REGULAR GRAPHS ARE 'DIFFICULT' FOR COLOURING

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Abstract: *Let k be 3 or 4. In this two cases we prove that the decision problem of k-colourability when restricted to Δ-regular graphs is NP-complete for any $\Delta \geq k+1$.*

1 Introduction

In this note we consider the time complexity of the decision problem of (vertex) k-colourability restricted to regular graphs.

It is known that 'almost all k-colourable graphs are easy to colour', namely the proportion of 'difficult' graphs for the usual backtrack algorithm vanishes with growing problem size [9]. Knowing this it is not surprising that there are algorithms with average polynomial time complexity [1], when average is taken over all graphs and even when the average is taken over all 3-colourable graphs with a given number of vertices[5].

If P≠NP, then for every algorithm there has to be a class of 'counterexamples', i.e. graphs on which the algorithm either has superpolynomial time complexity or it fails to produce a correct answer.

For example, Petford and Welsh noticed that one of the situations in which the 3-colourable graphs were not efficiently coloured by their randomised algorithm is when graphs are approximately regular of a low vertex degree [8]. Similarly, approximately regular graphs of a relative low vertex degree are 'difficult' also for the k-colouring generalisation of their algorithm [10]. Petford and Welsh conjectured that 'dense' graphs are easy. Indeed, Edwards showed that, when restricted to class of graphs with lowest vertex degree $\delta > \alpha n$ for arbitrary $\alpha > 0$, the decision problem of 3-colouring is polynomial [4].

This may be understood that the 'difficult' graphs are likely to be found among 'sparse' graphs. It is known that the problem of 3-colouring is NP-complete (even) when restricted to graphs of maximal vertex degree 4 [6]. Here we show that the problem can be further 'simplified', proving that the decision problem of 3-colouring is NP-complete when restricted to Δ-regular graphs (for $\Delta \geq 4$). We also show that the decision problem of 4-colouring is NP-complete when restricted to Δ-regular graphs (for $\Delta \geq 5$).

We assume that the reader is familiar with some standard definitions of graph theory and of computational complexity theory (given, for example, in [2] and [7]).

2 3-colourability of 4-regular graphs is NP-complete

Let us define the problem $\Pi(k, \Delta)$ as follows:
Input: Δ-regular graph G
Question: Is G k-colourable?

Lemma 1 *For any graph G there is a graph G' with no vertex of degree 1 or 2 such that:*
G is 3-colourable iff G' is 3-colourable

Remark: G' in the Lemma is either a graph with minimal vertex degree ≥ 3 or the empty graph, which is the case when G is, for example, a cycle. If G' is empty, it is trivially 3-colourable, and from the proof of the Lemma 1 it follows that also G is 3-colourable.

Proof (of Lemma 1): It is easy to see that the following assertions are true:

(a) If there is a vertex $v \in V(G)$ with degree 1, then G is 3-colourable if and only if the

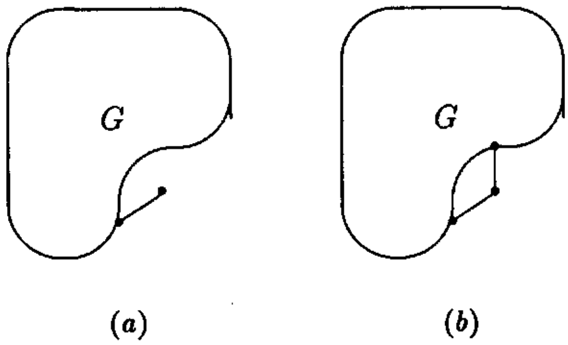

Figure 1: Vertices of degree 1 or 2 may be omitted

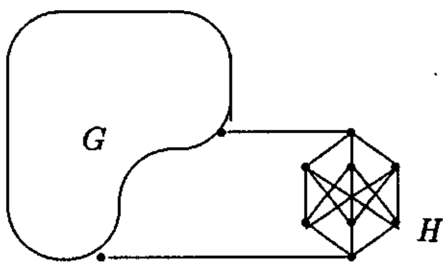

Figure 2: G is 3-colourable iff G' is 3-colourable

induced graph on $V \setminus \{v\}$ is 3-colourable (see Fig. 1(a)).

