable vibration is the vibration of a free-free sample because of the relative ease of fulfilling the boundary condition of the theoretical solution. A widely used method, which is also suitable for high temperatures, is based on Föster's idea [1], later improved by Spinner and Tefft [2].

The solution for the three-dimensional form of the partial differential equation of flexural vibration is complex, but the mathematical approach can be simplified, and a reasonably exact solution can be obtained. For long samples the simplified partial differential equation of the flexural vibration can be used. This equation has the form [3]:

$$\frac{\partial^2 y}{\partial t^2} + c_0^2 i^2 \frac{\partial^4 y}{\partial x^4} = 0 \qquad (1),$$

where x and y are the coordinates of the mass element of the sample, t is time, c_0 is the velocity of sound (i.e., the velocity of longitudinal wave propagation in the sample), i is the radius of inertion of the cross-section. For a sample with a circular cross-section, the radius of inertion is $i = d/4$ and for a sample with a rectangular cross-section it is $i = d/\sqrt{12}$, where d is the diameter of a cylindrical sample or the thickness of a prismatic sample in the direction of the vibration. The frequency equation derived from Eq. (1) for the free-free sample is:

$$\cos(al)\ \mathrm{ch}(al) = 1 \qquad (2),$$

where l is length of the sample, $a = \sqrt{\omega/(ic_0)}$, ω is a resonant angular frequency. The formulae for the velocity of sound c_0 and the Young's modulus E derived from Eq. (2) have the form:

$$c_0 = K\frac{l^2 f}{d} \ , \ E = \left(K\frac{l^2 f}{d}\right)^2 \rho \qquad (3),$$

where f is the resonant frequency, ρ is the material density and the values of the constant K are:

K = 1.12336 for a cylindrical sample and the fundamental resonant frequency,
K = 0.97286 for a prismatic sample and the fundamental resonant frequency,
K = 0.40752 for a cylindrical sample and the 1st overtone,
K = 0.35292 for a prismatic sample and the 1st overtone.

If a relatively short sample is used (with the ratio l/d < 20) it is necessary to take into account the influence of the shear forces and the rotary inertia. There are two possible ways to do this: 1) solving the complicated frequency equation derived from the partial differential equation accounting for these influences, 2) using the simple formula (3) and multiplying the calculated Young's modulus (or measured resonant frequency) by a correction coefficient.

Formulae for calculating the correction coefficients for the fundamental mode as well as for the first overtone of the flexural vibration of a cylindrical or a prismatic sample and comparisons of these coefficients are given in this paper.

1 CORRECTION COEFFICIENTS DERIVED FROM AN ALTERNATIVE EQUATION

If the ratio of l/d > 20, the measured resonant frequency is in a good agreement with that calculated from Eq. (3). If the ratio of l/d < 20, the measured resonant frequency is less than the frequency calculated from Eq. (3) for the same sound velocity: the shorter the sample, the bigger the discrepancy between the theoretical and the measured frequencies. To avoid this disagreement, the effect of the shear forces and the rotary inertia has to be taken into account. This leads to Timoshenko's equation [1] or to the alternative equation [4]:

$$\frac{\partial^2 y}{\partial t^2} + c_0^2 i^2 \frac{\partial^4 y}{\partial x^4} - pi^2 \frac{\partial^4 y}{\partial t^2 \partial x^2} = 0 \qquad (4).$$

Here $p = 2(1+\mu)/\kappa$, where μ is Poisson's ratio, and $\kappa = 0.710$. The frequency equation for the "free-free sample" derived from Eq. (4) is:

$$\begin{aligned} &2R_a R_b P_a P_b[\mathrm{ch}(al)\cos(bl) - 1] - \\ &-[R_b^2 P_a^2 - R_a^2 P_b^2]\mathrm{sh}(al)\sin(bl) = 0 \end{aligned} \qquad (5),$$

where:

$$R_a = R + a^2,\ R_b = R - b^2,\ P_a = Pa + a^3,\ P_b = Pb - b^3,$$
$$R = p(\omega/c_0)^2\ ,\ P = (1+p)(\omega/c_0)^2,$$
$$a = (\omega/c_0)\sqrt{-p/2 + \sqrt{p^2/4 + (c_0/i\omega)^2}}\ ,$$
$$b = (\omega/c_0)\sqrt{+p/2 + \sqrt{p^2/4 + (c_0/i\omega)^2}}\ .$$

The frequency equation (5) is complex and can be solved only by a numerical method. This is the reason why correction coefficients are used.