(b) If there is a vertex $v \in V(G)$ with degree 2, then G is 3-colourable if and only if the induced graph on $V \setminus \{v\}$ is 3-colourable (see Fig. 1(b)).

In this way we can reduce any graph G to a graph G' with minimal vertex degree at least 3 which is 3-colourable exactly when G is. **Q.E.D.**

Remark: Extracting G' successively using (a) and (b) can clearly be done efficiently.

Remark: It is obvious that maximal vertex degree of the graph G' is not greater than maximal vertex degree of the original graph G.

Lemma 2 *G is 3-colourable iff G' is 3-colourable where G' is a graph, obtained from G by the construction given in Fig. 2.*

The construction, given in Fig. 2, can be done as follows. Take two sets, say M and N, of three vertices each. Connect every pair x,y; $x \in M$ and $y \in N$. Add two vertices, say u and v and connect u to all the vertices of N and v to all the vertices of M. Now choose arbitrary pair of distinct vertices of G, say w and z, and connect u with w and v with z to get the graph G'.

Proof: Since the graph H is bipartite, it is easy to see that 3-colouring of arbitrary graph (G on Fig. 2) can be extended to 3-colouring of graph G'. On the other hand, since G is subgraph of G', G is 3-colourable if G' is. **Q.E.D.**

Now we shall prove

Lemma 3 *The problem* $\Pi(3,4)$ *is NP-complete.*

Proof: We will reduce the problem of 3-colourability of graphs with vertex degree at most 4 (which is known to be NP-complete [6]) to the problem $\Pi(3,4)$.

Let G be arbitrary graph with maximal degree $\Delta \leq 4$. By Lemma 1 there is a graph G_1 (which has at most as many vertices as G) and G_1 is 3-colourable exactly when G is 3-colourable. If G_1 is empty, then we know that G is 3-colourable.

Now consider the case when G_1 is nonempty. By construction, G_1 is a graph with vertex degrees 3 and 4. Since the sum of all the vertex degrees is twice the number of edges ($\sum_{v \in V} \delta_v = 2|E|$), the number of vertices with degree 3 must be even.

Now couple vertices of degree 3 in G_1 arbitrarily. Connect a copy of the graph H to each couple of vertices of degree 3, as defined in Fig. 2. By Lemma 2, this construction gives a graph G_2 which is 3-colourable exactly when G_1 is 3-colourable. **Q.E.D.**

Remark: Graph H has 8 vertices. Since we added at most $\frac{|V|}{2}8$ new vertices, the resulting graph G_2 has at most a constant factor more vertices than G_1.

Remark: The construction can clearly be done efficiently.

Thus 3-colourability of 4-regular graphs is NP-complete. Now we reduce the problem of 3-colourability of Δ-regular graphs to the same problem on $\Delta + 1$-regular graphs.

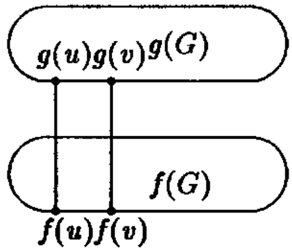

Figure 3: Joining two copies of a Δ-regular graph we get a $\Delta+1$-regular graph

3 3-colourability of Δ-regular graphs is NP-complete

Lemma 4 $\Pi(3,\Delta) \propto \Pi(3,\Delta+1)$

Proof: Let G be arbitrary Δ-regular graph. Now we give a construction of a graph G'.

Take two copies of G, $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$. Denote with $f: G \to G_1$ and $g: G \to G_2$ the corresponding isomorphisms.

Graph $G'=(V',E')$ is defined with: $V' = V_1 \cup V_2$ and $E' = E_1 \cup E_2 \cup \{\{f(v), g(v)\} \mid v \in V\}$ (see Fig. 3).

If G' is 3-colourable, then also G is 3-colourable, since it is isomorphic to a subgraph in G'.