If the measured resonant frequency f is multiplied by the correction coefficient Q then the correct value of the velocity of sound (or the Young's modulus) can be obtained from Eq. (3). Then Eq. (3) becomes:

$$c_0 = K\frac{l^2(Qf)}{d} \ , \ E = \left(K\frac{l^2(Qf)}{d}\right)^2 \rho \qquad (6),$$

where the coefficients K are the same as above. The correction coefficient Q was obtained from the resonant frequencies $f_{(2)}$ and $f_{(5)}$ computed using the numerical bisection method from the frequency equations (2) and (5) respectively. Then the correction coefficient is:

$$Q = \frac{f_{(2)}}{f_{(5)}} \qquad (7).$$

The correction coefficients Q are dependent on the ratio of l/d as well as on Poisson's ratio μ, but the dependence between Q and μ is weak. The values of Q could be considered constant for l/d > 6 and $0.15 < \mu < 0.4$ (see Fig. 1 and 2). The values of Q were computed for the fundamental mode and l/d > 2.5, and for l/d > 5 for the 1st overtone. The tabulated coefficients Q for the fundamental mode of flexural vibration and the cylindrical and prismatic samples are shown in [5] and [6] and for the 1st overtone of the flexural vibration and cylindrical and prismatic samples in [7].

It is often more convenient to have correction coefficients in the form of a formula than a table. Therefore, the formula for calculating the correction coefficients was suggested [8]:

$$Q = 1 + \frac{1}{l/d}\left\{\frac{A}{\mu} + \frac{1}{l/d}\left[B + \frac{1}{l/d}\left(C + \frac{D}{l/d}\right)\right]\right\} \quad (8).$$

Parameters *A, B, C, D* were obtained from a regression analysis of the correction coefficients listed in [5] to [7]. The values of the parameters *A, B, C, D* are in Tab. 1 and Tab. 2.

2 CORRECTION COEFFICIENTS DERIVED FROM TIMOSHENKO'S EQUATION

Timoshenko's equation describes the flexural vibration of samples with a uniform cross-section with a sufficient precision [3]. Resonant frequencies predicted by the frequency equation derived from Timoshenko's equation are in agreement with experimentally measured values. But this frequency equation is complex (even more than Eq. (5)) so correction coefficients together with the simple formula (3) are commonly used. The values of these coefficients calculated by Pickett are in table form in [9]. Pickett's values served as the basis for the formulae of correction coefficients for the fundamental mode of flexural vibration ([9] to [13]). The application of Pickett's coefficient *T* consists of its multiplication by the Young's modulus calculated from the simple formula (3). For the fundamental mode and a cylindrical sample:

$$T_{0c} = 1.000 + 4.939(d/l)^2, \quad \text{if} \quad l/d \geq 20$$

$$T_{0c} = 1 + 4.939(1 + 0.0752\mu + 0.8109\mu^2)(d/l)^2 - 0.4883(d/l)^4 - \left[\frac{4.691(1 + 0.2023\mu + 2.173\mu^2)(d/l)^4}{1.000 + 4.754(1 + 0.1408\mu + 1.536\mu^2)(d/l)^2}\right], \quad \text{if} \quad l/d < 20 \quad (9).$$

For the fundamental mode and the prismatic sample:

$$T_{0p} = 1.000 + 6.585(d/l)^2, \quad \text{if} \quad l/d \geq 20$$

$$T_{0p} = 1 + 6.585(1 + 0.0752\mu + 0.8109\mu^2)(d/l)^2 - 0.868(d/l)^4 - \left[\frac{8.340(1 + 0.2023\mu + 2.173\mu^2)(d/l)^4}{1.000 + 6.338(1 + 0.1408\mu + 1.536\mu^2)(d/l)^2}\right], \quad \text{if} \quad l/d < 20 \quad (10).$$

Then the correct value of the Young's modulus is $E = T_{0c}E_{(3)}$ or $E = T_{0p}E_{(3)}$, where $E_{(3)}$ is calculated by Eq. (3). Another, complicated formula was proposed by Martinček [14].

The relationship $T(\mu)$ or $Q(\mu)$ is weak, and for $l/d > 10$ the correction coefficients can be considered independent of the Poisson's ratio (see Fig. 1, Fig. 2). Therefore, a simpler formula for the correction coefficient can be used in such a case. For example, Acegorian and Chočian used:

$$T = 1 + 83(i/l)^2 - \frac{1370(i/l)^4}{1 + 83.4(i/l)^2} - 125(i/l)^4 \quad (11),$$

where i is a radius of inertia of the cross-section and l is the length of the sample [15].

3 CORRECTION COEFFICIENTS FOR SAMPLE WITH CHAMFERED EDGES

If the prismatic sample is not ideal but has chamfered or rounded edges an additional correction should be made. Correction factors *F* are in the range of 1.0031 to 1.0287 for the chamfer size of 0.08 to 0.25 mm. If the density of the sample is determined from its weight and dimensions then the density correction is made by multiplying Young's modulus by the factor $P \in (1.0011, 1.0105)$ for the same chamfer size. The

Table 1. *Parameters A, B, C, D for the fundamental mode*

Parameter	cross-section	
	circular	square
A	0.00002	-0.00002
B	2.5719	3.44347
C	-0.14069	-0.44952
D	-2.43588	-3.34228

Table 2. *Parameters A, B, C, D for the 1st overtone*

Parameter	cross-section	
	circular	square
A	-0.00131	-0.00249
B	7.44851	10.0605
C	-3.56057	-7.076
D	-4.50048	-2.02022

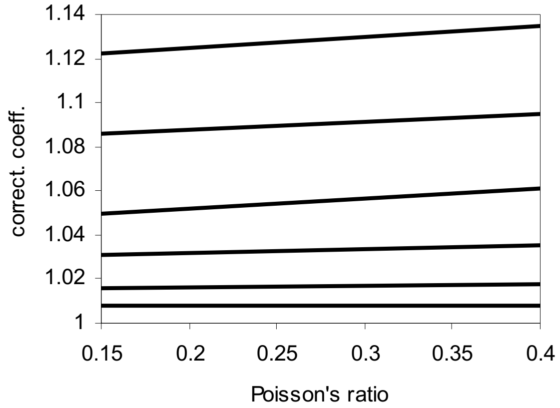

Fig. 1. *Correction coefficients Q_{0p} for the fundamental mode and prismatic sample. Graphs are (from the top) for: l/d = 5, 6, 7, 10, 15, 20*

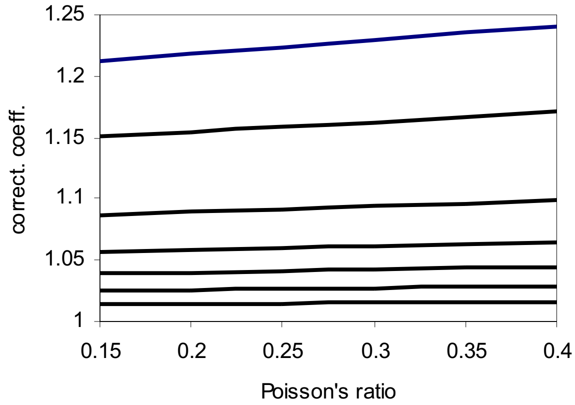

Fig. 2. *Correction coefficients Q_{1c} for 1st overtone and cylindrical sample. Graphs are (from the top) for: l/d = 5, 6, 8, 10, 12, 15, 20*

true Young's modulus $E = FPE_0$, where E_0 is the Young's modulus calculated for the ideal prismatic sample. The correction factors F and P as a function of the chamfer size were calculated by G. Quinn [16].

4 COMPARISON OF THE CORRECTION COEFFICIENTS

The correction coefficients calculated from Eq. (10) and Eq. (8) for the fundamental mode and a prismatic sample are shown in a graph in Fig. 3. The comparison of the coefficients Q_{1p} could not be made because there is not a formula for the correction coefficient for the 1st overtone and a prismatic sample in the ASTM standards ([10] to [13]), and correction coefficients for this case are not in [9]. An indirect verification of the correctness of the coefficients Q_{1p} calculated from the formula (8) is in good agreement between resonant frequencies calculated from commonly accepted Timoshenko's equation and Eq. (4), [4]. The graph of the correction coefficients Q_{1p} calculated from Eq. (8) is in Fig. 6.

The correction coefficients calculated from Eq. (9) and Eq. (8) for the fundamental mode and the cylindrical sample are in Fig. 4, and the correction coefficients for the 1st overtone and a cylindrical sample are plotted in Fig. 5. The coefficients T_{1c} tabulated in [9] served as a basis for the comparison.

As we can see in Fig. 3, 4 and 5, the correction coefficients calculated from Formula (8) show good agreement with those calculated with the help of Eq. (9) and Eq. (10), and with the coefficients presented in [9].