On the other hand, if we have a 3-colouring b of G, it is easy to construct a 3-colouring b' of G', for example with a 'shift' of colours: $b'(f(v)) = b(v)$ and $b'(g(v)) = (b(v) \bmod 3) + 1$.

Since for any Δ size blow up is a constant factor, the assertion of the Lemma follows. **Q.E.D.**

By induction, from Lemma 3 and Lemma 4 we have:

Proposition 1 *The decision problem of 3-colouring restricted to Δ-regular graphs $\Pi(3,\Delta)$ is NP-complete for $\Delta \geq 4$.*

It is known that for graphs of maximal vertex degree 3 the problem is polynomial [6]. Hence we know for all problems $\Pi(3,\Delta)$ whether they are polynomial or NP-complete.

4 4-colourability of Regular Graphs

Here we discuss an attempt to generalise the proposition 1 on the problem of k-colouring. With analogous proof as for the case of 3-colourings we prove a proposition for 4-colouring, while for $k > 4$ the time complexity of the decision problem of k-colouring of Δ-regular graphs remains open for some Δ.

Two of the previous lemmas are easily generalised:

Lemma 5 *Let G' be any subgraph of G obtained by the following process: if there is a vertex of degree less than k, delete it. Graph G is k-colourable if and only if graph G' is k-colourable.*

Proof: Assume we coloured the graph G' with k-colours. It is easy to see that there is algorithm, which properly extends the proper colouring of G' to a proper colouring of G. (Take, for example, vertices of G in opposite order as they were deleted from G. When a vertex was deleted, it had less than k neighbours, therefore there is at least one free colour for it.) **Q.E.D.**

Lemma 6 *For any graph G with vertex degrees k and $k+1$ there is a $k+1$-regular graph G', such that:*

G is k-colourable iff G' is k-colourable

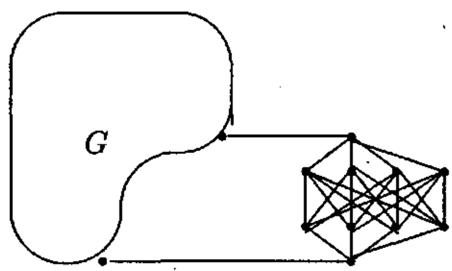

Figure 4: G is 4-colourable iff G' is 4-colourable

Proof: If there are at least two vertices of degree k in G, then we add a copy of graph H. For given k the graph H is defined as follows. Take a complete bipartite graph $K_{k,k}$. Add two vertices and connect one vertex with all the vertices of one independent set of the $K_{k,k}$ and the other vertex with the second independent set of the $K_{k,k}$ (for the case $k = 4$ see Fig. 4). In this way we reduce the number of vertices of degree k by two.

If there is only one vertex of degree k in G, then we construct a new graph as follows: Take two copies of G, connect the two vertices of degree k with an edge. The resulting graph is obviously $k + 1$-regular and it is easy to see that it is k-colourable exactly when G is k-colourable. **Q.E.D.**

For a proof of a generalization of the proposition 1 we need a lemma of the following type: decision problem of k-colouring on arbitrary graph can be reduced to the same problem on a graph of maximal vertex degree $k + 1$.

In the proof of the proposition for 3-colouring we used the result of Garey, Johnson and Stockmayer. Here we give the idea of a proof for $k = 4$. We were not able to generalise the idea for $k > 4$.

Lemma 7 *The decision problem of 4-colouring of graphs of vertex degree ≤ 5 is NP-complete.*