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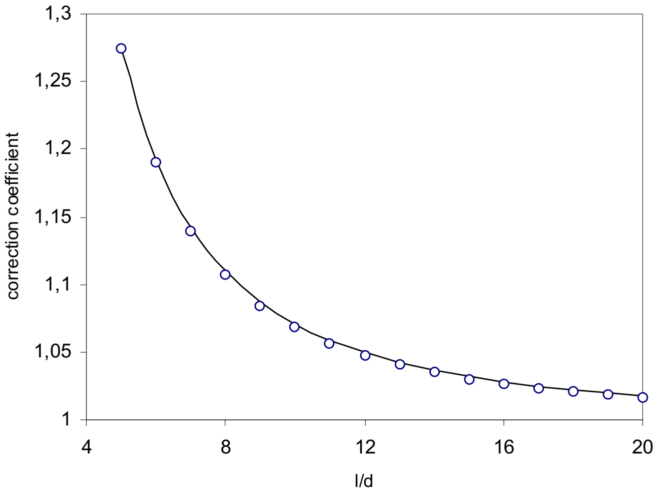

Fig. 3. *Correction coefficients for prismatic sample and fundamental mode: line - T_{0p} after ASTM, points - $(Q_{0p})^2$ calculated from Eq. (8), $\mu = 0.3$*

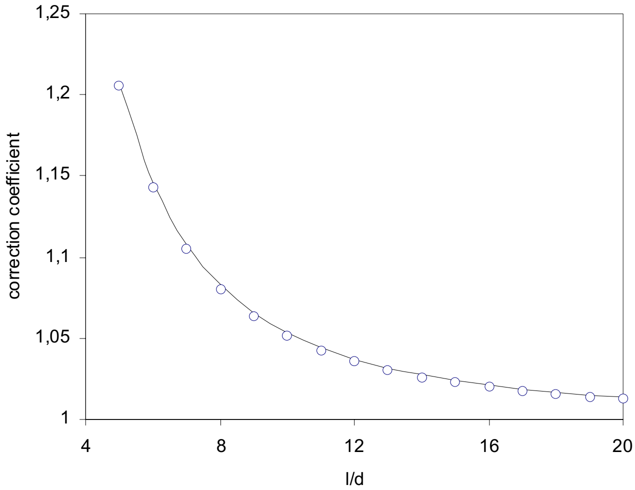

Fig. 4. *Correction coefficients for cylindrical sample and fundamental mode: line - T_{0c} after ASTM, points - $(Q_{0c})^2$ calculated from Eq. (8), $\mu = 0.3$*

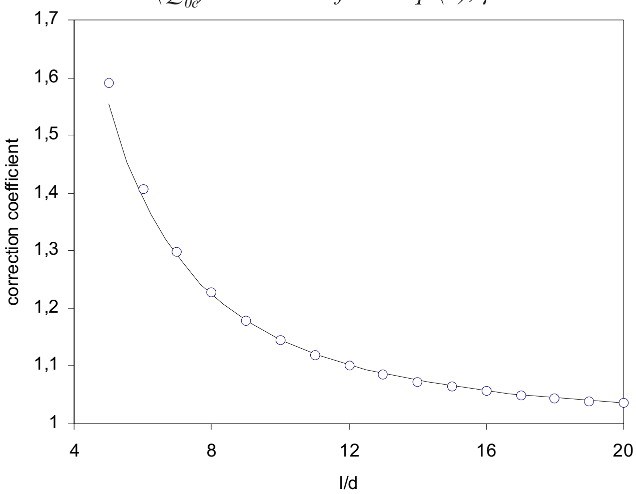

Fig. 5. *Correction coefficients for cylindrical sample and the1st overtone: line - T_{1c} after ASTM, points - $(Q_{1c})^2$ calculated from Eq. (8), $\mu = 0.3$*

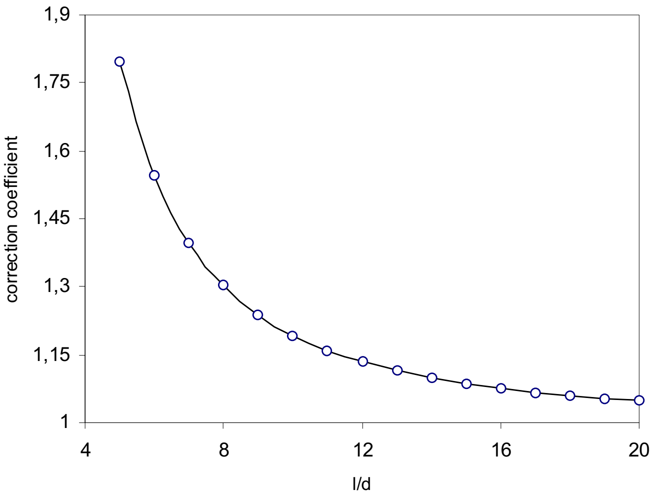

Fig. 6. *Correction coefficients for prismatic sample and the 1st overtone: $(Q_{1p})^2$ calculated from Eq. (8), $\mu = 0.3$*

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Authors' Address: Doc.Dr. Igor Štubňa
Dr. Anton Trník
Constantine the Philosopher University
Department of Physics
A. Hlinku 1
949 01 Nitra, Slovakia
istubna@ukf.sk

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