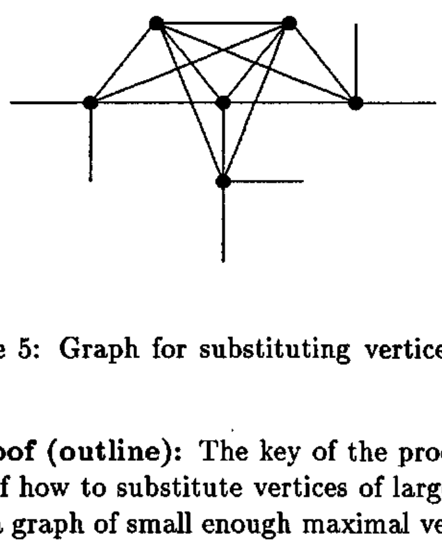

Figure 5: Graph for substituting vertices of degree 6

Proof (outline): The key of the proof is the idea of how to substitute vertices of large degree with a graph of small enough maximal vertex degree and with property that any 4-colouring of the resulting graph $G\prime$ defines a 4-colouring of the original graph G. Such graphs are given in Figures 5,6 and 7. The graphs in Fig. 5 and Fig. 6 are used for substituting vertices of degrees 6 and 7, respectively. For vertices of larger degrees, a longer chain is used, as indicated on Fig. 7. The graphs given have the property, that in any proper 4-colouring all the vertices with 'free edges' have to be coloured with the same colour. (This colour

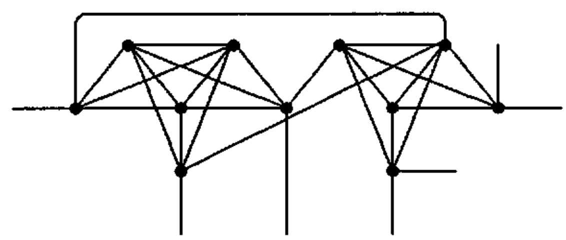

Figure 6: Graph for substituting vertices of degree 7

can be then assigned to the substituted vertex in the original graph. The other vertices of G can then be assigned the same colours as they had in the 4-colouring of $G\prime$.) We omit the details. **Q.E.D.**

With a straightforward generalization of the proof of Lemma 4 we have also:

Lemma 8 $\Pi(k, \Delta) \propto \Pi(k, \Delta + 1)$

Therefore:

Proposition 2 *The decision problem of 4-colouring of Δ-regular graphs $\Pi(4, \Delta)$ is NP-complete for any $\Delta \geq 5$.*

Again, because of the theorem of Brooks [3], the problem $\Pi(4, \Delta)$ has polynomial time complexity for $\Delta \leq 4$. Thus for all the problems $\Pi(4, \Delta)$ we know whether they are polynomial or NP-complete. Let us conclude with a couple of conjectures. Since we were unable to generalise the Lemma 7 we state

Conjecture 1 *The decision problem of k-colouring of graphs with vertex degree $\leq k + 1$ is NP-complete.*

If the first conjecture was true, then we would have a nice classification of time complexity for all the problems $\Pi(k, \Delta)$.

Conjecture 2 *For any $k > 2$, $\Delta > 2$ the decision problem of k-colourability of Δ-regular graphs $\Pi(k, \Delta)$ is NP-complete if $\Delta > k$ and is polynomial otherwise.*

Let us conclude with a simple consequence of the proposition. Assume we have an algorithm

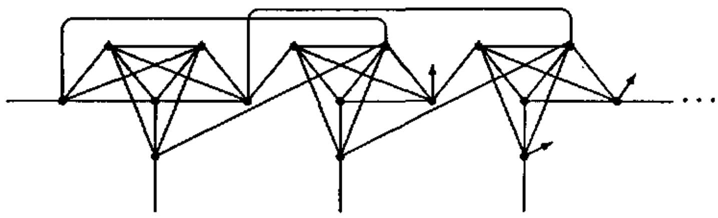

Figure 7: Graph for substituting vertices of degree > 5

$\mathcal{A}$ for 3-colouring and we want to characterise graphs, for which the algorithm does not provide the correct solution in polynomial time. If P$\neq$NP then for any algorithm $\mathcal{A}$ for each $\Delta \geq 4$ there exists an infinite family $F(\mathcal{A}, \Delta)$ of Δ-regular graphs such that the algorithm $\mathcal{A}$ has superpolynomial complexity on each family $F(\mathcal{A}, \Delta)$. If this were not the case for some Δ then $\mathcal{A}$ would be a polynomial algorithm for 3-colouring of Δ-regular graphs, which would imply P=NP!

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